Research Article

Jagmohan Tyagi

Nontrivial solutions for singular semilinear elliptic equations on the Heisenberg group

Abstract: In this article, we prove the existence of nontrivial weak solutions to the singular boundary value problem

$$\begin{cases} -\Delta_{\mathbb{H}^n} u = \mu \dfrac{g(\xi)u}{(|z|^4+t^2)^{\frac{1}{2}}} + \lambda f(\xi,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

on the Heisenberg group. We employ Bonanno's three critical point theorem to obtain the existence of weak solutions.

Keywords: Singular elliptic equations, variational methods, nontrivial solutions

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Jagmohan Tyagi: Indian Institute of Technology Gandhinagar, Vishwakarma Government Engineering College Complex, Chandkheda, Visat-Gandhinagar Highway, Ahmedabad, Gujarat, India – 382424, e-mail: jtyagi@iitgn.ac.in, jtyagi1@gmail.com

1 Introduction

Let us recall the briefs on the Heisenberg group $\mathbb{H}^n$. The Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \bullet)$ is the space $\mathbb{R}^{2n+1}$ with the noncommutative law of product

$$(x, y, t) \bullet (x', y', t') = (x + x', y + y', t + t' + 2(\langle y, x'\rangle - \langle x, y'\rangle)),$$

where $x, y, x', y' \in \mathbb{R}^n$, $t, t' \in \mathbb{R}$ and $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$. This operation endows $\mathbb{H}^n$ with the structure of a Lie group. The Lie algebra of $\mathbb{H}^n$ is generated by the left-invariant vector fields

$$T = \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad i = 1, 2, .3, \dots, n.$$

These generators satisfy the noncommutative formula

$$[X_i, Y_j] = -4\delta_{ij}T, \quad [X_i, X_j] = [Y_i, Y_j] = [X_i, T] = [Y_i, T] = 0.$$

Let $z = (x, y) \in \mathbb{R}^{2n}$ and $\xi = (z, t) \in \mathbb{H}^n$. The parabolic dilation

$$\delta_\lambda \xi = (\lambda x, \lambda y, \lambda^2 t)$$

satisfies

$$\delta_\lambda(\xi_0 \bullet \xi) = \delta_\lambda \xi \bullet \delta \xi_0$$

and

$$\|\xi\|_{\mathbb{H}^n} = (|z|^4 + t^2)^{\frac{1}{4}} = ((x^2 + y^2)^2 + t^2)^{\frac{1}{4}}$$

is a norm with respect to the parabolic dilation which is known as Korányi gauge norm $N(z, t)$. In other words, $\rho(\xi) = (|z|^4 + t^2)^{\frac{1}{4}}$ denotes the Heisenberg distance between ξ and the origin. Similarly, one can define the distance between (z, t) and (z', t') on $\mathbb{H}^n$ as follows:

$$\rho(z, t; z', t') = \rho((z', t')^{-1} . (z, t)).$$

It is clear that the vector fields X_i, Y_i, $i = 1, 2, \ldots, n$, are homogeneous of degree 1 under the norm $\|\cdot\|_{\mathbb{H}^n}$ and T is homogeneous of degree 2. The Lie algebra of Heisenberg group has the stratification $\mathbb{H}^n = V_1 \oplus V_2$, where the $2n$-dimensional horizontal space V_1 is spanned by $\{X_i, Y_i\}$, $i = 1, 2, \ldots, n$, while V_2 is spanned by T. The Korányi ball of center ξ_0 and radius r is defined by

$$B_{\mathbb{H}^n}(\xi_0, r) = \{\xi : \|\xi^{-1} \bullet \xi_0\| \leq r\}$$

and it satisfies

$$|B_{\mathbb{H}^n}(\xi_0, r)| = |B_{\mathbb{H}^n}(0, r)| = r^d |B_{\mathbb{H}^n}(\mathbf{0}, 1)|,$$

where $|\cdot|$ is the $(2n+1)$-dimensional Lebesgue measure on $\mathbb{H}^n$ and $d = 2n + 2$ is the so-called homogeneous dimension of the Heisenberg group $\mathbb{H}^n$. The Heisenberg gradient and the Heisenberg Laplacian or the Laplacian–Kohn operator on $\mathbb{H}^n$ are respectively given by

$$\nabla_{\mathbb{H}^n} = (X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n)$$

and

$$\Delta_{\mathbb{H}^n} = \sum_{i=1}^n X_i^2 + Y_i^2 = \sum_{i=1}^n \Big(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4 y_i \frac{\partial^2}{\partial x_i \partial t} - 4 x_i \frac{\partial^2}{\partial y_i \partial t} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial t^2} \Big).$$

