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Research Article

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Multiple solutions for a class of oscillatory discrete problems

Abstract: In this paper, we study a discrete nonlinear boundary value problem that involves a nonlinear term oscillating at infinity and a power-type nonlinearity u^p. By using variational methods, we establish the existence of a sequence of non-negative weak solutions that converges to $+\infty$ if $0 < p \leqslant 1$. In the superlinear case, we establish a sufficient condition for the existence of at least n solutions.

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1 Introduction and preliminary results

The study of discrete boundary value problems has captured special attention in the last decade. In this context, we point out the results obtained in the papers of Agarwal, Perera and O'Regan [1], Cabada, Iannizzotto and Tersian [4], Candito and Molica Bisci [5], Iannizzotto and Rădulescu [8], Rădulescu [20]. In all these papers, variational methods are applied to boundary value problems on "bounded" discrete intervals (that is, sets of the type $\{0, \dots, n\}$).

In [7], the continuous dependence on parameters for mountain pass solutions of second order discrete BVPs is investigated. Any such solution is reached via mountain pass approach apart from the limit solution which need not have mountain geometry. On the other hand, in [19] are presented some multiplicity results for a general class of nonlinear discrete problems with double-well potentials. Variational techniques are used to obtain the existence of saddle-point-type critical points. In addition to simple discrete boundary-value problems, partial difference equations as well as problems involving discrete p-Laplacian are considered. Also the boundedness of solutions is studied and possible applications, e.g. in image processing, are discussed. Similar models are studied and applied for the object identification, the so-called level set segmentation.

In many cases a problem in a continuous framework can be handled by using a suitable method from discrete mathematics and conversely (see [12]). The modeling simulation of certain nonlinear problems from economics, biological, neural networks, optimal control and others enforced in a natural manner the rapid development of the theory of difference equations. Interesting variational techniques have been used to get results for general partial difference operators; see Bereanu and Mawhin [2, 3]. Difference equations serve as mathematical models in diverse areas, such as economy, biology, physics, mechanics, computer science, finance. The studies regarding such type of problems can be placed at the interface of certain mathematical fields such as nonlinear partial differential equations and numerical analysis. For instance, we may consult the monographs of Kelley and Peterson [9], Lakshmikantham and Trigiante [11]. More details on this topic can also be found in [15, 16].

Let $n \geqslant 2$ be an integer number and denote $\mathbb{Z}[1, n] := \{1, \dots, n\}$. The discrete Laplace operator is defined by

$$\Delta u(k) = \nabla(\nabla u(k + 1)),$$

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where ∇ is the backward difference operator, namely

$$\nabla u(k) = u(k) - u(k-1) \quad \text{for all } k \in \mathbb{Z}[1, n].$$

In this paper, we are interested in the existence of solutions $u = (u(1), \dots, u(n)) \in \mathbb{R}^n_+$ of the problem

$$\begin{cases} -\Delta u(k) = \lambda a(k) u(k)^p + f(u(k)) & \text{for all } k \in \mathbb{Z}[1, n], \\ \quad u(0) = u(n+1) = 0, \end{cases} \tag{P_λ}$$

where $a = (a(1), \dots, a(n)) \in \mathbb{R}^n$ and $f : [0, +\infty) \to \mathbb{R}$ is continuous, while $p > 0$ and $\lambda \in \mathbb{R}$. We refer to Rădulescu [20] and Rădulescu and Repovš [21] for related nonlinear discrete problems.

We prove the existence of solutions for problem (P_λ) using variational methods, by considering an auxiliary problem and, under suitable assumptions on the data, we will prove the existence of solutions for this equation. The method consists of proving the fact that the corresponding energy functional admits a minimum, by using the direct methods of the calculus of variations.

We would like to emphasize that problem (P_λ) is the discrete version of the semilinear elliptic equation studied in [10], that is,

$$\begin{cases} -\Delta u = \lambda a(x) u^p + f(u) & \text{in } \Omega, \\ \quad u \geqslant 0, \quad u \neq 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ is a smooth bounded domain with boundary $\partial\Omega$, $a \in L^\infty(\Omega)$, $f : [0, \infty) \to \mathbb{R}$ is a continuous function, $p > 0$ and $\lambda \in \mathbb{R}$ are some parameters.

Moreover, problem (1.1) was recently extended by Molica Bisci, Rădulescu and Servadei in [14] to general classes of quasilinear elliptic equations. More exactly, they studied the problem

$$\begin{cases} -\operatorname{div} A(x, \nabla u) = \lambda \beta(x) u^q + f(u) & \text{in } \Omega, \\ \qquad u \geqslant 0 & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, is a bounded domain with smooth boundary $\partial\Omega$, $q > 0$ and $\lambda \in \mathbb{R}$ are parameters, while $\beta \in L^\infty(\Omega)$ and $f : [0, \infty) \to \mathbb{R}$ is a continuous function and $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a function satisfying some general assumptions.

Motivated by the studies in [10] and [14], we focus in the present paper on the case of nonlinear difference equations. We are concerned in the study of the number of solutions of problem (P_λ) and of their behavior in the case when f oscillates at infinity. The case of oscillation near the origin was studied in Mălin and Rădulescu [13]. Usually, equations involving oscillatory nonlinearities give infinitely many distinct solutions (see [17, 18]), but the presence of an additional term may alter the situation.

Define the vector space

$$H = \{v = (v(0), v(1), \dots, v(n), v(n+1)) \in \mathbb{R}^{n+2} : v(0) = v(n+1) = 0\}.$$

Then H is an n-dimensional Hilbert space (see [1]) with the inner product

$$\langle u, v \rangle = \sum_{k=1}^{n+1} \nabla u(k) \nabla v(k) \quad \text{for all } u, v \in H.$$

The associated norm is defined by

$$\|u\| = \left(\sum_{k=1}^{n+1} |\nabla u(k)|^2 \right)^{\frac{1}{2}}.$$

For all $u \in H$ we set

$$\|v\|_\infty = \max_{k \in \mathbb{Z}[1,n]} |v(k)|. \tag{1.3}$$

Since H is finite-dimensional, the norms $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent on H.

