Research Article

Massimiliano Ferrara, Shapour Heidarkhani and Pasquale F. Pizzimenti

Multiple solutions for Dirichlet impulsive equations

Abstract: In this paper we are interested to ensure the existence of multiple nontrivial solutions for some classes of problems under Dirichlet boundary conditions with impulsive effects. More precisely, by using a suitable analytical setting, the existence of at least three solutions is proved exploiting a recent three-critical points result for smooth functionals defined in a reflexive Banach space. Our approach generalizes some well-known results in the classical framework.

Keywords: Three solutions, Dirichlet boundary value problems with impulsive effects, critical point theory, variational methods

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Massimiliano Ferrara, Pasquale F. Pizzimenti: Department of Law and Economics, University Mediterranea of Reggio Calabria, Via dei Bianchi, 2, Palazzo Zani, I-89127, Reggio Calabria, Italy, e-mail: massimiliano.ferrara@unirc.it, ppizzimenti@unime.it
Shapour Heidarkhani: Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran, e-mail: s.heidarkhani@razi.ac.ir

1 Introduction

We deal with the following nonlinear Dirichlet problem

$$\begin{cases} -u''(x) = \lambda f(x, u(x)) + g(u(x)), & \text{a.e. } x \in [0, T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

with the impulsive conditions

$$\Delta u'(x_j) = I_j(u(x_j)), \quad j = 1, 2, \dots, p, \tag{1.2}$$

where $T > 0$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e.

$$|g(t_1) - g(t_2)| \le L|t_1 - t_2| \quad \text{for all } t_1, t_2 \in \mathbb{R},$$

satisfying $g(0) = 0$ and $x_0 = 0 < x_1 < x_2 < \cdots < x_p < x_{p+1} = T$ where $p \ge 1$,

$$\Delta u'(x_j) = u'(x_j^+) - u'(x_j^-) = \lim_{x \to x_j^+} u'(x) - \lim_{x \to x_j^-} u'(x)$$

and $I_j : \mathbb{R} \to \mathbb{R}$ are continuous satisfying the condition $\sum_{j=1}^{p}(I_j(t_1) - I_j(t_2))(t_1 - t_2) \ge 0$ for every $t_1, t_2 \in \mathbb{R}$ and λ is a positive real parameter. We refer to the impulsive problem (1.1)–(1.2) as (P).

Impulsive differential equations have become more important in recent years in some mathematical models of real processes and phenomena studied in spacecraft control, impact mechanics, physics, chemistry, chemical engineering, population dynamics, biotechnology, economics and inspection process in operations research. It is now recognized that the theory of impulsive differential equations is a natural framework for a mathematical modelling of many natural phenomena. For the background, theory and applications of impulsive differential equations, we refer the interested readers to [4, 5, 13, 15, 16, 22, 23, 28]. For a second order differential equation $u'' = f(t, u, u')$ one usually considers impulses in the position u and the velocity u'. However, in the motion of spacecraft one has to consider instantaneous impulses depending on the position that results in jump discontinuities in velocity, but with no change in position, see [8, 9, 19, 26]. The impulses only on the velocity occur also in impulsive mechanics, see [24, 25].

Impulsive differential equations have been studied extensively in the literature. There have been many approaches to study the existence of solutions of impulsive differential equations, such as fixed point theory, topological degree theory (including continuation methods and coincidence degree theory) and comparison method (including upper and lower solution methods and monotone iterative methods) and so on (see, for example, [2, 10, 12, 17, 18] and references therein). Recently, in [1, 3, 11, 14, 20, 21, 27, 30, 32, 33, 35–37] the existence and multiplicity of solutions of impulsive problems have been studied using critical point theory. In [21] and [37] the authors studied the following problem:

$$\begin{cases} -u''(x) + ku(x) = f(x, u(x)), \quad \text{a.e. } x \in [0, T], \\ u(0) = u(T) = 0, \end{cases}$$

with the impulsive conditions

$$\Delta u'(x_j) = I_j(u(x_j)), \quad j = 1, 2, \dots, p,$$

where k is a parameter and under different assumptions in which they established the existence of at least one solution and the existence of infinitely many weak solutions, respectively. In particular, in [3] the authors, using critical point theory, studied the multiplicity of solutions for problem (1.1) when $g = 0$.

