Research Article

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Nash-type equilibria and periodic solutions to nonvariational systems

Abstract: The paper deals with variational properties of fixed points for contraction-type operators. Under suitable conditions, the unique fixed point of a vector-valued operator is a Nash-type equilibrium of the corresponding energy functionals. This is achieved by an iterative scheme based on Ekeland's variational principle. An application to periodic solutions for second order differential systems is given to illustrate the theory.

Keywords: Nash-type equilibrium, fixed point, critical point, Ekeland principle, periodic solution, second order system

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1 Introduction

Most equations coming from mathematical modeling of real processes can be written as a fixed point equation

$$u = N(u)$$

associated to some operator N. In many cases, the equation also has a variational form, i.e. it is equivalent to an equation of the type

$$E'(u) = 0,$$

where E is the "energy" functional and E' is its derivative. Thus, the fixed points of the operator N appear as critical points of the functional E. The critical points could be minima, maxima or saddle points, conferring to the fixed points a variational property. Thus it makes sense to ask whether a fixed point of N is a minimum, or a maximum or a saddle point of E. Section 2 of this paper is dealing with this problem. The problem becomes even more interesting in case of a system

$$\begin{cases} u = N_1(u, v), \\ v = N_2(u, v), \end{cases}$$

which has not a variational form, but each of its component equations has, i.e. there exist the functionals E_1 and E_2 such that the system is equivalent to the equations

$$\begin{cases} E_{11}(u, v) = 0, \\ E_{22}(u, v) = 0, \end{cases}$$

where $E_{11}(u, v)$ is the partial derivative of E_1 with respect to u, and $E_{22}(u, v)$ is the partial derivative of E_2 with respect to v. How are connected the fixed points (u, v) of the operator $N = (N_1, N_2)$ with the variational properties of the two functionals? One possible situation, which fits to physical principles, is that a fixed point (u^*, v^*) of N is a Nash-type equilibrium of the functionals E_1, E_2 (see, e.g., [6]), that is,

$$E_1(u^*, v^*) = \min_u E_1(u, v^*),$$
$$E_2(u^*, v^*) = \min_v E_2(u^*, v).$$

In Section 3, the main part of the paper, we will focus on this problem. An iterative scheme for finding a Nash-type equilibrium is introduced and its convergence is studied. Finally in Section 4 we illustrate the theory by giving an application to periodic solutions for a second order differential system. The model is inspired by the oscillations of two pendulums under interconnection forces, where the solution, as a Nash-type equilibrium, is such that the motion of each pendulum is conformed to the minimum energy principle by taking into account the motion of the other.

The main tool of our approach is the Bishop–Phelps Theorem [1, 4], an equivalent statement of Ekeland's variational principle (see [2, 3, 7]).

Theorem 1.1 (Bishop–Phelps' theorem). *Let (M, d) be a complete metric space, let $E : M \to \mathbb{R}$ be lower semi-continuous and bounded from below and let $\varepsilon > 0$. Then for any $u_0 \in M$, there exists a point $u \in M$ such that*

$$E(u) \le E(u_0) - \varepsilon d(u_0, u)$$

and

$$E(u) < E(v) + \varepsilon d(u, v)$$

for every $v \ne u$.

For a C^1-functional E on a Banach space X, this result guarantees (take $\varepsilon = \frac{1}{n}$ and u_0 with $E(u_0) \le \inf_X E + \frac{1}{n}$) the existence of a sequence (u_n) with

$$E(u_n) \to \inf_X E \quad \text{and} \quad E'(u_n) \to 0.$$

2 A minimum property for classical contractions

In this section X is a Hilbert space with inner product and norm denoted by $(\cdot, \cdot)$ and $|\cdot|$, which is identified to its dual. We start with the case of contractions on the whole space.

Theorem 2.1. *Let $N : X \to X$ be a contraction with the unique fixed point u^* (guaranteed by Banach's contraction theorem). If there exists a C^1-functional E bounded from below such that*

$$E'(u) = u - N(u) \quad \text{for all } u \in X, \tag{2.1}$$

then u^* minimizes the functional E, i.e.

$$E(u^*) = \inf_X E.$$

Proof. As a consequence of Bishop–Phelps' theorem, there is a sequence (u_n) with

$$E(u_n) \to \inf_X E \quad \text{and} \quad E'(u_n) \to 0. \tag{2.2}$$

Let $v_n := E'(u_n) = u_n - N(u_n)$. We have $v_n \to 0$ and

$$|u_{n+p} - u_n| \le |N(u_{n+p}) - N(u_n)| + |v_{n+p} - v_n| \le a|u_{n+p} - u_n| + |v_{n+p} - v_n|.$$

Here $a \in [0, 1)$ is the contraction constant of N. Hence

$$|u_{n+p} - u_n| \le \frac{1}{1-a}|v_{n+p} - v_n|.$$

Since (v_n) is convergent (so Cauchy), this implies that (u_n) is Cauchy too. Hence $u_n \to \overline{u}$ for some $\overline{u}$. Now (2.2) yields $E(\overline{u}) = \inf_X E$ and $E'(\overline{u}) = 0$. The relation $E'(\overline{u}) = 0$ shows that $\overline{u}$ is a fixed point of N, and since N has a unique fixed point, $\overline{u} = u^*$.