Definition 1. We say that $u \in H$ is a weak solution for problem (P_λ) if

$$\sum_{k=1}^{n+1} \nabla u(k)\nabla v(k) - \lambda \sum_{k=1}^{n} a(k)u(k)^p v(k) - \sum_{k=1}^{n} f(u(k))v(k) = 0 \quad \text{for all } v \in H. \tag{1.4}$$

Remark 1. Note that (1.4) can be obtained by multiplying (P_λ) with $v(k)$ for all $k \in \mathbb{Z}[1, n]$ and summing up from $k = 0$ to $k = n + 1$. By taking into account that $v(0) = v(n + 1) = 0$ and using some simple computations we deduce the variational characterization of weak solutions from (1.4).

The paper is organized as follows. In Section 2, we will state the main results of the paper. In Section 3, we will consider an auxiliary problem and for it we will prove the existence of solutions by using direct minimization methods. Finally, in Section 4, we will study problem (P_λ) in presence of an oscillation term at infinity.

For an excellent overview of the most significant mathematical methods employed in this paper we refer to Ciarlet [6].

2 Main results

Throughout this paper, we assume that $f : [0, +\infty) \to \mathbb{R}$ is a continuous function and we denote by F the function defined as

$$F(s) := \int_0^s f(t)\,dt \quad \text{for any } s \in (0, +\infty).$$

We assume that f oscillates at infinity, namely the following assumptions are fulfilled:

(f_1^∞) $$-\infty < \liminf_{s\to+\infty} \frac{F(s)}{s^2}, \quad \limsup_{s\to+\infty} \frac{F(s)}{s^2} > \frac{1}{n},$$

(f_2^∞) $$l_\infty := \liminf_{s\to+\infty} \frac{f(s)}{s} < 0.$$

Example 1. Let $\alpha > 1$, $\beta \in \mathbb{R}$ and $\gamma > 0$. Define $f_\infty : [0, +\infty) \to \mathbb{R}$ by

$$f_\infty(s) = s(1 + \alpha \sin(\beta s^\gamma)).$$

Then f_∞ satisfies assumptions (f_1^∞) and (f_2^∞).

We point out that condition (f_1^∞) allows us to deduce some information about the number of solutions for problem (P_λ), while (f_2^∞) is used in order to prove the existence of the solutions.

Theorem 2. *Let $a = (a(1), \dots, a(n)) \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $0 < p \leqslant 1$. Assume that $f \in C([0, +\infty); \mathbb{R})$ satisfies conditions (f_1^∞) and (f_2^∞) with $f(0) = 0$. If either*
(i) *$p = 1$, $l_\infty \in (-\infty, 0)$ and $\lambda a(k) < \lambda_\infty$ for all $k \in \mathbb{Z}[1, n]$ and some $\lambda_\infty \in (0, -l_\infty)$, or*
(ii) *$p = 1$, $l_\infty = -\infty$ and $\lambda \in \mathbb{R}$ is arbitrary, or*
(iii) *$0 < p < 1$ and $\lambda \in \mathbb{R}$ is arbitrary,*
then there exists a sequence $\{u_i\}_i$ in H of non-negative, distinct weak solutions of problem (P_λ) such that

$$\lim_{i\to+\infty} \|u_i\| = \lim_{i\to+\infty} \|u_i\|_\infty = +\infty. \tag{2.1}$$

Theorem 3. *Let $a = (a(1), \dots, a(n)) \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $p > 1$. Assume that $f \in C([0, +\infty); \mathbb{R})$ satisfies conditions (f_1^∞) and (f_2^∞) with $f(0) = 0$. Then, for every $n \in \mathbb{N}$, there exists $\Lambda_n > 0$ such that problem (P_λ) has at least n distinct weak solutions $u_{1,\lambda}, \dots, u_{n,\lambda} \in H$ such that*

$$\|u_{i,\lambda}\| > i - 1 \quad \text{and} \quad \|u_{i,\lambda}\|_\infty > i - 1 \quad \text{for any } i = 1, \dots, n, \tag{2.2}$$

provided $\lambda \in [-\Lambda_n, \Lambda_n]$.

3 An auxiliary problem

Consider the problem

$$\begin{cases} -\Delta u(k) + c(k)u(k) = g(k, u(k)), & k \in \mathbb{Z}[1, n], \\ u(0) = u(n+1) = 0. \end{cases} \tag{P_g^c}$$

Here, we assume that $c = (c(1), \dots, c(n)) \in \mathbb{R}^n$ is such that

$$\min_{k \in \mathbb{Z}[1,n]} c(k) > 0, \tag{3.1}$$

while $g : \mathbb{Z}[1, n] \times [0, +\infty) \to \mathbb{R}$ is a Carathéodory function satisfying the following conditions:

(C1) $g(k, 0) = 0$ for every $k \in \mathbb{Z}[1, n]$,

(C2) there exists an $M_g > 0$ such that $|g(k, s)| \leqslant M_g$ for every $k \in \mathbb{Z}[1, n]$ and all $s \geqslant 0$,

(C3) there exist δ and η with $0 < \delta < \eta$ such that $g(k, s) \leqslant 0$ for every $k \in \mathbb{Z}[1, n]$ and all $s \in [\delta, \eta]$.

We extend the function g by taking $g(k, s) = 0$ for every $k \in \mathbb{Z}[1, n]$ and $s \leqslant 0$.

Definition 4. By a weak solution for problem (P_g^c) we understand a vector $u \in H$ such that for all $v \in H$,

$$\sum_{k=1}^{n+1} \nabla u(k) \nabla v(k) + \sum_{k=1}^{n} c(k)u(k)v(k) - \sum_{k=1}^{n} g(k, u(k))v(k) = 0.$$

Let $E_{c,g} : H \to \mathbb{R}$ be the energy functional associated to problem (P_g^c), namely

$$E_{c,g}(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\sum_{k=1}^{n} c(k)u(k)^2 - \sum_{k=1}^{n} G(k, u(k)), \quad u \in H, \tag{3.2}$$

where $G(k, s) := \int_0^s g(k, t)\,dt$ for any $s \in \mathbb{R}$ and $k \in \mathbb{Z}[1, n]$. Then $E_{c,g}$ is well-defined, of class $C^1(H; \mathbb{R})$ and

$$\langle E'_{c,g}(u), v\rangle = \langle u, v\rangle + \sum_{k=1}^{n} c(k)u(k)v(k) - \sum_{k=1}^{n} g(k, u(k))v(k) \quad \text{for all } u, v \in H.$$

Thus, the weak solutions of (P_g^c) coincide with the critical points of $E_{c,g}$.

Finally, we introduce the set W^η defined as

$$W^\eta := \{u \in H : \|u\|_\infty \leqslant \eta\},$$

where η is a positive parameter given in (C3).

