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Research Article

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N-Laplacian critical problem with discontinuous nonlinearities

Abstract: In this paper, we study the existence of a solution of the N-Laplacian critical problem with discontinuous nonlinearity of Heaviside type in a smooth bounded domain with respect to a positive parameter λ.

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1 Introduction

In this paper we discuss the following elliptic problems on smooth and bounded domain with the discontinuous nonlinearity:

$$\begin{cases} -\Delta_N u = \lambda(h(u)e^{|u|^{N/N-1}} + \chi_{\{u\le a\}}u^q) & \text{in } \Omega, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $\lambda, a > 0$ and $0 < q < N-1$. The N-Laplacian operator is defined as $\Delta_N u := |\nabla u|^{N-2}\nabla u$. Here we study the elliptic problem with discontinuous nonlinearity motivated from the following Trudinger–Moser inequality given in [21, 27]:

$$\sup_{\|u\|_{W_0^{1,N}}\le 1} \int_\Omega e^{\alpha_N |u|^{N/(N-1)}} < \infty \tag{1.2}$$

where $\alpha_N = N\omega_N^{1/(N-1)}$, ω_N is the volume of S^{N-1} and $\|u\|_{W_0^{1,N}} = \int_\Omega |\nabla u|^N$. We also have the following assumptions on $h(t)$.

(A1) One has $h(t) \in C^1(0,\infty)$, $h(0) = 0$, $h(t) > 0$ for $t > 0$ and $f(t) = h(t)e^{|t|^{N/N-1}}$ is nondecreasing in t.

(A2) For all $\epsilon > 0$,

$$\liminf_{t\to\infty} h(t)e^{\epsilon|t|^{N/N-1}} = \infty \quad \text{and} \quad \limsup_{t\to\infty} h(t)e^{-\epsilon|t|^{N/N-1}} = 0.$$

(A3) There exists a constant $K > 0$ such that

$$F(t) = \int_0^t f(s)\,ds < K(f(t)+1).$$

Problems with discontinuous nonlinearities have widely been studied in the last two decades because of their occurrence in mathematical physics. Such problems occur in various branches of mathematical physics, for example the obstacle problem, seepage surface problem and the Elenbass equations ([11–13]). Starting with the pioneering work of Ambrosetti, Brezis and Cerami in [7], there are many references where the existence and multiplicity results for the elliptic equation with continuous nonlinearity of concave-convex type are addressed. Subsequently, the quasilinear elliptic equations of concave-convex type continuous nonlinearities are studied in [16].

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The main difficulty in studying the elliptic equations with discontinuous nonlinearities is the non-smoothness of the associated functional. Therefore, we need to use the notion of generalized gradients as discussed in the pioneering work of Clarke [14] which was later applied to the differential equation setting by Chang [13]. Adopting the generalized critical point theory in [8], the authors have studied the semilinear elliptic equation with concave-convex type discontinuous nonlinearity in bounded domain. In [18], the existence and multiplicity results for the semilinear elliptic equations with discontinuous and sublinear nonlinearities are studied in $\mathbb{R}^N$. In [26], the authors have proved the existence and multiplicity of solutions of (1.1) for $N = 2$. Also in [15], the existence of multiple solutions of the critical problem with discontinuous and singular nonlinearity is obtained. The quasilinear elliptic equations with discontinuous nonlinearities have been studied in [5] and [24].

With this introduction we state the main result of this paper.

Theorem 1.1. *There exists a constant $\Lambda > 0$ such that for all $\lambda \in (0, \Lambda]$, there exists a solution of* (1.1).

The rest of the paper is organized as follows. In Section 2 we discuss some of the preliminary definitions and results which are required for setting the variational framework and for subsequent analysis. In Section 3 we prove the existence of the solution of (1.1). Lastly we conclude the paper in Section 4 quoting some open problems in this direction.

2 Preliminaries

We consider the following preliminary notions. Let X be a real Banach space, let X^* be its dual space and $\langle \cdot, \cdot \rangle$ denotes its duality action. Then for $x, v \in X$ and a locally Lipschitz continuous function f, the generalized directional derivative $f^0(x, v)$ (of f at x along v) is defined as

$$f^0(x, v) = \limsup_{\substack{h \to 0 \\ \lambda \downarrow 0}} \frac{f(x + h + \lambda v) - f(x + h)}{\lambda}.$$

The generalized gradient of f at x, denoted by $\partial f(x)$, is defined as

$$\partial f(x) \subseteq X^*, \quad w \in \partial f(x) \text{ if and only if } \langle w, v \rangle \leq f^0(x, v) \text{ for all } v \in X.$$

We recall some properties of ∂f which are proved in [13].

Lemma 2.1. *The generalized gradient of a locally Lipschitz continuous function f has the following properties.*
(i) *For all $k \in \mathbb{R}$, $\partial(kf)(x) = k\partial f(x)$.*
(ii) *If f, g are locally Lipschitz functions, then $\partial(f + g)(x) \subset \partial f(x) + \partial g(x)$.*
(iii) *If f is convex, then $\partial f(x)$ coincides with the sub-differential of f in the sense of convex analysis.*
(iv) *For each $v \in X$, $f^0(x, v) = \max\{\langle \xi, v \rangle : \xi \in \partial f(x)\}$.*
(v) *If x is a local minimum point for f, then $0 \in \partial f(x)$.*
(vi) *If f is differentiable in x with differential $f'(x) \in X^*$, then $\partial f(x) = \{f'(x)\}$.*

Definition 2.1. A point $x \in X$ is said to be a critical point of f if $0 \in \partial f(x)$.

The functional $I_\lambda : W_0^{1,N}(\Omega) \to \mathbb{R}$ associated to (1.1) is given by

$$I_\lambda(u) = \frac{1}{N} \int_\Omega |\nabla u|^N - \lambda \int_\Omega F(u) - \lambda \int_\Omega G(u),$$

where

$$G(s) = \int_0^s \chi_{\{t \leq a\}} t^q \, dt.$$

Now in view of [13, Theorem 2.1 and Theorem 2.2], $w \in \partial I_\lambda(u)$ if and only if there exists a function $\overline{w} \in L^\infty(\Omega)$ such that

$$\langle w, v\rangle = \int_\Omega |\nabla u|^{N-2}\nabla u \cdot \nabla v - \lambda \int_\Omega f(u)v - \lambda \int_\Omega u^q \overline{w} v \quad \text{for all } v \in W_0^{1_N}(\Omega)$$

and

$$\overline{w}(x) \in [\underline{\chi}_{\{u(x)\le a\}}, \overline{\chi}_{\{u(x)\le a\}}] \quad \text{for a.e. } x \in \Omega,$$

where

$$\underline{\chi}_{\{u(x)\le a\}} = \liminf_{s(x)\to u(x)} \chi_{\{s(x)\le a\}} \quad \text{and} \quad \overline{\chi}_{\{u(x)\le a\}} = \limsup_{s(x)\to u(x)} \chi_{\{s(x)\le a\}}.$$

