Research Article

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General decay in a Timoshenko-type system with thermoelasticity with second sound

Abstract: In this article, we consider a vibrating nonlinear Timoshenko system with thermoelasticity with second sound. We discuss the well-posedness and the regularity of solutions using the semi-group theory. Moreover, we establish an explicit and general decay result for a wide class of relaxation functions, which depend on a stability number μ.

Keywords: Timoshenko system, well-posedness, general decay, stability

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1 Introduction and setting of the problem

Beams represent the most common structural component found in civil and mechanical structures. Because of their ubiquity they are extensively studied, from an analytical viewpoint, in mechanics of materials. A widely used mathematical model for describing the transverse vibrations of beams is based on *Timoshenko beam theory* (TBT) developed by Timoshenko in the 1920s. The TBT accounts for both the effect of rotational inertia and shear deformation that occur within a beam during the vibration. These factors are neglected when applied to *Euler–Bernoulli beam theory* (EBT), which is appropriate for beams with small cross-sectional dimensions compared to the length. In fact, a fundamental assumption in EBT is that cross sections remain plane and normal to the deformed longitudinal axis throughout deformation, while in TBT cross sections remain plane but do not remain normal to the deformed longitudinal axis as the shear deformation is taken into account. The cross section rotation from the reference to the current configuration is denoted by φ in both models. In the EB model, this is the same as the rotation of the longitudinal axis. In the Timoshenko model, the difference is used as measure of mean shear distortion.

In 1921, Timoshenko [28] gave the following system of coupled hyperbolic equations:

$$\begin{cases} \rho u_{tt} = \big(K(u_x - \varphi)\big)_x & \text{in } (0, L) \times \mathbb{R}_+, \\ I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(u_t - \varphi) & \text{in } (0, L) \times \mathbb{R}_+, \end{cases} \tag{1.1}$$

together with boundary conditions of the form

$$EI\varphi_x\big|_{x=0}^{x=L} = 0, \quad (u_x - \varphi)\big|_{x=0}^{x=L} = 0,$$

as a simple model describing the transverse vibrations of a beam, where t denotes the time variable and x is the space variable along the beam of length L, in its equilibrium configuration, u is the transverse displacement of the beam and φ is the rotation angle of the filament of the beam. The coefficients ρ, I_ρ, E, I and K are

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respectively the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus.

System (1.1), with the above boundary conditions, is conservative and the natural energy of the beam, given by

$$\mathcal{E}(t) = \frac{1}{2}\int_0^L (\rho|u_t|^2 + I_\rho|\varphi_t|^2 + EI|\varphi_x|^2 + K|u_x - \varphi|^2)\,dx,$$

remains constant in time.

Vibration has long been known for its capacity of disturbance, discomfort, damage and destruction. Since a long time, many researchers have been investigating ways and means to control this phenomenon. However, with the development of control theory for partial differential equations over the last few decades, it is not surprising that the issue of stability and controllability of Timoshenko-type systems has received a great attention of many mathematicians. One effective method for vibration control is passive damping. Damping is most beneficial when used to reduce the amplitude of dynamic instabilities, or resonances, in a structure.

Kim and Renardy [7] considered (1.1) together with two boundary controls of the form

$$\begin{aligned} K\varphi(L,t) - K\frac{\partial u}{\partial x}(L,t) &= \alpha\frac{\partial u}{\partial t}(L,t) \quad \text{for all } t \geq 0,\\ EI\frac{\partial \varphi}{\partial x}(L,t) &= -\beta\frac{\partial \varphi}{\partial t}(L,t) \quad \text{for all } t \geq 0, \end{aligned}$$

and used the multiplier techniques to establish an exponential decay result for the natural energy of (1.1). They also provided numerical estimates to the eigenvalues of the operator associated with system (1.1). An analogous result was also established by Feng, Shi and Zhang [3]. Raposo, Ferreira, Santos and Castro [20] looked into the following system:

$$\begin{cases} \rho_1 u_{tt} - K(u_x - \varphi) + u_t = 0 & \text{in } (0,L)\times\mathbb{R}_+,\\ \rho_2\varphi_{tt} - b\varphi_{xx} + K(u_x-\varphi) + \varphi_t = 0 & \text{in } (0,L)\times\mathbb{R}_+,\\ u(0,t) = u(L,t) = \varphi(0,t) = \varphi(L,t) = 0 & \text{for all } t > 0, \end{cases} \tag{1.2}$$

and proved that the energy associated with (1.2) decays exponentially. Soufyane and Wehbe [27] considered

$$\begin{cases} \rho u_{tt} = \big(K(u_x - \varphi)\big)_x & \text{in } (0,L)\times\mathbb{R}_+,\\ I_\rho\varphi_{tt} = (EI\varphi_x)_x + K(u_x-\varphi) - b\varphi_t & \text{in } (0,L)\times\mathbb{R}_+,\\ u(0,t) = u(L,t) = \varphi(0,t) = \varphi(L,t) = 0 & \text{for all } t > 0, \end{cases} \tag{1.3}$$