Since $g(k, 0) = 0$ for every $k \in \mathbb{Z}[1, n]$ by condition (C1), it follows that $u \equiv 0$ is clearly a weak solution of problem (P_g^c).

Theorem 5. *Assume that $c = (c(1), \dots, c(n)) \in \mathbb{R}^n$ satisfies condition* (3.1) *and that $g : \mathbb{Z}[1, n] \times [0, +\infty) \to \mathbb{R}$ is a Carathéodory function satisfying* (C1), (C2) *and* (C3). *Then:*

(a) *the functional $E_{c,g}$ is bounded from below on W^η attaining its infimum at some $\tilde{u} \in W^\eta$,*

(b) *$\tilde{u}(k) \in [0, \delta]$ for every $k \in \mathbb{Z}[1, n]$, where δ is the positive parameter given in* (C3),

(c) *$\tilde{u}$ is a non-negative weak solution of problem (P_g^c).*

Proof. (a) Since the norms $\|\cdot\|_\infty$ and $\|\cdot\|$ are equivalent in the finite-dimensional space H, the set W^η is compact in H. Combining this fact with the continuity of $E_{c.g}$, we infer that $E_{c,g}|_{W^\eta}$ attains its infimum at $\tilde{u} \in W^\eta$.

(b) Let δ be as in assumption (C3) and let M be the following set:

$$M := \{k \in \mathbb{Z}[1, n] : \tilde{u}(k) \notin [0, \delta]\}.$$

Hence, arguing by contradiction, we suppose that $M \neq \emptyset$.

Define the truncation function $\gamma : \mathbb{R} \to \mathbb{R}$ by

$$\gamma(s) := \min\{s_+, \delta\},$$

where $s_+ = \max\{s, 0\}$. Now, set $w := \gamma \circ \tilde{u}$, that is,

$$w(k) = \begin{cases} \delta & \text{if } \tilde{u}(k) > \delta, \\ \tilde{u}(k) & \text{if } 0 \leqslant \tilde{u}(k) \leqslant \delta, \\ 0 & \text{if } \tilde{u}(k) < 0, \end{cases}$$

for every $k \in \mathbb{Z}[1, T]$. Since $\gamma(0) = 0$, we have $w(0) = w(n+1) = 0$, so $w \in H$. Besides, $0 \leqslant w(k) \leqslant \delta$ for every $k \in \mathbb{Z}[1, n]$. By assumption (C3) we know that $\delta < \eta$, and so $w \in W^\eta$.

We introduce the sets

$$M_- := \{k \in M : \tilde{u}(k) < 0\} \quad \text{and} \quad M_+ := \{k \in M : \tilde{u}(k) > \delta\}.$$

Thus, $M = M_- \cup M_+$ and we have that

$$w(k) = \begin{cases} \tilde{u}(k) & \text{for all } k \in \mathbb{Z}[1, n] \setminus M, \\ 0 & \text{for all } k \in M_-, \\ \delta & \text{for all } k \in M_+. \end{cases}$$

Moreover, we have

$$\begin{aligned} E_{c,g}(w) - E_{c,g}(\tilde{u}) &= \frac{1}{2}(\|w\|^2 - \|\tilde{u}\|^2) + \frac{1}{2}\sum_{k=1}^{n} c(k)[(w(k))^2 - (\tilde{u}(k))^2] - \sum_{k=1}^{n}[G(k, w(k)) - G(k, \tilde{u}(k))] \\ &=: \frac{1}{2}J_1 + \frac{1}{2}J_2 - J_3. \end{aligned} \tag{3.3}$$

Since γ is a Lipschitz function with Lipschitz constant 1, and $w = \gamma \circ \tilde{u}$, we have

$$\begin{aligned} J_1 &= \|w\|^2 - \|\tilde{u}\|^2 \\ &= \sum_{k=1}^{n+1}[|\nabla w(k)|^2 - |\nabla \tilde{u}(k)|^2] \\ &= \sum_{k=1}^{n+1}[|w(k) - w(k-1)|^2 - |\tilde{u}(k) - \tilde{u}(k-1)|^2] \leqslant 0. \end{aligned} \tag{3.4}$$

Since $\min_{k \in \mathbb{Z}[1,n]} c(k) > 0$ by (3.1), one has

$$\begin{aligned} J_2 &= \sum_{k=1}^{n} c(k)[(w(k))^2 - (\tilde{u}(k))^2] \\ &= \sum_{k \in M} c(k)[(w(k))^2 - (\tilde{u}(k))^2] \\ &= -\sum_{k \in M_-} c(k)(\tilde{u}(k))^2 + \sum_{k \in M_+} c(k)[\delta^2 - (\tilde{u}(k))^2] \leqslant 0. \end{aligned} \tag{3.5}$$

Next, we estimate J_3. Due to the fact that $g(k, s) = 0$ for all $s \leqslant 0$ and for every $k \in \mathbb{Z}[1, n]$, we have

$$\sum_{k \in M_-}[G(k, w(k)) - G(k, \tilde{u}(k))] = 0. \tag{3.6}$$

Moreover, by using the mean value theorem, for every $k \in M_+$, there exists $\theta(k) \in [\delta, \tilde{u}(k)] \subset [\delta, \eta]$ such that

$$G(k, w(k)) - G(k, \tilde{u}(k)) = G(k, \delta) - G(k, \tilde{u}(k)) = g(k, \theta(k))(\delta - \tilde{u}(k)).$$

Thus, taking into account hypothesis (C3) and definition of M_+, we have

$$\sum_{k \in M_+}[G(k, w(k)) - G(k, \tilde{u}(k))] \geqslant 0. \tag{3.7}$$

Hence, by (3.6) and (3.7), we obtain

$$J_3 = \sum_{k\in M} [G(k, w(k)) - G(k, \tilde{u}(k))] = \sum_{k\in M_+} [G(k, w(k)) - G(k, \tilde{u}(k))] \geqslant 0. \tag{3.8}$$

Combining relations (3.4), (3.5), (3.8) with (3.3), we get

$$E_{c,g}(w) - E_{c,g}(\tilde{u}) \leqslant 0. \tag{3.9}$$

On the other hand, since $w \in W^\eta$, it is easy to see that

$$E_{c,g}(w) \geqslant E_{c,g}(\tilde{u}) = \inf_{u\in W^\eta} E_{c,g}(u).$$

By this and (3.9) we get that every term in $E_{c,g}(w) - E_{c,g}(\tilde{u})$ should be zero. In particular, from J_2 and due to (3.1), we have

$$\sum_{k\in M_-} c(k)(\tilde{u}(k))^2 = \sum_{k\in M_+} c(k)[\delta^2 - (\tilde{u}(k))^2] = 0,$$

which implies that

$$\tilde{u}(k) = \begin{cases} 0 & \text{for every } k \in M_-, \\ \delta & \text{for every } k \in M_+. \end{cases}$$

In view of the definition of the sets M_- and M_+, we deduce that

$$M_- = M_+ = \emptyset,$$

which contradicts $M_- \cup M_+ = M \neq \emptyset$.