By [4], the fundamental solution on $\mathbb{H}^n$ of $-\Delta_{\mathbb{H}^n}$ with pole at the origin is

$$\Gamma(\xi) = \frac{c_d}{\rho(\xi)^{d-2}},$$

where c_d is a positive constant and $d = 2n + 2$ is the homogeneous dimension of the group. The fundamental solution on $\mathbb{H}^n$ of $-\Delta_{\mathbb{H}^n}$ with pole at ξ_0 is

$$\Gamma(\xi, \xi_0) = \frac{c_d}{\rho(\xi, \xi_0)^{d-2}}.$$

Let $\Omega \subset \mathbb{H}^n$, $n \geq 1$, be an open set. We define the associated Sobolev space as following:

$$H^1(\Omega, \mathbb{H}^n) = \{f \in L^2(\Omega) : \text{the distributional derivatives } X_i f, Y_i f \in L^2(\Omega),\ i = 1, 2, \ldots, n\}$$

and $H_0^1(\Omega, \mathbb{H}^n)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega, \mathbb{H}^n)$ under the norm

$$\|u\|_{H^1(\Omega, \mathbb{H}^n)} = \Big(\int_\Omega (|\nabla_{\mathbb{H}^n} u|^2 + |u|^2) d\xi \Big)^{\frac{1}{2}},$$

where $u : \Omega \subset \mathbb{H}^N \to \mathbb{R}$. The dual space of $H_0^1(\Omega, \mathbb{H}^n)$ will be denoted by $H^{-1}(\Omega, \mathbb{H}^n)$. In this work, we assume that $\Omega \subset \mathbb{H}^N$ is an open, bounded subset containing the origin with smooth boundary. Let us consider the following singular boundary value problem:

$$\begin{cases} -\Delta_{\mathbb{H}^n} u = \mu \dfrac{g(\xi) u}{(|z|^4 + t^2)^{\frac{1}{2}}} + \lambda f(\xi, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

For a precise survey on semilinear elliptic equations on Heisenberg group which is related to our problem, we consider the following prototype problem

$$\begin{cases} -\Delta_{\mathbb{H}^n} u = \mu V(\xi) u + f(\lambda, \xi, u) & \text{in } \Omega \subset \mathbb{H}^n, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

When $\mu = 0$ and $f(\lambda, \xi, u) = f_1(u)$ in (1.2), using the Mountain Pass Theorem and compact embedding, Garofalo and Lanconelli [5] proved the existence of a nonnegative solution, and using some integral identities of Rellich–Pohozaev type, they proved the nonexistence results to (1.2). Using the Mountain Pass Theorem and the Linking Theorem of Rabinowitz, Mokrani [8] studied the existence of a nontrivial solution to (1.2), where $f(\lambda, \xi, u) = \lambda u + |u|^{p-2}u$, $2 < p < 2 + \frac{2}{n}$, and V has a singularity on Ω which is controlled by an application of Hardy's inequality. Chen, Wei and Zhou [3] studied the same problem in a compact C^{∞}-manifold with conical singularity. Very recently, Balogh and Kristály [1] have obtained the existence of multiple solutions to

$$\begin{cases} -\Delta_{\mathbb{H}^n} u = \nu V(z,t)u - u + \lambda K(z,t) f(u) & \text{in } \Omega_\Psi \subset \mathbb{H}^n, \\ u = 0 & \text{on } \partial\Omega_\Psi, \end{cases} \tag{1.3}$$

where Ω_Ψ is an unbounded domain and V is measurable, cylindrically symmetric, i.e. $V(z,t) = V(|z|,t)$ and there exists a constant $C_V > 0$ such that

$$0 \le V(z,t) \le C_V \frac{|z|^2}{|z|^4 + t^2},$$

and assume certain assumptions on K and f. Since problem (1.1) has a singularity at the origin as in (1.3), so to establish the existence of nontrivial solutions to (1.1) is harder than (1.2).