An analogue result holds for contractions on a ball $\overline{B}_R = \{u \in X : |u| \le R\}$ of the Hilbert space X. □

Theorem 2.2. *Let $N : \overline{B}_R \to X$ be a contraction satisfying the Leray–Schauder condition*

$$u \ne \lambda N(u) \quad \text{for } |u| = R \text{ and } \lambda \in (0, 1), \tag{2.3}$$

and let u^* be the unique fixed point of N (guaranteed by the nonlinear alternative). If there exists a C^1-functional E bounded from below on $\overline{B}_R$ such that

$$E'(u) = u - N(u) \quad \text{for all } u \in \overline{B}_R,$$

then u^* minimizes the functional E on $\overline{B}_R$, i.e.

$$E(u^*) = \inf_{\overline{B}_R} E.$$

Proof. As a consequence of Schechter's critical point theorem in a ball (see [11, 12]), there is a sequence (u_n) of elements from $\overline{B}_R$ with

$$E(u_n) \to \inf_{\overline{B}_R} E$$

and either

$$E'(u_n) \to 0$$

or

$$E'(u_n) - \frac{(E'(u_n), u_n)}{R^2} u_n \to 0, \quad |u_n| = R, \quad (E'(u_n), u_n) \le 0.$$

In the first case, when $E'(u_n) \to 0$, we repeat the argument from the proof of Theorem 2.1. In the second case, since $(E'(u_n), u_n) = R^2 - (N(u_n), u_n)$ and N is bounded as a contraction, we may pass to a subsequence in order to assume the convergence $\mu_n := -\frac{(E'(u_n), u_n)}{R^2} \to \mu \ge 0$. Furthermore, if $v_n := E'(u_n) + \mu_n u_n$, then

$$(1 + \mu)u_n = v_n + N(u_n) + z_n$$

with $z_n = (\mu - \mu_n)u_n$. Clearly $z_n \to 0$. Next

$$(1 + \mu)|u_{n+p} - u_n| \le |v_{n+p} - v_n| + a|u_{n+p} - u_n| + |z_{n+p} - z_n|$$

and so

$$|u_{n+p} - u_n| \le \frac{1}{1 + \mu - a}(|v_{n+p} - v_n| + |z_{n+p} - z_n|).$$

This implies that (u_n) is Cauchy. Let $\overline{u}$ be its limit. Then

$$E(\overline{u}) = \inf_{\overline{B}_R} E \quad \text{and} \quad E'(\overline{u}) + \mu\overline{u} = 0,$$

where $|\overline{u}| = R$ and $\mu \ge 0$. We claim that the case $\mu > 0$ is not possible. Indeed, if we assume that $\mu > 0$, then from $\overline{u} - N(\overline{u}) + \mu\overline{u} = 0$ we would have $\overline{u} = \frac{1}{1+\mu}N(\overline{u})$ which has been excluded by (2.3). Hence $\mu = 0$, $E'(\overline{u}) = 0$, i.e. $\overline{u} = N(\overline{u})$. Again the uniqueness of the fixed point guarantees $\overline{u} = u^*$. □

In the setting of critical point theory, a sequence (u_n) with the properties that $(E(u_n))$ converges and $E'(u_n) \to 0$ is called a *Palais–Smale sequence*. A functional is said to satisfy the *Palais–Smale condition* if each Palais–Smale sequence has a convergent subsequence. Thus Theorem 2.1 says that if E' is represented by (2.1), then E satisfies more than the Palais–Smale condition, in the sense that the Palais–Smale sequences are entirely (not only some of their subsequences) convergent.

Remark 2.3. If in Theorem 2.2 the operator N is assumed to be more general condensing, then the minimizing sequence (u_n) has a subsequence converging to the absolute minimum of E on $\overline{B}_R$. Indeed, if (u_n) satisfies $E'(u_n) \to 0$, then using a measure α of noncompactness with respect to whom N is condensing, we find

$$\alpha(\{u_n\}) = \alpha(\{E'(u_n) + N(u_n)\}) \le \alpha(\{E'(u_n)\}) + \alpha(\{N(u_n)\}) = \alpha(\{N(u_n)\}). \tag{2.4}$$

If $\{u_n\}$ is not relatively compact, that is, $\alpha(\{u_n\}) > 0$, then by the condensing property, $\alpha(\{N(u_n)\}) < \alpha(\{u_n\})$, which in view of (2.4) yields the contradiction $\alpha(\{u_n\}) < \alpha(\{u_n\})$. Hence $\{u_n\}$ is relatively compact, as desired.

Note that the remark is also true for Theorem 2.1 if any minimizing sequence is bounded.