(c) Fix $v \in H$ arbitrarily and let

$$\varepsilon_0 := \frac{\eta - \delta}{\|v\|_\infty + 1} > 0,$$

where δ and η are given as in (C3). Moreover, let $I : [-\varepsilon_0, \varepsilon_0] \to \mathbb{R}$ be the function defined as

$$I(\varepsilon) := E_{c,g}(\tilde{u} + \varepsilon v).$$

First of all, thanks to (b), for any $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ we have

$$|\tilde{u}(k) + \varepsilon v(k)| \leqslant \tilde{u}(k) + \frac{\eta - \delta}{\|v\|_\infty + 1}\|v\|_\infty \leqslant \eta$$

for every $k \in \mathbb{Z}[1, n]$. Thus, $\tilde{u} + \varepsilon v \in W^\eta$.

Consequently, due to (a), we have $I(\varepsilon) \geqslant I(0)$ for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, that is, 0 is an interior minimum point for I. Then, $I'(0) = 0$ and $\langle E'_{c,g}(\tilde{u}), v\rangle = 0$. Taking into account that $v \in H$ is arbitrary and using the definition of $E_{c,g}$, we obtain that $\tilde{u}$ is a weak solution of problem (P_g^c). Moreover, due to (b), $\tilde{u}$ is non-negative in $\mathbb{Z}[1, n]$. □

Theorem 5 does not guarantee that the solution $\tilde{u}$ of problem (P_g^c) is not the trivial one. In spite of this, by Theorem 5 we will derive the existence of nontrivial solutions for the original problem (P_λ), provided the nonlinear term f is chosen appropriately.

Finally, we define the continuous truncation function $\tau_\eta : [0, +\infty) \to \mathbb{R}$ as

$$\tau_\eta(s) := \min\{\eta, s\} \quad \text{for every } s \geqslant 0, \tag{3.10}$$

where η is the positive constant given in assumption (C3).

4 Oscillation at infinity

In order to prove Theorems 2 and 3, we consider problem (P_g^c), where $c = (c(1), \dots, c(n)) \in \mathbb{R}^n$ fulfills (3.1) and $g : \mathbb{Z}[1, n] \times [0, +\infty) \to \mathbb{R}$ is a Carathéodory function which satisfies the following assumptions:

(A1) $g(k, 0) = 0$ for all $k \in \mathbb{Z}[1, n]$, and for any $s \geqslant 0$, there exists an $M > 0$ such that $\max_{t\in[0,s]} |g(k, t)| \leqslant M$ for all $k \in \mathbb{Z}[1, n]$, (A2) there exist two sequences $\{\delta_i\}_i$ and $\{\eta_i\}_i$ with $0 < \delta_i < \eta_i < \delta_{i+1}$ such that

$$\lim_{i\to+\infty} \delta_i = +\infty \quad \text{and} \quad g(k,s) \leqslant 0$$

for every $k \in \mathbb{Z}[1,n]$ and for all $s \in [\delta_i, \eta_i]$, $i \in \mathbb{N}$,

(A3) uniformly for all $k \in \mathbb{Z}[1,n]$,

$$-\infty < \liminf_{s\to+\infty} \frac{G(k,s)}{s^2} \quad \text{and} \quad \limsup_{s\to+\infty} \frac{G(k,s)}{s^2} > \frac{1}{n},$$

where $G(k,s) = \int_0^s g(k,t)\,dt$.

Proof of Theorem 2. We start by proving assertion (i). In this case when $p = 1$ and $l_\infty \in (-\infty, 0)$, we fix $\lambda \in \mathbb{R}$ such that $\lambda a(k) < \lambda_\infty$ for all $k \in \mathbb{Z}[1,n]$ and some $0 < \lambda_\infty < -l_\infty$.

Let us choose $\overline{\lambda}_\infty \in (\lambda_\infty, -l_\infty)$ and let

$$c(k) := \overline{\lambda}_\infty - \lambda a(k) \quad \text{and} \quad g(k,s) := f(s) + \overline{\lambda}_\infty s \quad \text{for all } (k,s) \in \mathbb{Z}[1,n] \times [0,+\infty). \tag{4.1}$$

Firstly, we show that the functions c and g given in (4.1) satisfy the assumptions (3.1), (A1), (A2) and (A3). It is clear that $\min_{k\in\mathbb{Z}[1,n]} c(k) > \overline{\lambda}_\infty - \lambda_\infty > 0$ and $c \in \mathbb{R}^n$ thanks to the fact that $a \in \mathbb{R}^n$, so (3.1) is satisfied.

Since $f(0) = 0$ by assumption and using the regularity of f, it is easy to see that g is a continuous function in $\mathbb{Z}[1,n] \times [0,+\infty)$ and $g(k,0) = 0$ for all $k \in \mathbb{Z}[1,n]$. Also, the continuity of $s \mapsto g(\cdot\,,s)$ and the Weierstrass theorem yield (A1).

Note that

$$\frac{G(k,s)}{s^2} = \frac{\overline{\lambda}_\infty}{2} + \frac{F(s)}{s^2} \quad \text{for any } k \in \mathbb{Z}[1,n] \text{ and } s > 0.$$

Thus, hypothesis (f_1^∞) implies (A3).