Note that $\underline{\chi}_{\{u(x)\le a\}}$ and $\overline{\chi}_{\{u(x)\le a\}}$ are the characteristic functions of the set theoretical upper and lower limits of sub level sets. Hence u defines a critical point for I_λ if and only if there exists a function $\overline{w} \in L^\infty(\Omega)$ such that

$$\int_\Omega |\nabla u|^{N-2}\nabla u \cdot \nabla v - \lambda \int_\Omega f(u)v - \lambda \int_\Omega u^q \overline{w} v = 0 \quad \text{for all } v \in W_0^{1,N}(\Omega) \tag{2.1}$$

with

$$\overline{w}(x) \in [\underline{\chi}_{\{u(x)\le a\}}, \overline{\chi}_{\{u(x)\le a\}}] \quad \text{for a.e. } x \in \Omega. \tag{2.2}$$

Definition 2.2. A solution of (1.1) is a function $u \in W_0^{1,N}(\Omega)$ such that $u > 0$ in Ω and u satisfies (2.1)–(2.2).

Definition 2.3. The non-smooth functional $I_\lambda : W_0^{1,N}(\Omega) \to \mathbb{R}$ is said to satisfy the Palais–Smale condition at $c \in \mathbb{R}$ if for every sequence $\{u_n\} \subset W_0^{1,N}(\Omega)$ such that $I(u_n) \to c$ and $\|w_n\|_{W^{-1,N'}} = \min_{w\in\partial I(u_n)} \|w\|_{W^{-1,N'}} \to 0$, there exists a subsequence of $\{u_n\}$ which converges strongly in $W_0^{1,N}(\Omega)$, where $W^{-1,N'}(\Omega)$ is the dual space of $W_0^{1,N}(\Omega)$.

Remark 2.1. By elliptic regularity, such a solution $u \in W^{2,p}(\Omega)$ for any $p \ge 1$. Thus on $\Omega_a = \{x \in \Omega : u(x) = a\}$ we get $(-f(x,a)) \in [0,1]$ which is a contradiction as $f(x,a) > 0$. Hence $|\Omega_a| = 0$, where $|\Omega_a|$ is the N-dimensional Lebesgue measure of Ω_a.

Now we recall the following result proved by Brezis–Lieb in [9].

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\{u_n\} \subset L^p(\Omega)$, $1 \le p < \infty$, be such that $\|u_n\|_{L^p} \le C$ for some $C > 0$ and $u_n \to u$ a.e. in Ω. Then*

$$\lim_{n\to\infty} [\|u_n - u\|_{L^p}^p - \|u_n\|_{L^p}^p + \|u\|_{L^p}^p] = 0.$$

Also we recall the following inequalities of [19, Lemma 4.1]. For $\xi, \zeta \in \mathbb{R}^N$,

$$\begin{cases} [(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)\cdot(\xi-\zeta)](|\xi|^p + |\zeta|^p)^{(2-p)/p} \ge (p-1)|\xi-\zeta|^p, & 1 < p < 2, \\ (|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)\cdot(\xi-\zeta) \ge \left(\frac{1}{2}\right)^p |\xi-\zeta|^p, & p \ge 2. \end{cases} \tag{2.3}$$

3 Existence of a solution of (1.1)

In this section, we prove the existence of a solution of (1.1). First we note that the functional I_λ is locally Lipschitz and non-differentiable. We also prove the following result about I_λ.

Lemma 3.1. *The functional I_λ satisfies the mountain pass geometry near* 0.

Proof. First we prove that there exist $\delta_0, r, \lambda_0 > 0$ such that $I_\lambda(u) > \delta_0$ for all $u \in W_0^{1,N}(\Omega)$ with $\|u\|_{W_0^{1,N}} = r$ and $\lambda \in (0, \lambda_0)$. Note that

$$\left|\int_\Omega G(u)\right| \le C_1 \|u\|_{W_0^{1,N}}^{q+1},$$

and from (A2) and (1.2) we have

$$\int_\Omega F(u) = \int_\Omega \int_0^u h(s)e^{|s|^{N/N-1}}\,ds \le C_2 + \int_\Omega e^{(2|u|)^{N/N-1}} \le C_3$$

for $\|u\|_{W_0^{1,N}} \le \frac{1}{2}$. Thus for $r < \frac{1}{2}$, λ_0 sufficiently small and $\|u\|_{W_0^{1,N}} = r$, we get $\delta_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$,

$$I_\lambda(u) \ge \frac{1}{N}\|u\|^N_{W_0^{1,N}} - \lambda C_3 - \lambda C_1 \|u\|^{q+1}_{W_0^{1,N}} > \delta_0.$$

Also for $\phi \in C_c^\infty(\Omega)$, $I_\lambda(t\phi) \to -\infty$ as $t \to \infty$ and for $t \to 0^+$ we have $I_\lambda(u) < 0$. Thus I_λ satisfies the mountain pass geometry near 0. □

We need the following lemma to prove Theorem 1.1.

Lemma 3.2. *Let $\{u_n\}$ be a Palais–Smale sequence for I_λ. Then there exists some $u \in W_0^{1,N}(\Omega)$ such that $u_n \rightharpoonup u$ and u is a critical point of I_λ.*

Proof. Here we follow the approach as in [23] and [25]. Let $\{u_n\}$ be a Palais–Smale sequence at $c \in \mathbb{R}$. Then $I_\lambda(u_n) \to c$ and $\|w_n\|_{W^{-1,N'}} = \min_{w \in \partial I_\lambda(u_n)} \|w\|_{W^{-1,N'}} \to 0$, where $W^{-1,N'}(\Omega)$ denotes the dual of $W_0^{1,N}(\Omega)$. This implies

$$\frac{1}{N}\int_\Omega |\nabla u_n|^N - \lambda \int_\Omega F(u_n) - \lambda \int_\Omega G(u_n) = c + o(1) \tag{3.1}$$

and there exists some $\overline{w_n}(x) \in [\underline{\chi}_{\{u_n(x)\le a\}}, \overline{\chi}_{\{u_n(x)\le a\}}]$ such that

$$\langle w_n, v\rangle = \int_\Omega |\nabla u_n|^{N-2}\nabla u_n \cdot \nabla v - \lambda \int_\Omega f(u_n)v - \lambda \int_\Omega \overline{w_n} u_n^q v \tag{3.2}$$

for all $v \in W_0^{1,N}(\Omega)$, where $\langle w, v\rangle$ denotes the duality action of $W^{-1,N'}(\Omega)$ on $W_0^{1,N}(\Omega)$.