If we focus on critical points of a functional E, and not on the fixed points of an operator N, then we can state the following more general results:

Theorem 2.4. *Let X be a Banach space with norm $|\cdot|$ and let E be a C^1-functional bounded from below with E' strongly monotone, that is,*

$$(E'(u) - E'(v), u - v) \geq a|u - v|^2 \quad \text{for all } u, v \in X,$$

and some $a > 0$. Then there is $u^* \in X$ with

$$E(u^*) = \inf_X E \quad \text{and} \quad E'(u^*) = 0.$$

Proof. As in the proof of Theorem 2.1, let (u_n) satisfy

$$E(u_n) \to \inf_X E \quad \text{and} \quad E'(u_n) \to 0.$$

Denote $v_n := E'(u_n)$. We have $v_n \to 0$ in X' and

$$a|u_{n+p} - u_n|^2 \leq (E'(u_{n+p}) - E'(u_n), u_{n+p} - u_n) = (v_{n+p} - v_n, u_{n+p} - u_n) \leq |v_{n+p} - v_n||u_{n+p} - u_n|.$$

It follows that

$$|u_{n+p} - u_n| \leq \frac{1}{a}|v_{n+p} - v_n|$$

and the assertion follows as above. □

Similarly we have the following generalization of Theorem 2.2.

Theorem 2.5. *Let X be a Hilbert space and let E be a C^1 functional such that E' is strongly monotone on $\overline{B}_R$ and*

$$E'(u) + \mu u \neq 0 \quad \text{for } |u| = R \text{ and } \mu > 0.$$

Then there exists $u^* \in \overline{B}_R$ with

$$E(u^*) = \inf_{\overline{B}_R} E \quad \text{and} \quad E'(u^*) = 0.$$

Proof. With the notations from the proof of Theorem 2.2, we have

$$\begin{aligned} a|u_{n+p} - u_n|^2 &\leq (E'(u_{n+p}) - E'(u_n), u_{n+p} - u_n) \\ &= (v_{n+p} - v_n, u_{n+p} - u_n) - \mu(u_{n+p} - u_n, u_{n+p} - u_n) \\ &\qquad + (\mu - \mu_{n+p})(u_{n+p}, u_{n+p} - u_n) - (\mu - \mu_n)(u_n, u_{n+p} - u_n). \end{aligned}$$

Hence

$$(a + \mu)|u_{n+p} - u_n| \leq |v_{n+p} - v_n| + R(|\mu - \mu_{n+p}| + |\mu - \mu_n|),$$

which implies that (u_n) is Cauchy. □

3 Nash-type equilibrium for Perov contractions

Let $(X_i, |\cdot|_i)$, $i = 1, 2$, be Hilbert spaces identified with their duals and let $X = X_1 \times X_2$. Consider the system

$$\begin{cases} u = N_1(u, v), \\ v = N_2(u, v), \end{cases}$$

where $(u, v) \in X$. Assume that each equation of the system has a variational form, i.e. that there exist continuous functionals $E_1, E_2 : X \to \mathbb{R}$ such that $E_1(\cdot, v)$ is Fréchet differentiable for every $v \in X_2$, $E_2(u, \cdot)$ is Fréchet differentiable for every $u \in X_1$, and

$$\begin{aligned} E_{11}(u, v) &= u - N_1(u, v), \\ E_{22}(u, v) &= v - N_2(u, v). \end{aligned} \tag{3.1}$$

Here $E_{11}(\cdot, v)$, $E_{22}(u, \cdot)$ are the Fréchet derivatives of $E_1(\cdot, v)$ and $E_2(u, \cdot)$, respectively.

We say that the operator $N : X \to X$, $N(u,v) = (N_1(u,v), N_2(u,v))$, is a *Perov contraction* [8] if there exists a matrix $M = [m_{ij}] \in \mathcal{M}_{2,2}(\mathbb{R}_+)$ such that M^n tends to the zero matrix 0 and the following matricial Lipschitz condition is satisfied:

$$\begin{bmatrix} |N_1(u,v) - N_1(\overline{u},\overline{v})|_1 \\ |N_2(u,v) - N_2(\overline{u},\overline{v})|_2 \end{bmatrix} \le M \begin{bmatrix} |u-\overline{u}|_1 \\ |v-\overline{v}|_2 \end{bmatrix} \tag{3.2}$$

for every $u, \overline{u} \in X_1$ and $v, \overline{v} \in X_2$.

Notice that the property $M^n \to 0$ is equivalent to $\rho(M) < 1$, where $\rho(M)$ is the spectral radius of the matrix M, and also to the fact that $I - M$ is nonsingular and all the entries of the matrix $(I-M)^{-1}$ are nonnegative (see, e.g., [9, 10]).

Theorem 3.1. *Assume that the above conditions are satisfied. In addition assume that $E_1(\cdot, v)$ and $E_2(u,\cdot)$ are bounded from below for every $u \in X_1$, $v \in X_2$, and that there are $R, a > 0$ such that one of the following conditions holds:*

$$\begin{aligned} E_1(u,v) &\ge \inf_{X_1} E_1(\cdot, v) + a \quad \text{for } |u|_1 \ge R \text{ and all } v \in X_2, \\ E_2(u,v) &\ge \inf_{X_2} E_2(u,\cdot) + a \quad \text{for } |v|_2 \ge R \text{ and all } u \in X_1. \end{aligned} \tag{3.3}$$

Then the unique fixed point (u^*, v^*) of N (guaranteed by Perov's fixed point theorem) is a Nash-type equilibrium of the pair of functionals (E_1, E_2), i.e.

$$\begin{aligned} E_1(u^*, v^*) &= \inf_{X_1} E_1(\cdot, v^*), \\ E_2(u^*, v^*) &= \inf_{X_2} E_2(u^*, \cdot). \end{aligned}$$

