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Carleman estimates and observability inequalities for parabolic equations with interior degeneracy

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Abstract. We consider a parabolic problem with degeneracy in the interior of the spatial domain, and we focus on Carleman estimates for the associated adjoint problem. The novelty of interior degeneracy does not let us adapt previous Carleman estimates to our situation. As an application, observability inequalities are established.

Keywords. Degenerate equation, interior degeneracy, Carleman estimates, observability inequalities.

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1 Introduction

In this paper we focus on two subjects that in the last years have been object of a large number of papers, i.e. *degenerate* problems and *Carleman estimates*. Indeed, as pointed out by several authors, many problems coming from physics (boundary layer models in [8], models of Kolmogorov type in [6], models of Grushin type in [5], etc.), biology (Wright–Fisher models in [30] and Fleming–Viot models in [18]), and economics (Black–Merton–Scholes equations in [15]) are described by degenerate parabolic equations.

On the other hand, the fields of applications of Carleman estimates are so wide that it is not surprising that also several papers are concerned with such a topic. A first example is their application to *global null controllability* (see [1–4, 10, 11, 13, 17, 19, 20, 24–27, 32] and the references therein): for all $T > 0$ and for all initial data $u_0 \in L^2((0,T)\times(0,1))$ there is a suitable control $h \in L^2((0,T)\times(0,1))$ such that the solution of

$$\begin{cases} u_t - (au_x)_x = h(t,x)\chi_\omega(x), & (t,x) \in (0,T)\times(0,1), \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

with some boundary conditions, satisfies $u(T,x) = 0$ for all $x \in [0,1]$. Here χ_ω

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denotes, as usual, the characteristic function of the set ω, i.e. $\chi(x) = 1$ if $x \in \omega$ and $\chi(x) = 0$ if $x \notin \omega$.

Moreover, Carleman estimates may be a fundamental tool in inverse problems, in parabolic, hyperbolic and fractional settings, e.g. see [14, 23, 28, 29, 33, 34] and their references.

However, all the previous papers deal with problems like (1.1) which are non degenerate or admit that the function a degenerates at the boundary of the domain, for example

$$a(x) = x^k(1-x)^\alpha, \quad x \in [0,1],$$

where k and α are positive constants.

To our best knowledge, [31] is the first paper treating a problem with a degeneracy which may occur in the interior of the spatial domain. In particular, Stahel considers a parabolic problem in a bounded smooth domain Ω of $\mathbb{R}^N$ with Dirichlet, Neumann or mixed boundary conditions, associated to an $N \times N$ matrix a, which is positive definite and symmetric, but whose smallest eigenvalue might converge to 0 as the space variable approaches a singular set contained in the closure of the spatial domain. In this case, he proves that the corresponding abstract Cauchy problem has a solution, provided that $\underline{a}^{-1} \in L^q(\Omega, \mathbb{R})$ for some $q > 1$, where

$$\underline{a}(x) := \min\{a(x)\xi \cdot \xi \mid \|\xi\| = 1\}.$$

Moreover, while in [31] only the existence of a solution for the parabolic problem is considered, in [22] the authors analyze in detail the degenerate operator

$$\mathcal{A}u := (au_x)_x$$

in the space $L^2(0,1)$, with or without weight, proving that in some cases it is nonpositive and selfadjoint, hence it generates a cosine family and, as a consequence, an analytic semigroup. In [22] the well-posedness of (1.1) with Dirichlet boundary conditions is also treated, but nothing is said about controllability properties.

This paper is then concerned with several inequalities (Carleman estimates, observability inequalities, Hardy–Poincaré inequalities) related to the parabolic equation with interior degeneracy

$$\begin{cases} u_t - (au_x)_x = h(t,x)\chi_\omega(x), & (t,x) \in (0,T) \times (0,1), \\ u(t,0) = u(t,1) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.2}$$

where $(t,x) \in Q_T := (0,T) \times (0,1)$, $u_0 \in L^2(0,1)$, a degenerates at $x_0 \in (0,1)$ and the control $h \in L^2(Q_T)$ acts on a nonempty subdomain ω of $(0,1)$ such that

$$x_0 \in \omega.$$

We shall admit two types of degeneracy for a, namely weak and strong degeneracy. In particular, we make the following assumptions (see also [9, 12] for related ones):

Hypothesis 1.1 (Weakly degenerate case (WD)). There exists an $x_0 \in (0,1)$ such that $a(x_0) = 0$, $a > 0$ on $[0,1] \setminus \{x_0\}$, $a \in C^1([0,1] \setminus \{x_0\})$ and there exists a constant $K \in (0,1)$ such that $(x - x_0)a' \le Ka$ in $[0,1] \setminus \{x_0\}$.

Hypothesis 1.2 (Strongly degenerate case (SD)). There exists an $x_0 \in (0,1)$ such that $a(x_0) = 0$, $a > 0$ on $[0,1] \setminus \{x_0\}$, $a \in C^1([0,1] \setminus \{x_0\}) \cap W^{1,\infty}(0,1)$ and there exists a constant $K \in [1,2)$ such that $(x - x_0)a' \le Ka$ in $[0,1] \setminus \{x_0\}$.

Typical examples for weak and strong degeneracies are $a(x) = |x - x_0|^\alpha$ for $0 < \alpha < 1$ and $a(x) = |x - x_0|^\alpha$ for $1 \le \alpha < 2$, respectively.

The starting point of the paper is actually the analysis of the adjoint problem to (1.2)

$$\begin{cases} v_t + (a(x)v_x)_x = h, & (t,x) \in (0,T) \times (0,1), \\ v(t,1) = v(t,0) = 0, & t \in (0,T). \end{cases} \tag{1.3}$$

In particular, for any (sufficiently regular) solution v of such a system we derive the new fundamental Carleman estimate having the form

$$\begin{aligned} &\int_0^T \int_0^1 \left(s\Theta a (v_x)^2 + s^3 \Theta^3 \frac{(x - x_0)^2}{a} v^2 \right) e^{2s\varphi} \, dx dt \\ &\quad \le C \left(\int_0^T \int_0^1 h^2 e^{2s\varphi} \, dx dt + s \int_0^T \left[a\Theta e^{2s\varphi} (x - x_0)(v_x)^2 \, dt \right]_{x=0}^{x=1} \right), \end{aligned} \tag{1.4}$$

for all $s \ge s_0$, where s_0 is a suitable constant. Here $\Theta(t) := [t(T-t)]^{-4}$, and $\varphi(t,x) := \Theta(t)\psi(x)$, with $\psi(x) < 0$ given explicitly in terms of a, see (3.2). Of course, for the Carleman inequality, the location of x_0 with respect to the control set ω is irrelevant, since ω plays no role at all.

For the proof of the previous Carleman estimate a fundamental role is played by the second basic result of this paper, that is, a general Hardy–Poincaré type inequality proved in Proposition 2.6, of independent interest, that we establish for all functions w which are only *locally absolutely continuous* in $[0,1] \setminus \{x_0\}$ and such that

$$w(0) = w(1) = 0 \quad \text{and} \quad \int_0^1 p(x) |w'(x)|^2 \, dx < +\infty,$$

and which reads

$$\int_0^1 \frac{p(x)}{(x - x_0)^2} w^2(x) \, dx \le C \int_0^1 p(x) |w'(x)|^2 \, dx.$$

Here p is *any* continuous function in $[0,1]$, with $p>0$ on $[0,1]\setminus\{x_0\}$, $p(x_0)=0$ and there exists $q\in(1,2)$ such that the function

$$x\mapsto\frac{p(x)}{|x-x_0|^q}$$

is nonincreasing on the left of $x=x_0$ and nondecreasing on the right of $x=x_0$.

Applying estimate (1.4) to any solution v of the adjoint problem (1.3) with $h=0$, we shall obtain the observability inequality

$$\int_0^1 v^2(0,x)\,dx\le C\int_0^T\int_\omega v^2(t,x)\,dx\,dt,$$

where now we consider the fact that $x_0\in\omega$.

Such a result is then extended to the complete linear problem

$$\begin{cases} u_t-(a(x)u_x)_x+c(t,x)u=h(t,x)\chi_\omega(x), & (t,x)\in(0,T)\times(0,1),\\ u(t,1)=u(t,0)=0, & t\in(0,T),\\ u(0,x)=u_0(x), & x\in(0,1), \end{cases}\tag{1.5}$$

where c is a bounded function, previously proving a Carleman estimate associated to this problem, see Corollary 5.1 and Proposition 5.2.

Finally, observe that on a we require that there exists a $K\in(0,2)$ such that $(x-x_0)a'\le Ka$ in $[0,1]$, and $K\ge 2$ is excluded. This technical assumption, which is essential in all our results, is the same made, for example, in [3], where the degeneracy occurs at the boundary of the domain and the problem fails to be null controllable on the whole interval $[0,1]$. But since the null controllability for the parabolic problem and the observability inequality for the adjoint problem are equivalent ([24]), it is not surprising that we require $K<2$.

The paper is organized as follows. In Section 2 we give the precise setting for the weak and the strong degenerate cases and some general tools we shall use several times; in particular a general weighted Hardy–Poincaré inequality is established. In Section 3 we provided the main result of this paper, i.e. a new Carleman estimate for degenerate operators with interior degeneracy. In Section 4 we apply the previous Carleman estimates together with a Caccioppoli type inequality to prove an observability inequality. Finally, in Section 5 we extend the previous results to complete linear problems.

We conclude this introduction with the following

Remark 1.3. At a first glance, one may think that our results can be obtained just by a "translation" of the ones obtained in [3], but this is not the case. Indeed, in [3] the degeneracy point was the origin, where the authors put suitable homogeneous

boundary conditions (Dirichlet for the (WD) case and weighted Neumann in the (SD) case) which coincide exactly with the ones obtained by the characterizations of the domains of the operators given in Propositions 2.2 and 2.3 below. In this way they can control *a priori* the possible uncontrolled behavior of the solution at the degeneracy point, while here we do not impose any condition on the solution in the interior point x_0.

2 Preliminary results

In order to study the well-posedness of problem (1.2), we introduce the operator

$$\mathcal{A}u := (au_x)_x$$

and we consider two different classes of weighted Hilbert spaces, which are suitable to study two different situations, namely the *weakly degenerate* (WD) and the *strongly degenerate* (SD) cases:

Case (WD): we have

$$H_a^1(0,1) := \big\{u \text{ is absolutely continuous in } [0,1], \\ \sqrt{a}u' \in L^2(0,1) \text{ and } u(0) = u(1) = 0\big\},$$

and

$$H_a^2(0,1) := \big\{u \in H_a^1(0,1) \mid au' \in H^1(0,1)\big\}.$$

Case (SD): we have

$$H_a^1(0,1) := \big\{u \in L^2(0,1) \mid u \text{ locally absolutely continuous in } [0,x_0) \cup (x_0,1], \\ \sqrt{a}u' \in L^2(0,1) \text{ and } u(0) = u(1) = 0\big\}$$

and

$$H_a^2(0,1) := \big\{u \in H_a^1(0,1) \mid au' \in H^1(0,1)\big\}.$$

In both cases we consider the norms

$$\|u\|^2_{H_a^1(0,1)} := \|u\|^2_{L^2(0,1)} + \|\sqrt{a}u'\|^2_{L^2(0,1)},$$

and

$$\|u\|^2_{H_a^2(0,1)} := \|u\|^2_{H_a^1(0,1)} + \|(au')'\|^2_{L^2(0,1)}$$

and we set

$$D(\mathcal{A}) = H_a^2(0,1).$$

The function a playing a crucial role, it is not surprising that the following lemma is crucial as well.

Lemma 2.1. *Assume that Hypothesis* 1.1 *or* 1.2 *is satisfied.*

(i) *For all* $\gamma \geq K$ *the map*

$$x \mapsto \frac{|x - x_0|^\gamma}{a}$$

is nonincreasing on the left of $x = x_0$ *and nondecreasing on the right of* $x = x_0$*, so that*

$$\lim_{x \to x_0} \frac{|x - x_0|^\gamma}{a} = 0 \quad \textit{for all } \gamma > K.$$

(ii) *If* $K < 1$*, then* $\frac{1}{a} \in L^1(0, 1)$.

(iii) *If* $K \in [1, 2)$*, then* $\frac{1}{\sqrt{a}} \in L^1(0, 1)$ *and* $\frac{1}{a} \notin L^1(0, 1)$.