By assumption (A3), there exist $K_1 > 0$ and $K_2 > N$ such that

$$F(t) \le K_1 + \frac{tf(t)}{K_2} \quad \text{for all } t > 0.$$

Thus (3.1) and (3.2) give

$$\begin{aligned} c + \lambda \int_\Omega G(u_n) + K_1|\Omega| + o(1) &\ge \frac{1}{N}\int_\Omega |\nabla u_n|^N - \frac{\lambda}{K_2}\int_\Omega f(u_n)u_n \\ &\ge \Big(\frac{1}{N} - \frac{1}{K_2}\Big)\|u_n\|^N_{W_0^{1,N}} + \frac{\lambda}{K_2}\int_\Omega \overline{w_n} u_n^{q+1} + \frac{1}{K_2}\langle w_n, u_n\rangle. \end{aligned}$$

This implies $\{u_n\}$ is bounded in $W_0^{1,N}(\Omega)$. Let $u_n \rightharpoonup u$ weakly in $W_0^{1,N}(\Omega)$. Also as $\|\overline{w}_n\|_{L^\infty} \le 1$, we can assume $\overline{w}_n \to \overline{w}$ in $L^\infty(\Omega)$ in weak* topology, where $\overline{w}(x) \in [\underline{\chi}_{\{u(x)\le a\}}, \overline{\chi}_{\{u(x)\le a\}}]$. Now note that $\{|\nabla u_n|^{N-2}\nabla u_n\}$ is bounded in $(L^{N/N-1}(\Omega))^N$. Thus

$$\begin{aligned} |\nabla u_n|^N &\rightharpoonup \mu \quad \text{in } D'(\Omega), \\ |\nabla u_n|^{N-2}\nabla u_n &\rightharpoonup \nu \quad \text{weakly in } (L^{N/N-1}(\Omega))^N, \end{aligned}$$

where μ is a nonnegative regular measure and $D'(\Omega)$ are the distributions on Ω. Let $\sigma > 0$ and

$$A_\sigma = \{x \in \overline{\Omega} : \text{ for all } r > 0, \mu(B_r(x) \cap \overline{\Omega}) \ge \sigma\}.$$

Note that A_σ is a finite set. If not, then there exists a sequence of distinct points $\{x_k\}$ in A_σ. Since for all $r > 0$, $\mu(B_r(x) \cap \overline{\Omega}) \ge \sigma$, we get $\mu(\{x_k\}) \ge \sigma$. This implies $\mu(A_\sigma) = \infty$. But

$$\mu(A_\sigma) = \lim_{n\to\infty} \int_{A_\sigma} |\nabla u_n|^N \le C < \infty.$$

Thus $A_\sigma = \{x_1, x_2, \dots, x_m\}$. Let $u \in W_0^{1,N}(\mathcal{O})$ and $\mathcal{O} \subset \mathbb{R}^N$ be a bounded domain. Then there exist r_1 and C_1 depending only on N such that

$$\int_{\mathcal{O}} e^{r_1\left(\frac{|u|}{\|\nabla u\|_{L^N(\mathcal{O})}}\right)^{N/N-1}} \leq C_1. \tag{3.3}$$

Assertion 1. *For any relatively compact subset $\mathcal{K} \subset \overline{\Omega} \setminus A_\sigma$ and $\sigma > 0$ such that $\frac{\sigma^{1/N-1}}{r_1} < 1$, we have*

$$\lim_{n\to\infty} \int_{\mathcal{K}} f(u_n)u_n = \int_{\mathcal{K}} f(u)u.$$

For this let $x_0 \in \mathcal{K}$ and $r_0 > 0$ such that $\mu(B_{r_0}(x_0) \cap \overline{\Omega}) < \sigma$. Let $\phi \in C_0^\infty(\Omega)$ such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ in $B_{\frac{r_0}{2}}(\Omega) \cap \overline{\Omega}$ and $\phi \equiv 0$ in $\overline{\Omega} \setminus B_{r_0}(x_0)$. Thus

$$\lim_{n\to\infty} \int_{B_{r_0}(x_0)\cap\overline{\Omega}} |\nabla u_n|^N \phi \, dx = \int_{B_{r_0}(x_0)\cap\overline{\Omega}} \phi \, d\mu \leq \mu(B_{r_0}(x_0) \cap \overline{\Omega}) \leq (1-\epsilon)\sigma.$$

Therefore for $n \in \mathbb{N}$ sufficiently large and $\epsilon > 0$ sufficiently small, we have

$$\lim_{n\to\infty} \int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} |\nabla u_n|^N \leq (1-\epsilon)\sigma. \tag{3.4}$$

Using this in (3.3) we get

$$\int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} |f(u_n)|^q \leq C, \tag{3.5}$$

where $q > 1$ such that $\frac{q\sigma^{1/N-1}}{r_1} < 1$.

Now we estimate

$$\int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} |f(u_n)u_n - f(u)u| \leq I_1 + I_2, \tag{3.6}$$

where

$$I_1 = \int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} |f(u_n) - f(u)||u|, \quad I_2 = \int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} |f(u_n)||u_n - u|.$$

By (3.5) and Hölder's inequality, $I_2 \to 0$ as $n \to \infty$. To estimate I_1, let $\phi \in C_c^\infty(\Omega)$ such that $\|u - \phi\|_{L^{q'}} < \epsilon$. Then

$$I_1 \leq \int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} |f(u_n)||u-\phi| + \int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} |f(u_n) - f(u)||\phi| + \int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} |f(u)||u-\phi|. \tag{3.7}$$

Now using (3.5) and Hölder's inequality, it is easy to check that the first and the last terms of the right-hand side of (3.7) go to zero as $n \to \infty$. Now we claim that the middle term also goes to zero. Note that for $u_n \to u$ in $L^1(\Omega)$ with $f(u_n) \in L^1(\Omega)$ and $\int_\Omega f(u_n)u_n \leq C_1$ we have $f(u_n) \to f(u)$ in $L^1(\Omega)$. Indeed for $\epsilon > 0$, there exists a constant $\delta > 0$ such that for any $A \subset \Omega$, $|A| \leq \delta$, we have

$$\int_A |u| < \epsilon \quad \text{and} \quad \int_A |f(u)| < \epsilon.$$

Now let $M_1 > 0$ such that $|\{x \in \Omega : |u(x)| \geq M_1\}| \leq \delta$. Take $M = \max\{M_1, \frac{C_1}{\epsilon}\}$. Then

$$\left| \int_\Omega f(u_n) - f(u) \right| \leq T_1 + T_2 + T_3,$$

where

$$T_1 = \int_{\{|u_n|\geq M\}} |f(u_n)|, \quad T_2 = \int_{\{|u|\geq M\}} |f(u)|, \quad T_3 = \left| \int_{\{|u_n|<M\}} |f(u_n)| - \int_{\{|u|<M\}} |f(u)| \right|.$$