Proof. Assume that (3.3) holds for E_2. We shall construct recursively two sequences (u_n) and (v_n) based on Bishop–Phelps' theorem. Let v_0 be any element of X_2. At any step n $(n \ge 1)$ we may find elements $u_n \in X_1$ and $v_n \in X_2$ such that

$$E_1(u_n, v_{n-1}) \le \inf_{X_1} E_1(\cdot, v_{n-1}) + \frac{1}{n}, \quad |E_{11}(u_n, v_{n-1})|_1 \le \frac{1}{n}, \tag{3.4}$$

$$E_2(u_n, v_n) \le \inf_{X_2} E_2(u_n, \cdot) + \frac{1}{n}, \qquad |E_{22}(u_n, v_n)|_2 \le \frac{1}{n}. \tag{3.5}$$

For $\frac{1}{n} < a$, from (3.3) and (3.5) we have $|v_n|_2 < R$. Hence the sequence (v_n) is bounded. Let $\alpha_n := E_{11}(u_n, v_{n-1})$ and $\beta_n := E_{22}(u_n, v_n)$. Clearly $\alpha_n, \beta_n \to 0$. Also

$$u_n - N_1(u_n, v_{n-1}) = \alpha_n \quad \text{and} \quad v_n - N_2(u_n, v_n) = \beta_n.$$

It follows that

$$\begin{aligned} |u_{n+p} - u_n|_1 &\le |N_1(u_{n+p}, v_{n+p-1}) - N_1(u_n, v_{n-1})|_1 + |\alpha_{n+p} - \alpha_n|_1 \\ &\le m_{11}|u_{n+p} - u_n|_1 + m_{12}|v_{n+p-1} - v_{n-1}|_2 + |\alpha_{n+p} - \alpha_n|_1 \\ &\le m_{11}|u_{n+p} - u_n|_1 + m_{12}|v_{n+p} - v_n|_2 + |\alpha_{n+p} - \alpha_n|_1 + m_{12}(|v_{n+p-1} - v_{n-1}|_2 - |v_{n+p} - v_n|_2). \end{aligned}$$

Denote $a_{n,p} = |u_{n+p} - u_n|_1$, $b_{n,p} = |v_{n+p} - v_n|_2$, $c_{n,p} = |\alpha_{n+p} - \alpha_n|_1$, $d_{n,p} = |\beta_{n+p} - \beta_n|_2$. Then

$$a_{n,p} \le m_{11}a_{n,p} + m_{12}b_{n,p} + c_{n,p} + m_{12}(b_{n-1,p} - b_{n,p}). \tag{3.6}$$

Similarly

$$b_{n,p} \le m_{21}a_{n,p} + m_{22}b_{n,p} + d_{n,p}.$$

Hence

$$\begin{bmatrix} a_{n,p} \\ b_{n,p} \end{bmatrix} \le M \begin{bmatrix} a_{n,p} \\ b_{n,p} \end{bmatrix} + \begin{bmatrix} c_{n,p} + m_{12}(b_{n-1,p} - b_{n,p}) \\ d_{n,p} \end{bmatrix}.$$

Consequently, since $I - M$ is invertible and its inverse contains only nonnegative entries, we may write

$$\begin{bmatrix} a_{n,p} \\ b_{n,p} \end{bmatrix} \le (I-M)^{-1} \begin{bmatrix} c_{n,p} + m_{12}(b_{n-1,p} - b_{n,p}) \\ d_{n,p} \end{bmatrix}.$$

Let $(I - M)^{-1} = [\gamma_{ij}]$. Then

$$a_{n,p} \le \gamma_{11}(c_{n,p} + m_{12}(b_{n-1,p} - b_{n,p})) + \gamma_{12} d_{n,p} \quad \text{and} \quad b_{n,p} \le \gamma_{21}(c_{n,p} + m_{12}(b_{n-1,p} - b_{n,p})) + \gamma_{22} d_{n,p}. \tag{3.7}$$

From the second inequality, one has

$$b_{n,p} \le \frac{\gamma_{21} m_{12}}{1 + \gamma_{21} m_{12}} b_{n-1,p} + \frac{\gamma_{21} c_{n,p} + \gamma_{22} d_{n,p}}{1 + \gamma_{21} m_{12}}. \tag{3.8}$$

Clearly $(b_{n,p})$ is bounded uniformly with respect to p. Lemma 3.2 below shows that $b_{n,p} \to 0$ uniformly for $p \in \mathbb{N}$, and hence (v_n) is Cauchy. The first inequality in (3.7) implies that (u_n) is also Cauchy. Let u^*, v^* be the limits of (u_n), (v_n) respectively. The conclusion now follows if we pass to the limit in (3.4) and (3.5).

In case that E_1 satisfies (3.3), we interchange E_1, E_2 in the construction of the two sequences, more exactly we will obtain

$$\begin{aligned} E_2(u_{n-1}, v_n) &\le \inf_{X_2} E_2(u_{n-1}, \cdot) + \frac{1}{n}, \quad & |E_{22}(u_{n-1}, v_n)|_2 &\le \frac{1}{n}, \\ E_1(u_n, v_n) &\le \inf_{X_1} E_1(\cdot, v_n) + \frac{1}{n}, \quad & |E_{11}(u_n, v_n)|_1 &\le \frac{1}{n}. \end{aligned}$$

Furthermore, the reasoning is similar. □

The lemma that was used is more or less similar to other results guaranteeing the convergence of iterative schema (see, for instance, [13]).