In the present paper, motivated by [3], by using variational methods we ensure an exact collection of the parameter λ for which problem (P) admits at least three weak solutions.

For a thorough account on the subject we refer the reader to [6, 31].

2 Preliminaries

Our analysis is based on the following three-critical points theorem to transfer the existence of three solutions of problem (P) into the existence of critical points of the Euler functional.

Theorem 2.1 ([7, Theorem 2.6]). *Let X be a reflexive real Banach space, let $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a sequentially weakly upper semicontinuous and continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exist $r \in \mathbb{R}$ and $u_1 \in X$ with $0 < r < \Phi(u_1)$, such that*

(a$_1$)
$$\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u) < r \frac{\Psi(u_1)}{\Phi(u_1)},$$

(a$_2$) *for each* $\lambda \in \Lambda_r := \left] \frac{\Phi(u_1)}{\Psi(u_1)}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[$ *the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Let $X := H_0^1(0, T)$. In the Sobolev space X, consider the inner product

$$\langle u, v \rangle = \int_0^T u'(x)v'(x)dx$$

and the corresponding norm

$$\|u\| = \Big(\int_0^T |u'(x)|^2 dx \Big)^{\frac{1}{2}}.$$

Let us denote $H^2(0, T) = \{u \in C^1[0, T] : u'' \in L^2[0, T]\}$. By a classical solution of problem (P), we mean a function $u \in \{u(x) \in H^1(0, T) : u(x) \in H^2(x_j, x_{j+1}),\ j = 0, 1, \dots, p\}$ such that u satisfies (1.1)–(1.2). We say that a function $u \in X$ is a weak solution of problem (P) if

$$\int_0^T u'(x)v'(x)dx + \sum_{j=1}^p I_j(u(x_j))v(x_j) - \int_0^T g(u(x))v(x)dx - \lambda \int_0^T f(x, u(x))v(x)dx = 0$$

for every $v \in X$. Using standard methods, if f is continuous, we see that a weak solution of (P) is indeed a classical solution (see [3, Lemma 5]).

In this paper, we assume throughout, and without further mention, that the following condition holds:

(A1) The impulsive functions I_j have sublinear growth, i.e., there exist constants $a_j > 0$, $b_j > 0$ and $\gamma_j \in [0, 1)$ for $j = 1, 2, \dots, p$ such that

$$|I_j(t)| \leq a_j + b_j|t|^{\gamma_j} \quad \text{for every } t \in \mathbb{R},\ j = 1, 2, \dots, p.$$

For the sake of convenience, in the sequel, we define

$$F(x,t) = \int_0^t f(x,\xi)d\xi \text{ for all } (x,t) \in [0,T] \times \mathbb{R}, \quad G(t) = -\int_0^t g(\xi)d\xi \text{ for all } t \in \mathbb{R},$$

and

$$C_1 = \frac{1}{2} - \sum_{j=1}^{p} \frac{b_j}{\gamma_j + 1}\Big(\frac{\sqrt{T}}{2}\Big)^{\gamma_j+1}, \quad C_2 = \frac{1}{2} + \sum_{j=1}^{p} \frac{b_j}{\gamma_j + 1}\Big(\frac{\sqrt{T}}{2}\Big)^{\gamma_j+1}, \quad C_3 = \sum_{j=1}^{p} a_j\frac{\sqrt{T}}{2} + \sum_{j=1}^{p} \frac{b_j}{\gamma_j + 1}\Big(\frac{\sqrt{T}}{2}\Big)^{\gamma_j+1}.$$

Suppose that the Lipschitz constant $L > 0$ of the function g satisfies $LT^2 < 4$.