Now

$$T_1 = \int_{\{|u_n|\geq M\}} \frac{|f(u_n)u_n|}{|u_n|} \leq \frac{C_1}{M_1} \leq \epsilon$$

and

$$T_2 = \int_{\{|u|\geq M\}} |f(u)| < \epsilon$$

since $|\{x \in \Omega : |u(x)| \geq M_1\}| \leq \delta$. Also

$$\begin{aligned} T_3 &= \Big| \int_{\{|u_n|<M\}} |f(u_n)| - \int_{\{|u|<M\}} |f(u)| \Big| \\ &\leq \Big| \int_\Omega \chi_{\{|u_n|<M\}}(|f(u_n)| - |f(u)|) \Big| + \Big| \int_\Omega (\chi_{\{|u_n|<M\}} - \chi_{\{|u|<M\}})|f(u)| \Big|. \end{aligned} \tag{3.8}$$

Note that $\chi_{\{|u_n|<M\}}(|f(u_n)| - |f(u)|) \to 0$ a.e. in Ω. Also

$$|\chi_{\{|u_n|<M\}}(|f(u_n)| - |f(u)|)| \leq \begin{cases} |f(u)| & \text{if } |u_n| \geq M, \\ C + |f(u)| & \text{if } |u_n| < M, \end{cases}$$

where $C = \sup_{\{|u_n|\}<M} |f(u_n)|$. Thus by the Lebesgue Dominated Convergence Theorem, the first term on the right-hand side of (3.8) goes to zero as $n \to \infty$. Also since

$$\{x \in \Omega : |u_n(x)| < M\} \setminus \{x \in \Omega : |u(x)| < M\} \subset \{x \in \Omega : |u(x)| \geq M\},$$

this implies

$$\int_\Omega (\chi_{\{|u_n|<M\}} - \chi_{\{|u|<M\}})|f(u)| \leq \int_{\{|u|\geq M\}} |f(u)| < \epsilon.$$

Thus the second term on the right-hand side of (3.8) also goes to zero as $n \to \infty$. Using this in (3.6) and (3.7) we get

$$\int_{B_{\frac{r_0}{2}}(x_0)\cap\overline{\Omega}} |f(u_n)u_n - f(u)u| \to 0 \quad \text{as } n \to \infty$$

and as $\mathcal{K}$ is compact, Assertion 1 follows.

Assertion 2. *Let $\epsilon_0 > 0$ be such that $B_{\epsilon_0}(x_i) \cap B_{\epsilon_0}(x_j) = \emptyset$ if $i \neq j$ and*

$$\Omega_{\epsilon_0} = \{x \in \overline{\Omega} : |x - x_j| \geq \epsilon_0,\ j = 1, 2, \ldots, m\}.$$

Then

$$\int_{\Omega_{\epsilon_0}} (|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u) \cdot (\nabla u_n - \nabla u) \to 0. \tag{3.9}$$

Indeed, let $0 < \epsilon < \epsilon_0$ and $\phi \in C_c^\infty(\mathbb{R}^N)$ such that $\phi \equiv 1$ in $B_{1/2}(0)$ and $\phi \equiv 0$ in $\overline{\Omega} \setminus B_1(0)$. Take

$$\psi_\epsilon = 1 - \sum_{j=1}^m \phi\Big(\frac{x - x_j}{\epsilon}\Big).$$

Then $0 \leq \psi_\epsilon \leq 1$, $\psi_\epsilon \equiv 1$ in $\overline{\Omega}_\epsilon = \overline{\Omega} \setminus \bigcup_{j=1}^m B_\epsilon(x_j)$, $\psi_\epsilon \equiv 0$ in $\bigcup_{j=1}^m B_{\epsilon/2}(x_j)$ and $\{\psi_\epsilon u_n\}$ is bounded in $W_0^{1,N}(\Omega)$. Now taking $v = \psi_\epsilon u_n$ in (3.2) we get

$$\int_\Omega [|\nabla u_n|^N \psi_\epsilon + |\nabla u_n|^{N-2}\nabla u_n \cdot \nabla\psi_\epsilon u_n - \lambda f(u_n)u_n\psi_\epsilon - \lambda\overline{w_n}u_n^{q+1}\psi_\epsilon] \leq \epsilon_n \|\psi_\epsilon u_n\|_{W_0^{1,N}}. \tag{3.10}$$

Again taking $v = -\psi_\epsilon u$ in (3.2) we get

$$\int_\Omega [-|\nabla u_n|^{N-2}\nabla u_n \cdot \nabla u\psi_\epsilon - |\nabla u_n|^{N-2}\nabla u_n \cdot \nabla\psi_\epsilon u + \lambda f(u_n)u\psi_\epsilon + \lambda\overline{w_n}u_n^q u\psi_\epsilon] \leq \epsilon_n \|\psi_\epsilon u\|_{W_0^{1,N}}. \tag{3.11}$$

Also from the convexity of $t \mapsto |t|^N$ for $t \in \mathbb{R}^N$, we have

$$\begin{aligned}0 &\le \int_{\Omega_{\epsilon_0}} (|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u) \cdot (\nabla u_n - \nabla u) \\ &\le \int_{\Omega} (|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u) \cdot (\nabla u_n - \nabla u)\psi_\epsilon.\end{aligned}$$

Rewriting this as

$$0 \le \int_{\Omega} |\nabla u_n|^N \psi_\epsilon - |\nabla u_n|^{N-2}\nabla u_n \cdot \nabla u \psi_\epsilon - |\nabla u|^{N-2}\nabla u \cdot \nabla u_n \psi_\epsilon + |\nabla u|^N \psi_\epsilon, \tag{3.12}$$

from (3.10), (3.11) and (3.12) we get

$$\begin{aligned}0 \le \int_{\Omega} |\nabla u_n|^{N-2}\nabla u_n \cdot \nabla\psi_\epsilon(u_n - u) + \int_{\Omega} \psi_\epsilon |\nabla u|^{N-2}\nabla u \cdot (\nabla u - \nabla u_n) \\ + \lambda \int_{\Omega} f(u_n)(u_n - u)\psi_\epsilon + \lambda \int_{\Omega} \overline{w_n} u_n^q (u_n - u)\psi_\epsilon + \epsilon_n \|\psi_\epsilon u_n\| + \epsilon_n \|\psi_\epsilon u\|_{W_0^{1,N}}.\end{aligned}$$

Now as by Young's inequality, for given $\delta > 0$, there exists a constant $C_\delta > 0$ such that

$$\begin{aligned}\int_{\Omega} |\nabla u_n|^{N-2}\nabla u_n \cdot \nabla\psi_\epsilon(u_n - u) &\le \delta \int_{\Omega} |\nabla u_n|^N + C_\delta \int_{\Omega} |\nabla\psi_\epsilon|^N |u_n - u|^N \\ &\le \delta C + C_\delta \Big(\int_{\Omega} |\nabla\psi_\epsilon|^{Nr}\Big)^{1/r} \Big(\int_{\Omega} |u_n - u|^{Ns}\Big)^{1/s},\end{aligned} \tag{3.13}$$

where C, r and $s > 0$ with $\frac{1}{r} + \frac{1}{s} = 1$. Thus we get

$$\limsup_{n\to\infty} \int_{\Omega} |\nabla u_n|^{N-2}\nabla u_n \cdot \nabla\psi_\epsilon(u_n - u) \le 0. \tag{3.14}$$