Lemma 3.2. *Let $(x_{n,p})$ and $(y_{n,p})$ be two sequences of real numbers depending on a parameter p such that $(x_{n,p})$ is bounded uniformly with respect to p, and*

$$0 \le x_{n,p} \le \lambda x_{n-1,p} + y_{n,p} \quad \text{for all } n, p \text{ and some } \lambda \in [0,1). \tag{3.9}$$

If $y_{n,p} \to 0$ uniformly with respect to p, then $x_{n,p} \to 0$ uniformly with respect to p too.

Proof. Let $\varepsilon > 0$ be any number. Since $y_{n,p} \to 0$ uniformly with respect to p, there exists an n_1 (not depending on p) such that $y_{n,p} \le \varepsilon$ for all $n \ge n_1$. From $x_{n,p} \le \lambda x_{n-1,p} + \varepsilon$ $(n \ge n_1)$ we deduce

$$x_{n,p} \le \lambda^{n-n_1} x_{n_1} + \varepsilon(\lambda + \lambda^2 + \cdots + \lambda^{n-n_1}) \le \lambda^{n-n_1} c + \varepsilon \frac{\lambda}{1-\lambda},$$

where c is a bound for $x_{n,p}$. This yields $x_{n,p} \to 0$ uniformly in p. □

Remark 3.3. Condition (3.3) is not necessary if one can prove that one of the “minimizing” sequences (u_n) and (v_n) is bounded.

Remark 3.4. If instead of condition (3.3) we assume that there exist convergent subsequences (u_{n_j}) and (v_{n_j}) of the sequences (u_n) and (v_n) given by (3.4) and (3.5), then the conclusion of Theorem 3.1 remains true. To prove this, we first show that the sequence $(b_{n_j-1,1})$, $b_{n_j-1,1} = |v_{n_j} - v_{n_j-1}|_2$, is bounded. Indeed, from (3.8), we obtain that

$$b_{n_j-1,1} \le \lambda b_{n_j-2,1} + (1-\lambda)^2, \quad j \ge j_1.$$

This yields

$$b_{n_j-1,1} \le \lambda^{n_j - n_{j-1}} b_{n_{j-1}-1,1} + 1 - \lambda, \quad j \ge j_2,$$

whence

$$b_{n_j-1,1} - 1 \le \lambda(b_{n_{j-1}-1,1} - 1), \quad j \ge j_2. \tag{3.10}$$

Denote $z_j = b_{n_j-1,1}$. The case $z_j > 1$ for all $j \ge j_2$ is not possible, since otherwise $0 \le z_j - 1 \le \lambda^{j-j_2}(z_{j_2} - 1)$, whence $z_j \to 1$. However, by (3.9) this would imply the contradiction $1 \le 1 - \lambda$. Therefore, there is $j_3 \ge j_2$ with $z_{j_3} \le 1$. Then (3.10) implies that $z_j \le 1$ for all $j \ge j_3$, that is, (z_j) is bounded as claimed.

Next, from Lemma 3.2, applied for $p = 1$ and $x_{j,1} := b_{n_j-1,1}$, we find that $b_{n_j-1,1} \to 0$ as $j \to \infty$. Hence the sequence (v_{n_j-1}) is convergent to the limit v^* of (v_{n_j}). The conclusion follows if we let $j \to \infty$ in (3.4) and (3.5) with $n = n_j$.

An analogue result holds for Perov contractions on the cartesian product $\overline{B}_{R_1} \times \overline{B}_{R_2}$ of two balls of X_1 and X_2.

Theorem 3.5. *Let $N : \overline{B}_{R_1} \times \overline{B}_{R_2} \to X$, $N = (N_1, N_2)$, be a Perov generalized contraction, i.e.* (3.2) *is satisfied for $u, \overline{u} \in \overline{B}_{R_1}$ and $v, \overline{v} \in \overline{B}_{R_2}$. Assume that for every $\lambda \in (0, 1)$,*

$$\begin{aligned} u &\neq \lambda N_1(u, v) \quad \text{if } |u|_1 = R_1,\ v \in \overline{B}_{R_2}, \\ v &\neq \lambda N_2(u, v) \quad \text{if } |v|_2 = R_2,\ u \in \overline{B}_{R_1}. \end{aligned}$$

In addition assume that the representation (3.1) *holds on $\overline{B}_{R_1} \times \overline{B}_{R_2}$ for two continuous functionals $E_1, E_2 : X \to \mathbb{R}$ such that $E_1(\cdot, v)$ is Fréchet differentiable for every $v \in X_2$, $E_2(u, \cdot)$ is Fréchet differentiable for every $u \in X_1$, and that $E_1(\cdot, v)$, $E_2(u, \cdot)$ are bounded from below on $\overline{B}_{R_1}$ and $\overline{B}_{R_2}$ respectively, for every $u \in \overline{B}_{R_1}, v \in \overline{B}_{R_2}$. Then the unique fixed point $(u^*, v^*) \in \overline{B}_{R_1} \times \overline{B}_{R_2}$ of N (guaranteed by the nonlinear alternative for Perov contractions* [9]*) is a Nash-type equilibrium in $\overline{B}_{R_1} \times \overline{B}_{R_2}$ of the pair of functionals (E_1, E_2), i.e.*

$$E_1(u^*, v^*) = \inf_{\overline{B}_{R_1}} E_1(\cdot, v^*) \quad \textit{and} \quad E_2(u^*, v^*) = \inf_{\overline{B}_{R_2}} E_2(u^*, \cdot).$$

The proof uses arguments from the proofs of Theorems 2.2 and 3.1.