Proof. The first point is an easy consequence of the assumption. Now, we prove the second point: by the first part, it follows that

$$\frac{|x - x_0|^K}{a(x)} \leq \max\left\{\frac{x_0^K}{a(0)}, \frac{(1 - x_0)^K}{a(1)}\right\}.$$

Thus

$$\frac{1}{a(x)} \leq \max\left\{\frac{x_0^K}{a(0)}, \frac{(1 - x_0)^K}{a(1)}\right\} \frac{1}{|x - x_0|^K}.$$

Since $K < 1$, the right-hand side of the last inequality is integrable, and then $\frac{1}{a} \in L^1(0, 1)$. Analogously, one obtains the first part of the third point.

On the contrary, the fact that $a \in C^1([0, 1])$, and $\frac{1}{\sqrt{a}} \in L^1(0, 1)$ implies that $\frac{1}{a} \notin L^1(0, 1)$. Indeed, the assumptions on a imply that

$$a(x) = \int_{x_0}^{x} a'(s)\, ds \leq C\, |x - x_0|$$

for a positive constant C. Thus for all $x \neq x_0$, $\frac{1}{a(x)} \geq C \frac{1}{|x - x_0|} \notin L^1(0, 1)$. □

We immediately start using the lemma above, giving the following characterizations for the (SD) case which are already given in [22, Propositions 2.3 and 2.4], but whose proofs we repeat here to make precise some calculations.

Proposition 2.2. *Let*

$$\begin{aligned} X := \{u \in L^2(0, 1) \mid\ & u \textit{ locally absolutely continuous in } [0, 1] \setminus \{x_0\}, \\ & \sqrt{a}u' \in L^2(0, 1), au \in H_0^1(0, 1) \textit{ and} \\ & (au)(x_0) = u(0) = u(1) = 0\}. \end{aligned}$$

Then, under Hypothesis 1.2 *we have*

$$H_a^1(0, 1) = X.$$

Proof. Obviously, $X \subseteq H_a^1$. Now we take $u \in H_a^1$, and we prove that $u \in X$. First, observe that $(au)(0) = (au)(1) = 0$. Moreover, since $a \in W^{1,\infty}(0,1)$, it follows that $(au)' = a'u + au' \in L^2(0,1)$. Thus, for $x < x_0$, one has

$$(au)(x) = \int_0^x (au)'(t)\,dt$$

This implies that there exists

$$\lim_{x \to x_0^-} (au)(x) = (au)(x_0) = \int_0^{x_0} (au)'(t)\,dt = L \in \mathbb{R}.$$

If $L \neq 0$, then there exists a constant $C > 0$ such that

$$|(au)(x)| \geq C$$

for all x in a neighborhood of x_0, $x \neq x_0$. Thus, setting

$$C_1 := \frac{C^2}{\max_{[0,1]} a(x)} > 0,$$

it follows that

$$|u^2(x)| \geq \frac{C^2}{a^2(x)} \geq \frac{C_1}{a(x)},$$

for all x in a left neighborhood of x_0, $x \neq x_0$. But, since the operator is strongly degenerate, $\frac{1}{a} \notin L^1(0,1)$, thus $u \notin L^2(0,1)$. Hence $L = 0$. Analogously, starting from

$$(au)(x) = -\int_x^1 (au)'(t)\,dt \quad \text{for } x > x_0,$$

one can prove that

$$\lim_{x \to x_0^+} (au)(x) = (au)(x_0) = 0$$

and thus $(au)(x_0) = 0$. From this it also easily follows that $(au)'$ is the distributional derivative of au, and so $au \in H_0^1(0,1)$, i.e. $u \in X$. □

Using the previous result, one can prove the following additional characterization.

Proposition 2.3. *Let*

$$\begin{aligned} D := \{u \in L^2(0,1) \mid\ & u \text{ locally absolutely continuous in } [0,1] \setminus \{x_0\}, \\ & au \in H_0^1(0,1), au' \in H^1(0,1), au \text{ is continuous at } x_0 \\ & \text{and } (au)(x_0) = (au')(x_0) = u(0) = u(1) = 0\}. \end{aligned}$$

Then, under Hypothesis 1.2 *we have*

$$H_a^2(0,1) = D(\mathcal{A}) = D.$$

Proof. For $D \subseteq D(\mathcal{A})$: Let $u \in D$. It is sufficient to prove that $\sqrt{a}u' \in L^2(0,1)$. Since $au' \in H^1(0,1)$ and $u(1) = 0$ (recall that $a > 0$ in $[0,1] \setminus \{x_0\}$), for any $x \in (x_0, 1]$ we have

$$\int_x^1 [(au')'u](s)\,ds = [au'u]_x^1 - \int_x^1 (a(u')^2)(s)\,ds$$
$$= -(au'u)(x) - \int_x^1 (a(u')^2)(s)\,ds.$$

Thus

$$(au'u)(x) = -\int_x^1 [(au')'u](s)\,ds - \int_x^1 (a(u')^2)(s)\,ds.$$

Since $u \in D$, $(au')'u \in L^1(0,1)$. Hence, there exists

$$\lim_{x \to x_0^+} (au'u)(x) = L \in [-\infty, +\infty),$$

since no integrability is known about $a(u')^2$ and such a limit could be $-\infty$. If $L \neq 0$, there exists a constant $C > 0$ such that

$$|(au'u)(x)| \geq C$$

for all x in a right neighborhood of x_0, $x \neq x_0$. Thus, by [22, Lemma 2.5], there exists a constant $C_1 > 0$ such that

$$|u(x)| \geq \frac{C}{|(au')(x)|} \geq \frac{C_1}{\sqrt{(x - x_0)}}$$

for all x in a right neighborhood of x_0, $x \neq x_0$. This implies that $u \notin L^2(0,1)$. Hence $L = 0$ and

$$\int_{x_0}^1 [(au')'u](s)\,ds = -\int_{x_0}^1 (a(u')^2)(s)\,ds. \tag{2.1}$$

If $x \in [0, x_0)$, proceeding as before and using the condition $u(0) = 0$, it follows that

$$\int_0^{x_0} [(au')'u](s)\,ds = -\int_0^{x_0} (a(u')^2)(s)\,ds. \tag{2.2}$$

By (2.1) and (2.2), it follows that

$$\int_0^1 [(au')'u](s)\,ds = -\int_0^1 (a(u')^2)(s)\,ds.$$

Since $(au')'u \in L^1(0,1)$, we get $\sqrt{a}u' \in L^2(0,1)$. Hence, $D \subseteq D(\mathcal{A})$.

For $D(\mathcal{A}) \subseteq D$: Let $u \in D(\mathcal{A})$. From Proposition 2.2 above, we know that $au \in H_0^1(0,1)$ and $(au)(x_0) = 0$. Thus, it is sufficient to prove that $(au')(x_0) = 0$. Toward this end, observe that, since $au' \in H^1(0,1)$, there exists an $L \in \mathbb{R}$ such that

$$\lim_{x \to x_0} (au')(x) = (au')(x_0) = L.$$

If $L \neq 0$, there exists a constant $C > 0$ such that

$$|(au')(x)| \geq C,$$

for all x in a neighborhood of x_0. Thus

$$|(a(u')^2)(x)| \geq \frac{C^2}{a(x)},$$

for all x in a neighborhood of x_0, $x \neq x_0$. By Lemma 2.1, this implies that $\sqrt{a}u' \notin L^2(0,1)$. Hence $L = 0$, that is, $(au')(x_0) = 0$. □

Now, let us go back to problem (1.2), recalling the following

Definition 2.4. If $u_0 \in L^2(0,1)$ and $h \in L^2(Q_T)$, then a function u is said to be a (weak) solution of (1.2) if

$$u \in C([0,T]; L^2(0,1)) \cap L^2(0,T; H_a^1(0,1))$$

and

$$\int_0^1 u(T,x)\varphi(T,x)\,dx - \int_0^1 u_0(x)\varphi(0,x)\,dx - \int_{Q_T} u\varphi_t\,dxdt$$
$$= -\int_{Q_T} au_x\varphi_x\,dxdt + \int_{Q_T} h\varphi\chi_\omega\,dxdt$$

for all $\varphi \in H^1(0,T; L^2(0,1)) \cap L^2(0,T; H_a^1(0,1))$.

As proved in [22] (see Theorems 2.2, 2.7 and 4.1), problem (1.2) is well posed in the sense of the following theorem:

Theorem 2.5. *For all $h \in L^2(Q_T)$ and $u_0 \in L^2(0,1)$, there exists a unique weak solution $u \in C([0,T]; L^2(0,1)) \cap L^2(0,T; H_a^1(0,1))$ of (1.2) and there exists a universal positive constant C such that*

$$\sup_{t\in[0,T]} \|u(t)\|^2_{L^2(0,1)} + \int_0^T \|u(t)\|^2_{H_a^1(0,1)}\,dt \leq C\left(\|u_0\|^2_{L^2(0,1)} + \|h\|^2_{L^2(Q_T)}\right). \tag{2.3}$$

Moreover, if $u_0 \in H^1_a(0,1)$*, then*

$$u \in H^1(0,T;L^2(0,1)) \cap C([0,T];H^1_a(0,1)) \cap L^2(0,T;H^2_a(0,1)), \tag{2.4}$$

and there exists a universal positive constant C *such that*

$$\begin{aligned} &\sup_{t\in[0,T]} \big(\|u(t)\|^2_{H^1_a(0,1)}\big) + \int_0^T \big(\|u_t\|^2_{L^2(0,1)} + \|(au_x)_x\|^2_{L^2(0,1)}\big)\,dt \\ &\qquad \le C\big(\|u_0\|^2_{H^1_a(0,1)} + \|h\|^2_{L^2(Q_T)}\big). \end{aligned} \tag{2.5}$$

Moreover, $\mathcal{A}$ *generates an analytic semigroup on* $L^2(0,1)$.

So far we have introduced all the tools which will let us deal with solutions of problem (1.2), also with additional regularity. Now, we conclude this section with an essential tool for proving Carleman estimates and observability inequalities, that is, a new weighted Hardy–Poincaré inequality for functions which may *not be* globally absolutely continuous in the domain, but whose irregularity point is "controlled" by the fact that the weight degenerates exactly there.

Proposition 2.6 (Hardy–Poincaré inequality). *Assume that* $p \in C([0,1])$*,* $p > 0$ *on* $[0,1] \setminus \{x_0\}$*,* $p(x_0) = 0$ *and there exists a* $q \in (1,2)$ *such that the function*

$$x \mapsto \frac{p(x)}{|x - x_0|^q}$$

is nonincreasing on the left of $x = x_0$ *and nondecreasing on the right of* $x = x_0$*. Then, there exists a constant* $C_{\mathrm{HP}} > 0$ *such that for any function* w*, locally absolutely continuous on* $[0,x_0) \cup (x_0,1]$ *and satisfying*

$$w(0) = w(1) = 0 \quad \textit{and} \quad \int_0^1 p(x)|w'(x)|^2\,dx < +\infty,$$

the following inequality holds:

$$\int_0^1 \frac{p(x)}{(x-x_0)^2} w^2(x)\,dx \le C_{\mathrm{HP}} \int_0^1 p(x)|w'(x)|^2\,dx. \tag{2.6}$$

Proof. Fix any $\beta \in (1,q)$ and $\varepsilon > 0$ small. Since $w(1) = 0$, applying Hölder's inequality and Fubini's theorem, we have

$$\begin{aligned} &\int_{x_0+\varepsilon}^1 \frac{p(x)}{(x-x_0)^2} w^2(x)\,dx \\ &= \int_{x_0+\varepsilon}^1 \frac{p(x)}{(x-x_0)^2} \left(\int_x^1 ((y-x_0)^{\beta/2} w'(y))(y-x_0)^{-\beta/2}\,dy \right)^2 dx \\ &\le \int_{x_0+\varepsilon}^1 \frac{p(x)}{(x-x_0)^2} \left(\int_x^1 (y-x_0)^{\beta} |w'(y)|^2\,dy \int_x^1 (y-x_0)^{-\beta}\,dy \right) dx \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\beta-1}\int_{x_0+\varepsilon}^{1}\frac{p(x)}{(x-x_0)^{1+\beta}}\left(\int_x^1 (y-x_0)^{\beta}|w'(y)|^2\,dy\right)dx\\
&= \frac{1}{\beta-1}\int_{x_0+\varepsilon}^{1}(y-x_0)^{\beta}|w'(y)|^2\left(\int_{x_0+\varepsilon}^{y}\frac{p(x)}{(x-x_0)^{1+\beta}}\,dx\right)dy\\
&= \frac{1}{\beta-1}\int_{x_0+\varepsilon}^{1}(y-x_0)^{\beta}|w'(y)|^2\left(\int_{x_0+\varepsilon}^{y}\frac{p(x)}{(x-x_0)^{q}}(x-x_0)^{q-1-\beta}\,dx\right)dy.
\end{aligned}
$$