Also noting that $u_n \rightharpoonup u$ weakly in $W_0^{1,N}(\Omega)$ we have

$$\lim_{n\to\infty} \int_{\Omega} \psi_\epsilon |\nabla u|^{N-2}\nabla u \cdot (\nabla u - \nabla u_n) = 0 \tag{3.15}$$

and

$$\lim_{n\to\infty} \int_{\Omega} \overline{w_n} u_n^q (u_n - u)\psi_\epsilon = 0. \tag{3.16}$$

Also by Assertion 1, taking $\mathcal{K} = \overline{\Omega}_{\epsilon/2}$ one can check that

$$\lim_{n\to\infty} \int_{\Omega} f(u_n)(u_n - u)\psi_\epsilon = 0. \tag{3.17}$$

Now from (3.14)–(3.17), (3.9) follows.

Since ϵ_0 is arbitrary, we get $\nabla u_n(x) \to \nabla u(x)$ a.e. in Ω and hence $|\nabla u_n|^{N-2}\nabla u_n \rightharpoonup |\nabla u|^{N-2}\nabla u$ weakly in $(L^{N/N-1}(\Omega))^N$. Thus passing to the limit in (3.2), for any $v \in W_0^{1,N}(\Omega)$, we get

$$\int_{\Omega} |\nabla u|^{N-2}\nabla u \cdot \nabla v - \lambda \int_{\Omega} f(u)v - \lambda \int_{\Omega} \overline{w}v = 0.$$

This implies that u is a critical point of I_λ. □

Now we prove the following compactness result.

Lemma 3.3. *Let $\{u_n\}$ be a Palais–Smale sequence at $c \in \mathbb{R}$ with*

$$\liminf_{n\to\infty} \|u_n\|_{W_0^{1,N}} < (\alpha_N)^{N-1},$$

where $\alpha_N = N\omega_N^{1/(N-1)}$ and ω_N is the volume of S^{N-1}. Then, up to a subsequence, $\{u_n\}$ converges strongly in $W_0^{1,N}(\Omega)$.

Proof. As $\{u_n\}$ is a Palais–Smale sequence at $c \in \mathbb{R}$, then as in Lemma 3.2 $u_n \rightharpoonup u_0$ weakly in $W_0^{1,N}(\Omega)$ for some $u_0 \in W_0^{1,N}(\Omega)$. Let $r_n = u_n - u_0$. Then we have $r_n \rightharpoonup 0$ and

$$\|u_n\|^N_{W_0^{1,N}} = \|r_n\|^N_{W_0^{1,N}} + \|u_0\|^N_{W_0^{1,N}} + o(1),$$

thanks to Lemma 2.2. Also,

$$\int_\Omega f(u_n)u_0 \to \int_\Omega f(u_0)u_0$$

and for each $n \in \mathbb{N}$, there exists some $\rho_n(x) \in [\underline{\chi}_{\{u_n(x)\le a\}}, \overline{\chi}_{\{u_n(x)\le a\}}]$ such that

$$\int_\Omega |\nabla u_n|^{N-2}\nabla u_n \nabla\phi - \lambda \int_\Omega f(u_n)\phi - \lambda \int_\Omega \rho_n u_n^q \phi \le \epsilon \|\phi\|_{W_0^{1,N}}$$

for all $\phi \in W_0^{1,N}(\Omega)$ and $\epsilon \downarrow 0$. Now taking $\phi = u_n$ and using Lemma 2.2, we get

$$\|r_n\|^N_{W_0^{1,N}} + \|u_0\|^N_{W_0^{1,N}} - \lambda \int_\Omega f(u_n)u_n - \lambda \int_\Omega \rho_n u_n^{q+1} \le \epsilon \|u_n\|_{W_0^{1,N}} + o(1). \tag{3.18}$$

Now as $\|\rho_n\|_{L^\infty} \le 1$, we can assume $\rho_n \to \rho$ in $L^\infty(\Omega)$ as $n \to \infty$ with respect to the weak* topology and $\rho(x) \in [\underline{\chi}_{\{u_0(x)\le a\}}, \overline{\chi}_{\{u_0(x)\le a\}}]$. Thus using the Lebesgue Dominated Convergence Theorem, equation (3.18) implies

$$\|r_n\|^N_{W_0^{1,N}} + \|u_0\|^N_{W_0^{1,N}} - \lambda \int_\Omega f(u_0)u_0 - \lambda \int_\Omega \rho u_0^{q+1} \le \epsilon \|u_n\|_{W_0^{1,N}} + \lambda \int_\Omega f(u_n) r_n + o(1). \tag{3.19}$$

Also as u_0 is a critical point of I_λ, we have $0 \in \partial I_\lambda(u_0)$. Thus for all $\phi \in W_0^{1,N}(\Omega)$,

$$\int_\Omega |\nabla u_0|^{N-2}\nabla u_0 \nabla\phi - \lambda \int_\Omega f(u_0)\phi - \lambda \int_\Omega \rho u_0^q \phi \ge 0. \tag{3.20}$$

Therefore (3.19) and (3.20) give

$$\|r_n\|^N_{W_0^{1,N}} \le \epsilon \|u_n\|_{W_0^{1,N}} + \lambda \int_\Omega f(u_n) r_n + o(1). \tag{3.21}$$

Also using Hölder's and Sobolev inequalities and choosing n large such that for $s > 1$, $s\|u_n\|^{N/N-1}_{W_0^{1,N}} \le (\alpha_N)^{(N-1)}$, we have

$$\begin{aligned}
\int_\Omega |f(u_n) r_n| &\le \Big(\int_\Omega |f(u_n)|^s\Big)^{1/s} \Big(\int_\Omega |r_n|^{s'}\Big)^{1/s'} \\
&\le \Big(\int_\Omega \exp\Big(s\|u_n\|^{N/N-1}_{W_0^{1,N}} \Big|\frac{u_n}{\|u_n\|_{W_0^{1,N}}}\Big|^{N/N-1}\Big)\Big)^{1/s} \Big(\int_\Omega |r_n|^{s'}\Big)^{1/s'} \\
&\le C\|r_n\|_{L^{s'}}
\end{aligned}$$

for some $C > 0$. Thus (3.21) implies $\|r_n\|_{W_0^{1,N}} \to 0$. Hence up to a subsequence $\{u_n\}$ converges strongly in $W_0^{1,N}(\Omega)$. □

Set $\Lambda = \sup\{\lambda > 0 : \text{problem (1.1) admits a solution}\}$.

Lemma 3.4. *We have* $0 < \Lambda < \infty$.