4 Application to periodic solutions of second order systems

To illustrate the theory, let us consider the periodic problem

$$\begin{cases} u''(t) = \nabla_x F(t, u(t), v(t)) & \text{a.e. on } (0, T), \\ v''(t) = \nabla_y G(t, u(t), v(t)) & \text{a.e. on } (0, T), \\ u(0) - u(T) = u'(0) - u'(T) = 0, & \\ v(0) - v(T) = v'(0) - v'(T) = 0, & \end{cases} \tag{4.1}$$

where $F, G : (0, T) \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \to \mathbb{R}$.

All functions of the type $H : (0, T) \times \mathbb{R}^n \to \mathbb{R}^m$, $H = H(t, x)$ $(n, m \geq 1)$, including F, G, $\nabla_x F$ and $\nabla_y G$, will be assumed to be L^1-Carathéodory, i.e. with $H(\cdot, x)$-measurable for each $x \in \mathbb{R}^n$, $H(t, \cdot)$-continuous for a.e. $t \in (0, T)$, and such that for each $R > 0$, there is $b_R \in L^1(0, T; \mathbb{R}_+)$ with $|H(t, x)| \leq b_R(t)$ for a.e. $t \in (0, T)$ and all $x \in \mathbb{R}^n$, $|x| \leq R$.

We refer to [5], for a survey on the periodic problem for systems having a variational form. In our case, the system has not a variational form, but it splits into two subsystems having each one a variational structure.

It is known (see [5]) that a pair $(u, v) \in H_p^1(0, T; \mathbb{R}^{k_1}) \times H_p^1(0, T; \mathbb{R}^{k_2})$ is a solution of (4.1) if and only if

$$E_{11}(u, v) = 0 \quad \text{and} \quad E_{22}(u, v) = 0,$$

where $E_1, E_2 : H_p^1(0, T; \mathbb{R}^{k_1}) \times H_p^1(0, T; \mathbb{R}^{k_2}) \to \mathbb{R}$ are given by

$$E_1(u, v) = \int_0^T \left(\frac{1}{2}|u'|^2 + F(t, u(t), v(t)) \right) dt \quad \text{and} \quad E_2(u, v) = \int_0^T \left(\frac{1}{2}|v'|^2 + G(t, u(t), v(t)) \right) dt. \tag{4.2}$$

Here $H_p^1(0, T; \mathbb{R}^k)$ is the space of functions of the form

$$u(t) = \int_0^t v(s) ds + c$$

with $u(0) = u(T)$, $c \in \mathbb{R}^k$ and $v \in L^2(0,T;\mathbb{R}^k)$. We shall define a scalar product in $H_p^1(0,T;\mathbb{R}^{k_i})$ ($i = 1,2$) by

$$(u,v)_i = \int_0^T [(u'(t), v'(t)) + m_i^2 (u(t), v(t))]dt,$$

where $m_i \neq 0$; the corresponding norm is

$$|u|_i = \left(\int_0^T (|u'(t)|^2 + m_i^2 |u(t)|^2) dt \right)^{1/2}.$$

We shall identify the dual $(H_p^1(0,T;\mathbb{R}^{k_i}))'$ with $H_p^1(0,T;\mathbb{R}^{k_i})$, via the mapping J_i, $h \in (H_p^1(0,T;\mathbb{R}^{k_i}))' \mapsto J_i h = u$, the unique weak solution of the problem

$$-u'' + m_i^2 u = h \quad \text{a.e. on } (0,T), \quad u(0) - u(T) = u'(0) - u'(T) = 0.$$

Then

$$E_{11}(u,v) = u - J_1(m_1^2 u - \nabla_x F(\cdot, u, v)) \quad \text{and} \quad E_{22}(u,v) = v - J_2(m_2^2 v - \nabla_y G(\cdot, u, v)).$$

Hence

$$N_1(u,v) = J_1(m_1^2 u - \nabla_x F(\cdot, u, v)) \quad \text{and} \quad N_2(u,v) = J_2(m_2^2 v - \nabla_y G(\cdot, u, v)).$$

In what follows we shall use the obvious inequality

$$|u|_{L^2} \leq \frac{1}{m_i} |u|_i, \quad u \in H_p^1(0,T;\mathbb{R}^{k_i}), \tag{4.3}$$

and the estimation of the norm of J_i, as a linear operator from $L^2(0,T;\mathbb{R}^{k_i})$ to $H_p^1(0,T;\mathbb{R}^{k_i})$. To obtain this, we start with the definition of the operator J_i, which gives

$$|J_i h|_i^2 = (J_i h, J_i h)_i = (h, J_i h)_{L^2} \leq |h|_{L^2} |J_i h|_{L^2} \leq \frac{1}{m_i} |h|_{L^2} |J_i h|_i.$$