Now, thanks to our hypothesis, we find

$$
\frac{p(x)}{(x-x_0)^q}\le\frac{p(y)}{(y-x_0)^q},\quad \text{for all } x,y\in[x_0+\varepsilon,1],\ x<y.
$$

Thus

$$
\begin{aligned}
&\int_{x_0+\varepsilon}^{1}\frac{p(x)}{(x-x_0)^2}w^2(x)\,dx\\
&\le \frac{1}{\beta-1}\int_{x_0+\varepsilon}^{1}\frac{p(y)}{(y-x_0)^q}(y-x_0)^{\beta}|w'(y)|^2\left(\int_{x_0+\varepsilon}^{y}(x-x_0)^{q-1-\beta}\,dx\right)dy\\
&= \frac{1}{(\beta-1)(q-\beta)}\int_{x_0+\varepsilon}^{1}\frac{p(y)}{(y-x_0)^q}(y-x_0)^{q-\beta}(y-x_0)^{\beta}|w'(y)|^2\,dy\\
&= \frac{1}{(\beta-1)(q-\beta)}\int_{x_0+\varepsilon}^{1}p(y)|w'(y)|^2\,dy. \qquad (2.7)
\end{aligned}
$$

Now, proceeding as before, and using the fact that $w(0)=0$, one has

$$
\begin{aligned}
&\int_0^{x_0-\varepsilon}\frac{p(x)}{(x_0-x)^2}w^2(x)\,dx\\
&\le \int_0^{x_0-\varepsilon}\frac{p(x)}{(x_0-x)^2}\left(\int_0^x (x_0-y)^{\beta}|w'(y)|^2\,dy\int_0^x (x_0-y)^{-\beta}\,dy\right)dx\\
&\le \frac{1}{\beta-1}\int_0^{x_0-\varepsilon}\frac{p(x)}{(x_0-x)^{1+\beta}}\left(\int_0^x (x_0-y)^{\beta}|w'(y)|^2\,dy\right)dx\\
&= \frac{1}{\beta-1}\int_0^{x_0-\varepsilon}(x_0-y)^{\beta}|w'(y)|^2\left(\int_y^{x_0-\varepsilon}\frac{p(x)}{(x_0-x)^q}(x_0-x)^{q-1-\beta}\,dx\right)dy.
\end{aligned}
$$

By assumption

$$
\frac{p(x)}{(x_0-x)^q}\le\frac{p(y)}{(x_0-y)^q},\quad \text{for all } x,y\in[0,x_0-\varepsilon],\ y<x.
$$

Hence,

$$\begin{aligned}
&\int_0^{x_0-\varepsilon} \frac{p(x)}{(x_0-x)^2} w^2(x)\,dx \\
&\le \frac{1}{\beta-1} \int_0^{x_0-\varepsilon} \frac{p(y)}{(x_0-y)^q} (x_0-y)^\beta |w'(y)|^2 \left(\int_y^{x_0-\varepsilon} (x_0-x)^{q-1-\beta}\,dx \right) dy \\
&= \frac{1}{(\beta-1)(q-\beta)} \int_0^{x_0-\varepsilon} p(y)|w'(y)|^2\,dy. \qquad (2.8)
\end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ and combining (2.7) and (2.8), the conclusion follows. □

3 Carleman estimate for degenerate parabolic problems

In this section we prove a crucial estimate of Carleman type that will be useful to prove an observability inequality for the adjoint problem of (1.2) in both the weakly and the strongly degenerate cases. Thus, let us consider the problem

$$\begin{cases} v_t + (a v_x)_x = h, & (t,x) \in (0,T) \times (0,1), \\ v(t,0) = v(t,1) = 0, & t \in (0,T), \end{cases} \qquad (3.1)$$

where a satisfies the following assumption:

Hypothesis 3.1. The function a satisfies Hypothesis 1.1 or Hypothesis 1.2 and there exists a constant $\vartheta \in (0, K]$ such that the function

$$x \mapsto \frac{a(x)}{|x-x_0|^\vartheta}$$

is nonincreasing on the left of $x = x_0$ and nondecreasing on the right of $x = x_0$. Here K is the constant appearing in Hypothesis 1.1 or 1.2, respectively.

Remark 3.2. Observe that if $x_0 = 0$, Hypothesis 3.1 is the same introduced in [3] only for the strongly degenerate case. On the other hand, here we have to require this additional assumption also in the weakly degenerate case. This is due to the fact that in this case we do not know if $u(x_0) = 0$ for all $u \in H_a^1(0,1)$, as it happens when $x_0 = 0$ and one imposes homogeneous Dirichlet boundary conditions, as in [3]; indeed, the choice of homogeneous Dirichlet boundary conditions in [3] helps in controlling the function at the degeneracy point, while here we do not require the corresponding condition $u(x_0) = 0$, so that some other condition is needed. However, in both cases, Hypothesis 3.1 is satisfied if $a(x) = |x-x_0|^K$, with $K \in (0,2)$.

Now, let us introduce the function $\varphi(t,x) := \Theta(t)\psi(x)$, where

$$\Theta(t) := \frac{1}{[t(T-t)]^4} \quad \text{and} \quad \psi(x) := c_1\left[\int_{x_0}^{x} \frac{y-x_0}{a(y)}\,dy - c_2\right], \tag{3.2}$$

with

$$c_2 > \max\left\{\frac{(1-x_0)^2}{a(1)(2-K)}, \frac{x_0^2}{a(0)(2-K)}\right\} \quad \text{and} \quad c_1 > 0.$$

A more precise restriction on c_1 will be needed later. Observe that $\Theta(t) \to +\infty$ as $t \to 0^+, T^-$, and by Lemma 2.1 we have that, if $x > x_0$,

$$\begin{aligned}
\psi(x) &\le c_1\left[\int_{x_0}^{x} \frac{(y-x_0)^K}{a(y)} \frac{1}{(y-x_0)^{K-1}}\,dy - c_2\right] \\
&\le c_1\left[\frac{(x-x_0)^K}{a(x)} \frac{(x-x_0)^{2-K}}{2-K} - c_2\right] \\
&\le c_1\left[\frac{(1-x_0)^K}{a(1)} \frac{(1-x_0)^{2-K}}{2-K} - c_2\right] \\
&= c_1\left[\frac{(1-x_0)^2}{(2-K)a(1)} - c_2\right] < 0.
\end{aligned} \tag{3.3}$$

In the same way one can treat the case $x \in [0, x_0)$, so that

$$\psi(x) < 0 \quad \text{for every } x \in [0,1].$$

Moreover, it is also easy to see that $\psi \ge -c_1c_2$.

Our main result is thus the following.

Theorem 3.3. *Assume Hypothesis* 3.1 *and let* $T > 0$. *Then, there exist two positive constants* C *and* s_0 *such that every solution* v *of* (3.1) *in*

$$\mathcal{V} := L^2(0,T; H_a^2(0,1)) \cap H^1\big(0,T; H_a^1(0,1)\big) \tag{3.4}$$

satisfies, for all $s \ge s_0$,

$$\begin{aligned}
&\int_0^T \int_0^1 \left(s\Theta a(v_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} v^2\right) e^{2s\varphi}\,dxdt \\
&\quad \le C\left(\int_0^T \int_0^1 h^2 e^{2s\varphi}\,dxdt + sc_1 \int_0^T \left[a\Theta e^{2s\varphi}(x-x_0)(v_x)^2\,dt\right]_{x=0}^{x=1}\right),
\end{aligned}$$

where c_1 *is the constant introduced in* (3.2).

3.1 Proof of Theorem 3.3

For $s > 0$, define the function

$$w(t,x) := e^{s\varphi(t,x)}v(t,x),$$

where v is any solution of (3.1) in $\mathcal{V}$; observe that, since $v \in \mathcal{V}$ and $\psi < 0$, we have $w \in \mathcal{V}$. Of course, w satisfies

$$\begin{cases} (e^{-s\varphi}w)_t + (a(e^{-s\varphi}w)_x)_x = h, & (t,x) \in (0,T)\times(0,1), \\ w(t,0) = w(t,1) = 0, & t \in (0,T), \\ w(T^-,x) = w(0^+,x) = 0, & x \in (0,1). \end{cases} \tag{3.5}$$

The previous problem can be recast as follows. Set

$$Lv := v_t + (av_x)_x$$

and

$$L_s w = e^{s\varphi}L(e^{-s\varphi}w), \quad s > 0.$$

Then (3.5) becomes

$$\begin{cases} L_s w = e^{s\varphi}h, \\ w(t,0) = w(t,1) = 0, & t \in (0,T), \\ w(T^-,x) = w(0^+,x) = 0, & x \in (0,1). \end{cases}$$

Computing $L_s w$, one has

$$L_s w = L_s^+ w + L_s^- w,$$

where

$$L_s^+ w := (aw_x)_x - s\varphi_t w + s^2 a(\varphi_x)^2 w,$$

and

$$L_s^- w := w_t - 2sa\varphi_x w_x - s(a\varphi_x)_x w.$$

Moreover,

$$\begin{aligned} 2\langle L_s^+ w, L_s^- w\rangle &\le 2\langle L_s^+ w, L_s^- w\rangle + \|L_s^+ w\|^2_{L^2(Q_T)} + \|L_s^- w\|^2_{L^2(Q_T)} \\ &= \|L_s w\|^2_{L^2(Q_T)} \\ &= \|he^{s\varphi}\|^2_{L^2(Q_T)}, \end{aligned} \tag{3.6}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $L^2(Q_T)$. As usual, we will separate the scalar product $\langle L_s^+ w, L_s^- w \rangle$ in distributed terms and boundary terms.

Lemma 3.4. *The following identity holds:*

$$\left.\begin{aligned}\langle L_s^+ w, L_s^- w\rangle = {} & \frac{s}{2}\int_0^T\int_0^1 \varphi_{tt} w^2\,dx dt \\ & + s^3\int_0^T\int_0^1 (2a\varphi_{xx} + a'\varphi_x)a(\varphi_x)^2 w^2\,dx dt \\ & - 2s^2\int_0^T\int_0^1 a\varphi_x\varphi_{tx} w^2\,dx dt \\ & + s\int_0^T\int_0^1 (2a^2\varphi_{xx} + aa'\varphi_x)(w_x)^2\,dx dt\end{aligned}\right\} \text{(D.T.)} \tag{3.7}$$

$$\text{(B.T.)}\quad \left\{\begin{aligned} & + \int_0^T [a w_x w_t]_{x=0}^{x=1}\,dt - \frac{s}{2}\int_0^1 [w^2\varphi_t]_{t=0}^{t=T}\,dx \\ & + \frac{s^2}{2}\int_0^1 [a(\varphi_x)^2 w^2]_{t=0}^{t=T}\,dt \\ & + \int_0^T \left[-s\varphi_x(a w_x)^2 + s^2 a\varphi_t\varphi_x w^2 - s^3 a^2(\varphi_x)^3 w^2\right]_{x=0}^{x=1} dt \\ & + \int_0^T [-sa(a\varphi_x)_x w w_x]_{x=0}^{x=1}\,dt - \frac{1}{2}\int_0^1 [a(w_x)^2]_0^T\,dx.\end{aligned}\right.$$