Recently, using Bonanno's three critical point theorem [2], Kristály and Varga [6] have obtained the existence of multiple weak solutions to the problem

$$\begin{cases} -\Delta u = \mu \dfrac{u}{|x|^2} + \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega,\ N \ge 3, \end{cases} \tag{1.4}$$

where $f : \mathbb{R} \to \mathbb{R}$ is superlinear at zero and sublinear at ∞, i.e.

$$\lim_{s\to 0} \frac{f(s)}{s} = 0, \quad \lim_{|s|\to\infty} \frac{f(s)}{s} = 0,$$

respectively, and assume that

$$\sup_{s\in\mathbb{R}} F(s) > 0, \quad \text{where} \quad F(s) = \int_0^s f(t)dt.$$

We recall that (1.4) is a counterpart of (1.1) in $\mathbb{R}^n$ and the aim of this paper is to establish the existence of nontrivial solutions to (1.1) using similar ideas of [6]. We remark that Bonanno's three critical point theorem is already applied to p-Laplacian [10] and N-Laplace equations with singular weights [11], see also the references cited therein and many other research papers in Euclidean setting. But we are not aware of its applications on the Heisenberg group. In this paper we apply Bonanno's three critical point theorem to singular elliptic equations on the Heisenberg group. The main difficulty in this problem arises due to the singularity at the origin which is handled by using Hardy's inequality on the Heisenberg group [8].

Motivated by the above assumptions, let us also make the following hypotheses:

(H1) $\displaystyle\lim_{s\to 0} \frac{f(\xi,s)}{s} = 0$ uniformly $\xi \in \Omega$,

(H2) $\displaystyle\lim_{|s|\to\infty} \frac{f(\xi,s)}{s} = 0$ uniformly $\xi \in \Omega$,

(H3) $\displaystyle\sup_{s\in\mathbb{R}} F(\xi,s) > 0$ for all $\xi \in \Omega$, where $F(\xi,s) = \displaystyle\int_0^s f(\xi,t)dt$.

We state now the theorem we will prove in Section 4:

Theorem 1.1. *Suppose* (H1)–(H3) *hold. Suppose there exists an $M > 0$ such that $-M \le g(\xi) \le 1$. Then for every $\mu \in [0, (\frac{n^2}{n+1})^2)$, there exist an open interval $\Lambda_\mu \subset (0,\infty)$ and a real number $\eta_\mu > 0$ such that for every $\lambda \in \Lambda_\mu$, problem* (1.1) *has two nontrivial weak solutions $u \in H_0^1(\Omega, \mathbb{H}^n)$ such that $\|u\|_{H_0^1(\Omega,\mathbb{H}^n)} \le \eta_\mu$.*

We organize the paper as follows. Section 2 deals with the preliminaries. Section 3 deals with auxiliary lemmas which have been used in the proof of the main theorem. The main result is proved in Section 4.

2 Preliminaries

We begin this section with short preliminaries which have been used in this article. Let us recall the following Hardy inequality and a compact embedding theorem on the Heisenberg group.

Lemma 2.1 ([8]). *For $n \geq 1$ and for any $u \in H_0^1(\Omega, \mathbb{H}^n)$, we have*

$$\int_\Omega \frac{|u|^2}{(|z|^4 + |t|^2)^{\frac{1}{2}}} d\xi \leq \left(\frac{n+1}{n^2}\right)^2 \int_\Omega |\nabla_{\mathbb{H}^n} u|^2 d\xi.$$

For our convenience, we write the above inequality as follows:

$$\int_\Omega \frac{|u|^2}{(|z|^4 + |t|^2)^{\frac{1}{2}}} d\xi \leq \frac{1}{C_n} \int_\Omega |\nabla_{\mathbb{H}^n} u|^2 d\xi,$$

where $C_n = (\frac{n^2}{n+1})^2$.