A special case of our main result is the following theorem.

Theorem 2.2. *Suppose that $C_1 - \frac{LT^2}{8} > 0$ and there exist positive constants η and δ with $\eta + \delta < T$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $F(t) = \int_0^t f(\xi)d\xi$ for each $t \in \mathbb{R}$. Assume that $F(d) > 0$ for some $d > 0$ and $F(\xi) \geq 0$ in $[0, d]$ and*

$$\liminf_{\xi \to 0} \frac{F(\xi)}{\frac{4}{T}(C_1 - \frac{LT^2}{8})\xi^2 - \frac{2}{\sqrt{T}}C_3\xi} = \limsup_{\xi \to +\infty} \frac{F(\xi)}{\frac{4}{T}(C_1 - \frac{LT^2}{8})\xi^2 - \frac{2}{\sqrt{T}}C_3\xi} = 0.$$

Then, there is $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ the problem

$$\begin{cases} -u''(x) = f(u(x)) + g(u(x)), \\ u(0) = u(T) = 0, \end{cases}$$

with the impulsive conditions

$$\Delta u'(x_j) = I_j(u(x_j)), \quad j = 1, 2, \dots, p,$$

where the function g is given as in problem (P), *admits at least three classical solutions.*

We need the following proposition in the proof of the main result.

Proposition 2.3. *Let $T : X \to X^*$ be the operator defined by*

$$T(u)v = \int_0^T u'(x)v'(x)dx + \sum_{j=1}^{p} I_j(u(x_j))v(x_j) - \int_0^T g(u(x))v(x)dx$$

for every $u, v \in X$. Then T admits a continuous inverse on X^*.

Proof. In view of assumption (A1), for $u \in X$ when $u(x_j) \geq 0$,

$$I_j(u(x_j))u(x_j) \geq (-a_j - b_j|u(x_j)|^{\gamma_j})u(x_j) = -a_ju(x_j) - b_j(u(x_j))^{\gamma_j+1};$$

when $u(x_j) < 0$,

$$I_j(u(x_j))u(x_j) \geq (a_j + b_j|u(x_j)|^{\gamma_j})u(x_j) = a_ju(x_j) + b_j(u(x_j))^{\gamma_j+1}.$$

Therefore, since

$$\|u\|_\infty = \max_{t\in[0,T]} |u(t)| \leq \frac{\sqrt{T}}{2}\|u\|, \tag{2.3}$$

for every $u(x_j)$, we have

$$I_j(u(x_j))u(x_j) \geq -a_j|u(x_j)| - b_j|u(x_j)|^{\gamma_j+1} \geq -a_j\frac{\sqrt{T}}{2}\|u\| - b_j\|u\|^{\gamma_j+1}. \tag{2.4}$$

So, recalling that

$$|g(t_1) - g(t_2)| \leq L|t_1 - t_2|$$

for all $t_1, t_2 \in \mathbb{R}$ and $g(0) = 0$, from (2.3) and (2.4), for any $u \in X \setminus \{0\}$, we obtain

$$\begin{aligned}
\lim_{\|u\|\to\infty} \frac{\langle T(u), u\rangle}{\|u\|} &= \lim_{\|u\|\to\infty} \frac{\int_0^T (u'(x))^2 dx + \sum_{j=1}^p I_j(u(x_j))u(x_j) - \int_0^T g(u(x))u(x)dx}{\|u\|} \\
&\geq \lim_{\|u\|\to\infty} \frac{\int_0^T (u'(x))^2 dx + \sum_{j=1}^p I_j(u(x_j))u(x_j) - \frac{LT^2}{4}\|u\|^2}{\|u\|} \\
&\geq \lim_{\|u\|\to\infty} \frac{(1 - \frac{LT^2}{4})\|u\|^2 - \sum_{j=1}^p (a_j \frac{\sqrt{T}}{2}\|u\| + b_j\|u\|^{\gamma_j+1})}{\|u\|} = \infty.
\end{aligned}$$