Proof. First, let us note that all integrals which appear in $\langle L_s^+ w, L_s^- w\rangle$ are well defined both in the weakly and the strongly degenerate case by Theorem 2.1, as simple calculations show, recalling that $w = e^{s\varphi} v$. Moreover, we remark that all the following integrations by parts are justified by Proposition 2.3 and the fact that $w \in \mathcal{V}$. Hence

$$\begin{aligned}\int_0^T\int_0^1 L_s^+ w w_t\,dx dt &= \int_0^T\int_0^1 \{(a w_x)_x - s\varphi_t w + s^2 a(\varphi_x)^2 w\} w_t\,dx dt \\ &= \int_0^T [a w_x w_t]_{x=0}^{x=1}\,dt - \int_0^T \frac{1}{2}\frac{d}{dt}\left(\int_0^1 a(w_x)^2\,dx\right) dt \\ &\quad - \frac{s}{2}\int_0^T dt\int_0^1 \varphi_t (w^2)_t\,dx \\ &\quad + \frac{s^2}{2}\int_0^T dt\int_0^1 a(\varphi_x)^2 (w^2)_t\,dx\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T [a w_x w_t]_{x=0}^{x=1}\,dt - \frac{s}{2}\int_0^1 [w^2\varphi_t]_{t=0}^{t=T}\,dx \\
&\quad + \frac{s^2}{2}\int_0^1 [a(\varphi_x)^2 w^2]_{t=0}^{t=T}\,dt \\
&\quad - \int_0^T \frac{1}{2}\frac{d}{dt}\left(\int_0^1 a(w_x)^2\,dx\right)dt \\
&\quad + \frac{s}{2}\int_0^T\int_0^1 \varphi_{tt} w^2\,dx dt \\
&\quad - s^2\int_0^T\int_0^1 a\varphi_x\varphi_{xt} w^2\,dx dt. \qquad (3.8)
\end{aligned}
$$

In addition, we have

$$
\begin{aligned}
&\int_0^T\int_0^1 L_s^+ w(-2sa\varphi_x w_x)\,dx dt \\
&\quad = -2s\int_0^T\int_0^1 \varphi_x\left[\frac{(aw_x)^2}{2}\right]_x dx dt + 2s^2\int_0^T\int_0^1 a\varphi_t\varphi_x\left(\frac{w^2}{2}\right)_x dx dt \\
&\qquad\quad - 2s^3\int_0^T\int_0^1 a^2(\varphi_x)^3 w w_x\,dx dt \\
&\quad = \int_0^T \left[-s\varphi_x(aw_x)^2 + s^2 a\varphi_t\varphi_x w^2 - s^3 a^2(\varphi_x)^3 w^2\right]_{x=0}^{x=1} dt \\
&\qquad\quad + s\int_0^T\int_0^1 \varphi_{xx}(aw_x)^2\,dx dt - s^2\int_0^T\int_0^1 (a\varphi_t\varphi_x)_x w^2\,dx dt \\
&\qquad\quad + s^3\int_0^T\int_0^1 \{(a(\varphi_x)^2)_x a\varphi_x + a(\varphi_x)^2(a\varphi_x)_x\} w^2\,dx dt. \qquad (3.9)
\end{aligned}
$$

At this point, note that $(a\varphi_x)_x = c_1\Theta$, so that $(a\varphi_x)_{xx} = 0$.

Moreover

$$
\begin{aligned}
&\int_0^T\int_0^1 L_s^+ w(-s(a\varphi_x)_x w)\,dx dt \\
&= \int_0^T [-saw_x w(a\varphi_x)_x]_{x=0}^{x=1}\,dt + s\int_0^T\int_0^1 aw_x(a\varphi_x)_x w_x\,dx dt \qquad (3.10) \\
&\quad + s^2\int_0^T\int_0^1 (a\varphi_x)_x\varphi_t w^2\,dx dt - s^3\int_0^T\int_0^1 a(\varphi_x)^2(a\varphi_x)_x w^2\,dx dt.
\end{aligned}
$$

Adding (3.8)–(3.10), (3.7) follows immediately. □

Now, the crucial step is to prove the following estimate:

Lemma 3.5. *Assume Hypothesis* 3.1. *Then there exists a positive constant* s_0 *such that for all* $s \geq s_0$ *the distributed terms of* (3.7) *satisfy the estimate*

$$\begin{aligned}
&\frac{s}{2}\int_0^T\int_0^1 \varphi_{tt} w^2\,dxdt + s^3\int_0^T\int_0^1 (2a\varphi_{xx} + a'\varphi_x)a(\varphi_x^2)w^2\,dxdt \\
&\qquad - 2s^2\int_0^T\int_0^1 a\varphi_x\varphi_{tx}w^2\,dxdt \\
&\qquad + s\int_0^T\int_0^1 (2a^2\varphi_{xx} + aa'\varphi_x)(w_x)^2\,dxdt \\
&\geq \frac{C}{2}s\int_0^T\int_0^1 \Theta a(w_x)^2\,dxdt + \frac{C^3}{2}s^3\int_0^T\int_0^1 \Theta^3\frac{(x-x_0)^2}{a}w^2\,dxdt,
\end{aligned}$$

for a positive constant C.

Proof. Using the definition of φ, the distributed terms of $\int_0^T\int_0^1 L_s^+ wL_s^- w\,dxdt$ take the form

$$\begin{aligned}
&\frac{s}{2}\int_0^T\int_0^1 \ddot{\Theta}\psi w^2\,dxdt + s^3\int_0^T\int_0^1 \Theta^3 a(2a\psi'' + a'\psi')(\psi')^2w^2\,dxdt \\
&\qquad - 2s^2\int_0^T\int_0^1 \Theta\dot{\Theta}a(\psi')^2w^2\,dxd \qquad (3.11)\\
&\qquad + s\int_0^T\int_0^1 \Theta a(2a\psi'' + a'\psi')(w_x)^2\,dxdt.
\end{aligned}$$

Because of the choice of $\psi(x)$, one has

$$2a(x)\psi''(x) + a'(x)\psi'(x) = c_1\frac{2a(x) - a'(x)(x-x_0)}{a(x)}.$$

Thus (3.11) becomes

$$\begin{aligned}
&\frac{s}{2}\int_0^T\int_0^1 \ddot{\Theta}\psi w^2\,dxdt - 2s^2\int_0^T\int_0^1 \Theta\dot{\Theta}a(\psi')^2w^2\,dxdt \\
&\qquad + sc_1\int_0^T\int_0^1 \Theta(2a(x) - a'(x)(x-x_0))(w_x)^2\,dxdt \\
&\qquad + s^3c_1\int_0^T\int_0^1 \Theta^3(\psi')^2(2a(x) - a'(x)(x-x_0))w^2\,dxdt.
\end{aligned}$$

By assumption, one can estimate the previous terms in the following way:

$$\begin{aligned}
&\frac{s}{2}\int_0^T\int_0^1 \ddot{\Theta}\psi w^2\,dxdt - 2s^2\int_0^T\int_0^1 \Theta\dot{\Theta}a(\psi')^2w^2\,dxdt\\
&\qquad + sc_1\int_0^T\int_0^1 \Theta(2a(x)-a'(x)(x-x_0))(w_x)^2\,dxdt\\
&\qquad + s^3c_1\int_0^T\int_0^1 \Theta^3(\psi')^2(2a(x)-a'(x)(x-x_0))w^2\,dxdt\\
&\geq -2s^2\int_0^T\int_0^1 \Theta\dot{\Theta}a(\psi')^2w^2\,dxdt + \frac{s}{2}\int_0^T\int_0^1 \ddot{\Theta}\psi w^2\,dxdt\\
&\qquad + sC\int_0^T\int_0^1 \Theta a(w_x)^2\,dxdt\\
&\qquad + s^3C^3\int_0^T\int_0^1 \Theta^3\frac{(x-x_0)^2}{a}w^2\,dxdt,
\end{aligned}$$

where $C>0$ is some universal positive constant. Observing that

$$|\Theta\dot{\Theta}|\leq c\Theta^{9/4}\leq c\Theta^3 \quad\text{and}\quad |\ddot{\Theta}|\leq c\Theta^{3/2},$$

for a positive constant c, we conclude that, for s large enough,

$$\begin{aligned}
\left|-2s^2\int_0^T\int_0^1 \Theta\dot{\Theta}a(\psi')^2w^2\,dxdt\right| &\leq 2cs^2\int_0^T\int_0^1 \Theta^3a(\psi')^2w^2\,dxdt\\
&= 2cs^2\int_0^T\int_0^1 \Theta^3c_1^2\frac{(x-x_0)^2}{a}w^2\,dxdt\\
&\leq \frac{C^3}{4}s^3\int_0^T\int_0^1 \Theta^3\frac{(x-x_0)^2}{a}w^2\,dxdt.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\left|\frac{s}{2}\int_0^T\int_0^1 \ddot{\Theta}\psi w^2\,dxdt\right| &\leq \frac{s}{2}c_1c\left|\int_0^T\int_0^1 \Theta^{3/2}bw^2\,dxdt\right|\\
&\quad + s\frac{c_1c_2}{2}c\left|\int_0^T\int_0^1 \Theta^{3/2}w^2\,dxdt\right|,
\end{aligned}\tag{3.12}$$

where

$$b(x)=\int_{x_0}^x\frac{y-x_0}{a(y)}\,dy\geq 0.$$

Now, since the function $x \mapsto \frac{|x-x_0|^K}{a(x)}$ is nonincreasing on $[0, x_0)$ and nondecreasing on $(x_0, 1]$ (see Lemma 2.1), one has

$$b(x) \le \frac{(x-x_0)^2}{(2-K)a(x)},$$

see (3.3). Hence

$$\frac{s}{2}c_1 c \int_0^T \int_0^1 \Theta^{3/2} b w^2 \, dx dt \le \frac{C^3}{8} s^3 \int_0^T \int_0^1 \Theta^3 \frac{(x-x_0)^2}{a} w^2 \, dx dt,$$

for s large enough.

It remains to bound the term $|\int_0^T \int_0^1 \Theta^{3/2} w^2 \, dx dt|$. Using the Young inequality, we find

$$\begin{aligned} & s\frac{c_1 c_2}{2} c \left| \int_0^1 \Theta^{3/2} w^2 \, dx \right| \\ & = s\frac{c_1 c_2}{2} c \left| \int_0^1 \left(\Theta \frac{a^{1/3}}{|x-x_0|^{2/3}} w^2 \right)^{3/4} \left(\Theta^3 \frac{|x-x_0|^2}{a} w^2 \right)^{1/4} \right| \\ & \le s\frac{3c_1 c_2}{4} c \int_0^1 \Theta \frac{a^{1/3}}{|x-x_0|^{2/3}} w^2 \, dx + s\frac{c_1 c_2}{4} c \int_0^1 \Theta^3 \frac{|x-x_0|^2}{a} w^2 \, dx. \end{aligned} \tag{3.13}$$

Now, consider the function $p(x) = (a(x)|x-x_0|^4)^{1/3}$. It is clear that, setting

$$C_1 := \max \left\{ \left(\frac{x_0^2}{a(0)} \right)^{2/3}, \left(\frac{(1-x_0)^2}{a(1)} \right)^{2/3} \right\},$$

by Lemma 2.1 we have

$$p(x) = a(x) \left(\frac{(x-x_0)^2}{a(x)} \right)^{2/3} \le C_1 a(x) \quad \text{and} \quad \frac{a^{1/3}}{|x-x_0|^{2/3}} = \frac{p(x)}{(x-x_0)^2}.$$

Moreover, using Hypothesis 3.1, one has that the function $x \mapsto \frac{p(x)}{|x-x_0|^q}$, where $q := \frac{4+\vartheta}{3} \in (1,2)$, is nonincreasing on the left of $x = x_0$ and nondecreasing on the right of $x = x_0$. The Hardy–Poincaré inequality (see Proposition 2.6) implies

$$\begin{aligned} \int_0^1 \Theta \frac{a^{1/3}}{|x-x_0|^{2/3}} w^2 \, dx &= \int_0^1 \Theta \frac{p}{(x-x_0)^2} w^2 \, dx \\ &\le C_{\mathrm{HP}} \int_0^1 \Theta p(w_x)^2 \, dx \\ &\le C_{\mathrm{HP}} C_1 \int_0^1 \Theta a(w_x)^2 \, dx, \end{aligned} \tag{3.14}$$

where C_{HP} and C_1 are the Hardy–Poincaré constant and the constant introduced before, respectively. Thus, for s large enough, by (3.13) and (3.14), we have

$$s\frac{c_1c_2}{2}c\int_0^1 \Theta^{3/2}w^2\,dx \le \frac{C}{2}s\int_0^1 \Theta a(w_x)^2\,dx + \frac{C^3}{8}s^3\int_0^1 \Theta^3\frac{(x-x_0)^2}{a}w^2\,dxdt,$$

for a positive constant C. Using the estimates above, from (3.12) we finally obtain

$$\left|\frac{s}{2}\int_0^T\int_0^1 \ddot{\Theta}\psi w^2\,dxdt\right| \le \frac{C}{2}s\int_0^T\int_0^1 \Theta a(w_x)^2\,dxdt + \frac{C^3}{4}s^3\int_0^T\int_0^1 \Theta^3\frac{(x-x_0)^2}{a}w^2\,dxdt.$$