where b is a positive and continuous function satisfying

$$b(x) \geq b_0 > 0 \quad \text{for all } x \in [a_0, a_1] \subset [0, L]$$

and proved that the uniform stability of (1.3) holds if and only if the wave speeds are equal ($K/\rho = EI/I_\rho$); otherwise only the asymptotic stability has been proved. Rivera and Racke [18] obtained a similar result in a work, where the damping function $b = b(x)$ is allowed to change sign. They also treated in [17] a nonlinear Timoshenko-type system of the form

$$\begin{cases} \rho_1\varphi_{tt} - \sigma_1(\varphi_x, \psi)_x = 0,\\ \rho_2\psi_{tt} - \chi(\psi_x)_x + \sigma_2(\varphi_x,\psi) + d\psi_t = 0 \end{cases}$$

in a one-dimensional bounded domain and gave an alternative proof for a sufficient and necessary condition for exponential stability in the linear case and then proved a polynomial stability in general. Moreover, they investigated the global existence of small smooth solutions and exponential stability in the nonlinear case.

Shi and Feng [24] used the frequency multiplier method to investigate a nonuniform Timoshenko beam and showed that, under some locally distributed controls, the vibration of the beam decays exponentially. The nonuniform Timoshenko beam has also been studied by Ammar-Khodja, Kerbal and Soufyane [2] and a similar result to that in [24] has been established.

Ammar-Khodja, Benabdallah, Muñoz Rivera and Racke [1] considered a linear Timoshenko-type system with memory of the form

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \int_0^t g(t-s)\psi_{xx}(s)\,ds + K(\varphi_x + \psi) = 0, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0 \end{cases} \tag{1.4}$$

in $(0,L) \times \mathbb{R}_+$, and used the multiplier techniques to prove that the system is uniformly stable if and only if the wave speeds are equal ($K/\rho_1 = b/\rho_2$) and g decays uniformly. More precisely, they proved an exponential decay if g decays in an exponential rate and polynomially if g decays in a polynomial rate. They also required some extra technical conditions on both g' and g'' to obtain their results. This result has been later improved by Messaoudi and Mustafa [10] and Guesmia and Messaoudi [5], where the technical conditions on g'' have been removed and those on g' have been weakened. Also, Guesmia and Messaoudi [6] considered the following system:

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - \kappa\psi_{xx} + \int_0^t g(t-\tau)\big(a(x)\psi_x(\tau)\big)_x\,d\tau + K(\varphi_x + \psi) + b(x)h(\psi_t) = 0, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0 \end{cases} \tag{1.5}$$

in $(0,1) \times \mathbb{R}_+$ and proved under conditions on the relaxation function g similar to those in [4] and by assuming that

$$a(x) + b(x) \geq \rho > 0 \quad \text{for all } x \in (0,1),$$

an exponential stability for g decaying exponentially and h linear, and polynomial stability when g decays polynomially and h is nonlinear.

Concerning stabilization via classical heat effect, Rivera and Racke [16] investigated the system

$$\begin{cases} \rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x = 0 & \text{in } (0,L) \times \mathbb{R}_+, \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \gamma\theta_x = 0 & \text{in } (0,L) \times \mathbb{R}_+, \\ \rho_3\theta_t - K\theta_{xx} + \gamma\psi_{xt} = 0 & \text{in } (0,L) \times \mathbb{R}_+, \end{cases}$$

where φ, ψ and θ are functions of (x,t) which model the transverse displacement of the beam, the rotation angle of the filament and the difference temperature, respectively. Under appropriate conditions on $\sigma, \rho_i, b, K, \gamma$, they proved several exponential decay results for the linearized system and non-exponential stability result for the case of different wave speeds.

Concerning Timoshenko systems of thermoelasticity with second sound, Messaoudi, Pokojovy and Said-Houari [12] studied

$$\begin{cases} \rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x + \mu\varphi_t = 0 & \text{in } (0,L) \times \mathbb{R}_+, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta\theta_x = 0 & \text{in } (0,L) \times \mathbb{R}_+, \\ \rho_3\theta_t + \gamma q_x + \delta\psi_{tx} = 0 & \text{in } (0,L) \times \mathbb{R}_+, \\ \tau_0 q_t + q + \kappa\theta_x = 0 & \text{in } (0,L) \times \mathbb{R}_+, \end{cases}$$

where $\varphi = \varphi(x,t)$ is the displacement vector, $\psi = \psi(x,t)$ is the rotation angle of the filament, $\theta = \theta(x,t)$ is the temperature difference, $q = q(x,t)$ is the heat flux vector and ρ_1, ρ_2, ρ_3, b, k, γ, δ, κ, μ, τ_0 are positive constants.