Proof. Let $\phi \in C_c^\infty(\Omega)$. Then by Lemma 3.1, for $\lambda \in (0, \lambda_0)$ and $\rho > 0$ small we have

$$-\infty < c_0 = \inf_{\overline{B_\rho}} I_\lambda < 0.$$

Now by applying Ekeland's variational principle, there exists some $u_\epsilon \in \overline{B_\rho}$ such that

(i) $I_\lambda(u_\epsilon) < c_0 + \epsilon$,

(ii) $I_\lambda(u_\epsilon) < I_\lambda(u) + \epsilon\|u - u_\epsilon\|_{W_0^{1,N}}$, for $u \not\equiv u_\epsilon$, $u \in \overline{B_\rho}$.

As $I_\lambda(u_\epsilon) < c_0 + \epsilon < \inf_{\partial B_\rho} I_\lambda$ for $0 < \epsilon < \inf_{\partial B_\rho} I_\lambda - \inf_{\overline{B_\rho}} I_\lambda$, we have $u_\epsilon \in B_\rho$. Let $t > 0$ with $v_t = u_\epsilon + tv \in B_\rho$ for $v \in W_0^{1,N}(\Omega)$. Thus from (ii) we get

$$\frac{I_\lambda(u_\epsilon + tv) - I_\lambda(u_\epsilon)}{t} \geq -\epsilon\|v\|_{W_0^{1,N}}.$$

Now for $t \to 0^+$, this gives

$$-\epsilon\|v\|_{W_0^{1,N}} \leq (I_\lambda)^0(u_\epsilon, v) = \max_{w \in \partial I_\lambda(u_\epsilon)} \langle w, v\rangle \quad \text{for all } v \in W_0^{1,N}(\Omega).$$

Interchanging v and $-v$ we get

$$\min_{w \in \partial I_\lambda(u_\epsilon)} \langle w, v\rangle \leq \epsilon\|v\|_{W_0^{1,N}} \quad \text{for all } v \in W_0^{1,N}(\Omega)$$

and hence

$$\sup_{\|v\|_{W_0^{1,N}}=1} \min_{w \in \partial I_\lambda(u_\epsilon)} \langle w, v\rangle \leq \epsilon.$$

This on applying Ky Fan's min-max theorem ([10, Proposition 1.8]) implies $\min_{w \in \partial I_\lambda(u_\epsilon)}\langle w, v\rangle \leq \epsilon$ which ensures the existence of a Palais–Smale sequence $\{u_n\}$ with $I_\lambda(u_n) \to c_0$ and

$$\|w_n\|_{W^{-1,N'}} = \min_{w \in \partial I_\lambda(u_n)} \|w\|_{W^{-1,N'}} \to 0 \quad \text{as } n \to \infty.$$

Then the existence of the solution u_λ follows from Lemma 3.2. Also using Lemma 3.3 we have $u_n \to u_\lambda$ strongly in $W_0^{1,N}(\Omega)$. This implies $I_\lambda(u_\lambda) = c_0 < 0$ and hence $u_\lambda \not\equiv 0$. Now the strong maximum principle implies that $u_\lambda > 0$. This proves $\Lambda > 0$.

Also by the isolatedness of the first eigenvalue of $-\Delta_N$ on $W_0^{1,N}(\Omega)$, it can be proved that $\Lambda < \infty$. Indeed, for $\lambda_n \to \infty$ such that (1.1) admits a solution u_{λ_n} for $\lambda = \lambda_n$, we can find $\lambda^* > 0$ such that $\lambda f(t) > (\lambda_1 + \epsilon)t^{N-1}$ for all $\lambda > \lambda^*$ and $t > 0$. Then u_{λ_n} is a super-solution and for $\mu < \lambda_1 + \epsilon$ such that $\mu\phi_1 \leq u_{\lambda_n}$, $\mu\phi_1$ is a sub-solution of

$$\begin{cases} -\Delta_N u = (\lambda_1 + \epsilon)u^{N-1} & \text{in } \Omega, \\ u = 0 & \text{in } \Omega, \end{cases}$$

where λ_1 is the first eigenvalue of $-\Delta_N$ in $W_0^{1,N}(\Omega)$ and ϕ_1 is the first eigenfunction. Then by the monotone iteration method, for each $\epsilon > 0$, the above problem admits a solution. This contradicts that λ_1 is isolated. Thus $\Lambda < \infty$. □

We prove the following comparison principle for the N-Laplacian problem with discontinuous nonlinearity.

Lemma 3.5. *Let $v, w \in W_0^{1,N}(\Omega)$ with $v > 0$, $w \geq 0$ such that*

$$\begin{cases} -\Delta_N v \geq \chi_{\{v \leq a\}} v^q & \text{in } \Omega, \\ v = 0 & \text{on } \Omega, \end{cases} \tag{3.22}$$

and

$$\begin{cases} -\Delta_N w \leq \chi_{\{w \leq a\}} w^q & \text{in } \Omega, \\ w = 0 & \text{on } \Omega. \end{cases} \tag{3.23}$$

Then $v \geq w$ a.e. in Ω.

Proof. Let $\mathcal{A} = \{x \in \Omega : w(x) > v(x)\}$. Then multiplying (3.22) by $\frac{w^N}{v^{N-1}}\chi_{\mathcal{A}}$ and (3.23) by $w\chi_{\mathcal{A}}$, where $\chi_{\mathcal{A}}$ is the characteristic function of $\mathcal{A}$, we get

$$\int_{\mathcal{A}} |\nabla v|^{N-2}\nabla v \cdot \nabla\Big(\frac{w^N}{v^{N-1}}\Big) \geq \int_{\mathcal{A}} \chi_{\{v \leq a\}} v^q \frac{w^N}{v^{N-1}} \tag{3.24}$$

and

$$\int_{\mathcal{A}} |\nabla w|^N \leq \int_{\mathcal{A}} \chi_{\{w\leq a\}} w^q w. \tag{3.25}$$

Now as by Picone's identity for N-Laplacian in [4],

$$\int_{\mathcal{A}} |\nabla w|^N \geq \int_{\mathcal{A}} |\nabla v|^{N-2}\nabla v \cdot \nabla\Big(\frac{w^N}{v^{N-1}}\Big),$$

from (3.24) and (3.25) we get

$$0 \geq \int_{\mathcal{A}} \Big(\frac{\chi_{\{v\leq a\}} v^q}{v^{N-1}} - \frac{\chi_{\{w\leq a\}} w^q}{w^{N-1}}\Big) w^N. \tag{3.26}$$

Note that $\mathcal{A} = A_1 \cup A_2 \cup A_3$, where

$$\begin{aligned} A_1 &= \{x \in \Omega : w(x) > v(x)\} \cap \{x \in \Omega : a \geq w(x)\}, \\ A_2 &= \{x \in \Omega : w(x) > v(x)\} \cap \{x \in \Omega : w(x) > a \geq v(x)\}, \\ A_3 &= \{x \in \Omega : w(x) > v(x)\} \cap \{x \in \Omega : v(x) > a\}. \end{aligned}$$