Hence

$$|J_i h|_i \leq \frac{1}{m_i} |h|_{L^2}, \quad h \in L^2(0,T;\mathbb{R}^{k_i}). \tag{4.4}$$

We shall say that a function $H : (0,T) \times \mathbb{R}^k \to R$ is of *coercive-type* if the functional $E : H_p^1(0,T;\mathbb{R}^k) \to \mathbb{R}$ given by

$$E(u) = \int_0^T \left(\frac{1}{2} |u'(t)|^2 + H(t, u(t)) \right) dt \tag{4.5}$$

is *coercive*, i.e. $E(u) \to +\infty$ as $|u| \to \infty$. Here

$$|u| = \left(\int_0^T (|u'(t)|^2 + |u(t)|^2) dt \right)^{1/2}.$$

Lemma 4.1. *If for some $\gamma \in \mathbb{R} \setminus \{0\}$, $\nabla(H - \gamma^2|x|^2)$ is bounded by an L^1-function for all $x \in \mathbb{R}^k$ and the average of $H(t,x) - \gamma^2|x|^2$ with respect to t is bounded from bellow, more exactly*

$$|\nabla(H(t,x) - \gamma^2|x|^2)| \leq a(t)$$

for a.e. $t \in (0,T)$, all $x \in \mathbb{R}^k$, some $a \in L^1(0,T;\mathbb{R}_+)$, and

$$\int_0^T H(t,x)dt - T\gamma^2|x|^2 \geq C > -\infty$$

for all $x \in \mathbb{R}^k$ and some constant C, then the functional (4.5) *is coercive.*

Proof. We follow the idea from [5, proof of Theorem 1.5]. Denote

$$H_\gamma(t,x) = H(t,x) - \gamma^2|x|^2.$$

For $u \in H^1_p(0,T;\mathbb{R}^k)$, we have $u = \overline{u} + \widehat{u}$, where $\overline{u} = \frac{1}{T}\int_0^T u(t)dt$ and $\widehat{u} = u - \overline{u}$. Then

$$E(u) = \int_0^T \Big(\frac{1}{2}|u'(t)|^2 + \gamma^2|u(t)|^2\Big)dt + \int_0^T H_\gamma(t,\overline{u})dt + \int_0^T [H_\gamma(t,u(t)) - H_\gamma(t,\overline{u})]dt$$
$$\geq \min\{1,2\gamma^2\}|u|^2 + C + \int_0^T\int_0^1 (\nabla H_\gamma(t,\overline{u}+s\widehat{u}(t)),\widehat{u}(t))dsdt$$
$$\geq \min\{1,2\gamma^2\}|u|^2 + C - |a|_{L^1}|\widehat{u}|_\infty.$$

Since $|\widehat{u}|_\infty \leq c|\widehat{u}| \leq c|u|$, we deduce that

$$E(u) \geq \min\{1,2\gamma^2\}|u|^2 + C - c|a|_{L^1}|u| \to \infty \quad \text{as } |u| \to \infty. \qquad \square$$

Notice that if H is of coercive-type, then the functional (4.5) is bounded from below. Indeed, the coercivity property implies that there exists a number $R > 0$ such that $E(u) \geq 0$ if $|u| > R$. As the injection of $H^1_p(0,T;\mathbb{R}^k)$ into $C(0,T;\mathbb{R}^k)$ is continuous, there is $c > 0$ such that

$$|u|_\infty \leq c|u| \quad \text{for every } u \in H^1_p(0,T;\mathbb{R}^k).$$

Then, for $|u| \leq R$, we have $|u|_\infty \leq cR$, and since H is L^1-Carathéodory, $H(t,u(t)) \geq -b(t)$ for a.e. $t \in (0,T)$. As a result, for $|u| \leq R$, one has $E(u) \geq -|b|_{L^1}$. Hence $E(u) \geq -|b|_{L^1}$ for all $u \in H^1_p(0,T;\mathbb{R}^k)$ as we claimed.

Our hypotheses are as follows:

(H1) for each $R > 0$, there exist $\sigma_1, \sigma_2 \in L^1(0,T;\mathbb{R}_+)$ and $\gamma \neq 0$ such that

$$F(t,x,y) \geq \gamma^2|x|^2 - \sigma_1(t)|x| - \sigma_2(t)$$

for a.e. $t \in (0,T)$, all $x \in \mathbb{R}^{k_1}$ and $y \in \mathbb{R}^{k_2}$ with $|y| \leq R$,

(H2) there exist $g, g_1 : (0,T) \times \mathbb{R}^{k_2} \to \mathbb{R}$ of coercive-type with

$$g(t,y) \leq G(t,x,y) \leq g_1(t,y) \tag{4.6}$$

for all $x \in \mathbb{R}^{k_1}$, $y \in \mathbb{R}^{k_2}$ and a.e. $t \in (0,T)$,