Several exponential decay results for both linear and nonlinear cases have been established in the presence of the extra frictional damping $\mu\varphi_t$. Fernández Sare and Racke [4] considered

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0 & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \delta\theta_x = 0 & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_3\theta_t + \gamma q_x + \delta\psi_{tx} = 0 & \text{in } (0, L) \times \mathbb{R}_+, \\ \tau q_t + q + \kappa\theta_x = 0 & \text{in } (0, L) \times \mathbb{R}_+, \end{cases} \tag{1.6}$$

and showed that, in the absence of the extra frictional damping ($\mu = 0$), the coupling via Cattaneo's law causes loss of the exponential decay usually obtained in the case of coupling via Fourier's law [16]. This surprising property holds even for systems with history of the form

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0 & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \int_0^{+\infty} g(s)\psi_{xx}(\cdot, t - s)\, ds + \delta\theta_x = 0 & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_3\theta_t + \gamma q_x + \delta\psi_{tx} = 0 & \text{in } (0, L) \times \mathbb{R}_+, \\ \tau q_t + q + \kappa\theta_x = 0 & \text{in } (0, L) \times \mathbb{R}_+. \end{cases} \tag{1.7}$$

Precisely, it has been shown that both systems (1.6) and (1.7) are no longer exponentially stable even for equal-wave speeds ($k/\rho_1 = b/\rho_2$). However, no other rate of decay has been discussed. Recently, Santos, Almeida Júnior and Muñoz Rivera [22] considered (1.6) and introduced a new stability number

$$\mu = \Big(\tau - \frac{\rho_1}{k\rho_3}\Big)\Big(\frac{\rho_2}{b} - \frac{\rho_1}{k}\Big) - \frac{\tau\delta^2\rho_1}{bk\rho_3}$$

and used the semi-group method to obtain exponential decay result for $\mu = 0$ and a polynomial decay for $\mu \neq 0$.

The boundary feedback of memory type has also been used by Santos [21]. He considered a Timoshenko system and showed that the presence of two feedbacks of memory type at a portion of the boundary stabilizes the system uniformly. He also obtained the rate of decay of the energy, which is exactly the rate of decay of the relaxation functions. This last result has been improved and generalized by Messaoudi and Soufyane [14]. For more results concerning well-posedness and controllability of Timoshenko systems, we refer the reader to [9, 11, 13, 15, 23, 25, 26].

In this paper, we consider the following Timoshenko system:

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0 \quad \text{in } (0, 1) \times \mathbb{R}_+, \tag{1.8a}$$
$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \delta\theta_x + \alpha(t)h(\psi_t) = 0 \quad \text{in } (0, 1) \times \mathbb{R}_+, \tag{1.8b}$$
$$\rho_3\theta_t + q_x + \delta\psi_{xt} = 0 \quad \text{in } (0, 1) \times \mathbb{R}_+, \tag{1.8c}$$
$$\tau q_t + \beta q + \theta_x = 0 \quad \text{in } (0, 1) \times \mathbb{R}_+, \tag{1.8d}$$
$$\varphi_x(0, t) = \varphi_x(1, t) = \psi(0, t) = \psi(1, t) = q(0, t) = q(1, t) = 0 \quad \text{for all } t \geq 0, \tag{1.8e}$$
$$\varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x) \quad \text{for all } x \in (0, 1), \tag{1.8f}$$
$$\psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x) \quad \text{for all } x \in (0, 1), \tag{1.8g}$$
$$\theta(x, 0) = \theta_0(x), \quad q(x, 0) = q_0(x) \quad \text{for all } x \in (0, 1), \tag{1.8h}$$

where $\rho_1, \rho_2, \rho_3, b, k, \delta, \beta$ and τ are positive constants, $\varphi = \varphi(x, t)$ is the displacement vector, $\psi = \psi(x, t)$ is the rotation angle of the filament, $\theta = \theta(x, t)$ is the temperature difference and $q = q(x, t)$ is the heat flux vector. Also, α and h are two functions to be specified later.

Using (1.8a), (1.8c) and the boundary conditions (1.8e), we have

$$\frac{d^2}{dt^2}\int_0^1 \varphi(x, t)\, dx = 0 \quad \text{and} \quad \frac{d}{dt}\int_0^1 \theta(x, t)\, dx = 0.$$

Consequently, we obtain

$$\int_0^1 \varphi(x,t)\,dx = \Big(\int_0^1 \varphi_1(x)\,dx\Big)t + \int_0^1 \varphi_0(x)\,dx \quad \text{and} \quad \int_0^1 \theta(x,t)\,dx = \int_0^1 \theta_0(x)\,dx.$$

If we set

$$\overline{\varphi}(x,t) = \varphi(x,t) - \Big(\Big(\int_0^1 \varphi_1(x)\,dx\Big)t + \int_0^1 \varphi_0(x)\,dx\Big) \quad \text{and} \quad \overline{\theta}(x,t) = \theta(x,t) - \int_0^1 \theta_0(x)\,dx,$$

then $(\overline{\varphi}, \psi, \overline{\theta}, q)$ also satisfy system (1.8) and we have

$$\int_0^1 \overline{\varphi}(x,t)\,dx = 0 \quad \text{and} \quad \int_0^1 \overline{\theta}(x,t)\,dx = 0.$$

From now on, we use the new variables $(\overline{\varphi}, \psi, \overline{\theta}, q)$, but we denote them by $(\varphi, \psi, \theta, q)$, for simplicity.