In the sequel, since $l_\infty < -\overline{\lambda}_\infty$ and using (f_2^∞), there exists a sequence $\{s_i\}_i \subset (0,+\infty)$ converging to $+\infty$ as $i \to +\infty$ such that

$$f(s_i) < -\overline{\lambda}_\infty s_i \quad \text{for all } i \in \mathbb{N} \text{ large enough.}$$

Thus, we have

$$g(k,s_i) = f(s_i) + \overline{\lambda}_\infty s_i < 0.$$

Consequently, by using the continuity of f, we may fix two sequences $\{\delta_i\}_i, \{\eta_i\}_i \subset (0,+\infty)$ such that $0 < \delta_i < s_i < \eta_i < \delta_{i+1}$, $\lim_{i\to+\infty} \delta_i = +\infty$ and $g(k,s) = \overline{\lambda}_\infty s + f(s) \leqslant 0$ for any $k \in \mathbb{Z}[1,n]$ and all $s \in [\delta_i, \eta_i]$ and $i \geqslant i^*$, $i^* \in \mathbb{N}$. Therefore, hypothesis (A2) is also fulfilled for g on every interval $[\delta_i, \eta_i]$, $i \in \mathbb{N}$. For any $i \in \mathbb{N}$, we consider the truncation function $g_i : \mathbb{Z}[1,n] \times [0,+\infty) \to \mathbb{R}$,

$$g_i(k,s) := g(k, \tau_{\eta_i(s)}) \quad \text{and} \quad G_i(k,s) := \int_0^s g_i(k,t)\,dt \tag{4.2}$$

for every $k \in \mathbb{Z}[1,n]$ and $s \geqslant 0$, where τ_{η_i} is the function defined in (3.10) with $\eta = \eta_i$.

Let $E_i : H \to \mathbb{R}$ be the energy functional associated with problem $(P^c_{g_i})$, that is, $E_i := E_{c,g_i}$, where E_{c,g_i} is the functional given in (3.2) with $g = g_i$. Taking into account hypotheses (A1) and (A2), it is easily seen that the function g_i fulfills all the assumptions of Theorem 5 for any $i \in \mathbb{N}$. Thus, for every $i \in \mathbb{N}$, there exists $u_i \in W^{\eta_i}$ such that

$$\min_{u\in W^{\eta_i}} E_i(u) = E_i(u_i), \tag{4.3}$$

$$u_i(k) \in [0, \delta_i] \quad \text{for every } k \in \mathbb{Z}[1,n], \tag{4.4}$$

$$u_i \text{ is a non-negative weak solution of } (P^c_{g_i}). \tag{4.5}$$

Taking into account the definition of g_i, (A2) and (4.4), it is easily seen that

$$g_i(k, u_i(k)) = g(k, \tau_{\eta_i}(u_i(k))) = g(k, u_i(k)) \quad \text{for every } k \in \mathbb{Z}[1,n].$$

Thus, by the above relation and (4.5), u_i is also a non-negative weak solution for problem (P_g^c). In the sequel, we need to show that there are infinitely many distinct elements in the sequence $\{u_i\}_i$. To this end, first of all we claim that, up to a subsequence,

$$\lim_{i\to+\infty} E_i(u_i) = -\infty. \tag{4.6}$$

Indeed, due to (f_1^∞) and (4.1), we have that

$$\limsup_{s\to+\infty} \frac{G(k,s)}{s^2} = \frac{\overline{\lambda}_\infty}{2} + \limsup_{s\to+\infty} \frac{F(s)}{s^2} > \frac{\overline{\lambda}_\infty}{2} + \frac{1}{n}.$$

In particular, for a small $\varepsilon_\infty > 0$, there exists a sequence $\{\tilde{s}_i\}_i$ tending to $+\infty$ such that

$$G(k,\tilde{s}_i) > \Big(\frac{1}{n} + \frac{\overline{\lambda}_\infty}{2} + \varepsilon_\infty\Big)\tilde{s}_i^2. \tag{4.7}$$

Since $\delta_i \nearrow +\infty$ by (A2), we can choose a subsequence of $\{\delta_i\}_i$ still denoted by $\{\delta_i\}_i$ such that

$$0 < \tilde{s}_i \leqslant \delta_i \quad \text{for all } i \in \mathbb{N}. \tag{4.8}$$

Let $i \in \mathbb{N}$ be fixed and let us define the function $w_i \in H$ by

$$w_i(k) = \tilde{s}_i \quad \text{for every } k \in \mathbb{Z}[1,n].$$

Then $\|w_i\|_\infty = \tilde{s}_i \leqslant \delta_i < \eta_i$ by (A2) and (4.8). Hence, $w_i \in W^{\eta_i}$. This yields that for every $k \in \mathbb{Z}[1,n]$, we have

$$G_i(k, w_i(k)) = G_i(k,\tilde{s}_i) = \int_0^{\tilde{s}_i} g_i(k,t)\,dt = \int_0^{\tilde{s}_i} g(k,\tau_{\eta_i}(t))\,dt = \int_0^{\tilde{s}_i} g(k,t)\,dt = G(k,\tilde{s}_i). \tag{4.9}$$

Then, by using (3.1), (4.1), (4.7) and (4.9), for i sufficiently large we have

$$\begin{aligned} E_i(w_i) &= \frac{1}{2}\sum_{k=1}^{n+1} |\nabla w_i(k-1)|^2 + \frac{1}{2}\sum_{k=1}^{n} c(k)(w_i(k))^2 - \sum_{k=1}^{n} G_i(k,w_i(k)) \\ &< (\tilde{s}_i)^2 + \frac{1}{2}\overline{\lambda}_\infty n(\tilde{s}_i)^2 - nG(k,\tilde{s}_i) \\ &< (\tilde{s}_i)^2 + \frac{1}{2}\overline{\lambda}_\infty n(\tilde{s}_i)^2 - n\Big(\frac{1}{n} + \frac{\overline{\lambda}_\infty}{2} + \varepsilon_\infty\Big)(\tilde{s}_i)^2 \\ &= -n\varepsilon_\infty(\tilde{s}_i)^2. \end{aligned}$$

By construction, we know that $w_i \in W^{\tilde{s}_i} \subset W^{\eta_i}$. Consequently, by the above relations and (4.3), we have

$$E_i(u_i) = \min_{u\in W^{\eta_i}} E_i(u) \leqslant E_i(w_i) < -n\varepsilon_\infty(\tilde{s}_i)^2 \quad \text{for all } i \in \mathbb{N}. \tag{4.10}$$

Since $\lim_{i\to+\infty} \tilde{s}_i = +\infty$, by relation (4.10) it easily follows claim (4.6). As a consequence of (4.6) we get that the sequence $\{u_i\}_i$ has infinitely many distinct elements (and, in particular, $u_i \neq 0$ in $\mathbb{Z}[1,n]$, being $E_i(0) = 0$). Indeed, let us assume that in the sequence $\{u_i\}_i$ there is only a finite number of elements, say $\{u_1,\dots,u_n\}$ for some $n \in \mathbb{N}$. Consequently, due to (4.2), the sequence $\{E_i(u_i)\}_i$ reduces to at most the finite set $\{E_1(u_1),\dots,E_n(u_n)\}$ which contradicts relation (4.6). Hence problem (P_g^c) admits infinitely many distinct weak solutions.