Summing up, we obtain

$$\begin{aligned}
&\frac{s}{2}\int_0^T\int_0^1 \varphi_{tt}w^2\,dxdt + s\int_0^T\int_0^1 a(a\varphi_x)_{xx}ww_x\,dxdt\\
&\qquad - 2s^2\int_0^T\int_0^1 a\varphi_x\varphi_{tx}w^2\,dxdt\\
&\qquad + s\int_0^T\int_0^1 (2a^2\varphi_{xx} + aa'\varphi_x)(w_x)^2\,dxdt\\
&\qquad + s^3\int_0^T\int_0^1 (2a\varphi_{xx} + a'\varphi_x)a(\varphi_x)^2w^2\,dxdt\\
&\ge \frac{C}{2}s\int_0^T\int_0^1 \Theta a(w_x)^2\,dxdt + \frac{C^3}{2}s^3\int_0^T\int_0^1 \Theta^3\frac{(x-x_0)^2}{a}w^2\,dxdt. \qquad \square
\end{aligned}$$

For the boundary terms in (3.7), it holds:

Lemma 3.6. *The boundary terms in* (3.7) *reduce to*

$$-s\int_0^T \left[\Theta(aw_x)^2\psi'\right]_{x=0}^{x=1}dt.$$

Proof. Using the definition of φ, we have that the boundary terms become

$$\begin{aligned}
(\text{B.T.}) = \int_0^T \Big[&aw_xw_t - sa\Theta(a\psi')'ww_x + s^2\dot{\Theta}\Theta a\psi\psi'w^2\\
&- s^3a^2\Theta^3(\psi')^3w^2 - s\Theta(aw_x)^2\psi'\Big]_{x=0}^{x=1}dt \qquad (3.15)\\
+\int_0^1 \Big[&-\frac{s}{2}w^2\psi\dot{\Theta} + \frac{s^2}{2}aw^2(\psi')^2\Theta^2 - \frac{1}{2}a(w_x)^2\Big]_{t=0}^{t=T}dx.
\end{aligned}$$

Since $w \in \mathcal{V}$, we have $w \in C([0,T]; H_a^1(0,1))$. Thus $w(0,x)$, $w(T,x)$, $w_x(0,x)$, $w_x(T,x)$ and $\int_0^1 [a(w_x)^2]_0^T\,dx$ are indeed well defined. Using the boundary conditions of w and the definition of w, we get that

$$\int_0^1 \left[-\frac{s}{2} w^2 \psi \dot{\Theta} + \frac{s^2}{2} a w^2 (\psi')^2 \Theta^2 - \frac{1}{2} a (w_x)^2 \right]_{t=0}^{t=T} dx = 0.$$

Moreover, since $w \in \mathcal{V}$, we have that $w_t(t,0)$ and $w_t(t,1)$ make sense. Therefore, also $a(0)w_x(t,0)$ and $a(1)w_x(t,1)$ are well defined. In fact

$$w(t,\cdot) \in H_a^2(0,1) \quad \text{and} \quad a(\cdot)w_x(t,\cdot) \in W^{1,2}(0,1) \subset C([0,1]).$$

Thus $\int_0^T [a w_x w_t]_{x=0}^{x=1} dt$ is well defined and actually equals 0, as we get using the boundary conditions on w.

Now, consider the second, the third and the fourth terms of (3.15). By definition of ψ and using the hypothesis on a, the functions $(a\psi')'$, $a\psi\psi'$ and $a^2(\psi')^3$ are bounded on $[0,1]$. Thus, by the boundary conditions on w, one has

$$\begin{aligned} s \int_0^T \left[a\Theta (a\psi')' w w_x \right]_{x=0}^{x=1} dt &= s^2 \int_0^T \left[\dot{\Theta} \Theta a \psi \psi' w^2 \right]_{x=0}^{x=1} dt \\ &= s^3 \int_0^T \left[a^2 \Theta^3 (\psi')^3 w^2 \right]_{x=0}^{x=1} dt = 0. \qquad \square \end{aligned}$$

From Lemma 3.4, Lemma 3.5, and Lemma 3.6, we deduce immediately that there exist two positive constants C and s_0 such that all solutions w of (3.5) satisfy, for all $s \geq s_0$,

$$\begin{aligned} \int_0^T \int_0^1 L_s^+ w L_s^- w \, dxdt \geq\ & Cs \int_0^T \int_0^1 \Theta a (w_x)^2 \, dxdt \\ & + Cs^3 \int_0^T \int_0^1 \Theta^3 \frac{(x-x_0)^2}{a} w^2 \, dxdt \\ & - s \int_0^T \left[\Theta a^2 (w_x)^2 \psi' \right]_{x=0}^{x=1} dt. \end{aligned} \tag{3.16}$$

Thus, a straightforward consequence of (3.6) and of (3.16) is the next result.

Proposition 3.7. *Assume Hypothesis* 3.1 *and let $T > 0$. Then, there exist two positive constants C and s_0 such that all solutions w of* (3.5) *in $\mathcal{V}$ satisfy, for all $s \geq s_0$,*

$$\begin{aligned} & s \int_0^T \int_0^1 \Theta a (w_x)^2 \, dxdt + s^3 \int_0^T \int_0^1 \Theta^3 \frac{(x-x_0)^2}{a} w^2 \, dxdt \\ & \quad \leq C \left(\int_0^T \int_0^1 h^2 e^{2s\varphi(t,x)} \, dxdt + s \int_0^T \left[\Theta a^2 (w_x)^2 \psi' \right]_{x=0}^{x=1} dt \right). \end{aligned}$$

Recalling the definition of w, we have

$$v = e^{-s\varphi} w \quad \text{and} \quad v_x = -s\Theta\psi' e^{-s\varphi} w + e^{-s\varphi} w_x.$$

Thus, by the Cauchy–Schwarz inequality

$$\int_0^T \int_0^1 \left(s\Theta a(v_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\varphi}\, dxdt$$
$$\leq \int_0^T \int_0^1 \left(s\Theta a(w_x)^2\, dxdt + s^3 c_1^2 \Theta^3 \frac{(x-x_0)^2}{a} w^2 \right) dxdt,$$

and by Proposition 3.7, Theorem 3.3 follows.

4 Application of Carleman estimates to observability inequalities

In this section we provide a possible application of the Carleman estimates established in the previous section, considering the control problem (1.2). In particular, we consider the situation in which x_0 is inside the control interval

$$x_0 \in \omega = (\alpha, \beta) \subset (0, 1). \tag{4.1}$$

Now, we associate to the linear problem (1.2) the homogeneous adjoint problem

$$\begin{cases} v_t + (av_x)_x = 0, & (t, x) \in Q_T, \\ v(t, 0) = v(t, 1) = 0, & t \in (0, T), \\ v(T, x) = v_T(x) \in L^2(0, 1), \end{cases} \tag{4.2}$$

where $T > 0$ is given. By the Carleman estimate in Theorem 3.3, we will deduce the following observability inequality for both the weakly and the strongly degenerate cases:

Proposition 4.1. *Assume Hypothesis* 3.1 *and* (4.1). *Then there exists a positive constant* C_T *such that every solution* $v \in C([0, T]; L^2(0, 1)) \cap L^2(0, T; H_a^1(0, 1))$ *of* (4.2) *satisfies*

$$\int_0^1 v^2(0, x)\, dx \leq C_T \int_0^T \int_w v^2(t, x)\, dxdt. \tag{4.3}$$

4.1 Proof of Proposition 4.1

In this subsection we will prove, as a consequence of the Carleman estimate proved in Section 3, the observability inequality (4.3). For this purpose, we will give some preliminary results. As a first step, we consider the adjoint problem with more

regular final-time datum

$$\begin{cases} v_t + (av_x)_x = 0, & (t,x) \in Q_T, \\ v(t,0) = v(t,1) = 0, & t \in (0,T), \\ v(T,x) = v_T(x) \in D(\mathcal{A}^2), \end{cases} \tag{4.4}$$

where

$$D(\mathcal{A}^2) = \{u \in D(\mathcal{A}) \mid \mathcal{A}u \in D(\mathcal{A})\}$$

and $\mathcal{A}u := (au_x)_x$. Observe that $D(\mathcal{A}^2)$ is densely defined in $D(\mathcal{A})$ (see, for example, [7, Lemma 7.2]) and hence in $L^2(0,1)$. As in [10], [11] or [20], letting v_T vary in $D(\mathcal{A}^2)$, we define the following class of functions:

$$\mathcal{W} := \{v \text{ is a solution of (4.4)}\}.$$

Obviously (see, for example, [7, Theorem 7.5])

$$\mathcal{W} \subset C^1\big([0,T]\,;\, H_a^2(0,1)\big) \subset \mathcal{V} \subset \mathcal{U},$$

where $\mathcal{V}$ is defined in (3.4) and

$$\mathcal{U} := C([0,T]; L^2(0,1)) \cap L^2(0,T; H_a^1(0,1)). \tag{4.5}$$

We start with

Proposition 4.2 (Caccioppoli's inequality). *Let ω' and ω be two open subintervals of $(0,1)$ such that $\omega' \subset\subset \omega \subset (0,1)$ and $x_0 \notin \bar{\omega}'$. Let $\varphi(t,x) = \Theta(t)\Upsilon(x)$, where Θ is defined in* (3.2) *and*

$$\Upsilon \in C([0,1],(-\infty,0)) \cap C^1([0,1] \setminus \{x_0\},(-\infty,0))$$

is such that

$$|\Upsilon_x| \le \frac{c}{\sqrt{a}} \quad \text{in } [0,1] \setminus \{x_0\} \tag{4.6}$$

for some $c > 0$. Then, there exist two positive constants C and s_0 such that every solution $v \in \mathcal{W}$ of the adjoint problem (4.4) *satisfies*

$$\int_0^T \int_{\omega'} (v_x)^2 e^{2s\varphi}\, dxdt \le C \int_0^T \int_{\omega} v^2\, dxdt, \tag{4.7}$$

for all $s \ge s_0$.

Remark 4.3. Of course, our prototype for Υ is the function ψ defined in (3.2). Indeed,

$$|\psi'(x)| = c_1 \left|\frac{x-x_0}{a(x)}\right| = c_1 \sqrt{\frac{|x-x_0|^2}{a(x)}} \frac{1}{\sqrt{a(x)}} \le c \frac{1}{\sqrt{a(x)}}$$

by Lemma 2.1.

Proof of Proposition 4.2. Let us consider a smooth function $\xi : [0,1] \to \mathbb{R}$ such that

$$\begin{cases} 0 \le \xi(x) \le 1, & x \in [0,1], \\ \xi(x) = 1, & x \in \omega', \\ \xi(x) = 0, & x \in [0,1] \setminus \omega. \end{cases}$$

Since v solves (4.4) and has homogeneous boundary conditions, by the choice of φ, we have

$$\begin{aligned} 0 &= \int_0^T \frac{d}{dt}\left(\int_0^1 \xi^2 e^{2s\varphi} v^2\,dx\right) dt \\ &= \int_0^T \int_0^1 (2s\xi^2\varphi_t e^{2s\varphi} v^2 + 2\xi^2 e^{2s\varphi} v v_t)\,dx dt \\ &= 2s\int_0^T \int_0^1 \xi^2\varphi_t e^{2s\varphi} v^2\,dx dt + 2\int_0^T \int_0^1 \xi^2 e^{2s\varphi} v(-(a v_x)_x)\,dx dt \\ &= 2s\int_0^T \int_0^1 \xi^2\varphi_t e^{2s\varphi} v^2\,dx dt + 2\int_0^T \int_0^1 (\xi^2 e^{2s\varphi} v)_x a v_x\,dx dt \\ &= 2s\int_0^T \int_0^1 \xi^2\varphi_t e^{2s\varphi} v^2\,dx dt + 2\int_0^T \int_0^1 (\xi^2 e^{2s\varphi})_x a v v_x\,dx dt \\ &\qquad + 2\int_0^T \int_0^1 \xi^2 e^{2s\varphi} a (v_x)^2\,dx dt \\ &= 2s\int_0^T \int_\omega \xi^2\varphi_t e^{2s\varphi} v^2\,dx dt + 2\int_0^T \int_\omega (\xi^2 e^{2s\varphi})_x a v v_x\,dx dt \\ &\qquad + 2\int_0^T \int_\omega \xi^2 e^{2s\varphi} a (v_x)^2\,dx dt. \end{aligned}$$