The article is organized as follows. First, in Section 2, we use the semi-group theory to prove the existence and uniqueness of solutions of system (1.8). Next, in Section 3, we study the asymptotic behavior of the energy of solutions of system (1.8) using the multiplier method. For that purpose, we assume some hypotheses on α and h. The optimal exponential and polynomial decay rate estimates can be obtained in some special cases with explicit nonlinear terms.

2 Well-posedness and regularity

In this section, we discuss the well-posedness of the problem (1.8), using the semi-group theory. We consider the following hypotheses on α and h:

(A_1) $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable and non-increasing,

(A_2) $h\colon \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function satisfying $h(0) = 0$.

We introduce the Hilbert spaces

$$\begin{aligned} L^2_\star(0,1) &= \Big\{ v \in L^2(0,1) : \int_0^1 v(s)\,ds = 0 \Big\}, \\ H^1_\star(0,1) &= H^1(0,1) \cap L^2_\star(0,1), \\ H^2_\star(0,1) &= \{ v \in H^2(0,1) : v_x(0) = v_x(1) = 0 \}. \end{aligned}$$

The energy associated with system (1.8) is defined by

$$E(\varphi,\psi,\theta,q)(t) = \frac{1}{2}\int_0^1 (\rho_1\varphi_t^2 + \rho_2\psi_t^2 + b\psi_x^2 + k(\varphi_x+\psi)^2 + \rho_3\theta^2 + \tau q^2)\,dx.$$

Let

$$H = H^1_\star(0,1) \times L^2_\star(0,1) \times H^1_0(0,1) \times L^2(0,1) \times L^2_\star(0,1) \times L^2(0,1)$$

be the Hilbert space endowed with the inner product defined, for $U = (u_1,u_2,u_3,u_4,u_5,u_6)^t \in H$ and $V = (v_1,v_2,v_3,v_4,v_5,v_6)^t \in H$, by

$$\begin{aligned} \langle U, V\rangle_H = \rho_1\langle u_2, v_2\rangle_{L^2(0,1)} + \rho_2\langle u_4, v_4\rangle_{L^2(0,1)} + k\langle u_{1x}+u_3, v_{1x}+v_3\rangle_{L^2(0,1)} \\ + b\langle u_{3x}, v_{3x}\rangle_{L^2(0,1)} + \rho_3\langle u_5, v_5\rangle_{L^2(0,1)} + \tau\langle u_6, v_6\rangle_{L^2(0,1)}. \end{aligned}$$

For $\Phi = (\varphi, u, \psi, v, \theta, q)^t$ and $\Phi_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0)^t$, where $u = \varphi_t$ and $v = \psi_t$, system (1.8) is equivalent to the abstract first order Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\Phi(t) + (A + B)\Phi(t) = 0 & \text{for all } t \in \mathbb{R}_+, \\ \Phi(0) = \Phi_0, \end{cases} \tag{2.1}$$

where $A\colon D(A) \subset H \to H$ is the linear operator defined by

$$A\Phi = \begin{pmatrix} -u \\ -\frac{k}{\rho_1}\varphi_{xx} - \frac{k}{\rho_1}\psi_x \\ -v \\ -\frac{b}{\rho_2}\psi_{xx} + \frac{k}{\rho_2}(\varphi_x + \psi) + \frac{\delta}{\rho_2}\theta_x \\ \frac{1}{\rho_3}q_x + \frac{\delta}{\rho_3}v_x \\ \frac{\beta}{\tau}q + \frac{1}{\tau}\theta_x \end{pmatrix} \tag{2.2}$$

and $B\colon D(B) \subset H \to H$ is the nonlinear operator defined by

$$B\Phi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \alpha(t)h(v) \\ 0 \\ 0 \end{pmatrix}.$$

The domain of the operator A is given by $D(A) = \{\Phi \in H : A\Phi \in H\}$, is endowed with the graph norm

$$\|\Phi\|_{D(A)} = \|\Phi\|_H + \|A\Phi\|_H$$

and it can be characterized by

$$D(A) = (H^2_*(0,1) \cap H^1_*(0,1)) \times H^1_*(0,1) \times (H^2(0,1) \cap H^1_0(0,1)) \times H^1_0(0,1) \times H^1_*(0,1) \times H^1_0(0,1).$$

The domain of the operator B is given by $D(B) = \{\Phi \in H : B\Phi \in H\} = H$.

