

## Research Article

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# Solutions of nonlinear problems involving $p(x)$ -Laplacian operator

**Abstract:** In the present paper, by using variational principle, we obtain the existence and multiplicity of solutions of a nonlocal problem involving  $p(x)$ -Laplacian. The problem is settled in the variable exponent Sobolev space  $W_0^{1,p(x)}(\Omega)$ , and the main tools are the Mountain-Pass theorem and Fountain theorem.

**Keywords:**  $p(x)$ -Laplacian, nonlocal problem, variational methods, Mountain Pass theorem, Fountain theorem

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## 1 Introduction

In this paper, we are concerned with the problem

$$\begin{cases} -\operatorname{div}(a(x, \nabla u)) = f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (\text{P})$$

where  $\Omega \subset \mathbb{R}^N$  ( $N \geq 3$ ) is a smooth bounded domain,  $p \in C(\overline{\Omega})$  with  $1 < p(x)$  for any  $x \in \overline{\Omega}$ ,  $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function and  $\operatorname{div}(a(x, \nabla u))$  is  $p(x)$ -Laplacian type operator.

The operator  $\operatorname{div}(a(x, \nabla u))$ , which appears in (P), is a more general operator than the  $p(x)$ -Laplacian operator

$$\Delta_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u),$$

where  $p(x) > 1$ . This causes some difficulties in calculations and requires more general conditions. Moreover, the nonlinear problems involving  $p(x)$ -Laplacian operator have become extremely attractive over the last two decades [3, 7, 15, 19]. These problems appear in the modelling of stationary thermo-rheological viscous flows of non-Newtonian fluids, in the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium and in the study of electrorheological fluids or elastic mechanics [1, 2, 11, 16, 22].

Problems of type (P) have intensively been studied, but to the best of our knowledge, such kinds of equations were firstly studied in [6, 18]. The authors investigated the standard growth condition problem

$$-\operatorname{div}(a(x, \nabla u)) = f(x, u), \quad (1.1)$$

where  $f$  satisfies the Ambrosetti–Rabinowitz type condition. They obtained the existence of a weak solution by using a variation of the Mountain Pass theorem which was introduced in [5]. Moreover, in [17] the author showed the existence of at least two weak solutions of a similar problem (1.1) for the case  $p(x) = p$  (constant).

In [14], the authors dealt with the problem

$$-\operatorname{div}(a(x, \nabla u)) = \lambda(u^{\gamma-1} - u^{\beta-1}),$$

with  $1 < \beta < \gamma < \inf_{x \in \overline{\Omega}} p(x)$ . They proved the existence of at least two distinct nonnegative weak solutions by using critical point theory. We must point out that assumptions (A1)–(A5) established in this article are

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similar with the assumptions accepted in [14], but not the same; for example, in [14] the authors assume that

$$(a(x, \xi) - a(x, \psi)) \cdot (\xi - \psi) \geq 0 \quad \text{for all } x \in \overline{\Omega} \text{ and } \xi, \psi \in \mathbb{R}^N,$$

which we do not use. Furthermore, in [13], the authors obtained the existence of infinitely many nontrivial weak solutions for  $f(x, u) = m(x)|u|^{r(x)-2}u + n(x)|u|^{s(x)-2}u$  with  $1 < r(x) < p(x) < s(x)$  in problem (1.1) involving  $p(x)$ -Laplacian operator by a variation of Mountain Pass lemma and critical point theory. Motivated from the above papers, we consider problem (1.1) for  $p(x)$ -Laplacian operator under the assumptions (A1)–(A5), and establish some suitable conditions which assure the existence and multiplicity of solutions for problem (P).

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on variable exponent Lebesgue–Sobolev spaces. In Section 3, we get some existence results for weak solutions of problem (P).

## 2 Preliminaries

We state some properties of the variable exponent Lebesgue–Sobolev spaces  $L^{p(x)}(\Omega)$  and  $W^{1,p(x)}(\Omega)$ , where  $\Omega \subset \mathbb{R}^N$  is a smooth bounded domain (for details, see [8, 10, 12, 13]).

Set

$$C_+(\overline{\Omega}) = \{p : p \in C(\overline{\Omega}), \inf p(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

Let  $p(x) \in C_+(\overline{\Omega})$ , and denote

$$p^- := \inf_{x \in \overline{\Omega}} p(x) \leq p(x) \leq p^+ := \sup_{x \in \overline{\Omega}} p(x) < \infty.$$

For any  $p(x) \in C_+(\overline{\Omega})$ , we define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u : \text{the function } u : \Omega \rightarrow \mathbb{R} \text{ is measurable, } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\},$$

with the norm

$$|u|_{p(x)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Then  $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$  becomes a Banach spaces.