Clearly,

$$\int_{A_3} \Big(\frac{\chi_{\{v\leq a\}} v^q}{v^{N-1}} - \frac{\chi_{\{w\leq a\}} w^q}{w^{N-1}}\Big) w^N = 0.$$

Also as $q < N-1$, using the definition of $\chi_{\{t\leq a\}}$ one can check that

$$\int_{A_i} \Big(\frac{\chi_{\{v\leq a\}} v^q}{v^{N-1}} - \frac{\chi_{\{w\leq a\}} w^q}{w^{N-1}}\Big) w^N \geq 0$$

for $i = 1, 2$. Using this in (3.26) we infer that $|\mathcal{A}| = 0$. Hence $v \geq w$ a.e. in Ω. □

Lemma 3.6. *For all $\lambda \in (0, \Lambda)$, problem (1.1) admits a solution.*

Proof. Let $\overline{u}$ be the minimal solution of

$$\begin{cases} -\Delta_N u = \lambda(h(u)e^{|u|^{N/N-1}} + u^q) & \text{in } \Omega, \\ u = 0 & \text{on } \Omega, \end{cases} \tag{3.27}$$

with $\|\overline{u}\|_{L^\infty} \to 0$ as $\lambda \to 0$. The existence of $\overline{u}$ follows from [17]. Also by minimizing the corresponding functional over $W_0^{1,N}(\Omega)$ one can easily show the existence of the unique solution $\underline{u}$ for $\lambda > 0$ of the following problem:

$$\begin{cases} -\Delta_N u = \lambda\chi_{\{u\leq a\}} u^q & \text{in } \Omega, \\ u = 0 & \text{on } \Omega. \end{cases} \tag{3.28}$$

Then $\underline{u}$ and $\overline{u}$ are sub-solution and super-solution for (1.1), respectively. Also by Lemma 3.5, one has $\underline{u} \leq \overline{u}$. Let $M_\lambda = \{u \in W_0^{1,N}(\Omega) : \underline{u} \leq u \leq \overline{u}\}$. As I_λ is coercive and weakly lower semi-continuous over M_λ, there exists some $u_\lambda \in M_\lambda$ such that $I_\lambda(u_\lambda) = \inf_{u\in M_\lambda} I_\lambda(u)$. Now we claim that u_λ solves problem (1.1). For this, let $\phi \in W_0^{1,N}(\Omega)$ and $v_\epsilon = u_\lambda + \epsilon\phi - \phi^\epsilon + \phi_\epsilon$, where $\phi^\epsilon = (u_\lambda + \epsilon\phi - \overline{u})^+$ and $\phi_\epsilon = (u_\lambda + \epsilon\phi - \underline{u})^-$. Let $0 < t < 1$ such that $u_\lambda + t(v_\epsilon - u_\lambda) \in M_\lambda$. Then as

$$0 \leq \lim_{t\to 0^+} \frac{I_\lambda(u_\lambda + t(v_\epsilon - u_\lambda)) - I_\lambda(u_\lambda)}{t} \leq (I_\lambda)^0(u_\lambda, v_\epsilon - u_\lambda),$$

there exists some $w_\epsilon \in L^\infty(\Omega)$, $w_\epsilon(x) \in [\underline{\chi}_{\{u_\lambda(x)\leq a\}}, \overline{\chi}_{\{u_\lambda(x)\leq a\}}]$ such that

$$\int_\Omega |\nabla u_\lambda|^{N-2}\nabla u_\lambda \cdot \nabla(v_\epsilon - u_\lambda) - \lambda \int_\Omega f(u_\lambda)(v_\epsilon - u_\lambda) - \lambda \int_\Omega \overline{w_\epsilon} u_\lambda^q (v_\epsilon - u_\lambda) \geq 0.$$

This implies

$$\int_\Omega |\nabla u_\lambda|^{N-2}\nabla u_\lambda \cdot \nabla\phi - \lambda \int_\Omega f(u_\lambda)\phi - \lambda \int_\Omega \overline{w_\epsilon} u_\lambda^q \phi \geq \frac{1}{\epsilon}(E^\epsilon - E_\epsilon), \tag{3.29}$$

where

$$E^{\epsilon} = \int_{\Omega} |\nabla u_{\lambda}|^{N-2}\nabla u_{\lambda} \cdot \nabla \phi^{\epsilon} - \lambda \int_{\Omega} f(u_{\lambda})\phi^{\epsilon} - \lambda \int_{\Omega} \overline{w_{\epsilon}} u_{\lambda}^{q} \phi^{\epsilon}$$

and

$$E_{\epsilon} = \int_{\Omega} |\nabla u_{\lambda}|^{N-2}\nabla u_{\lambda} \cdot \nabla \phi_{\epsilon} - \lambda \int_{\Omega} f(u_{\lambda})\phi_{\epsilon} - \lambda \int_{\Omega} \overline{w_{\epsilon}} u_{\lambda}^{q} \phi_{\epsilon}.$$

Now using (A1) we get

$$\begin{aligned}
\frac{E^{\epsilon}}{\epsilon} &\geq \frac{1}{\epsilon}\int_{\Omega^{\epsilon}} (|\nabla u_{\lambda}|^{N-2}\nabla u_{\lambda} \cdot \nabla \phi^{\epsilon} - \lambda f(u_{\lambda})\phi^{\epsilon} - \lambda \overline{w_{\epsilon}} u_{\lambda}^{q}\phi^{\epsilon}) \\
&= \frac{1}{\epsilon}\int_{\Omega^{\epsilon}} (|\nabla u_{\lambda}|^{N-2}\nabla u_{\lambda} - |\nabla \overline{u}|^{N-2}\nabla \overline{u}) \cdot \nabla \phi^{\epsilon} + \frac{1}{\epsilon}\int_{\Omega^{\epsilon}} (|\nabla \overline{u}|^{N-2}\nabla \overline{u}\nabla \phi^{\epsilon} - \lambda f(u_{\lambda})\phi^{\epsilon} - \lambda \overline{w_{\epsilon}} u_{\lambda}^{q}\phi^{\epsilon}) \\
&\geq \frac{1}{\epsilon}\int_{\Omega^{\epsilon}} (|\nabla u_{\lambda}|^{N-2}\nabla u_{\lambda} - |\nabla \overline{u}|^{N-2}\nabla \overline{u}) \cdot \nabla \phi^{\epsilon} + \frac{\lambda}{\epsilon}\int_{\Omega^{\epsilon}} ((f(\overline{u}) - f(u_{\lambda}))\phi^{\epsilon} + \frac{\lambda}{\epsilon}\int_{\Omega^{\epsilon}} (\overline{u}^{q} - \overline{w_{\epsilon}} u_{\lambda}^{q})\phi^{\epsilon} \\
&\geq \frac{1}{\epsilon}\int_{\Omega^{\epsilon}} (|\nabla u_{\lambda}|^{N-2}\nabla u_{\lambda} - |\nabla \overline{u}|^{N-2}\nabla \overline{u}) \cdot \nabla(\overline{u} - u_{\lambda}) + \int_{\Omega^{\epsilon}} (|\nabla u_{\lambda}|^{N-2}\nabla u_{\lambda} - |\nabla \overline{u}|^{N-2}\nabla \overline{u}) \cdot \nabla \phi \\
&\geq \int_{\Omega^{\epsilon}} (|\nabla u_{\lambda}|^{N-2}\nabla u_{\lambda} - |\nabla \overline{u}|^{N-2}\nabla \overline{u}) \cdot \nabla \phi = o_{\epsilon}(1),
\end{aligned}$$