We first state and prove the following lemmas which will be useful to deduce the well-posedness result.

Lemma 2.1. *For $\Phi \in D(A)$, we have $(A\Phi, \Phi)_H \geq 0$.*

Proof. For any $\Phi = (\varphi, u, \psi, v, \theta, q)^t \in D(A)$, we have

$$\begin{aligned}(A\Phi, \Phi)_H = k\int_0^1 -(u_x + v)(\varphi_x + \psi)\,dx + \int_0^1 (-k\varphi_{xx} - k\psi_x)u\,dx + b\int_0^1 -v_x\psi_x\,dx \\ + \int_0^1 (-b\psi_{xx} + k(\varphi_x + \psi) + \delta\theta_x)v\,dx + \int_0^1 (q_x + \delta v_x)\theta\,dx + \int_0^1 (\beta q + \theta_x)q\,dx.\end{aligned}$$

Using integration by parts and the boundary conditions in (1.8), we obtain

$$(A\Phi, \Phi)_H = \beta\int_0^1 q^2\,dx \geq 0. \qquad \square$$

Lemma 2.2. *The operator $I + A$ is surjective.*

Proof. For any $W = (w_1, w_2, w_3, w_4, w_5, w_6) \in H$, we prove that there exists some $V = (v_1, v_2, v_3, v_4, v_5, v_6)$ in $D(A)$ satisfying

$$(I + A)V = W.$$

That is,

$$-v_2 + v_1 = w_1, \tag{2.3a}$$
$$-kv_{1xx} - kv_{3x} + \rho_1 v_1 = \rho_1(w_1 + w_2), \tag{2.3b}$$
$$-v_4 + v_3 = w_3, \tag{2.3c}$$
$$-bv_{3xx} + k(v_{1x} + v_3) + \delta v_{5x} + \rho_2 v_4 = \rho_2 w_4, \tag{2.3d}$$
$$v_{6x} + \delta v_{4x} + \rho_3 v_5 = \rho_3 w_5, \tag{2.3e}$$
$$(\beta + \tau)v_6 + v_{5x} = \tau w_6. \tag{2.3f}$$

Then, (2.3a), (2.3c) and (2.3e) yield

$$v_2 = v_1 - w_1 \in H^1_*(0,1), \tag{2.4}$$
$$v_4 = v_3 - w_3 \in H^1_0(0,1), \tag{2.5}$$
$$v_{6x} = \rho_3 w_5 + \delta w_{3x} - \delta v_{3x} - \rho_3 v_5.$$

By integration over $(0, x)$ and using $v_6(0) = w_3(0) = v_3(0) = 0$, we obtain

$$v_6 = \rho_3 \int_0^x w_5\,ds + \delta w_3 - \delta v_3 - \rho_3 \int_0^x v_5\,ds. \tag{2.6}$$

We substitute (2.6) into (2.3f) and we get

$$v_{5x} + (\beta + \tau)\left[\rho_3 \int_0^x w_5\,ds + \delta w_3 - \delta v_3 - \rho_3 \int_0^x v_5\,ds\right] = \tau w_6.$$

Hence, we deduce that

$$-v_{5x} + (\beta + \tau)\delta v_3 + \rho_3(\beta + \tau)\int_0^x v_5\,ds = (\beta + \tau)\delta w_3 + (\beta + \tau)\rho_3 \int_0^x w_5\,ds - \tau w_6. \tag{2.7}$$

Again, we substitute (2.7) into (2.3d), to get

$$-bv_{3xx} + kv_{1x} + kv_3 + \delta\left[(\beta + \tau)\delta v_3 + \rho_3(\beta + \tau)\int_0^x v_5\,ds - (\beta + \tau)\delta w_3 - (\beta + \tau)\rho_3 \int_0^x w_5\,ds - \tau w_6\right] + \rho_2 v_3$$
$$= \rho_2(w_3 + w_4)$$

and we infer that

$$\begin{aligned} -bv_{3xx} + kv_{1x} + kv_3 + \delta^2(\beta + \tau)\delta v_3 + \rho_3\delta(\beta + \tau)\int_0^x v_5\,ds + \rho_2 v_3 \\ = (\beta + \tau)\delta^2 w_3 + (\beta + \tau)\delta\rho_3 \int_0^x w_5\,ds - \delta\tau w_6 + \rho_2(w_3 + w_4). \end{aligned} \tag{2.8}$$