Lemma 2.2 ([5]). *Let $\Omega \subset \mathbb{H}^N$ be a bounded open set. Then the following inclusion is compact:*

$$H_0^1(\Omega, \mathbb{H}^n) \subset\subset L^p(\Omega) \quad \text{for } 1 \leq p < \frac{2Q}{Q-2}, \; Q = 2n + 2.$$

The exponent $Q^* = \frac{2Q}{Q-2}$ is called critical since the embedding

$$H_0^1(\Omega, \mathbb{H}^n) \subset L^{Q^*}(\Omega)$$

is continuous but not compact for every domain Ω. We denote the Sobolev embedding constant of the above compact embedding by $S_p > 0$, i.e.

$$\|u\|_{L^p(\Omega)} \leq S_p \|u\|_{H_0^1(\Omega, \mathbb{H}^n)} \quad \text{for all } u \in H_0^1(\Omega, \mathbb{H}^n).$$

Let us introduce the energy functional

$$E_{\mu,\lambda} : H_0^1(\Omega, \mathbb{H}^n) \to \mathbb{R}$$

associated with (1.1), defined by

$$E_{\mu,\lambda}(u) = \Phi_\mu(u) - \lambda J(u), \quad u \in H_0^1(\Omega, \mathbb{H}^n),$$

where

$$\Phi_\mu(u) = \frac{1}{2} \int_\Omega |\nabla_{\mathbb{H}^n} u|^2 d\xi - \frac{\mu}{2} \int_\Omega \frac{g(\xi)|u|^2}{(|z|^4 + t^2)^{\frac{1}{4}}} d\xi, \quad J(u) = \int_\Omega F(\xi, u) d\xi.$$

Using (H2) and standard arguments, it is easy to see that $E_{\mu,\lambda}$ is of class C^1 and the critical points of $E_{\mu,\lambda}$ are exactly the weak solutions of (1.1). Therefore, it is sufficient to obtain the existence of multiple critical points of $E_{\mu,\lambda}$ for certain values of μ and λ. To establish the existence of critical points of $E_{\mu,\lambda}$, we use the following Bonanno's three critical point theorem [2].

Theorem 2.3 (Bonanno's three critical point theorem). *Let X be a separable and reflexive real Banach space and $\Phi, J : X \to \mathbb{R}$ be two continuously Gâteau differentiable functionals. Assume that there exists an $x_0 \in X$ such that $\Phi(x_0) = 0 = J(x_0)$ and $\Phi(x) \geq 0$ for every $x \in X$ and suppose there exist $x_1 \in X$ and $r > 0$ such that*
(i) $r < \Phi(x_1)$,
(ii) $\sup_{\Phi(x)<r} J(x) < r \frac{J(x_1)}{\Phi(x_1)}$.
Further, put

$$\bar{a} = \frac{hr}{r \frac{J(x_1)}{\Phi(x_1)} - \sup_{\Phi(x)<r} J(x)}, \quad h > 1,$$

and assume that the functional $\Phi - \lambda J$ is sequentially weakly lower semicontinuous, satisfies the Palais–Smale condition and
(iii) $\lim_{\|x\| \to +\infty} (\Phi(x) - \lambda J(x)) = +\infty$ *for every $\lambda \in [0, \bar{a}]$.*

Then there exist an open interval $\Lambda \subseteq [0, \bar{a}]$ and a positive real number η such that for each $\lambda \in \Lambda$, the equation

$$\Phi'(x) - \lambda J'(x) = 0$$

admits at least three solutions in X whose norms are less than η.

3 Auxiliary lemmas

We assume that the hypotheses of Theorem 1.1 are fulfilled throughout this section. We begin this section with some lemmas which have been used in the proof of the main theorem.