The variable exponent Sobolev space  $W^{1,p(x)}(\Omega)$  is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)} \quad \text{for all } u \in W^{1,p(x)}(\Omega).$$

The space  $W_0^{1,p(x)}(\Omega)$  is denoted as the closure of  $C_0^\infty(\Omega)$  in  $W^{1,p(x)}(\Omega)$  with respect to the norm  $\|u\|_{1,p(x)}$ . For  $u \in W_0^{1,p(x)}(\Omega)$ , we can define an equivalent norm

$$\|u\| = |\nabla u|_{p(x)},$$

since the Poincaré inequality holds, i.e. there exists a positive constant  $C > 0$  such that

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)} \quad \text{for all } u \in W_0^{1,p(x)}(\Omega).$$

**Proposition 2.1** ([8, 12]). *The conjugate space of  $L^{p(x)}(\Omega)$  is  $L^{p'(x)}(\Omega)$ , where  $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1$ . We have*

$$\left| \int_{\Omega} uv \, dx \right| \leq \left( \frac{1}{p^-} + \frac{1}{(p^-)'} \right) |u|_{p(x)} |v|_{p'(x)} \quad \text{for all } u \in L^{p(x)}(\Omega) \text{ and } v \in L^{p'(x)}(\Omega).$$

**Proposition 2.2** ([8, 12]). Denote

$$\rho(u) = \int_{\Omega} |u(x)|^{p(x)} dx \quad \text{for all } u, u_n \in L^{p(x)}(\Omega), n = 1, 2, \dots$$

Then

$$|u|_{p(x)} > 1 \implies |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}, \quad (\text{i})$$

$$|u|_{p(x)} < 1 \implies |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}, \quad (\text{ii})$$

$$\lim_{n \rightarrow \infty} |u_n|_{p(x)} = 0 \iff \lim_{n \rightarrow \infty} \rho(u_n) = 0, \quad (\text{iii})$$

$$\lim_{n \rightarrow \infty} |u_n|_{p(x)} \rightarrow \infty \iff \lim_{n \rightarrow \infty} \rho(u_n) \rightarrow \infty. \quad (\text{iv})$$

**Proposition 2.3** ([8, 12]). If  $u, u_n \in L^{p(x)}(\Omega)$ ,  $n = 1, 2, \dots$ , then the following statements are equivalent:

- (i)  $\lim_{n \rightarrow \infty} |u_n - u|_{p(x)} = 0$ ,
- (ii)  $\lim_{n \rightarrow \infty} \rho(u_n - u) = 0$ ,
- (iii)  $u_n \rightarrow u$  measured in  $\Omega$  and  $\lim_{n \rightarrow \infty} \rho(u_n) = \rho(u)$ .

**Proposition 2.4** ([8, 12]). The following statements hold.

- (i) If  $1 < p^- \leq p^+ < \infty$ , then the three spaces  $L^{p(x)}(\Omega)$ ,  $W^{1,p(x)}(\Omega)$  and  $W_0^{1,p(x)}(\Omega)$  are separable and reflexive Banach spaces.
- (ii) If  $q \in C_+(\bar{\Omega})$  and  $q(x) < p^*(x)$  for all  $x \in \bar{\Omega}$ , then the embedding  $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$  is compact and continuous, where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } N \leq p(x). \end{cases}$$

**Definition 2.5.** Let  $X$  be a Banach spaces and let  $J \in C^1(X, \mathbb{R})$ . We say that  $J$  satisfies the Palais–Smale condition (PS) in  $X$  if any sequence  $\{u_n\}$  in  $X$  such that

- (i)  $\{J(u_n)\}$  is bounded,
  - (ii)  $J'(u_n) \rightarrow 0$  as  $n \rightarrow \infty$ ,
- has a convergent subsequence.

**Lemma 2.6** (Mountain-Pass theorem, [4, 17, 20]). Let  $X$  be a Banach spaces and let  $J \in C^1(X, \mathbb{R})$  satisfy the Palais–Smale condition. Assume that  $J(0) = 0$ , and that there exist two positive real numbers  $\eta$  and  $r$  such that

- (i) there exist two positive real numbers  $\eta$  and  $r$  such that  $J(u) \geq r$  with  $\|u\| = \eta$ ,
- (ii) there exists  $u_1 \in X$  such that  $\|u_1\| > \rho$  and  $J(u) < 0$ .