It remains to prove (2.1). Since the norms $\|\cdot\|_\infty$ and $\|\cdot\|$ are equivalent, it is enough to prove that

$$\lim_{i\to+\infty} \|u_i\|_\infty = +\infty.$$

Arguing by contradiction, we assume that for a subsequence of $\{u_i\}_i$, still denoted by $\{u_i\}_i$, there exists a constant $C > 0$ such that

$$\|u_i\|_\infty \leqslant C \quad \text{for all } i \in \mathbb{N}.$$

Therefore, we have

$$\begin{aligned}E_i(u_i) &= \frac{1}{2}\sum_{k=1}^{n+1}|\nabla u_i(k-1)|^2 + \frac{1}{2}\sum_{k=1}^{n} c(k)(u_i(k))^2 - \sum_{k=1}^{n} G_i(k,u_i(k))\\ &\geqslant -\sum_{k=1}^{n} G_i(k,u_i(k)) = -\sum_{k=1}^{n} G(k,u_i(k))\\ &= -\sum_{k=1}^{n}\int_0^{u_i(k)} g(k,s)\,ds\\ &\geqslant -\sum_{k=1}^{n}\max_{s\in[0,C]}|g(k,s)|u_i(k)\\ &\geqslant -\delta_i n\max_{s\in[0,C]}|g(\cdot,s)|.\end{aligned}$$

Since $\lim_{i\to+\infty}\delta_i = +\infty$, by (A2) the above inequality contradicts relation (4.6). Thus, we get the existence of infinitely many distinct nontrivial non-negative solutions $\{u_i\}_i$ for problem (P_g^c) satisfying condition (2.1).

Due to the choice of c and g in (4.1) and taking into account that $p = 1$, it is easy to see that (P_g^c) is equivalent to problem (P_λ). So, u_i is a weak solution of problem (P_λ) which concludes the proof of assertion (i).

Now, let us consider assertion (ii). In this case when $p = 1$ and $l_\infty = -\infty$, we take $\lambda \in \mathbb{R}$ be arbitrary fixed, $\overline{\lambda}_\infty \in (0, -l_\infty)$ and

$$c(k) := \overline{\lambda}_\infty \quad\text{and}\quad g(k,s) = f(s) + (\lambda a(k) + \overline{\lambda}_\infty)s \quad \text{for all } (k,s) \in \mathbb{Z}[1,n]\times[0,+\infty). \tag{4.11}$$

In this setting, the arguments are the same of the ones used in the previous case.

Finally, let us prove assertion (iii). In this case when $0 < p < 1$, let $\lambda \in \mathbb{R}$ be arbitrary fixed and we choose $\overline{\lambda}_\infty \in (0, -l_\infty)$ and

$$c(k) := \overline{\lambda}_\infty \quad\text{and}\quad g(k,s) := \lambda a(k)s^p + \overline{\lambda}_\infty s + f(s) \quad \text{for all } (k,s) \in \mathbb{Z}[1,T]\times[0,+\infty). \tag{4.12}$$

Also in this setting our aim is to prove that c and g given in (4.12) satisfy the conditions (3.1), (A1)–(A3). Hypothesis (3.1) is clearly satisfied. By assumption, we know that $f(0) = 0$ and thus $g(k,0) = 0$ for all $k \in \mathbb{Z}[1,n]$. Due to the fact that $a \in \mathbb{R}^n$, the continuity of $s \mapsto g(\cdot,s)$ and the Weierstrass theorem yield that (A1) hold too. Furthermore, since $p < 1$ and

$$\frac{G(k,s)}{s^2} = \lambda\frac{a(k)}{p+1}s^{p-1} + \frac{\overline{\lambda}_\infty}{2} + \frac{F(s)}{s^2} \quad \text{for all } k \in \mathbb{Z}[1,n] \text{ and } s \in (0,+\infty),$$

hypothesis (f_1^∞) implies (A3). Next, for all $k \in \mathbb{Z}[1,n]$ and every $s \in [0,+\infty)$, we have

$$g(k,s) \leqslant |\lambda|\cdot\|a\|_\infty s^p + \overline{\lambda}_\infty s + f(s). \tag{4.13}$$

Due to (f_2^∞) and (4.13) we have

$$\liminf_{s\to+\infty}\frac{g(k,s)}{s} \leqslant \liminf_{s\to+\infty}\Big(|\lambda|\cdot\|a\|_\infty s^{p-1} + \overline{\lambda}_\infty + \frac{f(s)}{s}\Big) = \overline{\lambda}_\infty + l_\infty < 0 \tag{4.14}$$

for all $k \in \mathbb{Z}[1,n]$, thanks to the choice of p, i.e. $p < 1$.

Therefore, we can fix a sequence $\{s_i\}_i \subset (0,+\infty)$ converging to $+\infty$ as $i \to +\infty$ such that $g(k,s_i) < 0$ for all $i \in \mathbb{N}$ large enough and for every $k \in \mathbb{Z}[1,n]$. Thus, by using the continuity of $s \mapsto g(\cdot,s)$, there exist two sequences $\{\delta_i\}_i, \{\eta_i\}_i \subset (0,+\infty)$ such that $0 < \delta_i < s_i < \eta_i < \delta_{i+1}$, $\lim_{i\to+\infty}\delta_i = +\infty$ and $g(k,s) \leqslant 0$ for every $k \in \mathbb{Z}[1,n]$ and all $s \in [\delta_i,\eta_i]$ and $i \in \mathbb{N}$ large enough. Thus, hypothesis (A2) hold true.