Lemma 3.1. *For every $\mu \in [0, C_n)$ and $\lambda \in \mathbb{R}$, the functional $E_{\mu,\lambda}$ is sequentially weakly lower semicontinuous on $H_0^1(\Omega, \mathbb{H}^n)$.*

Proof. By (H1), there exists a constant $C > 0$ such that

$$|f(\xi, s)| \leq C(1 + |s|) \quad \text{for all } \xi \in \Omega \text{ and all } s \in \mathbb{R}. \tag{3.1}$$

Using standard arguments with the compactness of the embedding $H_0^1(\Omega, \mathbb{H}^n) \subset\subset L^2(\Omega)$, one can see easily the sequentially weak continuity of J. Again using the compactness of the above embedding with the concentration compactness principle [7], in view of the work of Montefusco [9], it is not difficult to see that $\Phi_\mu(u)$ is sequentially weakly lower semicontinuous for every $\mu \in [0, C_n)$. □

Lemma 3.2. *For every $\mu \in [0, C_n)$ and $\lambda \in \mathbb{R}$, the functional $E_{\mu,\lambda}$ is coercive and satisfies the Palais–Smale condition.*

Proof. Let us fix $\mu \in [0, C_n)$ and let $\lambda \in \mathbb{R}$ be arbitrary. By (H1), there exists a constant $\delta = \delta(\mu, \lambda) > 0$ such that

$$|f(\xi, s)| < \frac{1}{2}(1 + |\lambda|)^{-1}\Big(1 - \frac{\mu}{C_n}\Big)S_2^{-2}|s| \quad \text{for all } \xi \in \Omega,\ |s| > \delta. \tag{3.2}$$

An integration yields

$$|F(\xi, s)| \leq \frac{1}{2}(1 + |\lambda|)^{-1}\Big(1 - \frac{\mu}{C_n}\Big)S_2^{-2}|s|^2 + \max_{\bar{\Omega} \times \{v : |v| \leq \delta\}} f(\xi, v)|s| \quad \text{for all } (\xi, s) \in \bar{\Omega} \times \mathbb{R}. \tag{3.3}$$

Since

$$E_{\mu,\lambda}(u) = \frac{1}{2}\int_\Omega |\nabla_{\mathbb{H}^n} u|^2 d\xi - \frac{\mu}{2}\int_\Omega \frac{g(\xi)|u|^2}{(|z|^4 + |t|^2)^{\frac{1}{2}}} d\xi - \lambda \int_\Omega F(\xi, u) d\xi$$

and $-M \leq g(\xi) \leq 1$, so from (3.3), we get

$$\begin{aligned}
E_{\mu,\lambda}(u) &\geq \frac{1}{2}\int_\Omega |\nabla_{\mathbb{H}^n} u|^2 d\xi - \frac{\mu}{2}\int_\Omega \frac{|u|^2}{(|z|^4 + |t|^2)^{\frac{1}{2}}} d\xi - \frac{|\lambda|}{2(1 + |\lambda|)}\Big(1 - \frac{\mu}{C_n}\Big)S_2^{-2}\int_\Omega |u|^2 d\xi - |\lambda| \max_{\bar{\Omega} \times \{t : |t| \leq \delta\}} |F(\xi, t)| \int_\Omega |u| d\xi \\
&\geq \frac{1}{2}\int_\Omega |\nabla_{\mathbb{H}^n} u|^2 d\xi - \frac{\mu}{2C_n}\int_\Omega |\nabla_{\mathbb{H}^n} u|^2 d\xi - \frac{|\lambda|}{2(1 + |\lambda|)}\Big(1 - \frac{\mu}{C_n}\Big)S_2^{-2}\int_\Omega |u|^2 d\xi - |\lambda| \max_{\bar{\Omega} \times \{t : |t| \leq \delta\}} |F(\xi, t)| \int_\Omega |u| d\xi \\
&\geq \frac{1}{2}\Big(1 - \frac{\mu}{C_n}\Big)\|u\|^2_{H_0^1(\Omega,\mathbb{H}^n)} - \frac{1}{2}\Big(1 - \frac{\mu}{C_n}\Big)S_2^{-2}\frac{|\lambda|}{1 + |\lambda|}S_2\|u\|^2_{H_0^1(\Omega,\mathbb{H}^n)} - |\lambda| \max_{\bar{\Omega} \times \{t : |t| \leq \delta\}} |F(\xi, t)| S_1 \|u\|_{H_0^1(\Omega,\mathbb{H}^n)} \\
&\geq \frac{1}{2(1 + |\lambda|)}\Big(1 - \frac{\mu}{C_n}\Big)\|u\|^2_{H_0^1(\Omega,\mathbb{H}^n)} - |\lambda| \max_{\bar{\Omega} \times \{t : |t| \leq \delta\}} |F(\xi, t)| S_1 \|u\|_{H_0^1(\Omega,\mathbb{H}^n)}.
\end{aligned} \tag{3.4}$$