Thus, the map T is coercive. Now, taking into account [29, (2.2)], since

$$\sum_{j=1}^p (I_j(t_1) - I_j(t_2))(t_1 - t_2) \geq 0$$

for $t_1, t_2 \in \mathbb{R}$, we see that

$$\langle T(u) - T(v), u - v\rangle \geq \Big(1 - \frac{LT^2}{4}\Big)\|u - v\|^2,$$

so T is uniformly monotone. Therefore, since T is hemicontinuous in X, by [34, Theorem 26.A (d)], T^{-1} exists and is continuous on X^*. □

3 Main results

We formulate our main result as follows:

Theorem 3.1. *Assume that $C_1 - \frac{LT^2}{8} > 0$, and there exist positive constants η, δ, c and d with $\eta + \delta < T$ and*

$$d > c > \frac{\sqrt{T}C_3}{4(C_1 - \frac{LT^2}{8})}$$

such that

(A2) $$F(x,t) \geq 0 \text{ for all } (x,t) \in ([0,\eta] \cup [T - \delta]) \times [0,d],$$

(A3) $$\int_0^T \sup_{t\in[-c,c]} F(x,t)dx < \frac{\frac{4}{T}(C_1 - \frac{LT^2}{8})c^2 - \frac{2}{\sqrt{T}}C_3 c}{\frac{\eta+\delta}{\eta\delta}(C_2 + \frac{LT^2}{8})d^2 + \sqrt{\frac{\eta+\delta}{\eta\delta}}C_3 d} \int_\eta^{T-\delta} F(x,d)dx,$$

(A4) $$\limsup_{|t|\to+\infty} \frac{F(x,t)}{t^2} < \frac{4(C_1 - \frac{LT^2}{8})}{T^2} \int_0^T \sup_{t\in[-c,c]} F(x,t)dx, \text{ uniformly with respect to } x \in [0,T].$$

Then, for each

$$\lambda \in \Lambda_1 := \left] \frac{\frac{\eta+\delta}{\eta\delta}(C_2 + \frac{LT^2}{8})d^2 + \sqrt{\frac{\eta+\delta}{\eta\delta}}C_3 d}{\int_\eta^{T-\delta} F(x,d)dx}, \frac{\frac{4}{T}(C_1 - \frac{LT^2}{8})c^2 - \frac{2}{\sqrt{T}}C_3 c}{\int_0^T \sup_{t\in[-c,c]} F(x,t)dx} \right[$$

problem (P) *admits at least three distinct weak solutions in* X.

We now exhibit an example in which the hypotheses of Theorem 3.1 are satisfied.

Example 3.1. Consider the problem

$$\begin{cases} -u''(x) = \lambda e^{x-u}u^{17}(18-u) + \frac{1}{16}u, \quad \text{a.e. } x \in [0,4], \\ u(0) = u(4) = 0, \\ \Delta u'(x_1) = \frac{1}{16} + \frac{1}{32}|u(x_1)|^{\frac{1}{2}}, \quad \Delta u'(x_2) = \frac{1}{32} + \frac{1}{64}|u(x_2)|^{\frac{1}{2}}, \quad 0 < x_1 < x_2 < 4, \end{cases} \tag{3.4}$$

such that $\frac{1}{32}(|u(x_1)|^{\frac{1}{2}} - |v(x_1)|^{\frac{1}{2}})(u(x_1) - v(x_1)) + \frac{1}{64}(|u(x_2)|^{\frac{1}{2}} - |v(x_2)|^{\frac{1}{2}})(u(x_2) - v(x_2)) \geq 0$ for all $u, v \in H_0^1(0,4)$. It is obvious that $C_1 = \frac{15}{32}, C_2 = \frac{17}{32}$ and $C_3 = \frac{1}{8}$. Also a direct calculation shows that $F(x,t) = e^{x-t}t^{18}$ for all $(x,t) \in [0,4] \times \mathbb{R}$. Choose $\eta = \delta = 1$, $c = 5$ and $d = 10$. Clearly assumption (A2) is fulfilled, and a straightforward computation shows that assumption (A3) is satisfied. In particular, since