Put

$$G = \{\phi \in C([0, 1], X) : \phi(0) = 0, \phi(1) = u_1\}.$$

Set

$$\beta = \inf\{\max J(\phi([0, 1])) : \phi \in G\}.$$

Then  $\beta \geq r$  and  $\beta$  is a critical value of  $J$ .

Since  $X$  is a reflexive and separable Banach space, there exist (see [21, Section 17])  $e_j \in X$  and  $e_j^* \in X^*$  such that

$$X = \overline{\text{span}\{e_j : j = 1, 2, \dots\}}, \quad X^* = \overline{\text{span}\{e_j^* : j = 1, 2, \dots\}},$$

and

$$\langle e_i^*, e_j \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

where  $\langle \cdot, \cdot \rangle$  denotes the duality product between  $X$  and  $X^*$ . For convenience, we write

$$X_j = \text{span}\{e_j\}, \quad Y_k = \bigoplus_{j=1}^k X_j, \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}. \quad (2.1)$$

**Lemma 2.7** (Fountain theorem, [20]). Assume that  $X$  is a Banach spaces,  $J \in C^1(X, \mathbb{R})$  is an even functional and  $X_k, Y_k$  and  $Z_k$  are defined as in (2.1). If there exist  $\rho_k > \gamma_k > 0$  for each  $k = 1, 2, \dots$  such that

(i)  $\inf_{u \in Z_k, \|u\|=\gamma_k} J(u) \rightarrow \infty$  as  $k \rightarrow \infty$ ,

(ii)  $\max_{u \in Y_k, \|u\|=\rho_k} J(u) \leq 0$ ,

(iii)  $J$  satisfies condition (PS) for every  $c > 0$ ,

then  $J$  has a sequence of critical values tending to  $+\infty$ .

### 3 Main results and proofs

Let  $X$  denote the variable exponent Sobolev space  $W_0^{1,p(x)}(\Omega)$ . The energy functional corresponding to problem (P) is defined as  $J : X \rightarrow \mathbb{R}$ ,

$$J(u) = \int_{\Omega} A(x, \nabla u) dx - \int_{\Omega} F(x, u) dx := \Lambda(u) - I(u),$$

where

$$\Lambda(u) = \int_{\Omega} A(x, \nabla u) dx \quad \text{and} \quad I(u) = \int_{\Omega} F(x, u) dx$$

and

$$F(x, t) = \int_0^t f(x, s) ds.$$

It is well known that standard arguments imply that  $I \in C^1(X, \mathbb{R})$  and the derivative of  $I$  is

$$\langle I'(u), v \rangle = \int_{\Omega} f(x, u)v dx \quad \text{for all } u, v \in X.$$

We say that  $u \in X$  is a weak solution of (P) if

$$\int_{\Omega} a(x, \nabla u) \nabla \varphi dx - \int_{\Omega} f(x, u) \varphi dx = 0 \quad \text{for all } \varphi \in X,$$

where  $a(x, \xi) : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$  is the continuous derivative with respect to  $\xi$  of the mapping  $A : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ ,  $A = A(x, \xi)$ , i.e.  $a(x, \xi) = \nabla_{\xi} A(x, \xi)$ .

In this article, we assume that  $f$ ,  $A$  and  $a$  satisfy the following conditions:

(f<sub>0</sub>)  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory condition and

$$|f(x, t)| \leq c(1 + |t|^{\alpha(x)-1}),$$

$$\alpha(x) \in C_+(\overline{\Omega}) \text{ and } \alpha(x) < p^*(x) \text{ for all } x \in \Omega.$$

**Assumption (A1).** The following inequality holds:

$$|a(x, \xi)| \leq c_0(h_0(x) + |\xi|^{p(x)-1}) \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^N,$$

where  $h_0(x) \in L^{p'(x)}(\Omega)$  is a nonnegative measurable function.

**Assumption (A2).** The mapping  $A$  is  $p(x)$ -uniformly convex: There exists a constant  $k > 0$  such that

$$A\left(x, \frac{\xi + \psi}{2}\right) \leq \frac{1}{2}A(x, \xi) + \frac{1}{2}A(x, \psi) - k|\xi - \psi|^{p(x)} \quad \text{for all } x \in \Omega \text{ and } \xi, \psi \in \mathbb{R}^N.$$

**Assumption (A3).** The following inequalities hold true:

$$|\xi|^{p(x)} \leq a(x, \xi) \cdot \xi \leq p(x)A(x, \xi) \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^N.$$