Now, if $\|u\|_{H_0^1(\Omega,\mathbb{H}^n)} \to +\infty$, we conclude that $E_{\mu,\lambda}(u) \to +\infty$ as well, i.e. $E_{\mu,\lambda}(u)$ is coercive. For the Palais–Smale condition, let $\{u_n\}$ be a sequence in $H_0^1(\Omega, \mathbb{H}^n)$ such that $E_{\mu,\lambda}(u_n)$ is bounded and $\|E'_{\mu,\lambda}(u_n)\|_{H^{-1}(\Omega,\mathbb{H}^n)} \to 0$. Since $E_{\mu,\lambda}$ is coercive, so $\{u_n\}$ is bounded. Up to a subsequence, we may suppose that, as $n \to \infty$,

$$\begin{aligned}
u_n &\rightharpoonup u \quad &&\text{in } H_0^1(\Omega, \mathbb{H}^n),\\
u_n &\to u &&\text{in } L^2(\Omega),\\
u_n(\xi) &\to u(\xi) &&\text{for a.e. } \xi \in \Omega.
\end{aligned}$$

Now again using Lemma 2.1 and the fact that $-M \le g(\xi) \le 1$, using the same lines of proof as in [6], one can see easily that

$$\begin{aligned}\left(1-\frac{\mu}{C_n}\right)\|u_n-u\|_{H_0^1(\Omega,\mathbb{H}^n)} &\le \|u_n-u\|^2_{H_0^1(\Omega,\mathbb{H}^n)} - \mu\int_\Omega \frac{|u_n-u|^2}{(|z|^4+|t|^2)^{\frac{1}{2}}}d\xi \\ &= E'_{\mu,\lambda}(u_n)(u_n-u) + E'_{\mu,\lambda}(u)(u-u_n) + \lambda\int_\Omega (f(\xi,u_n(\xi)) - f(u(\xi)))(u_n(\xi)-u(\xi))d\xi. \end{aligned} \tag{3.5}$$

It is easy to see that $E'_{\mu,\lambda}(u_n)(u_n-u)$ and $E'_{\mu,\lambda}(u)(u-u_n)$ tend to 0 as $n\to\infty$ and using (3.1), we can see that

$$\int_\Omega (f(\xi,u_n(\xi)) - f(u(\xi)))(u_n(\xi)-u(\xi))d\xi \le C\big[2|\Omega|^{\frac{1}{2}} + \|u_n\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big]\|u_n-u\|_{L^2(\Omega)} \to 0 \quad \text{as } n\to\infty$$

and therefore the conclusion follows. □

Lemma 3.3. *For every $\mu\in[0,C_n)$,*

$$\lim_{m\to 0^+}\frac{\sup\{J(u):\Phi_\mu(u)<m\}}{m} = 0.$$

Proof. We fix $\mu\in[0,C_n)$. By (H1), for any given $\epsilon>0$ there exists a constant $\delta(\epsilon)>0$ such that

$$|f(\xi,s)| < \frac{\epsilon}{4}\left(1-\frac{\mu}{C_n}\right)S_2^{-2}|s| \quad \text{for all } \xi\in\Omega,\ |s|<\delta. \tag{3.6}$$

We fix a $\gamma_1\in(2,Q^*)$ and combining (3.1) with (3.6) yields

$$|F(\xi,s)| \le \frac{\epsilon}{4}\left(1-\frac{\mu}{C_n}\right)S_2^{-2}|s|^2 + C(1+\delta)\delta^{1-\gamma_1}|s|^{\gamma_1} \quad \text{for all } \xi\in\Omega,\ s\in\mathbb{R}. \tag{3.7}$$