$$\limsup_{|t| \to +\infty} \frac{F(x,t)}{t^2} = 0$$

for every $x \in [0,4]$, assumption (A4) is satisfied. So, Theorem 3.1 is applicable to problem (3.4) for every

$$\lambda \in \left]\frac{e^9(105 + \sqrt{2})}{8 \cdot 10^{17}(e^2 - 1)}, \frac{51e^5}{32 \cdot 5^{17}(e^4 - 1)}\right[.$$

Proof of Theorem 2.2. Fix

$$\lambda > \lambda^* := \frac{\frac{\eta+\delta}{\eta\delta}(C_2 + \frac{LT^2}{8})d^2 + \sqrt{\frac{\eta+\delta}{\eta\delta}}C_3 d}{(T - \delta - \eta)F(d)}$$

for some $d > 0$. Taking into account that

$$\liminf_{\xi \to 0} \frac{F(\xi)}{\frac{4}{T}(C_1 - \frac{LT^2}{8})\xi^2 - \frac{2}{\sqrt{T}}C_3\xi} = 0,$$

there is a sequence $\{c_n\} \subset]0, +\infty[$ such that $\lim_{n\to+\infty} c_n = 0$ and

$$\lim_{n\to\infty} \frac{\sup_{|\xi| \leq c_n} F(\xi)}{\frac{4}{T}(C_1 - \frac{LT^2}{8})c_n^2 - \frac{2}{\sqrt{T}}C_3 c_n} = 0.$$

In fact, one has

$$\lim_{n\to\infty} \frac{\sup_{|\xi| \leq c_n} F(\xi)}{\frac{4}{T}(C_1 - \frac{LT^2}{8})c_n^2 - \frac{2}{\sqrt{T}}C_3 c_n} = \lim_{n\to\infty} \frac{F(\xi_{c_n})}{\frac{4}{T}(C_1 - \frac{LT^2}{8})\xi_{c_n}^2 - \frac{2}{\sqrt{T}}C_3\xi_{c_n}} \cdot \frac{\frac{4}{T}(C_1 - \frac{LT^2}{8})\xi_{c_n}^2 - \frac{2}{\sqrt{T}}C_3\xi_{c_n}}{\frac{4}{T}(C_1 - \frac{LT^2}{8})c_n^2 - \frac{2}{\sqrt{T}}C_3 c_n} = 0,$$

where $F(\xi_{c_n}) = \sup_{|\xi| \leq c_n} F(\xi)$. Hence, there is $\bar{c} > 0$ such that

$$\frac{\sup_{|\xi| \leq \bar{c}} F(\xi)}{\frac{4}{T}(C_1 - \frac{LT^2}{8})\bar{c}^2 - \frac{2}{\sqrt{T}}C_3\bar{c}} < \min\left\{\frac{(T - \delta - \eta)F(d)}{T(\frac{\eta+\delta}{\eta\delta}(C_2 + \frac{LT^2}{8})d^2 + \sqrt{\frac{\eta+\delta}{\eta\delta}}C_3 d)}, \frac{1}{\lambda T}\right\},$$

and $d > \bar{c} > \frac{\sqrt{T}C_3}{2(C_1 - \frac{LT^2}{8})}$. Applying Theorem 3.1 the desired conclusion follows. □