**Assumption (A4).** We have  $A(x, 0) = 0$  for all  $x \in \Omega$ .

**Assumption (A5).** We have  $A(x, -\xi) = A(x, \xi)$  for all  $x \in \Omega$  and  $\xi \in \mathbb{R}^N$ .

**Lemma 3.1** ([13]). *The following statements hold.*

(i) *A verifies the growth condition:*

$$|A(x, \xi)| \leq c_0(h_0(x)|\xi| + |\xi|^{p(x)}) \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^N.$$

(ii) *A is  $p(x)$ -homogeneous:*

$$A(x, z\xi) \leq A(x, \xi)z^{p(x)} \quad \text{for all } z \geq 1, \xi \in \mathbb{R}^N \text{ and } x \in \Omega.$$

**Lemma 3.2** ([14]). *The following statements hold.*

(i) *The functional  $\Lambda$  is well-defined on  $X$ .*

(ii) *The functional  $\Lambda$  is of class  $C^1(X, \mathbb{R})$  and*

$$\langle \Lambda'(u), v \rangle = \int_{\Omega} a(x, \nabla u) \cdot \nabla v \, dx \quad \text{for all } u, v \in X.$$

(iii) *The functional  $\Lambda$  is weakly lower semi-continuous on  $X$ .*

**Lemma 3.3** ([13]). *The following statements hold.*

(i) *For all  $u, v \in X$ ,*

$$\Lambda\left(\frac{u+v}{2}\right) \leq \frac{1}{2}\Lambda(u) + \frac{1}{2}\Lambda(v) - k\|u-v\|^{p^-}.$$

(ii) *For all  $u, v \in X$ ,*

$$\Lambda(u) - \Lambda(v) \geq \langle \Lambda'(v), u-v \rangle.$$

(iii)  *$J$  is weakly lower semi-continuous on  $X$ .*

(iv)  *$J$  is well-defined on  $X$  and of class  $C^1(X, \mathbb{R})$ , and its derivative given by*

$$\langle J'(u), v \rangle = \int_{\Omega} a(x, \nabla u) \nabla v \, dx - \int_{\Omega} f(x, u)v \, dx$$

*for all  $u, v \in X$ .*

**Theorem 3.4.** *Suppose that (A3) and the following condition hold:*

(f<sub>1</sub>)  *$f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory condition and*

$$|f(x, t)| \leq c(1 + |t|^{\beta-1}) \quad \text{for all } (x, t) \in \Omega \times \mathbb{R},$$

*where  $1 \leq \beta < p^-$ .*

*Then problem (P) has a weak solution.*

*Proof.* Let  $\|u\| > 1$ . From condition (f<sub>1</sub>), we have

$$|F(x, u)| \leq c(|t| + |t|^{\beta}).$$

On the other hand, by (A3) and Proposition 2.2 (i), we get

$$J(u) \geq \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} \, dx - c \int_{\Omega} |u|^{\beta} \, dx - c \int_{\Omega} |u| \, dx \geq \frac{1}{p^+} \|u\|^{p^-} - c\|u\|^{\beta} - c\|u\| \rightarrow +\infty \quad \text{as } \|u\| \rightarrow +\infty.$$

So,  $J$  is coercive. Since  $J$  is weakly lower semi-continuous,  $J$  has a minimum point  $u$  in  $X$ , which is a weak solution of problem (P). The proof is completed.  $\square$

**Lemma 3.5.** *Suppose that (A3), (f<sub>0</sub>) and the following condition hold:*

(AR) *there exist  $t_* > 0$  and  $\theta > p^+$  such that*

$$0 < \theta F(x, t) \leq f(x, t)t, \quad |t| \geq t_*, \quad \text{a.e. } x \in \Omega,$$

*Then,  $J$  satisfies condition (PS).*

*Proof.* Let us assume that there exists a sequence  $\{u_n\} \subset X$  such that

$$|J(u_n)| \leq c \quad \text{and} \quad J'(u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.1)$$

Then, by using (A3) and (AR), we can write

$$\begin{aligned} c + \|u_n\| &\geq J(u_n) - \frac{1}{\theta} \langle J'(u_n), u_n \rangle \\ &= \int_{\Omega} A(x, \nabla u_n) dx - \int_{\Omega} F(x, u_n) dx - \frac{1}{\theta} \left( \int_{\Omega} a(x, \nabla u_n) \nabla u_n dx - \int_{\Omega} f(x, u_n) u_n dx \right) \\ &\geq \left(1 - \frac{p^+}{\theta}\right) \int_{\Omega} A(x, \nabla u_n) dx + \int_{\Omega} \left( \frac{1}{\theta} f(x, u_n) u_n - F(x, u_n) \right) dx \\ &\geq \left(1 - \frac{p^+}{\theta}\right) \|u_n\|^{p^-}. \end{aligned}$$