Hence, by definition of ξ and the Cauchy–Schwarz inequality, the previous identity gives

$$\begin{aligned} &2\int_0^T \int_\omega \xi^2 e^{2s\varphi} a (v_x)^2\,dx dt \\ &\quad = -2s\int_0^T \int_\omega \xi^2\varphi_t e^{2s\varphi} v^2\,dx dt - 2\int_0^T \int_\omega (\xi^2 e^{2s\varphi})_x a v v_x\,dx dt \\ &\quad \le -2s\int_0^T \int_\omega \xi^2\varphi_t e^{2s\varphi} v^2\,dx dt + \int_0^T \int_\omega \left(\sqrt{a}\xi e^{s\varphi} v_x\right)^2 dx dt \\ &\qquad + \int_0^T \int_\omega \left(\sqrt{a}\frac{(\xi^2 e^{2s\varphi})_x}{\xi e^{s\varphi}} v\right)^2 dx dt \end{aligned}$$

$$= -2s\int_0^T\int_\omega \xi^2\varphi_t e^{2s\varphi}v^2\,dxdt + \int_0^T\int_\omega \xi^2 e^{2s\varphi}a(v_x)^2\,dxdt + \int_0^T\int_\omega \frac{[(\xi^2 e^{2s\varphi})_x]^2}{\xi^2 e^{2s\varphi}}av^2\,dxdt.$$

Thus,

$$\int_0^T\int_\omega \xi^2 e^{2s\varphi}a(v_x)^2\,dxdt \le -2\int_0^T\int_\omega \xi^2 s\varphi_t e^{2s\varphi}v^2\,dxdt + \int_0^T\int_\omega \frac{[(\xi^2 e^{2s\varphi})_x]^2}{\xi^2 e^{2s\varphi}}av^2\,dxdt.$$

Since $x_0 \notin \bar{\omega}'$, we get

$$\begin{aligned}\inf_{x\in\omega'} a(x)\int_0^T\int_{\omega'} e^{2s\varphi}(v_x)^2\,dxdt &\le \int_0^T\int_{\bar{\omega}'} \xi^2 e^{2s\varphi}a(v_x)^2\,dxdt\\ &\le \int_0^T\int_\omega \xi^2 e^{2s\varphi}a(v_x)^2\,dxdt\\ &\le -2\int_0^T\int_\omega \xi^2 s\varphi_t e^{2s\varphi}v^2\,dxdt\\ &\quad + \int_0^T\int_\omega \frac{[(\xi^2 e^{2s\varphi})_x]^2}{\xi^2 e^{2s\varphi}}av^2\,dxdt.\end{aligned}$$

Calculations show that $s\varphi_t e^{2s\varphi}$ is uniformly bounded if $s \ge s_0 > 0$, since Υ is strictly negative, a rough estimate being

$$|s\varphi_t e^{2s\varphi}| \le c\frac{1}{s_0^{1/4}(-\max\Upsilon)^{1/4}}.$$

Indeed, $|\dot{\Theta}| \le c\Theta^{5/4}$ and

$$|s\varphi_t e^{2s\varphi}| \le cs(-\Upsilon)\Theta^{5/4}e^{2s\varphi} \le \frac{c}{\big(s(-\Upsilon)\big)^{5/4}}$$

for some constant $c > 0$ which may vary at every step.

On the other hand, we can estimate

$$\frac{[(\xi^2 e^{2s\varphi})_x]^2}{\xi^2 e^{2s\varphi}} \le C\big(e^{2s\varphi} + s^2(\varphi_x)^2 e^{2s\varphi}\big).$$

Of course, $e^{2s\varphi} < 1$, while $s^2(\varphi_x)^2 e^{2s\varphi}$ can be estimated as follows:

$$s^2(\varphi_x)^2 e^{2s\varphi} \le \frac{c}{(-\max\Upsilon)^2}(\Upsilon_x)^2 \le \frac{c}{a}$$

by (4.6), for some constant $c > 0$.

In conclusion, we can find a positive constant C such that

$$-2\int_0^T\int_\omega \xi^2 s\varphi_t e^{2s\varphi}v^2\,dxdt + \int_0^T\int_\omega \frac{[(\xi^2 e^{2s\varphi})_x]^2}{\xi^2 e^{2s\varphi}} a v^2\,dxdt \le C\int_0^T\int_\omega v^2\,dxdt,$$

and the claim follows. □

We shall need the following lemma:

Lemma 4.4. *Assume Hypothesis* 3.1 *and* (4.1). *Then there exist two positive constants C and s_0 such that every solution $v \in \mathcal{W}$ of* (4.4) *satisfies, for all $s \ge s_0$,*

$$\int_0^T\int_0^1 \left(s\Theta a v_x^2 + s^3\Theta^3\frac{(x-x_0)^2}{a}v^2\right)e^{2s\varphi}\,dxdt \le C\int_0^T\int_w v^2\,dxdt.$$

Here Θ and φ are as in (3.2).

For the proof of the previous lemma we need the following classical Carleman estimate (see, for example [3, Proposition 4.4]):

Proposition 4.5 (Classical Carleman estimates). *Let z be the solution of*

$$\begin{cases} z_t + (az_x)_x = h \in L^2((0,T)\times(A,B)), \\ z(t,A) = z(t,B) = 0, \quad t \in (0,T), \end{cases} \tag{4.8}$$

where $a \in C^1([A,B])$ is a strictly positive function. Then there exist positive constants c, r and s_0 such that for any $s \ge s_0$

$$\begin{aligned} &\int_0^T\int_A^B s\Theta e^{r\zeta}(z_x)^2 e^{-2s\Phi}\,dxdt + \int_0^T\int_A^B s^3\Theta^3 e^{3r\zeta}z^2 e^{-2s\Phi}\,dxdt \\ &\le c\int_0^T\int_A^B e^{-2s\Phi}h^2\,dxdt - c\int_0^T \left[\sigma(t,\cdot)e^{-2s\Phi(t,\cdot)}|z_x(t,\cdot)|^2\right]_{x=A}^{x=B} dt. \end{aligned} \tag{4.9}$$

Here the functions ζ, σ and Φ are defined in the following way:

$$\zeta(x) := \int_x^B \frac{1}{\sqrt{a(y)}}dy, \quad \sigma(t,x) := rs\Theta(t)e^{r\zeta(x)},$$

$$\Phi(t,x) := \Theta(t)\Psi(x) \quad \textit{with} \quad \Psi(x) := e^{2r\zeta(A)} - e^{r\zeta(x)} > 0,$$

where $(t,x) \in [0,T]\times[A,B]$ and Θ is defined in (3.2).

(Observe that $\Phi > 0$ and $\Phi(t,x) \to +\infty$, as $t \downarrow 0$, $t \uparrow T$.)

Proof of Lemma 4.4. By assumption, we can find two subintervals $\omega_1 \subset (0, x_0)$ and $\omega_2 \subset (x_0, 1)$ such that $(\omega_1 \cup \omega_2) \subset\subset \omega \setminus \{x_0\}$. Now, set $\lambda_i := \inf \omega_i$ and $\beta_i := \sup \omega_i$, $i = 1, 2$, and consider a smooth function $\xi : [0, 1] \to \mathbb{R}$ such that

$$\begin{cases} 0 \le \xi(x) \le 1, & x \in [0, 1], \\ \xi(x) = 1, & x \in [\lambda_1, \beta_2], \\ \xi(x) = 0, & x \in [0, 1] \setminus \omega. \end{cases}$$

Define $w := \xi v$, where v is the solution of (4.4). Hence, w satisfies

$$\begin{cases} w_t + (aw_x)_x = (a\xi_x v)_x + \xi_x a v_x =: f, & (t, x) \in (0, T) \times (0, 1), \\ w(t, 0) = w(t, 1) = 0, & t \in (0, T). \end{cases} \tag{4.10}$$

Applying Theorem 3.7 and using the fact that $w = 0$ in a neighborhood of $x = 0$ and $x = 1$, we have

$$\begin{aligned} \int_0^T \int_0^1 \left(s\Theta a (w_x)^2 + s^3 \Theta^3 \frac{(x - x_0)^2}{a} w^2 \right) e^{2s\varphi} \, dx dt \\ \le C \int_0^T \int_0^1 e^{2s\varphi} f^2 \, dx dt \end{aligned} \tag{4.11}$$

for all $s \ge s_0$. Then, using the definition of ξ and in particular the fact that ξ_x and ξ_{xx} are supported in $\tilde{\omega}$, where $\tilde{\omega} := [\inf \omega, \lambda_1] \cup [\beta_2, \sup \omega]$, we can write

$$f^2 = ((a\xi_x v)_x + \xi_x a v_x)^2 \le C(v^2 + (v_x)^2) \chi_{\tilde{\omega}},$$

since the function a' is bounded on $\tilde{\omega}$. Hence, applying Proposition 4.2 and (4.11), we get

$$\begin{aligned} \int_0^T \int_{\lambda_1}^{\beta_2} \left(s\Theta a (v_x)^2 + s^3 \Theta^3 \frac{(x - x_0)^2}{a} v^2 \right) e^{2s\varphi} \, dx dt \\ = \int_0^T \int_{\lambda_1}^{\beta_2} \left(s\Theta a (w_x)^2 + s^3 \Theta^3 \frac{(x - x_0)^2}{a} w^2 \right) e^{2s\varphi} \, dx dt \\ \le \int_0^T \int_0^1 \left(s\Theta a (w_x)^2 + s^3 \Theta^3 \frac{(x - x_0)^2}{a} w^2 \right) e^{2s\varphi} \, dx dt \\ \le C \int_0^T \int_{\tilde{\omega}} e^{2s\varphi} (v^2 + (v_x)^2) \, dx dt \\ \le C \int_0^T \int_{\omega} v^2 \, dx dt, \end{aligned} \tag{4.12}$$

for a positive constant C. Now, consider a smooth function $\eta : [0,1] \to \mathbb{R}$ such that

$$\begin{cases} 0 \le \eta(x) \le 1, & x \in [0,1], \\ \eta(x) = 1, & x \in [\beta_2, 1], \\ \eta(x) = 0, & x \in [0, \frac{\lambda_2+2\beta_2}{3}]. \end{cases}$$

Define $z := \eta v$, where v is the solution of (4.4). Then z satisfies (4.8) and (4.9), with

$$h := (a\eta_x v)_x + \eta_x a v_x, \quad A = \lambda_2 \quad \text{and} \quad B = 1.$$

Since h is supported in $[\frac{\lambda_2+2\beta_2}{3}, \beta_2]$, by Propositions 4.2 and 4.5 with

$$\zeta(x) = \zeta_1(x) := \int_x^1 \frac{1}{\sqrt{a(y)}}\, dy, \tag{4.13}$$

we get

$$\begin{aligned} &\int_0^T \int_{\lambda_2}^1 s\Theta e^{r\zeta_1} (z_x)^2 e^{-2s\Phi}\, dxdt + \int_0^T \int_{\lambda_2}^1 s^3\Theta^3 e^{3r\zeta_1} z^2 e^{-2s\Phi}\, dxdt \\ &\quad \le c \int_0^T \int_{\lambda_2}^1 e^{-2s\Phi} h^2\, dxdt \\ &\quad \le C \int_0^T \int_{\tilde{\omega}_1} v^2\, dxdt + C \int_0^T \int_{\tilde{\omega}_1} e^{-2s\Phi} (v_x)^2\, dxdt \\ &\quad \le C \int_0^T \int_{\omega} v^2\, dxdt, \end{aligned} \tag{4.14}$$

where $\tilde{\omega}_1 = (\lambda_2, \beta_2)$.