For $m>0$, we define the sets

$$A_m = \{u\in H_0^1(\Omega,\mathbb{H}^n) : \Phi_\mu(u)<m\},\quad B_m = \left\{u\in H_0^1(\Omega,\mathbb{H}^n) : \left(1-\frac{\mu}{C_n}\right)\|u\|^2_{H_0^1(\Omega,\mathbb{H}^n)} < 2m\right\}.$$

By an application of (2.1), it is not difficult to see that $A_m\subseteq B_m$. By (3.7), for every $u\in A_m$ and hence $u\in B_m$ we have

$$\begin{aligned} J(u) &\le \frac{\epsilon}{4}\left(1-\frac{\mu}{C_n}\right)S_2^{-2}\int_\Omega |u|^2 d\xi + C(1+\delta)\delta^{1-\gamma_1}\int_\Omega |u|^{\gamma_1}d\xi \\ &\le \frac{\epsilon}{4}\left(1-\frac{\mu}{C_n}\right)\int_\Omega |\nabla_{\mathbb{H}^n}u|^2 d\xi + C(1+\delta)^{1-\gamma_1}S_{\gamma_1}^{\gamma_1}2^{\frac{\gamma_1}{2}}m^{\frac{\gamma_1}{2}}\left(1-\frac{\mu}{C_n}\right)^{-\frac{\gamma_1}{2}} \\ &\le \frac{\epsilon}{2}m + C(1+\delta)^{1-\gamma_1}S_{\gamma_1}^{\gamma_1}2^{\frac{\gamma_1}{2}}m^{\frac{\gamma_1}{2}}\left(1-\frac{\mu}{C_n}\right)^{-\frac{\gamma_1}{2}} \\ &\le \frac{\epsilon}{2}m + C_1 m^{\frac{\gamma_1}{2}}, \end{aligned} \tag{3.8}$$

where

$$C_1 = C(1+\delta)^{1-\gamma_1}S_{\gamma_1}^{\gamma_1}2^{\frac{\gamma_1}{2}}\left(1-\frac{\mu}{C_n}\right)^{-\frac{\gamma_1}{2}}.$$

Thus there exists an $m(\epsilon)>0$ such that for every $0<m<m(\epsilon)$

$$0 \le \frac{\sup_{u\in A_m}J(u)}{m} \le \frac{\sup_{u\in B_m}J(u)}{m} \le \frac{\epsilon}{2} + C_1 m^{\frac{\gamma_1-2}{2}} < \epsilon,$$

which proves the lemma. □

Now we are ready to sketch the proof of the main result.

4 Proof of Theorem 1.1 is concluded

Proof. Let $t_0 \in \mathbb{R}$ such that $F(\xi, t_0) > 0$, by (H3). Since $F(\xi, t_0)$ is continuous, there exists $\beta > 0$ such that $F(\xi, t_0) \geq \beta > 0$. We choose $R_0 > 0$ such that $R_0 < \operatorname{dist}(0, \partial\Omega)$. For $\eta \in (0,1)$, as already defined in [6] for the Euclidean case, we also define

$$u_\eta(w) = \begin{cases} 0, & \text{if } w \in \mathbb{H}^n \setminus B_{2n+1}(0, R_0), \\ t_0, & \text{if } w \in B_{2n+1}(0, \eta R_0), \\ \frac{t_0}{R_0(1-\eta)}(R_0 - |w|), & \text{if } w \in B_{2n+1}(0, R_0) \setminus B_{2n+1}(0, \eta R_0), \end{cases}$$

where $B_n(0, r)$ denotes the n-dimensional open ball with center 0 and radius $r > 0$. One can compute

$$|\nabla_{\mathbb{H}^n} u_\eta(w)|^2 = \frac{s_0^2}{R_0^2(1-\eta)^2}\left(1 - \frac{t^2}{r^2}\right)(1 + 4t^2)$$