Since  $p^- > 1$ , the sequence  $\{u_n\}$  is bounded in  $X$ . Thus, there exists  $u \in X$ , up to a subsequence, such that  $u_n \rightharpoonup u$  in  $X$ . Thanks to the compact embedding  $X \hookrightarrow L^{\alpha(x)}(\Omega)$ , we get

$$u_n \rightarrow u \quad \text{in } L^{\alpha(x)}(\Omega) \quad \text{and} \quad u_n \rightarrow u \quad \text{a.e. in } \Omega. \quad (3.2)$$

From (3.1), we have

$$\langle J'(u_n), u_n - u \rangle \rightarrow 0,$$

that is,

$$\langle J'(u_n), u_n - u \rangle = \int_{\Omega} a(x, \nabla u_n) (\nabla u_n - \nabla u) dx - \int_{\Omega} f(x, u_n) (u_n - u) dx \rightarrow 0.$$

By using  $(f_0)$  and Proposition 2.1, it follows

$$\left| \int_{\Omega} f(x, u_n) (u_n - u) dx \right| \leq c_1 \|u_n\|^{\alpha(x)-1} |u_n - u|_{\alpha(x)}.$$

If we consider the relations given in (3.2), we obtain

$$\int_{\Omega} f(x, u_n) (u_n - u) dx \rightarrow 0.$$

Then, we have

$$\int_{\Omega} a(x, \nabla u_n) (\nabla u_n - \nabla u) dx \rightarrow 0,$$

that is,

$$\lim_{n \rightarrow \infty} \langle \Lambda'(u_n), u_n - u \rangle = 0.$$

From Lemma 3.3 (ii), we get

$$0 = \lim_{n \rightarrow \infty} \langle \Lambda'(u_n), u - u_n \rangle \leq \lim_{n \rightarrow \infty} (\Lambda(u) - \Lambda(u_n)) = \Lambda(u) - \lim_{n \rightarrow \infty} \Lambda(u_n),$$

or  $\lim_{n \rightarrow \infty} \Lambda(u_n) \leq \Lambda(u)$  and from Lemma 3.3 (iii) we obtain  $\lim_{n \rightarrow \infty} \Lambda(u_n) = \Lambda(u)$ .

Now, we assume by contradiction that  $\{u_n\}$  does not converge strongly to  $u$  in  $X$ . Then, there exists an  $\varepsilon > 0$  and a subsequence  $\{u_{n_m}\}$  of  $\{u_n\}$  such that  $\|u_{n_m} - u\| \geq \varepsilon$ . Moreover, from Lemma 3.3 (i), we can write the following inequality:

$$\frac{1}{2} \Lambda(u) + \frac{1}{2} \Lambda(u_{n_m}) - \Lambda\left(\frac{u_{n_m} + u}{2}\right) \geq k \|u_{n_m} - u\|^{p^-} \geq k \varepsilon^{p^-}.$$

Letting  $m \rightarrow \infty$  in the above inequality, we obtain

$$\lim_{n \rightarrow \infty} \sup \Lambda\left(\frac{u_{n_m} + u}{2}\right) \leq \Lambda(u) - k \varepsilon^{p^-}.$$

We also have  $\{\frac{u_{n_m} + u}{2}\}$  converges weakly to  $u$  in  $X$ . Using Lemma 3.2 (iii), we obtain

$$\Lambda(u) \leq \lim_{n \rightarrow \infty} \inf \Lambda\left(\frac{u_{n_m} + u}{2}\right),$$

and this is a contradiction. Hence, it follows that  $\{u_n\}$  converges strongly to  $u$  in  $X$ .  $\square$

**Theorem 3.6.** Assume that the conditions (A3), (A4), (AR) hold, that  $f$  satisfies  $(f_0)$  and that the following condition holds:

$(f_2)$   $f(x, t) = o(|t|^{p^+-1})$ ,  $t \rightarrow 0$ , for  $x \in \Omega$  uniformly.

If  $\alpha^- > p^+$ , then problem (P) has a nontrivial weak solution.

**Lemma 3.7.** Suppose that the conditions  $(f_0)$ ,  $(f_2)$ , (A3) and (AR) hold. Then the following statements hold:

- (i) There exist two positive real numbers  $\eta$  and  $r$  such that  $J(u) \geq r > 0$ ,  $u \in X$  with  $\|u\| = \eta$ .
- (ii) There exists  $u \in X$  such that  $\|u\| > \rho$ ,  $J(u) < 0$ .