By using (2.7), (2.8) and (2.3b), it can be shown that v_1, v_3 and v_5 satisfy

$$-kv_{1xx} - kv_{3x} + \rho_1 v_1 = h_1 \in L^2_*(0,1), \tag{2.9a}$$
$$-bv_{3xx} + kv_{1x} + kv_3 + (\delta^2(\beta + \tau) + \rho_2)v_3 + \rho_3\delta(\beta + \tau)\int_0^x v_5\,ds = h_2 \in L^2(0,1), \tag{2.9b}$$
$$-\rho_3 v_{5x} + \rho_3(\beta + \tau)\delta v_3 + \rho_3^2(\beta + \tau)\int_0^x v_5\,ds = h_3 \in L^2(0,1), \tag{2.9c}$$

where

$$h_1 = \rho_1(w_1 + w_2),$$
$$h_2 = (\beta + \tau)\delta^2 w_3 + (\beta + \tau)\delta\rho_3 \int_0^x w_5\,ds - \delta\tau w_6 + \rho_2(w_3 + w_4)$$

and

$$h_3 = \rho_3(\beta+\tau)\delta w_3 + (\beta+\tau)\rho_3^2 \int_0^x w_5\,ds - \rho_3 \tau w_6.$$

Let $u = (u_1, u_3, u_5)$ and $v = (v_1, v_3, v_5)$, a simple multiplication of (2.9a), (2.9b) and (2.9c), by u_1, u_3 and $\int_0^x u_5\,ds$, respectively, and integration over $(0,1)$ yields

$$\begin{aligned}
&-k\int_0^1 v_{1xx}u_1\,dx - k\int_0^1 v_{3x}u_1\,dx + \rho_1\int_0^1 v_1u_1\,dx = \int_0^1 h_1u_1\,dx, \\
&-b\int_0^1 v_{3xx}u_3\,dx + k\int_0^1 v_{1x}u_3\,dx + k\int_0^1 v_3u_3\,dx \\
&\qquad + (\delta^2(\beta+\tau)+\rho_2)\int_0^1 v_3u_3\,dx + \rho_3\delta(\beta+\tau)\int_0^1\Big(\int_0^x v_5\,ds\Big)u_3\,dx = \int_0^1 h_2u_3\,dx, \\
&-\rho_3\int_0^1 v_{5x}\Big(\int_0^x u_5\,ds\Big)\,dx + \rho_3(\beta+\tau)\delta\int_0^1 v_3\Big(\int_0^x u_5\,ds\Big)\,dx \\
&\qquad + \rho_3^2(\beta+\tau)\int_0^1\Big(\int_0^x v_5\,ds\Big)\Big(\int_0^x u_5\,ds\Big)\,dx = \int_0^1 h_3\Big(\int_0^x u_5\,ds\Big)\,dx.
\end{aligned} \tag{2.10}$$

Using integration by parts and the boundary conditions yields

$$\begin{aligned}
&k\int_0^1 v_{1x}u_{1x}\,dx + k\int_0^1 v_3u_{1x}\,dx + \rho_1\int_0^1 v_1u_1\,dx = \int_0^1 h_1u_1\,dx, \\
&b\int_0^1 v_{3x}u_{3x}\,dx + k\int_0^1 v_{1x}u_3\,dx + k\int_0^1 v_3u_3\,dx \\
&\qquad + (\delta^2(\beta+\tau)+\rho_2)\int_0^1 v_3u_3\,dx + \rho_3\delta(\beta+\tau)\int_0^1\Big(\int_0^x v_5\,ds\Big)u_3\,dx = \int_0^1 h_2u_3\,dx, \\
&\rho_3\int_0^1 v_{5x}u_5\,dx + \rho_3(\beta+\tau)\delta\int_0^1 v_3\Big(\int_0^x u_5\,ds\Big)\,dx \\
&\qquad + \rho_3^2(\beta+\tau)\int_0^1\Big(\int_0^x v_5\,ds\Big)\Big(\int_0^x u_5\,ds\Big)\,dx = \int_0^1 h_3\Big(\int_0^x u_5\,ds\Big)\,dx.
\end{aligned}$$

The sum of the previous equations gives the following variational formulation

$$b(v,u) = l(u) \tag{2.11}$$

for all $u = (u_1, u_3, u_5) \in H_*^1(0,1)\times H_0^1(0,1)\times L_*^2(0,1)$, where b is defined by

$$\begin{aligned}
b(v,u) = {}& k\int_0^1 (v_{1x}+v_3)(u_{1x}+u_3)\,dx + \rho_1\int_0^1 v_1u_1\,dx + b\int_0^1 v_{3x}u_{3x}\,dx \\
&+ (\delta^2(\beta+\tau)+\rho_2)\int_0^1 v_3u_3\,dx + \rho_3\delta(\beta+\tau)\int_0^1\Big(\int_0^x v_5\,ds\Big)u_3\,dx + \rho_3\int_0^1 v_{5x}u_5\,dx \\
&+ \rho_3(\beta+\tau)\delta\int_0^1 v_3\Big(\int_0^x u_5\,ds\Big)\,dx + \rho_3^2(\beta+\tau)\int_0^1\Big(\int_0^x v_5\,ds\Big)\Big(\int_0^x u_5\,ds\Big)\,dx
\end{aligned}$$

and l is defined by

$$l(u) = \int_0^1 h_1 u_1\,dx + \int_0^1 h_2 u_3\,dx + \int_0^1 h_3\Big(\int_0^x u_5\,ds\Big)\,dx.$$