4 Proofs of the main results

Proof of Theorem 3.1. We proceed by applying Theorem 2.1 for the functionals $\Phi, \Psi : X \to \mathbb{R}$ given by

$$\Phi(u) = \frac{1}{2}\int_0^T (u'(x))^2 dx + \sum_{j=1}^{p} \int_0^{u(x_j)} I_j(t)dt + \int_0^T G(u(x))dx \tag{4.1}$$

and

$$\Psi(u) = \int_0^T F(x, u(x))dx \tag{4.2}$$

for each $u \in X$. It is well known that Ψ is a Gâteaux differentiable functional and sequentially weakly lower semicontinuous whose Gâteaux derivative at the point $u \in X$ is the functional $\Psi'(u) \in X^*$, given by

$$\Psi'(u)(v) = \int_0^T f(x, u(x))v(x)dx$$

for every $v \in X$. We claim that $\Psi' : X \to X^*$ is a compact operator. Indeed, for fixed $u \in X$, assume $u_n \to u$ weakly in X as $n \to \infty$. Then $u_n \to u$ strongly in $C([0, T])$. Since $f(x, \cdot)$ is continuous in $\mathbb{R}$ for every $x \in [0, T]$, we get that $f(x, u_n) \to f(x, u)$ strongly as $n \to \infty$. By the Lebesgue Theorem, $\Psi'(u_n) \to \Psi'(u)$ strongly, which means that Ψ' is strongly continuous, then it is a compact operator. Hence the claim holds true. Moreover, Φ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_0^T u'(x)v'(x)dx + \sum_{j=1}^{p} I_j(u(x_j))v(x_j) - \int_0^T g(u(x))v(x)dx$$

for every $v \in X$. Furthermore, Proposition 2.3 gives that Φ' admits a continuous inverse on X^* and since Φ' is monotone, we obtain that Φ is sequentially weakly lower semicontinuous (see [34, Proposition 25.20]). Moreover, similar to the argument as given in [3], for all $u \in X$ we see that

$$\Big(C_1 - \frac{LT^2}{8}\Big)\|u\|^2 - C_3\|u\| \le \Phi(u) \le \Big(C_2 + \frac{LT^2}{8}\Big)\|u\|^2 + C_3\|u\|. \tag{4.3}$$

Choose

$$r := \frac{4}{T}\Big(C_1 - \frac{LT^2}{8}\Big)c^2 - \frac{2}{\sqrt{T}}C_3 c \quad \text{and} \quad u_1(x) = \begin{cases} \frac{d}{\eta}x, & x \in [0, \eta), \\ d, & x \in [\eta, T - \delta], \\ \frac{d}{\delta}(T - x), & x \in (T - \delta, T]. \end{cases}$$

It is clear that $u_1 \in X$ and

$$\|u_1\| = \sqrt{\frac{\eta + \delta}{\eta\delta}}d.$$

From assumption (A1) we observe that C_2 and C_3 are positive. Arguing as in [3], let

$$H(t) = \Big(C_1 - \frac{LT^2}{8}\Big)t^2 - C_3 t$$

for $t \ge 0$. Since $C_1 - \frac{LT^2}{8} > 0$ and $C_3 > 0$, one has

$$H(t) \le 0 \quad \text{for every } t \in \Big[0, \frac{C_3}{C_1 - \frac{LT^2}{8}}\Big]$$

and $H(t)$ is strictly increasing on $[\frac{C_3}{C_1 - \frac{LT^2}{8}}, +\infty)$. Since $\eta > 0$, $\delta > 0$ and $\eta + \delta < T$, we get

$$\frac{1}{\eta} + \frac{1}{\delta} > \frac{4}{\delta}.$$

Moreover, since $C_1 - \frac{LT^2}{8} > 0$ and $C_3 > 0$ and $c > \frac{\sqrt{T}C_3}{4(C_1 - \frac{LT^2}{8})}$, we have $c > 0$. Thus,

$$\sqrt{\frac{\eta + \delta}{\eta\delta}}c > \frac{2}{\sqrt{T}}c.$$

From $d > c$, we have

$$\sqrt{\frac{\eta + \delta}{\eta\delta}}d > \sqrt{\frac{\eta + \delta}{\eta\delta}}c.$$