*Proof.* (i) Let  $\|u\| < 1$ . Then, using (A3) we have

$$J(u) \geq \frac{1}{p^+} \|u\|^{p^+} - \int_{\Omega} F(x, u) dx.$$

Since  $p^+ < \alpha^- \leq \alpha(x) < p^*(x)$ , and since the embeddings  $X \hookrightarrow L^{p^+}(\Omega)$  and  $X \hookrightarrow L^{\alpha(x)}(\Omega)$  are continuous, there exist  $c_2, c_3 > 0$  such that

$$|u|_{\alpha(x)} \leq c_2 \|u\| \quad \text{and} \quad |u|_{p^+} \leq c_3 \|u\| \quad \text{for all } u \in X. \quad (3.3)$$

Let  $\varepsilon > 0$  be small enough such that  $\varepsilon c_3^{p^+} \leq \frac{1}{2p^+}$ . On the other hand, from  $(f_0)$  and  $(f_2)$ , we obtain that

$$F(x, t) \leq \varepsilon |t|^{p^+} + c |t|^{\alpha(x)} \quad \text{for all } (x, t) \in \Omega \times \mathbb{R}. \quad (3.4)$$

Then, by (3.3)–(3.4) and Proposition 2.2 (ii) for  $\|u\| < 1$ , we have

$$\begin{aligned} J(u) &\geq \frac{1}{p^+} \|u\|^{p^+} - \varepsilon \int_{\Omega} |u|^{p^+} dx - c \int_{\Omega} |u|^{\alpha(x)} dx \\ &\geq \frac{1}{p^+} \|u\|^{p^+} - \varepsilon c_3^{p^+} \|u\|^{p^+} - c c_2^{\alpha^-} \|u\|^{\alpha^-} \\ &\geq \frac{1}{2p^+} \|u\|^{p^+} - c_4 \|u\|^{\alpha^-}. \end{aligned}$$

Hence, there exist two positive real numbers  $\eta$  and  $r$  such that  $J(u) \geq r > 0$ ,  $u \in X$  with  $\|u\| = \eta \in (0, 1)$ .

(ii) From (AR), it follows that

$$F(x, t) \geq c_5 |t|^{\theta}, \quad |t| \geq t_*, \quad \text{a.e. } x \in \Omega.$$

Thus, for any fixed  $\omega \in X \setminus \{0\}$ ,  $t > 1$  and from Lemma 3.1 (ii), we have

$$J(t\omega) = \int_{\Omega} A(x, \nabla t\omega) dx - \int_{\Omega} F(x, t\omega) dx \leq t^{p^+} \int_{\Omega} A(x, \nabla \omega) dx - c_5 t^{\theta} \int_{\Omega} |\omega|^{\theta} dx,$$

which implies  $J(t\omega) \rightarrow -\infty$  as  $t \rightarrow +\infty$ . So (ii) holds.  $\square$

*Proof of Theorem 3.6.* From Lemma 3.5, Lemma 3.7, Lemma 3.3 (iii) and Lemma 3.3 (iv), (A4) and the fact that  $J(0) = 0$ ,  $J$  satisfies the all statements of Lemma 2.6. Therefore,  $J$  has at least one nontrivial critical point, i.e. problem (P) has a nontrivial weak solution. The proof is complete.  $\square$

**Theorem 3.8.** Assume that (A2), (A3), (A5), (AR) hold provided  $p^+ < \alpha^-$ . In addition, suppose that the following condition holds:

$(f_3)$   $f(x, -t) = -f(x, t)$  for  $(x, t) \in \Omega \times \mathbb{R}$ .

Then  $J$  has a sequence of critical points  $\{u_n\}$  such that  $J(u_n) \rightarrow +\infty$  and problem (P) has infinite many pairs of solutions.

**Lemma 3.9.** If  $\alpha(x) \in C_+(\overline{\Omega})$ ,  $\alpha(x) < p^*(x)$  for any  $x \in \overline{\Omega}$ , denote

$$\beta_k = \sup\{|u|_{\alpha(x)} : \|u\| = 1, u \in Z_k\}.$$

Then  $\lim_{k \rightarrow \infty} \beta_k = 0$ .

Since the proof of Lemma 3.9 is similar to that of [9, Lemma 4.9], we omit it.