(H3) there exist $m_{ij} \in \mathbb{R}_+$ $(i,j = 1,2)$ with

$$\begin{aligned} |m_1^2(x-\overline{x}) - \nabla_x(F(t,x,y) - F(t,\overline{x},\overline{y}))| &\leq m_{11}|x-\overline{x}| + m_{12}|y-\overline{y}|,\\ |m_2^2(y-\overline{y}) - \nabla_y(G(t,x,y) - G(t,\overline{x},\overline{y}))| &\leq m_{21}|x-\overline{x}| + m_{22}|y-\overline{y}| \end{aligned} \tag{4.7}$$

for all $x,\overline{x} \in \mathbb{R}^{k_1}$, $y,\overline{y} \in \mathbb{R}^{k_2}$ and a.e. $t \in (0,T)$ such that the spectral radius of the matrix

$$M = \begin{bmatrix} \frac{m_{11}}{m_1^2} & \frac{m_{12}}{m_1 m_2} \\ \frac{m_{21}}{m_1 m_2} & \frac{m_{22}}{m_2^2} \end{bmatrix} \tag{4.8}$$

is strictly less than one.

Theorem 4.2. *Under the assumptions* (H1)–(H3), *problem* (4.1) *has a unique solution*

$$(u,v) \in H^1_p(0,1;\mathbb{R}^{k_1}) \times H^1_p(0,1;\mathbb{R}^{k_2})$$

which is a Nash-type equilibrium of the pair of energy functionals (E_1, E_2) *given by* (4.2).

Proof. From (H1) we have that $E_1(\cdot, v)$ is bounded from below for each $v \in H^1_p(0,1;\mathbb{R}^{k_2})$. Indeed, if $R = |v|_\infty$, then

$$E_1(u,v) \geq \int_0^T \Big(\frac{1}{2}|u'(t)|^2 + \gamma^2|u(t)|^2 - \sigma_1(t)|u(t)| - \sigma_2(t)\Big)dt \geq C_1|u|_1^2 - C_2|u|_1 - C_3,$$

whence the desired conclusion. Next, using the first inequality in (4.6), we have

$$E_2(u,v) \geq \phi(v) := \int_0^T \Big(\frac{1}{2}|v'(t)|^2 + g(t,v(t))\Big)dt.$$

Since g is of coercive-type, ϕ is bounded from below. Thus $E_2(u,\cdot)$ is bounded from bellow even uniformly with respect to u.

Furthermore, if we also denote

$$\phi_1(v) = \int_0^T \Big(\frac{1}{2}|v'(t)|^2 + g_1(t, v(t))\Big)dt,$$

we fix any number $a > 0$, we use (4.6) and the coercivity of ϕ, then we may find a number $R > 0$ such that

$$\inf E_2(u, \cdot\,) + a \le \inf \phi_1 + a \le \phi(v)$$

for $|v|_2 \ge R$. Since $E_2(u, v) \ge \phi(v)$, this shows that condition (3.3) is satisfied by E_2.

Finally using (4.3), (4.4) and (4.7), we obtain

$$\begin{aligned}
|N_1(u, v) - N_1(\overline{u}, \overline{v})|_1 &= |J_1(m_1^2(u - \overline{u}) - \nabla_x(F(\,\cdot\,, u, v) - F(\,\cdot\,, \overline{u}, \overline{v})))|_1 \\
&\le \frac{1}{m_1}|m_1^2(u - \overline{u}) - \nabla_x(F(\,\cdot\,, u, v) - F(\,\cdot\,, \overline{u}, \overline{v}))|_{L^2} \\
&\le \frac{m_{11}}{m_1}|u - \overline{u}|_{L^2} + \frac{m_{12}}{m_1}|v - \overline{v}|_{L^2} \\
&\le \frac{m_{11}}{m_1^2}|u - \overline{u}|_1 + \frac{m_{12}}{m_1 m_2}|v - \overline{v}|_2.
\end{aligned}$$

A similar inequality holds for N_2 and so condition (3.2) is satisfied with the matrix M given by (4.8). Therefore all the hypotheses of Theorem 3.1 are fulfilled. □

Example 4.3. Consider the system of two scalar equations

$$\begin{cases} u'' = 2\gamma_1^2 u + a(t)\sin u(t) + b(t)\cos u(t)\cos v(t) + c(t), \\ v'' = 2\gamma_2^2 v + A(t)\sin u(t)\cos v(t) + B(t)\cos v(t), \end{cases} \tag{4.9}$$

where $\gamma_1, \gamma_2 \ne 0$ and $a, b, A, B \in L^\infty(0, T)$, $c \in L^1(0, T)$. In this case,

$$\begin{aligned}
F(t, x, y) &= \gamma_1^2 x^2 - a(t)\cos x + b(t)\sin x\cos y + c(t)x, \\
G(t, x, y) &= \gamma_2^2 y^2 + A(t)\sin x\sin y + B(t)\sin y,
\end{aligned}$$

and we let $m_i = \gamma_i\sqrt{2}$ $(i = 1, 2)$. If the spectral radius of the matrix

$$M = \begin{bmatrix} \frac{1}{2\gamma_1^2}(|a|_\infty + |b|_\infty) & \frac{|b|_\infty}{2\gamma_1\gamma_2} \\ \frac{|A|_\infty}{2\gamma_1\gamma_2} & \frac{1}{2\gamma_2^2}(|A|_\infty + |B|_\infty) \end{bmatrix}$$

is strictly less than one, then system (4.9) has a unique T-periodic solution which is a Nash-type equilibrium of the pair of energy functionals of the system.

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