Now, choose the constant c_1 in (3.2) so that

$$c_1 \ge \max\left\{ \frac{e^{2r\zeta_1(\lambda_2)} - 1}{c_2 - \frac{(1-x_0)^2}{a(1)(2-K)}}, \frac{e^{2r\zeta_1(\lambda_2)} - 1}{c_2 - \frac{x_0^2}{a(0)(2-K)}} \right\},$$

where ζ_1 is defined as before. Then, by the definition of φ, the choice of c_1 and by Lemma 2.1, one can prove that there exists a positive constant k, for example

$$k = \max\left\{ \max_{[\lambda_2,1]} a, \frac{(1-x_0)^2}{a(1)} \right\},$$

such that

$$a(x) e^{2s\varphi(t,x)} \le k e^{r\zeta_1(x)} e^{-2s\Phi(t,x)}$$

and

$$\frac{(x-x_0)^2}{a(x)} e^{2s\varphi(t,x)} \le k e^{r\zeta_1(x)} e^{-2s\Phi(t,x)} \le k e^{3r\zeta_1(x)} e^{-2s\Phi(t,x)}$$

for every $(t,x) \in [0,T] \times [\lambda_2, 1]$. Thus, by (4.14), one has

$$\begin{aligned}
&\int_0^T \int_{\lambda_2}^1 \left(s\Theta a(z_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} z^2 \right) e^{2s\varphi}\, dxdt \\
&\qquad \le k \int_0^T \int_{\lambda_2}^1 s\Theta e^{r\zeta_1} (z_x)^2 e^{-2s\Phi}\, dxdt \\
&\qquad\qquad + k \int_0^T \int_{\lambda_2}^1 s^3\Theta^3 e^{3r\zeta_1} z^2 e^{-2s\Phi}\, dxdt \\
&\qquad \le kC \int_0^T \int_\omega v^2\, dxdt,
\end{aligned}$$

for a positive constant C. As a trivial consequence,

$$\begin{aligned}
&\int_0^T \int_{\beta_2}^1 \left(s\Theta a(v_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\varphi}\, dxdt \\
&\qquad = \int_0^T \int_{\beta_2}^1 \left(s\Theta a(z_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} z^2 \right) e^{2s\varphi}\, dxdt \\
&\qquad \le \int_0^T \int_{\lambda_2}^1 \left(s\Theta a(z_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} z^2 \right) e^{2s\varphi}\, dxdt \\
&\qquad \le kC \int_0^T \int_\omega v^2\, dxdt,
\end{aligned} \tag{4.15}$$

for a positive constant C.

Thus (4.12) and (4.15) imply

$$\begin{aligned}
&\int_0^T \int_{\lambda_1}^1 \left(s\Theta a(v_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\varphi}\, dxdt \\
&\qquad \le C \int_0^T \int_\omega v^2\, dxdt,
\end{aligned} \tag{4.16}$$

for some positive constant C. To complete the proof it is sufficient to prove a similar inequality on the interval $[0, \lambda_1]$. To this end, we follow a reflection procedure introducing the functions

$$W(t,x) := \begin{cases} v(t,x), & x \in [0,1], \\ -v(t,-x), & x \in [-1,0], \end{cases} \tag{4.17}$$

where v solves (4.4), and

$$\tilde{a}(x) := \begin{cases} a(x), & x \in [0,1], \\ a(-x), & x \in [-1,0]. \end{cases} \tag{4.18}$$

Then W satisfies the problem

$$\begin{cases} W_t + (\tilde{a}W_x)_x = 0, & (t,x) \in (0,T) \times (-1,1), \\ W(t,-1) = W(t,1) = 0, & t \in (0,T). \end{cases} \tag{4.19}$$

Now, consider a cut off function $\rho : [-1,1] \to \mathbb{R}$ such that

$$\begin{cases} 0 \le \rho(x) \le 1, & x \in [-1,1], \\ \rho(x) = 1, & x \in (-\lambda_1, \lambda_1), \\ \rho(x) = 0, & x \in [-1, -\frac{\lambda_1+2\beta_1}{3}] \cup [\frac{\lambda_1+2\beta_1}{3}, 1]. \end{cases}$$

Define $Z := \rho W$, where W is the solution of (4.19). Then Z satisfies (4.8)–(4.9), with $h := (\tilde{a}\rho_x W)_x + \rho_x \tilde{a} W_x$, $A = -\beta_1$ and $B = \beta_1$. Now define

$$\zeta(x) = \zeta_2(x) := \int_x^{\beta_1} \frac{1}{\sqrt{\tilde{a}(y)}}\, dy.$$

Using Proposition 4.5 (which still holds true, since $\tilde{a} \in C^1([-\beta_1, \beta_1] \setminus \{0\})$ and $\tilde{a}'(0^+) = -\tilde{a}'(0^-) \in \mathbb{R}$, see [21]) with

$$\tilde{\Phi}(t,x) := \Theta(t)(e^{2r\zeta_2(-\beta_1)} - e^{r\zeta_2(x)}), \tag{4.20}$$

the fact that $Z_x(t,-\beta_1) = Z_x(t,\beta_1) = 0$, the definition of W and the fact that ρ is supported in $[-\frac{\lambda_1+2\beta_1}{3}, -\lambda_1] \cup [\lambda_1, \frac{\lambda_1+2\beta_1}{3}]$, give

$$\begin{aligned} &\int_0^T \int_{-\beta_1}^{\beta_1} s\Theta e^{r\zeta_2} (Z_x)^2 e^{-2s\tilde{\Phi}}\, dxdt + \int_0^T \int_{-\beta_1}^{\beta_1} s^3\Theta^3 e^{3r\zeta_2} Z^2 e^{-2s\tilde{\Phi}}\, dxdt \\ &\quad \le C \int_0^T \int_{-\beta_1}^{\beta_1} e^{-2s\tilde{\Phi}} h^2\, dxdt \\ &\quad \le C \int_0^T \int_{-\frac{\lambda_1+2\beta_1}{3}}^{-\lambda_1} e^{-2s\tilde{\Phi}} (W^2 + (W_x)^2)\, dxdt \\ &\qquad + C \int_0^T \int_{\lambda_1}^{\frac{\lambda_1+2\beta_1}{3}} e^{-2s\tilde{\Phi}} (W^2 + (W_x)^2)\, dxdt. \end{aligned} \tag{4.21}$$

Now, putting $\Xi(x) := e^{2r\zeta_2(-\beta_1)} - e^{r\zeta_2(x)}$ and

$$E := \frac{\Xi(-\beta_1)}{\Xi(\beta_1)} = \frac{e^{2r\zeta_2(-\beta_1)} - e^{r\zeta_2(-\beta_1)}}{e^{2r\zeta_2(-\beta_1)} - 1} \in (0,1),$$

we note that for any $x \in [0, \beta_1]$, $s \ge s_0$ and $t \in (0,T)$ we have

$$e^{-2s\Theta(t)\Xi(-x)} \le e^{-2Es\Theta(t)\Xi(x)}.$$

Hence, using the oddness of the involved functions,

$$\begin{aligned}
&\int_0^T \int_{-\frac{\lambda_1+2\beta_1}{3}}^{-\lambda_1} e^{-2s\tilde{\Phi}}(W^2+(W_x)^2)\,dxdt \\
&\quad\le \int_0^T \int_{\lambda_1}^{\frac{\lambda_1+2\beta_1}{3}} e^{-2Es\tilde{\Phi}}(W^2+(W_x)^2)\,dxdt \\
&\quad\le \int_0^T \int_{\lambda_1}^{\frac{\lambda_1+2\beta_1}{3}} v^2\,dxdt + C\int_0^T \int_{\lambda_1}^{\frac{\lambda_1+2\beta_1}{3}} e^{-2Es\Theta\Xi}(v_x)^2\,dxdt \\
&\quad\le \int_0^T \int_{\omega} v^2\,dxdt + C\int_0^T \int_{\lambda_1}^{\frac{\lambda_1+2\beta_1}{3}} e^{-2Es\Theta\Xi}(v_x)^2\,dxdt,
\end{aligned} \tag{4.22}$$

for some positive constant C. Now, after relabeling $\tilde{s} = Es$, (4.22) and Proposition 4.2 imply the existence of $C > 0$ and $s_1 > 0$ such that for all $s \ge s_1$ we get

$$\int_0^T \int_{-\frac{\lambda_1+2\beta_1}{3}}^{-\lambda_1} e^{-2s\tilde{\Phi}}(W^2+(W_x)^2)\,dxdt \le C\int_0^T \int_{\omega} v^2\,dxdt. \tag{4.23}$$

On the other hand, Proposition 4.2 immediately implies in an easier way that

$$\int_0^T \int_{\lambda_1}^{\frac{\lambda_1+2\beta_1}{3}} e^{-2s\tilde{\Phi}}(W^2+(W_x)^2)\,dxdt \le C\int_0^T \int_{\omega} v^2\,dxdt \tag{4.24}$$

for all s large enough and for a suitable $C > 0$.

In conclusion, (4.21)–(4.24) imply that there exist $s_0, C > 0$ such that

$$\begin{aligned}
&\int_0^T \int_{-\beta_1}^{\beta_1} s\Theta e^{r\zeta_2}(Z_x)^2 e^{-2s\tilde{\Phi}}\,dxdt + \int_0^T \int_{-\beta_1}^{\beta_1} s^3\Theta^3 e^{3r\zeta_2} Z^2 e^{-2s\tilde{\Phi}}\,dxdt \\
&\qquad\le C\int_0^T \int_{\omega} v^2\,dxdt
\end{aligned} \tag{4.25}$$

for all $s \ge s_0$.

Now, define $\tilde{\varphi}(t,x) := \Theta(t)\tilde{\psi}(x)$, where

$$\tilde{\psi}(x) := \begin{cases} \psi(x), & x \ge 0, \\ \psi(-x) = c_1\left[\displaystyle\int_{-x_0}^{x} \frac{t+x_0}{\tilde{a}(t)}\,dt - c_2\right], & x < 0. \end{cases} \tag{4.26}$$

and choose the constant c_1 so that

$$c_1 \ge \max\left\{ \frac{e^{2r\zeta_1(\lambda_2)}-1}{c_2 - \frac{(1-x_0)^2}{a(1)(2-K)}}, \frac{e^{2r\zeta_1(\lambda_2)}-1}{c_2 - \frac{x_0^2}{a(0)(2-K)}}, \frac{e^{2r\zeta_2(-\beta_1)}-1}{c_2 - \frac{(1-x_0)^2}{a(1)(2-K)}}, \frac{e^{2r\zeta_2(-\beta_1)}-1}{c_2 - \frac{x_0^2}{a(0)(2-K)}} \right\}.$$

Thus, by the definition of $\tilde{\varphi}$, one can prove as before that there exists a positive constant k, for example

$$k = \max\left\{ \max_{[-\beta_1,\beta_1]} \tilde{a}, \frac{x_0^2}{a(0)} \right\},$$

such that

$$\tilde{a}(x) e^{2s\tilde{\varphi}(t,x)} \leq k e^{r\zeta_2(x)} e^{-2s\tilde{\Phi}(t,x)}$$

and

$$\frac{(x-x_0)^2}{\tilde{a}(x)} e^{2s\tilde{\varphi}(t,x)} \leq k e^{r\zeta_2(x)} e^{-2s\tilde{\Phi}(t,x)} \leq k e^{3r\zeta_2(x)} e^{-2s\tilde{\Phi}(t,x)}$$

for every $(t,x) \in [0,T] \times [-\beta_1, \beta_1]$. Thus, by (4.25), one has

$$\begin{aligned}
&\int_0^T \int_{-\beta_1}^{\beta_1} \left(s\Theta \tilde{a} (Z_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{\tilde{a}} Z^2 \right) e^{2s\tilde{\varphi}} \, dx dt \\
&\qquad \leq k \int_0^T \int_{-\beta_1}^{\beta_1} s\Theta e^{r\zeta_2} (Z_x)^2 e^{-2s\tilde{\Phi}} \, dx dt \\
&\qquad\qquad + k \int_0^T \int_{-\beta_1}^{\beta_1} s^3\Theta^3 e^{3r\zeta_2} Z^2 e^{-2s\tilde{\Phi}} \, dx dt \\
&\qquad \leq kC \int_0^T \int_\omega v^2 \, dx dt.
\end{aligned} \tag{4.27}$$