We introduce the Hilbert space $\Lambda = H^1_*(0,1) \times H^1_0(0,1) \times L^2(0,1)$ equipped with the norm

$$\|v\|^2_\Lambda = \|v_{1x} + v_3\|_2^2 + \|v_1\|_2^2 + \|v_{3x}\|_2^2 + \|v_5\|_2^2.$$

It is clear that b is a bilinear and continuous form on $\Lambda \times \Lambda$ and l is a linear and continuous form on Λ. Furthermore, there exists a positive constant c_0 such that

$$\begin{aligned} b(v,v) &= k\|v_{1x} + v_3\|_2^2 + \rho_1\|v_1\|_2^2 + b\|v_{3x}\|_2^2 + (\delta^2(\beta+\tau) + \rho_2)\|v_3\|_2^2 + \rho_3\|v_5\|_2^2 \\ &\quad + 2\rho_3(\beta+\tau)\delta\int_0^1 v_3\Big(\int_0^x v_5\,ds\Big)\,dx + \rho_3^2(\beta+\tau)\int_0^1\Big(\int_0^x v_5\,ds\Big)^2\,dx \\ &\geq c_0\|v\|^2_\Lambda, \end{aligned}$$

which implies that b is coercive. Therefore, using the Lax–Milgram theorem we conclude that system (2.9) has a unique solution

$$(v_1, v_3, v_5) \in (H^1_*(0,1) \times H^1_0(0,1) \times L^2_*(0,1)),$$

and we deduce from (2.4)–(2.6) the existence of $v_2 \in H^1_*(0,1)$, $v_4 \in H^1_0(0,1)$ and $v_6 \in L^2_*(0,1)) \subset L^2(0,1))$.

Now, it remains to show that

$$v_1 \in H^2_*(0,1) \cap H^1_*(0,1), \quad v_3 \in H^2(0,1) \cap H^1_0(0,1), \quad v_5 \in H^1_*(0,1) \quad \text{and} \quad v_6 \in H^1_0(0,1).$$

From (2.9), we have

$$-kv_{1xx} = kv_{3x} - \rho_1 v_1 + h_1 \in L^2(0,1).$$

Consequently, it follows that

$$v_1 \in H^2(0,1) \cap H^1_*(0,1).$$

Moreover, (2.10) is also true for any $\varphi_1 \in \mathcal{C}^1([0,1])$. Hence, we have

$$k\int_0^1 v_{1x}\varphi_{1x}\,dx + k\int_0^1 v_3\varphi_{1x}\,dx + \rho_1\int_0^1 v_1\varphi_1\,dx = \int_0^1 h_1\varphi_1\,dx$$

for any $\varphi_1 \in \mathcal{C}^1([0,1])$. Thus, using integration by parts we obtain

$$v_{1x}(1)\varphi_1(1) - v_{1x}(0)\varphi_1(0) = 0 \quad \text{for all } \varphi_1 \in \mathcal{C}^1([0,1]).$$

Therefore, $v_{1x}(1) = v_{1x}(0) = 0$, and we deduce that

$$v_1 \in H^2_*(0,1) \cap H^1_*(0,1).$$

Now, if we substitute (2.3f) into (2.3d), we get

$$bv_{3xx} = kv_{1x} + kv_3 + \delta\tau w_6 - \delta(\beta+\tau)v_6 + \rho_2 v_3 - h_2 \in L^2(0,1).$$

Consequently, it follows that

$$v_3 \in H^2(0,1) \cap H^1_0(0,1).$$

On the other hand, from (2.3f) we get

$$v_{5x} = \tau w_6 - (\beta+\tau)v_6 \in L^2(0,1)$$

and we deduce that

$$v_5 \in H^1(0,1) \cap L^2_*(0,1).$$

Similarly, from (2.3) we have

$$v_{6x} = \rho_3 w_5 + \delta w_{3x} - \delta v_{3x} - \rho_3 v_5 \in L^2(0,1) \quad \text{which implies} \quad v_6 \in H^1_0(0,1),$$

as $v_6(0) = v_6(1) = 0$. Finally, the operator $I + A$ is surjective. □

Using Lemmas 2.1, 2.2 and the Lipshitz continuity of B we conclude that the operator $A + B$ is the infinitesimal generator of a nonlinear contraction C_0-semi-group on the Hilbert space H. Finally, by applying the semi-group theory to (2.1) (see [8, 19]), we easily get the following well-posedness result.