Finally, arguing as in the proof of assertion (i) we observe that problem (P_g^c) is equivalent to problem (P_λ) through the choice (4.12). This ends the proof of Theorem 2. □

Proof of Theorem 3. Let $\overline{\lambda}_\infty \in (0, -l_\infty)$, where $l_\infty < 0$ is given in assumption (f_2^∞), and let us choose

$$c(k) := \overline{\lambda}_\infty \quad\text{and}\quad g(k,s,\lambda) := \lambda a(k)s^p + \overline{\lambda}_\infty s + f(s) \tag{4.15}$$

for all $(k,s) \in \mathbb{Z}[1,n]\times[0,+\infty)$, $\lambda \in \mathbb{R}$. Note that for all $k \in \mathbb{Z}[1,n]$ and every $s \in [0,+\infty)$, we have

$$g(k,s,\lambda) \leqslant |\lambda|\cdot\|a\|_\infty s^p + \overline{\lambda}_\infty s + f(s).$$

In the sequel, since $l_\infty < -\lambda_\infty$ and using (f_2^∞), there exists a sequence $\{s_i\}_i \subset (0, +\infty)$ converging to $+\infty$ as $i \to +\infty$ such that

$$f(s_i) < -\overline{\lambda}_\infty s_i \quad \text{for } i \in \mathbb{N} \text{ large enough.}$$

Thus, we have

$$g(k, s_i, 0) = \overline{\lambda}_\infty s_i + f(s_i) < 0$$

for $i \in \mathbb{N}$ large enough and for all $k \in \mathbb{Z}[1, n]$. Due to the continuity of $s \mapsto g(\cdot, s, \cdot)$ we can fix three sequences $\{\delta_i\}_i, \{\eta_i\}_i \subset (0, +\infty)$, $\{\lambda_i\}_i \subset (0, 1)$ such that

$$0 < \delta_i < s_i < \eta_i < \delta_{i+1}, \qquad \lim_{i\to+\infty} \delta_i = +\infty, \tag{4.16}$$

and for $i \in \mathbb{N}$ large enough,

$$g(k, s, \lambda) \leqslant 0 \quad \text{for all } k \in \mathbb{Z}[1, n],\ \lambda \in [-\lambda_i, \lambda_i] \text{ and } s \in [\delta_i, \eta_i]. \tag{4.17}$$

Without any loss of generality, we may assume that

$$\delta_i \geqslant i, \quad i \in \mathbb{N}. \tag{4.18}$$

For any $i \in \mathbb{N}$ and $\lambda \in [-\lambda_i, \lambda_i]$, let $g_i : \mathbb{Z}[1, n] \times [0, +\infty) \times [-\lambda_i, \lambda_i] \to \mathbb{R}$ be the function defined by

$$g_i(k, s, \lambda) := g(k, \tau_{\eta_i}(s), \lambda) \tag{4.19}$$

and

$$G_i(k, s, \lambda) := \int_0^s g_i(k, t, \lambda)\, dt$$

for all $k \in \mathbb{Z}[1, n]$ and $s \geqslant 0$. Let $E_{i,\lambda} : H \to \mathbb{R}$ be the energy functional associated with problem $(P^c_{g_i(\cdot,\cdot,\lambda)})$, that is,

$$E_{i,\lambda} := E_{c,g_i(\cdot,\cdot,\lambda)},$$

where $E_{c,g_i(\cdot,\cdot,\lambda)}$ is the functional given in (3.2) with $g = g_i(\cdot, \cdot, \lambda)$.

Note that for every $i \in \mathbb{N}$ and $\lambda \in [-\lambda_i, \lambda_i]$, the functions c given in (4.15) and g_i fulfill all the hypotheses of Theorem 5. Consequently, applying Theorem 5 we get that, for $i \in \mathbb{N}$ sufficiently large and $\lambda \in [-\lambda_i, \lambda_i]$, there exists $u_{i,\lambda} \in W^{\eta_i}$ such that

$$\min_{u\in W^{\eta_i}} E_{i,\lambda}(u) = E_{i,\lambda}(u_{i,\lambda}), \tag{4.20}$$

$$u_{i,\lambda}(k) \in [0, \delta_i] \text{ for all } k \in \mathbb{Z}[1, n], \tag{4.21}$$

$$u_{i,\lambda} \text{ is a non-negative weak solution of } (P^c_{g_i(\cdot,\cdot,\lambda)}). \tag{4.22}$$

Now, by (4.16) and (4.21) for i sufficiently large and for all $k \in \mathbb{Z}[1, n]$, we have

$$0 \leqslant u_{i,\lambda}(k) \leqslant \delta_i < \eta_i, \tag{4.23}$$

and thus

$$g_i(k, u_{i,\lambda}(k), \lambda) = g(k, u_{i,\lambda}(k), \lambda). \tag{4.24}$$

On account of the definition of the functions g_i and c, and relations (4.22) and (4.24), $u_{i,\lambda}$ is also a non-negative weak solution of problem (P_λ), provided i is large and $|\lambda| \leqslant \lambda_i$.

It remains to prove that for any $n \in \mathbb{N}$ problem (P_λ) admits at least n distinct solutions, for suitable values of λ. At this purpose, thanks to the choices of c and g_i and (4.23), the functional $E_{i,\lambda}$ is given by

$$\begin{aligned} E_{i,\lambda}(u) &= \frac{1}{2}\|u\|^2 - \lambda \sum_{k=1}^n a(k)\frac{|u(k)|^{p+1}}{p+1} - \sum_{k=1}^n F(u(k)) \\ &= E_{i,0}(u) - \lambda \sum_{k=1}^n a(k)\frac{|u(k)|^{p+1}}{p+1} \quad \text{for any } u \in H. \end{aligned} \tag{4.25}$$

Note that, for $\lambda = 0$, the function $g_i(\cdot,\cdot,\lambda) = g_i(\cdot,\cdot,0)$ verifies the hypotheses (3.1), (A1), (A2) and (A3). In fact, $g_i(\cdot,\cdot,0)$ is the function appearing in (4.2) and $E_i := E_{i,0}$ is the energy functional associated with problem $(P^c_{g_i(\cdot,\cdot,0)})$. Denoting $u_i := u_{i,0}$, up to a subsequence we also have

$$E_i(u_i) = \min_{u\in W^{\eta_i}} E_i(u) \leqslant E_i(w_i) \quad \text{for all } i \in \mathbb{N}, \tag{4.26}$$

$$\lim_{i\to+\infty} E_i(u_i) = -\infty, \tag{4.27}$$

where $w_i \in W^{\eta_i}$ appear in the proof of Theorem 2, see relations (4.6) and (4.10), respectively. We fix a sequence $\{\theta_i\}_i$ with negative terms such that $\lim_{i\to+\infty}\theta_i = -\infty$. Due to (4.26) and (4.27), up to a subsequence, we may assume that

$$\theta_i < E_i(u_i) \leqslant E_i(w_i) < \theta_{i-1} \quad \text{for } i \geqslant i^* \text{ with } i^* \in \mathbb{N}. \tag{4.28}$$