Hence, by (4.27) and the definition of W and Z, we get

$$\begin{aligned}
&\int_0^T \int_0^{\lambda_1} \left(s^3\Theta^3 \frac{(x-x_0)^2}{a} v^2 + s\Theta a (v_x)^2 \right) e^{2s\varphi} \, dx dt \\
&\qquad = \int_0^T \int_0^{\lambda_1} \left(s^3\Theta^3 \frac{(x-x_0)^2}{a} W^2 + s\Theta a (W_x)^2 \right) e^{2s\varphi} \, dx dt \\
&\qquad \leq \int_0^T \int_{-\lambda_1}^{\lambda_1} \left(s^3\Theta^3 \frac{(x-x_0)^2}{\tilde{a}} W^2 + s\Theta \tilde{a} (W_x)^2 \right) e^{2s\tilde{\varphi}} \, dx dt \\
&\qquad = \int_0^T \int_{-\lambda_1}^{\lambda_1} \left(s^3\Theta^3 \frac{(x-x_0)^2}{\tilde{a}} Z^2 + s\Theta \tilde{a} (Z_x)^2 \right) e^{2s\tilde{\varphi}} \, dx dt \\
&\qquad \leq \int_0^T \int_{-\beta_1}^{\beta_1} \left(s^3\Theta^3 \frac{(x-x_0)^2}{\tilde{a}} Z^2 + s\Theta \tilde{a} (Z_x)^2 \right) e^{2s\tilde{\varphi}} \, dx dt \\
&\qquad \leq C \int_0^T \int_\omega v^2 \, dx dt,
\end{aligned} \tag{4.28}$$

for a positive constant C. Therefore, by (4.16) and (4.28), Lemma 4.4 follows. □

We shall also use the following

Lemma 4.6. *Assume Hypothesis* 3.1 *and* (4.1). *Then there exists a positive constant C_T such that every solution $v \in \mathcal{W}$ of* (4.4) *satisfies*

$$\int_0^1 v^2(0,x)\,dx \le C_T \int_0^T \int_\omega v^2(t,x)\,dxdt.$$

Proof. Multiplying the equation of (4.4) by v_t and integrating by parts over $(0,1)$, one has

$$\begin{aligned}
0 &= \int_0^1 (v_t + (av_x)_x)v_t\,dx \\
&= \int_0^1 (v_t^2 + (av_x)_x v_t)\,dx \\
&= \int_0^1 v_t^2\,dx + [av_x v_t]_{x=0}^{x=1} - \int_0^1 av_x v_{tx}\,dx \\
&= \int_0^1 v_t^2\,dx - \frac{1}{2}\frac{d}{dt}\int_0^1 a(v_x)^2\,dx \\
&\ge -\frac{1}{2}\frac{d}{dt}\int_0^1 a(v_x)^2\,dx.
\end{aligned}$$

Thus, the function $t \mapsto \int_0^1 a(v_x)^2\,dx$ is increasing for all $t \in [0,T]$. In particular,

$$\int_0^1 av_x(0,x)^2\,dx \le \int_0^1 av_x(t,x)^2\,dx.$$

Integrating the last inequality over $[\frac{T}{4}, \frac{3T}{4}]$, Θ being bounded therein, we find

$$\begin{aligned}
\int_0^1 a(v_x)^2(0,x)\,dx &\le \frac{2}{T}\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 a(v_x)^2(t,x)\,dxdt \\
&\le C_T \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 s\Theta a(v_x)^2(t,x)e^{2s\varphi}\,dxdt.
\end{aligned}$$

Hence, by Lemma 4.4 and the previous inequality, there exists a positive constant C such that

$$\int_0^1 a(v_x)^2(0,x)\,dx \le C\int_0^T \int_\omega v^2\,dxdt. \tag{4.29}$$

Proceeding again as in the proof of Lemma 3.5 and applying Hardy–Poincaré's inequality, by (4.29), one has

$$\begin{aligned}\int_0^1 \left(\frac{a}{(x-x_0)^2}\right)^{1/3} v^2(0,x)\,dx &= \int_0^1 \frac{p}{(x-x_0)^2} v^2(0,x)\,dx \\ &\le C_{\mathrm{HP}} \int_0^1 p(v_x)^2(0,x)\,dx \\ &\le C_1 C_{\mathrm{HP}} \int_0^1 a(v_x)^2(0,x)\,dx \\ &\le C \int_0^T \int_\omega v^2\,dxdt,\end{aligned}$$

for a positive constant C. Here $p(x) = (a(x)|x-x_0|^4)^{1/3}$, C_{HP} is the Hardy–Poincaré constant and

$$C_1 := \max\left\{\left(\frac{x_0^2}{a(0)}\right)^{2/3}, \left(\frac{(1-x_0)^2}{a(1)}\right)^{2/3}\right\},$$

as before.

By Lemma 2.1, the function

$$x \mapsto \frac{a(x)}{(x-x_0)^2}$$

is nondecreasing on $[0, x_0)$ and nonincreasing on $(x_0, 1]$. Then

$$\left(\frac{a(x)}{(x-x_0)^2}\right)^{1/3} \ge C_2 := \min\left\{\left(\frac{a(1)}{(1-x_0)^2}\right)^{1/3}, \left(\frac{a(0)}{x_0^2}\right)^{1/3}\right\} > 0.$$

Hence

$$C_2 \int_0^1 v(0,x)^2\,dx \le C \int_0^T \int_\omega v^2\,dxdt$$

and the thesis follows. □

Proof of Proposition 4.1. The proof is now standard, but we give it with some precise references: let $v_T \in L^2(0,1)$ and let v be the solution of (4.2) associated to v_T. Since $D(\mathcal{A}^2)$ is densely defined in $L^2(0,1)$, there exists a sequence $(v_T^n)_n \subset D(\mathcal{A}^2)$ which converges to v_T in $L^2(0,1)$. Now, consider the solution v_n associated to v_T^n.

As shown in Theorem 2.5, the semigroup generated by $\mathcal{A}$ is analytic, hence $\mathcal{A}$ is closed (for example, see [16, Theorem I.1.4]; thus, by [16, Theorem II.6.7], we get that $(v_n)_n$ converges to a certain v in $C(0,T;L^2(0,1))$, so that

$$\lim_{n\to+\infty}\int_0^1 v_n^2(0,x)\,dx = \int_0^1 v^2(0,x)\,dx,$$

and also

$$\lim_{n\to+\infty}\int_0^T\int_\omega v_n^2\,dxdt = \int_0^T\int_\omega v^2\,dxdt.$$

But, by Lemma 4.6 we know that

$$\int_0^1 v_n^2(0,x)\,dx \le C_T\int_0^T\int_\omega v_n^2\,dxdt.$$

Thus Proposition 4.1 is now proved. □

5 Linear extension

In this section we want to extend the observability inequality proved in the previous section starting from linear complete problems of the form

$$\begin{cases} u_t - (a(x)u_x)_x + c(t,x)u = h(t,x)\chi_\omega(x), & (t,x)\in(0,T)\times(0,1),\\ u(t,1) = u(t,0) = 0, & t\in(0,T),\\ u(0,x) = u_0(x), & x\in(0,1), \end{cases} \tag{5.1}$$

where $u_0\in L^2(0,1)$, $h\in L^2(Q_T)$, $c\in L^\infty(Q_T)$, ω is as in (4.1) and a satisfies Hypothesis 3.1. Observe that the well-posedness of (5.1) follows by [22, Theorem 4.1]. As for the previous case, we shall prove an observability inequality for the solution of the associated homogeneous adjoint problem

$$\begin{cases} v_t + (av_x)_x - cv = 0, & (t,x)\in(0,T)\times(0,1),\\ v(t,1) = v(t,0) = 0, & t\in(0,T),\\ v(T) = v_T\in L^2(0,1). \end{cases} \tag{5.2}$$

To obtain an observability inequality for (5.2) like the one in Proposition 4.1, we consider the problem

$$\begin{cases} v_t + (a(x)v_x)_x - cv = h, & (t,x)\in(0,T)\times(0,1),\\ v(t,1) = v(t,0) = 0, & t\in(0,T), \end{cases} \tag{5.3}$$

and we prove the following Carleman estimate as a corollary of Theorem 3.3.

Corollary 5.1. *Assume Hypothesis* 3.1 *and let $T > 0$. Then, there exist two positive constants C and s_0 such that every solution v in $\mathcal{V}$ of* (5.3) *satisfies, for all $s \geq s_0$,*

$$\int_0^T \int_0^1 \left(s\Theta a(v_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\varphi}\, dxdt \leq C\left(\int_0^T \int_0^1 h^2 e^{2s\varphi}\, dxdt + sc_1 \int_0^T \left[a\Theta e^{2s\varphi}(x-x_0)(v_x)^2\, dt \right]_{x=0}^{x=1} \right),$$

where c_1 is the constant introduced in (3.2).

Proof. Rewrite the equation of (5.3) as $v_t + (av_x)_x = \bar{h}$, where $\bar{h} := h + cv$. Then, applying Theorem 3.3, there exist two positive constants C and s_0 such that

$$\begin{aligned} &\int_0^T \int_0^1 \left(s\Theta a(v_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\varphi}\, dxdt \\ &\leq C\left(\int_0^T \int_0^1 \bar{h}^2 e^{2s\varphi}\, dxdt + sc_1 \int_0^T \left[a\Theta e^{2s\varphi}(x-x_0)(v_x)^2\, dt \right]_{x=0}^{x=1} \right) \end{aligned} \tag{5.4}$$

for all $s \geq s_0$. Using the definition of $\bar{h}$, the term $\int_0^T \int_0^1 \bar{h}^2 e^{2s\varphi(t,x)}\, dxdt$ can be estimated in the following way:

$$\begin{aligned} \int_0^T \int_0^1 \bar{h}^2 e^{2s\varphi}\, dxdt \leq{}& 2\int_0^T \int_0^1 h^2 e^{2s\varphi}\, dxdt \\ &+ 2\|c\|^2_{L^\infty(Q_T)} \int_0^T \int_0^1 e^{2s\varphi} v^2\, dxdt. \end{aligned} \tag{5.5}$$

Applying the Hardy–Poincaré inequality (see Proposition 2.6) to

$$w(t,x) := e^{s\varphi(t,x)} v(t,x)$$

and proceeding as in (3.13), recalling that $0 < \inf \Theta \leq \Theta \leq c\Theta^2$, one has

$$\begin{aligned} \int_0^1 e^{2s\varphi} v^2\, dx &= \int_0^1 w^2\, dx \\ &\leq C \int_0^1 a(w_x)^2\, dx + \frac{s}{2} \int_0^1 \frac{(x-x_0)^2}{a} w^2\, dx \\ &\leq C\Theta \int_0^1 a e^{2s\varphi}(v_x)^2\, dx + C\Theta^3 s^2 \int_0^1 e^{2s\varphi} v^2 \frac{(x-x_0)^2}{a}\, dx. \end{aligned}$$

Using this last inequality in (5.5), we have

$$\begin{aligned}\int_0^T \int_0^1 \bar{h}^2 e^{2s\varphi}\,dxdt &\\ \leq 2\int_0^T \int_0^1 h^2 e^{2s\varphi}\,dxdt &\\ + \|c\|^2_{L^\infty(Q_T)} C \int_0^T \int_0^1 \Theta a e^{2s\varphi}(v_x)^2\,dxdt &\\ + \|c\|^2_{L^\infty(Q_T)} C s^2 \int_0^T \int_0^1 \Theta^3 e^{2s\varphi}\frac{(x-x_0)^2}{a} v^2\,dxdt,&\end{aligned} \tag{5.6}$$

for a positive constant C. Using this inequality in (5.4), we obtain

$$\begin{aligned}\int_0^T \int_0^1 \left(s\Theta a(v_x)^2 + s^3\Theta^3\frac{(x-x_0)^2}{a}v^2\right) e^{2s\varphi}\,dxdt &\\ \leq C\Bigg(2\int_0^T \int_0^1 h^2 e^{2s\varphi}\,dxdt &\\ + \int_0^T \int_0^1 \Theta a e^{2s\varphi}(v_x)^2\,dxdt &\\ + s^2\int_0^T \int_0^1 e^{2s\varphi}\Theta^3\frac{(x-x_0)^2}{a}v^2\,dxdt &\\ + sc_1\int_0^T \left[a\Theta e^{2s\varphi}(x-x_0)(v_x)^2\,dt\right]_{x=0}^{x=1}\Bigg).&\end{aligned}$$

Hence, for all $s \geq s_0$, where s_0 is assumed sufficiently large, the thesis of the corollary follows. □

As a consequence of the previous corollary, one can deduce an observability inequality for the adjoint problem (5.2). In fact, without loss of generality we can assume that $c \geq 0$ (otherwise one can reduce the problem to this case introducing $\tilde{v} := e^{-\lambda t}v$ for a suitable λ). Using this assumption, we can prove that the analogous of Lemma 4.4 and of Lemma 4.7 still hold true. Thus, as before, one can prove the following observability inequality:

Proposition 5.2. *Assume Hypotheses* 3.1 *and* (4.1). *Then there exists a positive constant* C *such that every solution* $v \in C([0,T]; L^2(0,1)) \cap L^2(0,T; H_a^1(0,1))$ *of* (5.2) *satisfies*

$$\int_0^1 v^2(0,x)\,dx \leq C_T \int_0^T \int_\omega v^2(t,x)\,dxdt. \tag{5.7}$$

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