and therefore it is easy to see that $u_\eta \in H_0^1(\Omega, \mathbb{H}^n)$. Also,

$$\begin{aligned} J(u_\eta) &\geq [F(\xi, t_0)\eta^{2n+1} - \max_{\overline{\Omega} \times \{s : |s| \leq |t_0|\}} |F(\xi, s)|(1 - \eta^{2n+1})]V_{2n+1}R_0^{2n+1} \\ &\geq [\beta\eta^{2n+1} - \max_{\overline{\Omega} \times \{s : |s| \leq |t_0|\}} |F(\xi, s)|(1 - \eta^{2n+1})]V_{2n+1}R_0^{2n+1}, \end{aligned} \tag{4.1}$$

where V_n denotes the volume of the n-dimensional unit ball in $\mathbb{R}^n$.

For η close enough to 1, the right hand side of the last inequality becomes strictly positive, so we choose such a number, say η_0. We fix $\mu \in [0, C_n)$. By Lemma 3.3, we may choose m_0 such that

$$2m_0 < \left(1 - \frac{\mu}{C_n}\right)\|u_{\eta_0}\|^2_{H_0^1(\Omega, \mathbb{H}^n)},$$

$$\sup\{J(u) : \Phi_\mu(u) < m_0\} < \frac{2[F(\xi, t_0)\eta^{2n+1} - \max_{\overline{\Omega} \times \{s : |s| \leq |t_0|\}} |F(\xi, s)|(1 - \eta^{2n+1})]V_{2n+1}R_0^{2n+1}}{\|u_{\eta_0}\|^2_{H_0^1(\Omega, \mathbb{H}^n)}}.$$

By choosing $x_1 = u_{\eta_0}$, the hypotheses of Theorem 2.3 are satisfied. Define

$$\bar{A} = \bar{A}_\mu = \frac{1 + m_0}{\frac{J(u_{\eta_0})}{\Phi_\mu(u_{\eta_0})} - \frac{\sup\{J(u) : \Phi_\mu(u) < m_0\}}{m_0}}. \tag{4.2}$$

In view of Lemmas 3.1 and 3.2, all the hypotheses of Theorem 2.3 are satisfied after putting $x_0 = 0$. An application of Theorem 2.3 implies that there exist an open interval $\Lambda_\mu \subset [0, \bar{A}_\mu]$ and a number $\eta_\mu > 0$ such that for each $\lambda \in \Lambda_\mu$, the equation $E'_{\mu,\lambda} \equiv \Phi'_\mu(u) - \lambda J'(u) = 0$ admits at least three solutions in $H_0^1(\Omega, \mathbb{H}^n)$ which have $H_0^1(\Omega, \mathbb{H}^n)$-norm less than η_μ. Since (H1) implies that $f(\xi, 0) = 0$, so (H1) admits one trivial solution and hence there exist two nontrivial solutions to equation (1.1), which completes the proof. □

Remark 4.1. As in [6, 10, 11], we can find an explicit estimation of the intervals $\Lambda_\mu, \mu \in [0, C_n)$. In order to obtain the estimation, let us fix t_0, R_0 and η_0 as in the previous section. Let $\mu \in [0, C_n)$ and by Lemma 3.3, we can assume that $m_0 < 1$ and

$$\frac{\sup\{J(u) : \Phi_\mu(u) < m_0\}}{m_0} < \frac{J(u_{\eta_0})}{2\Phi_\mu(u_{\eta_0})}. \tag{4.3}$$

Now from (4.2) and (4.3), we have

$$\bar{A}_\mu < \frac{4\Phi_\mu(u_{\eta_0})}{J(u_{\eta_0})} \tag{4.4}$$

and therefore

$$\Lambda_\mu \subset \left[0, 2\left(1 - \frac{\mu}{C_n}\right)\frac{s_0^2(1 + 4R_0^2)}{R_0^2(1-\eta_0)^2[\beta\eta_0^{2n+1} - \max_{\overline{\Omega} \times \{s : |s| \leq |t_0|\}} |F(\xi, s)|(1 - \eta_0^{2n+1})]V_{2n+1}R_0^{2n+1}}\right].$$

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