Moreover, the condition

$$c > \frac{\sqrt{T}C_3}{4(C_1 - \frac{LT^2}{8})}$$

implies that

$$\frac{2}{\sqrt{T}}c > \frac{C_3}{C_1 - \frac{LT^2}{8}}.$$

Hence,

$$\|u_1\| > \frac{2}{\sqrt{T}}c > \frac{C_3}{(C_1 - \frac{LT^2}{8})},$$

which in conjunction with the fact that $H(t)$ is strictly increasing on $[\frac{C_3}{C_1 - \frac{LT^2}{8}}, +\infty)$, yields that

$$H(\|u_1\|) > H\Big(\frac{2}{\sqrt{T}}c\Big) > H\Big(\frac{C_3}{C_1 - \frac{LT^2}{8}}\Big),$$

which means

$$\Big(C_1 - \frac{LT^2}{8}\Big)\|u_1\|^2 - C_3\|u_1\| > \Big(C_1 - \frac{LT^2}{8}\Big)\Big(\frac{2}{\sqrt{T}}c\Big)^2 - C_3\Big(\frac{2}{\sqrt{T}}c\Big) = r.$$

It follows from $H(t) = r$ that

$$t = \frac{2c}{\sqrt{T}} \quad \text{or} \quad t = \frac{C_3 - \frac{2c(C_1 - \frac{LT^2}{8})}{\sqrt{T}}}{(C_1 - \frac{LT^2}{8})}.$$

Moreover, since $c > 0$, $C_1 - \frac{LT^2}{8} > 0$ and

$$c > \frac{\sqrt{T}C_3}{4(C_1 - \frac{LT^2}{8})},$$

we observe that the unique solution of $H(t) = r$ with $x \in [0, +\infty)$ is $t = \frac{2c}{\sqrt{T}}$. Thus, taking into account that $H(t)$ is strictly increasing on $[\frac{C_3}{C_1 - \frac{LT^2}{8}}, +\infty)$ and

$$c > \frac{\sqrt{T}C_3}{4(C_1 - \frac{LT^2}{8})},$$

we obtain

$$\Big[\frac{C_3}{C_1 - \frac{LT^2}{8}}, \frac{2}{\sqrt{T}}c\Big] \subseteq \{t : H(t) \leq r\} \tag{4.4}$$

and

$$\Big(\frac{2}{\sqrt{T}}c, +\infty\Big) \cap \{t : H(t) \leq r\} = \emptyset. \tag{4.5}$$

Recalling that $H(t) \leq 0$ for every $t \in [0, \frac{C_3}{C_1 - \frac{LT^2}{8}}]$ and $r > 0$, one has

$$\Big[0, \frac{C_3}{C_1 - \frac{LT^2}{8}}\Big] \subseteq \{t : H(t) \leq r\}. \tag{4.6}$$

Thus, from (4.4)–(4.6), we have

$$\{t : H(t) \leq r\} = \Big[0, \frac{2}{\sqrt{T}}c\Big]. \tag{4.7}$$

For any $u \in X$, from (4.3), we observe

$$H(\|u\|) \leq \Phi(u) \leq r.$$

Therefore, using (4.7) one has $\|u\| \leq \frac{2c}{\sqrt{T}}$. So, taking (2.3) into account, we have

$$\Phi^{-1}(]-\infty, r]) \subseteq \{u \in X : \|u\|_\infty \leq c\},$$

which leads to

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) = \sup_{u\in\Phi^{-1}(]-\infty,r])} \int_0^T F(x, u(x))dx \leq \int_0^T \sup_{t\in[-c,c]} F(x, t)dx.$$

Therefore, assumption (A3) implies that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)\leq\int_0^T\sup_{t\in[-c,c]}F(x,t)dx<\frac{\frac{4}{T}(C_1-\frac{LT^2}{8})c^2-\frac{2}{\sqrt{T}}C_3c}{\frac{\eta+\delta}{\eta\delta}(C_2+\frac{LT^2}{8})d^2+\sqrt{\frac{\eta+\delta}{\eta\delta}}C_3d}\int_\eta^{T-\delta}F(x,d)dx$$
$$<r\frac{\int_0^T F(x,u_1(x))dx}{\Phi(u_1)}$$
$$=r\frac{\Psi(u_1)}{\Phi(u_1)},$$