For any $i \geqslant i^*$ let

$$\lambda_i' := \frac{(p+1)(E_i(u_i)-\theta_i)}{(\|a\|_\infty+1)n\delta_i^{p+1}} \quad \text{and} \quad \lambda_i'' := \frac{(p+1)(\theta_{i-1}-E_i(w_i))}{(\|a\|_\infty+1)n\delta_i^{p+1}}. \tag{4.29}$$

Note that λ_i' and λ_i'' are strictly positive, due to (4.28) and they are independent of λ. Now, fix $n \in \mathbb{N}$ and let

$$\Lambda_n := \min\{\lambda_{i^*+1},\dots,\lambda_{i^*+n},\lambda'_{i^*+1},\dots,\lambda'_{i^*+n},\lambda''_{i^*+1},\dots,\lambda''_{i^*+n}\}.$$

On account of (4.28), $\Lambda_n > 0$ and it is independent of λ. Moreover, if $|\lambda| \leqslant \Lambda_n$, then we have $|\lambda| \leqslant \lambda_i$ for any $i = i^*+1,\dots,i^*+n$. Thus, for any $\lambda \in \mathbb{R}$ with $|\lambda| \leqslant \Lambda_n$, we have that $u_{i,\lambda}$ is a non-negative weak solution of problem (P_λ), for any $i = i^*+1,\dots,i^*+n$.

Next, we will show that these solutions are distinct. At this purpose, note that $u_{i,\lambda} \in W^{\eta_i}$ by (4.23) and so for any $\lambda \in \mathbb{R}$ with $|\lambda| \leqslant \Lambda_n$ we have

$$E_i(u_i) = \min_{u\in W^{\eta_i}} E_i(u) \leqslant E_i(u_{i,\lambda}). \tag{4.30}$$

Thus, by (4.25) and (4.30), for any λ with $|\lambda| \leqslant \Lambda_n$ we have

$$\begin{aligned}
E_{i,\lambda}(u_{i,\lambda}) &= E_i(u_{i,\lambda}) - \frac{\lambda}{p+1}\sum_{k=1}^{n} a(k)|u_{i,\lambda}(k)|^{p+1}\\
&\geqslant E_i(u_i) - \frac{|\lambda|}{p+1}\|a\|_\infty\delta_i^{p+1}n\\
&\geqslant E_i(u_i) - \frac{\Lambda_n}{p+1}\|a\|_\infty\delta_i^{p+1}n\\
&\geqslant E_i(u_i) - \frac{\lambda_i'}{p+1}\|a\|_\infty\delta_i^{p+1}n\\
&> \theta_i
\end{aligned} \tag{4.31}$$

for any $i = i^*+1,\dots,i^*+n$, thanks to (4.23), the choice of Λ_n and the definition of λ_i'.

On the other hand, by (4.25), (4.26) and using the fact that $\|w_i\|_\infty = \tilde{s}_i \leqslant \delta_i$ (see the proof of Theorem 2) and the definition of λ_i'', for any λ with $|\lambda| \leqslant \Lambda_n$ and for any $i = i^*+1,\dots,i^*+n$ we deduce that

$$\begin{aligned}
E_{i,\lambda}(u_{i,\lambda}) &= \min_{u\in W^{\eta_i}} E_{i,\lambda}(u)\\
&\leqslant E_{i,\lambda}(w_i)\\
&= E_i(w_i) - \frac{\lambda}{p+1}\sum_{k=1}^{n} a(k)|w_i(k)|^{p+1}\\
&\leqslant E_i(w_i) + \frac{|\lambda|}{p+1}\|a\|_\infty\delta_i^{p+1}n\\
&\leqslant E_i(w_i) + \frac{\Lambda_n}{p+1}\|a\|_\infty\delta_i^{p+1}n\\
&\leqslant E_i(w_i) + \frac{\lambda_i''}{p+1}\|a\|_\infty\delta_i^{p+1}n\\
&< \theta_{i-1}.
\end{aligned} \tag{4.32}$$

Consequently, for every $i = i^* + 1, \dots, i^* + n$ and $\lambda \in [-\Lambda_n, \Lambda_n]$, by (4.31) and (4.32) and the properties of $\{\theta_i\}_i$, we have

$$\theta_i < E_{i,\lambda}(u_{i,\lambda}) < \theta_{i-1} < 0, \tag{4.33}$$

and therefore

$$E_{n,\lambda}(u_{n,\lambda}) < \cdots < E_{1,\lambda}(u_{1,\lambda}) < 0. \tag{4.34}$$

Note that $u_{i,\lambda} \in W^{\eta_n}$ for every $i = i^* + 1, \dots, i^* + n$, so $E_{i,\lambda}(u_{i,\lambda}) = E_{n,\lambda}(u_{i,\lambda})$, see relation (4.19). From above, for every $\lambda \in [-\Lambda_n, \Lambda_n]$, we have

$$E_{n,\lambda}(u_{n,\lambda}) < \cdots < E_{n,\lambda}(u_{1,\lambda}) < 0 = E_{n,\lambda}(0).$$

In particular, the solutions $u_{1,\lambda}, \dots, u_{n,\lambda}$ are all distinct and nontrivial, whenever $\lambda \in [-\Lambda_n, \Lambda_n]$. Finally, it remains to prove conclusion (2.2). For this, we assume that $n \geqslant 2$ and fix $\lambda \in [-\Lambda_n, \Lambda_n]$. We prove that

$$\|u_{i,\lambda}\|_\infty > \delta_{i-1} \quad \text{for all } i \in \{2, \dots, n\}. \tag{4.35}$$

Let us assume that there exists an element $i_0 \in \{2, \dots, n\}$ such that $\|u_{i_0,\lambda}\|_\infty \leqslant \delta_{i_0-1}$. Since $\delta_{i_0-1} < \eta_{i_0-1}$, it follows that $u_{i_0,\lambda} \in W^{\eta_{i_0-1}}$. Thus, on account of (4.20) and (4.19), we have

$$E_{i_0-1,\lambda}(u_{i_0-1,\lambda}) = \min_{u \in W^{\eta_{i_0-1}}} E_{i_0-1,\lambda} \leqslant E_{i_0-1,\lambda}(u_{i_0,\lambda}) = E_{i_0,\lambda}(u_{i_0,\lambda}),$$

which contradicts (4.34). Therefore, (4.35) holds true.

Thus, from (4.18) we have $\|u_{i,\lambda}\|_\infty > i - 1$ for all $i \in \{1, \dots, n\}$. This concludes the proof of Theorem 3. □

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