*Proof of Theorem 3.8.* By the assumptions  $(f_3)$  and  $(A5)$ ,  $J$  is an even functional and  $J$  satisfies condition  $(PS)$  from Lemma 3.5. We only need to prove that if  $k$  is large enough, then there exist  $\rho_k > \gamma_k > 0$  such that

- (i)  $b_k := \inf\{J(u) : u \in Z_k, \|u\| = \gamma_k\} \rightarrow \infty$  as  $k \rightarrow \infty$ ,
- (ii)  $a_k := \max\{J(u) : u \in Y_k, \|u\| = \rho_k\} \leq 0$ .

Thus, the conclusion of Theorem 3.8 can be obtained from Lemma 2.7.

(i) For any  $u \in Z_k$  with  $\|u\| > 1$ , we have

$$\begin{aligned} J(u) &\geq \frac{1}{p^+} \|u\|^{p^-} - c \int_{\Omega} |u|^{\alpha(x)} dx - c \int_{\Omega} |u| dx \\ &\geq \frac{1}{p^+} \|u\|^{p^-} - c_6 \rho_{\alpha}(u) - c_7 \|u\| \\ &\geq \begin{cases} \frac{1}{p^+} \|u\|^{p^-} - c_6 - c_7 \|u\|, & \text{if } |u|_{\alpha(x)} \leq 1, \\ \frac{1}{p^+} \|u\|^{p^-} - c_6 \beta_k^{\alpha^+} \|u\|^{\alpha^+} - c_7 \|u\| & \text{if } |u|_{\alpha(x)} > 1, \end{cases} \\ &\geq \frac{1}{p^+} \|u\|^{p^-} - c_6 \beta_k^{\alpha^+} \|u\|^{\alpha^+} - c_7 \|u\| - c_8. \end{aligned}$$

Set

$$\|u\| = \gamma_k = (c_9 \alpha^+ \beta_k^{\alpha^+})^{\frac{1}{p^- - \alpha^+}}.$$

Since  $\beta_k \rightarrow 0$  and  $p^- < p^+ < \alpha^-$ , it reads  $\lim_{k \rightarrow \infty} \gamma_k = \infty$ . Thus,

$$\begin{aligned} J(u) &\geq \frac{1}{p^+} (c_9 \alpha^+ \beta_k^{\alpha^+})^{\frac{p^-}{p^- - \alpha^+}} - c_6 \beta_k^{\alpha^+} (c_9 \alpha^+ \beta_k^{\alpha^+})^{\frac{\alpha^+}{p^- - \alpha^+}} - c_7 \|u\| - c_8 \\ &\geq \left( \frac{1}{p^+} - \frac{1}{\alpha^+} \right) (c_9 \alpha^+ \beta_k^{\alpha^+})^{\frac{p^-}{p^- - \alpha^+}} - c_7 (c_9 \alpha^+ \beta_k^{\alpha^+})^{\frac{1}{p^- - \alpha^+}} - c_8 \rightarrow \infty \quad \text{as } k \rightarrow \infty. \end{aligned}$$

(ii) We prove that for each  $k = 1, 2, \dots$ , there exist  $\rho_k > \gamma_k > 0$  such that

$$\max_{u \in Y_k, \|u\| = \rho_k} J(u) \leq 0.$$

Let us define the functional  $\|\cdot\|_{\theta} : X \rightarrow \mathbb{R}$  by

$$\|u\|_{\theta} = \left( \int_{\Omega} |u(x)|^{\theta} dx \right)^{\frac{1}{\theta}},$$

which is a norm on  $X$ . Since the norms  $\|\cdot\|_{\theta}$  and  $\|\cdot\|$  are equivalent on the finite-dimensional subspace  $Y_k$ , there is a constant  $c > 0$  such that

$$\|u\|_{\theta} \geq c \|u\| \quad \text{for all } u \in Y_k. \quad (3.5)$$

By  $(AR)$ , there exists a constant  $c_{10} > 0$  such that

$$F(x, t) \geq c_{10} |t|^{\theta}. \quad (3.6)$$

On the other hand, for all  $u \in Y_k$  with  $\|u\| > 1$  and from  $(A3)$ , we write

$$J(u) \leq \frac{1}{p^-} \|u\|^{p^+} - \int_{\Omega} F(x, u) dx.$$

Moreover, by (3.5)–(3.6), we conclude that

$$J(u) \leq \frac{1}{p^-} \|u\|^{p^+} - c_{11}^{\theta} \|u\|^{\theta}. \quad (3.7)$$

Since  $\theta > p^+$ , we have

$$\lim_{t \rightarrow \infty} \left( \frac{1}{p^-} t^{p^+} - c_{11}^{\theta} t^{\theta} \right) = -\infty.$$