namely, assumption (a_1) of Theorem 2.1 is fulfilled. Furthermore, due to assumption (A4), there exist two constants $\gamma,\eta\in\mathbb{R}$ with

$$\gamma<\frac{\int_0^T\sup_{t\in[-c,c]}F(x,t)dx}{\frac{4}{T}(C_1-\frac{LT^2}{8})c^2-\frac{2}{\sqrt{T}}C_3c}$$

such that

$$\frac{T^2}{4(C_1-\frac{LT^2}{8})}F(x,t)\leq\gamma t^2+\eta\quad\text{for a.e. } x\in[0,T].$$

Fix $u\in X$. Then

$$F(x,u(x))\leq\frac{4(C_1-\frac{LT^2}{8})}{T^2}(\gamma|u(x)|^2+\eta)\quad\text{for a.e. } x\in[0,T].\tag{4.8}$$

Now, in order to prove the coercivity of the functional $\Phi-\lambda\Psi$, first we assume that $\gamma>0$. So, for any fixed $\lambda\in\Lambda_1$, from (2.3), (4.1)–(4.3) and (4.8) we have

$$\begin{aligned}\Phi(u)-\lambda\Psi(u)&=\frac{1}{2}\int_0^T(u'(x))^2dx+\sum_{j=1}^p\int_0^{u(x_j)}I_j(t)dt+\int_0^T G(u(x))dx-\lambda\int_0^T F(x,u(x))dx\\&\geq\Big(C_1-\frac{LT^2}{8}\Big)\|u\|^2-C_3\|u\|-\lambda\gamma\frac{4(C_1-\frac{LT^2}{8})}{T^2}\int_0^T|u(x)|^2dx-\lambda\frac{4(C_1-\frac{LT^2}{8})}{T^2}\eta\\&\geq\Big(C_1-\frac{LT^2}{8}\Big)\|u\|^2-C_3\|u\|-\lambda\gamma\frac{4(C_1-\frac{LT^2}{8})}{T^2}\cdot\frac{T^2}{4}\|u\|^2-\lambda\frac{4(C_1-\frac{LT^2}{8})}{T^2}\eta\\&=\Big(C_1-\frac{LT^2}{8}\Big)(1-\lambda\gamma)\|u\|^2-C_3\|u\|-\lambda\frac{4(C_1-\frac{LT^2}{8})}{T^2}\eta\\&\geq\Big(C_1-\frac{LT^2}{8}\Big)\Big(1-\gamma\frac{\frac{4}{T}(C_1-\frac{LT^2}{8})c^2-\frac{2}{\sqrt{T}}C_3c}{\int_0^T\sup_{t\in[-c,c]}F(x,t)dx}\Big)\|u\|^2\\&\quad-C_3\|u\|-\frac{\frac{4}{T}(C_1-\frac{LT^2}{8})c^2-\frac{2}{\sqrt{T}}C_3c}{\int_0^T\sup_{t\in[-c,c]}F(x,t)dx}\cdot\frac{4(C_1-\frac{LT^2}{8})}{T^2}\eta,\end{aligned}$$

and thus

$$\lim_{\|u\|\to+\infty}(\Phi(u)-\lambda\Psi(u))=+\infty,$$

which means that the functional $\Phi-\lambda\Psi$ is coercive. On the other hand, if $\gamma\leq0$, we get

$$\lim_{\|u\|\to+\infty}(\Phi(u)-\lambda\Psi(u))=+\infty.$$

Both cases lead to the coercivity of functional $\Phi-\lambda\Psi$. So, assumption (a_2) of Theorem 2.1 is satisfied. Now, we can apply Theorem 2.1. Hence, by using Theorem 2.1, taking into account that the weak solutions of (P) are exactly the solutions of the equation $\Phi'(u)-\lambda\Psi'(u)=0$, problem (P) admits at least three distinct weak solutions. □

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