where $\Omega^{\epsilon} = \{x \in \Omega : (u_{\lambda} + \epsilon\phi)(x) > \overline{u}(x)\}$ and the second last inequality follows from (2.3). Similarly, we can show that $\frac{E_{\epsilon}}{\epsilon} \leq o_{\epsilon}(1)$. Thus from (3.29) we get

$$\int_{\Omega} |\nabla u_{\lambda}|^{N-2}\nabla u_{\lambda} \cdot \nabla \phi - \lambda \int_{\Omega} f(u_{\lambda})\phi - \lambda \int_{\Omega} \overline{w_{\epsilon}} u_{\lambda}^{q}\phi \geq 0 \quad \text{for all } \phi \in W_0^{1,N}(\Omega).$$

Hence u_{λ} solves (1.1). □

Lemma 3.7. *For $\lambda = \Lambda$, problem* (1.1) *admits a solution.*

Proof. To show the existence of a solution to (1.1) for $\lambda = \Lambda$, we take $\{\lambda_n\} \subset (0, \Lambda)$ such that $\lambda_n \to \Lambda$ as $n \to \infty$. As there exists a solution u_{λ_n} of (1.1) for $\lambda = \lambda_n$, we have

$$\int_{\Omega} |\nabla u_{\lambda_n}|^{N-2}\nabla u_{\lambda_n} \cdot \nabla \phi = \lambda_n\Bigg(\int_{\Omega} f(u_{\lambda_n})\phi + \int_{\Omega} \chi_{\{u_{\lambda_n}(x) \leq a\}} u_{\lambda_n}^{q}\phi\Bigg) \quad \text{for all } \phi \in W_0^{1,N}(\Omega). \tag{3.30}$$

Taking $\phi = u_{\lambda_n}$ in (3.30),

$$\int_{\Omega} |\nabla u_{\lambda_n}|^{N} = \lambda_n\Bigg(\int_{\Omega} f(u_{\lambda_n})u_{\lambda_n} + \int_{\Omega} \chi_{\{u_{\lambda_n}(x) \leq a\}} u_{\lambda_n}^{q+1}\Bigg). \tag{3.31}$$

Moreover, by the definition of u_{λ_n} we have

$$I_{\lambda_n}(u_{\lambda_n}) = \min_{u \in M_{\lambda_n}} I_{\lambda_n}(u) \leq I_{\lambda_n}(\underline{u}) < \frac{1}{2}\|\underline{u}\|^{N}_{W_0^{1,N}}.$$

This implies

$$\frac{1}{N}\|u_{\lambda_n}\|^{N}_{W_0^{1,N}} < \frac{1}{N}\|\underline{u}\|^{N}_{W_0^{1,N}} + \lambda_n \int_{\Omega} F(u_{\lambda_n}) + \lambda_n \int_{\Omega} G(u_{\lambda_n}). \tag{3.32}$$

From (3.31) and (3.32) we get

$$\frac{1}{N}\int_{\Omega} f(u_{\lambda_n})u_{\lambda_n} - \int_{\Omega} F(u_{\lambda_n}) < \frac{1}{N\lambda_n}\|\underline{u}\|^{2}_{W_0^{1,N}} + \int_{\Omega} G(u_{\lambda_n}) - \frac{1}{N}\int_{\Omega} \chi_{\{u_{\lambda_n} \leq a\}} u_{\lambda_n}^{q+1}. \tag{3.33}$$

Also, from (A3), for any $K > 0$, there exists a constant $C > 0$ such that

$$\frac{1}{N}f(t)t - F(t) \geq \Big(\frac{1}{N} - \frac{1}{K}\Big)f(t)t - C \quad \text{for all } t > 0. \tag{3.34}$$

Thus, from (3.33) and (3.34), for $K > N$ we have

$$\left(\frac{1}{N}-\frac{1}{K}\right)\int_\Omega f(u_{\lambda_n})u_{\lambda_n} < \frac{1}{N\lambda_n}\|\underline{u}\|^2_{W_0^{1,N}} + \int_\Omega G(u_{\lambda_n}) - \frac{1}{N}\int_\Omega \chi_{\{u_{\lambda_n}\le a\}}u_{\lambda_n}^{q+1} + C. \tag{3.35}$$

Using (3.31), (3.35) and the Sobolev inequality, we estimate $\|u_{\lambda_n}\|_{W_0^{1,N}}$ as

$$\|u_{\lambda_n}\|^N_{W_0^{1,N}} < K_1\|\underline{u}\|^N_{W_0^{1,N}} + K_2\int_\Omega u_{\lambda_n}^{q+1} + C \le K_1\|\underline{u}\|^N_{W_0^{1,N}} + K_2 S\|u_{\lambda_n}\|^{q+1}_{W_0^{1,N}} + C, \tag{3.36}$$

where S is the best constant in Sobolev embedding. As $0 < q < N-1$, $\{u_{\lambda_n}\}$ is bounded in $W_0^{1,N}(\Omega)$. Thus $u_{\lambda_n} \rightharpoonup u_\Lambda$ weakly for some $u_\Lambda \in W_0^{1,N}(\Omega)$. Now it is easy to show that u_Λ solves (1.1) for $\lambda = \Lambda$. □

The proof of Theorem 1.1 follows from Lemma 3.4, Lemma 3.6 and Lemma 3.7.

4 Conclusion

In this paper, we have discussed the existence of the solution of the critical elliptic problem with the regular and discontinuous nonlinearities of Heaviside type. The non-smoothness of the associated functional is overcome by casting the sub-super solution method into a variational framework by a notion of generalized gradient for Lipschitz continuous functional.

The multiplicity results in case of N-Laplacian is an open problem in this direction. Here we would like to mention that for proving the multiplicity for N-Laplacian problem with continuous nonlinearity, the standard approach is to show that the first solution is a local minimum of the associated functional in C^1 topology and then use the "$W^{1,N}$ versus C^1" arguments for establishing the mountain pass geometry around this solution. But in case of the discontinuous nonlinearity, having a solution which is a local minimum of the associated functional in C^1 topology is itself an open question.

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