Therefore, we can choose  $\rho_k > \gamma_k > 0$  such that

$$\frac{1}{p^-} \|u\|^{p^+} - c_{11}^\theta \|u\|^\theta < 0 \quad \text{for all } u \in Y_k, \|u\| = \rho_k. \quad (3.8)$$

It is obvious from (3.7) and (3.8) that

$$\max_{u \in Y_k, \|u\| = \rho_k} J(u) \leq 0.$$

We now can apply Lemma 2.7 to the  $J \in C^1(X, \mathbb{R})$  and obtain a sequence of critical values of  $J$  converging to  $+\infty$ . Consequently, there is a sequence  $(\pm u_n)_{n \in \mathbb{N}}$  of critical points for  $J$  such that  $J(\pm u_n) \rightarrow +\infty$  as  $n \rightarrow \infty$ . The proof is complete.  $\square$

## References

- [1] D. Andrade and T. F. Ma, An operator equation suggested by a class of stationary problems, *Comm. Appl. Nonlinear Anal.* **4** (1997), 65–71.
- [2] S. N. Antontsev and S. I. Shmarev, A model porous medium equation with variable exponent of nonlinearity: Existence, uniqueness and localization properties of solutions, *Nonlinear Anal.* **60** (2005), 515–545.
- [3] M. Avci, Existence and multiplicity of solutions for Dirichlet problems involving the  $p(x)$ -Laplace operator, *Electron. J. Differential Equations* **14** (2013), 1–9.
- [4] K. C. Chang, *Critical Point Theory and Applications*, Shanghai Scientific and Technology Press, Shanghai, 1986.
- [5] D. M. Duc, Nonlinear singular elliptic equations, *J. Lond. Math. Soc. (2)* **40** (1989), 420–440.
- [6] D. M. Duc and N. T. Vu, Nonuniformly elliptic equations of  $p$ -Laplacian type, *Nonlinear Anal.* **61** (2005), 1483–1495.
- [7] X. L. Fan, Solutions for  $p(x)$ -Laplacian Dirichlet problems with singular coefficients, *J. Math. Anal. Appl.* **312** (2005), 464–477.
- [8] X. L. Fan, J. S. Shen and D. Zhao, Sobolev embedding theorems for spaces  $W^{k,p(x)}(\Omega)$ , *J. Math. Anal. Appl.* **262** (2001), 749–760.
- [9] X. L. Fan and Q. H. Zhang, Existence of solutions for  $p(x)$ -Laplacian Dirichlet problems, *Nonlinear Anal.* **52** (2003), 1843–1852.
- [10] X. L. Fan and D. Zhao, On the spaces  $L^{p(x)}(\Omega)$  and  $W^{m,p(x)}(\Omega)$ , *J. Math. Anal. Appl.* **263** (2001), 424–446.
- [11] T. C. Halsey, Electrorheological fluids, *Science* **258** (1992), 761–766.
- [12] O. Kováčik and J. Rákosník, On spaces  $L^{p(x)}$  and  $W^{k,p(x)}$ , *Czechoslovak Math. J.* **41** (1991), no. 116, 592–618.
- [13] R. A. Mashiyeve, B. Cekic, M. Avci and Z. Yucedag, Existence and multiplicity of weak solutions for nonuniformly elliptic equations with nonstandard growth condition, *Complex Var. Elliptic Equ.* **57** (2012), no. 5, 579–595.
- [14] M. Mihăilescu and V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **462** (2006), 2625–2641.
- [15] M. Mihăilescu and V. Rădulescu, On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, *Proc. Amer. Math. Soc.* **135** (2007), no. 9, 2929–2937.
- [16] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Math. 1748, Springer, Berlin, 2000.
- [17] H. Q. Toan and Q. A. Ngô, Multiplicity of weak solutions for a class of nonuniformly elliptic equations of  $p$ -Laplacian type, *Nonlinear Anal.* **70** (2009), no. 4, 1536–1546.
- [18] N. T. Vu, Mountain pass theorem and nonuniformly elliptic equations, *Vietnam J. Math.* **33** (2005), no. 4, 391–408.
- [19] Z. Yucedag, Existence of solutions for  $p(x)$ -Laplacian equations without Ambrosetti–Rabinowitz type condition, *Bull. Malaysian Math. Sci. Soc. (2)* **38** (2015), no. 3, 1023–1033.
- [20] M. Willem, *Minimax Theorems*, Birkhäuser, Basel, 1996.
- [21] J. F. Zhao, *Structure Theory of Banach Spaces* (in Chinese), Wuhan University Press, Wuhan, 1991.
- [22] V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Math. USSR. Izv.* **9** (1987), 33–66.