Theorem 2.3. *Assume that* (A_1) *and* (A_2) *are satisfied. Then for all initial data*

$$(\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0) \in (H^2_\star(0,1) \cap H^1_\star(0,1)) \times H^1_\star(0,1) \times (H^2(0,1) \cap H^1_0(0,1)) \times H^1_0(0,1) \times H^1_\star(0,1) \times H^1_0(0,1),$$

system (1.8) *has a unique solution* $(\varphi, \psi, \theta, q)$ *that verifies*

$$\begin{aligned}(\varphi, \psi) \in C^0(\mathbb{R}_+, (H^2_\star(0,1) \cap H^1_\star(0,1)) \times (H^2(0,1) \cap H^1_0(0,1))) \\ \cap C^1(\mathbb{R}_+, H^1_\star(0,1) \times H^1_0(0,1)) \cap C^2(\mathbb{R}_+, L^2_\star(0,1) \times L^2(0,1))\end{aligned}$$

and

$$(\theta, q) \in C^0(\mathbb{R}_+, H^1_\star(0,1) \times H^1_0(0,1)) \cap C^1(\mathbb{R}_+, L^2_\star(0,1) \times L^2(0,1)).$$

3 Stability results

In this section, we state and prove a stability result for the nonlinear Timoshenko system (1.8). For this purpose, we consider the following hypotheses:

(A_1) $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable and non-increasing function.

(A_2^*) $h\colon \mathbb{R} \to \mathbb{R}$ is a continuous non-decreasing function such that $h(0) = 0$ and there exists a continuous strictly increasing odd function $h_0 \in C([0, +\infty))$, continuously differentiable in a neighborhood of 0 and satisfying $h_0(0) = 0$,

$$\begin{aligned} h_0(|s|) \le |h(s)| \le h_0^{-1}(|s|) &\quad \text{for all } |s| \le \varepsilon, \\ c_1|s| \le |h(s)| \le c_2|s| &\quad \text{for all } |s| \ge \varepsilon, \end{aligned}$$

where $c_i > 0$ for $i = 1, 2$.

Moreover, we define a function H by

$$H(x) = \sqrt{x}\,h_0(\sqrt{x}). \tag{3.1}$$

Thanks to Assumption (A_2^*), H is of class C^1 and is strictly convex on $(0, r^2]$, where $r > 0$ is a sufficiently small number.

Remark 3.1. We denote by c a positive generic constant throughout this paper.

The hypothesis (A_1) implies that $\alpha(t) \le c$.

We recall here the stability number defined by

$$\mu = \Big(\tau - \frac{\rho_1}{k\rho_3}\Big)\Big(\frac{\rho_2}{b} - \frac{\rho_1}{k}\Big) - \frac{\tau\delta^2\rho_1}{bk\rho_3}.$$

3.1 The case $\mu = 0$

In this subsection, we state and prove the decay results which are not necessarily of exponential or polynomial type. For this purpose, we establish several lemmas. We recall that the energy associated with system (1.8) is defined by

$$E(t) := \frac{1}{2}\int_0^1 (\rho_1\varphi_t^2 + \rho_2\psi_t^2 + b\psi_x^2 + k(\varphi_x + \psi)^2 + \rho_3\theta^2 + \tau q^2)\,dx. \tag{3.2}$$

Throughout the rest of this paper we assume that conditions (A_1) and (A_2^*) hold.

Lemma 3.2. *Let $(\varphi, \psi, \theta, q)$ be a solution of system* (1.8). *Then, the functional E satisfies*

$$E'(t) = -\beta \int_0^1 q^2\, dx - \alpha(t) \int_0^1 \psi_t h(\psi_t)\, dx \le 0. \tag{3.3}$$

Proof. By multiplying the first fourth equations in (1.8), respectively, by φ_t, ψ_t, θ and q, using the integration by parts with respect to x over $(0, 1)$, the boundary conditions (1.8e) and the hypotheses (A_1) and (A_2^*), we obtain (3.3). □

Lemma 3.3. *Let $(\varphi, \psi, \theta, q)$ be a solution of system* (1.8). *Then, the functional*

$$K_1(t) := -\int_0^1 (\rho_1 \varphi\varphi_t + \rho_2 \psi\psi_t)\, dx \tag{3.4}$$

verifies the following estimate:

$$K_1'(t) \le -\rho_1 \int_0^1 \varphi_t^2\, dx - \rho_2 \int_0^1 \psi_t^2\, dx + c \int_0^1 \psi_x^2\, dx + k \int_0^1 (\varphi_x + \psi)^2\, dx + \frac{\delta}{2} \int_0^1 \theta^2\, dx + \frac{1}{2} \int_0^1 h^2(\psi_t)\, dx. \tag{3.5}$$

Proof. By differentiating (3.4) and using the first and second equations of (1.8), we get

$$K_1'(t) = -\rho_1 \int_0^1 \varphi_t^2\, dx - \rho_2 \int_0^1 \psi_t^2\, dx - \int_0^1 k(\varphi_x + \psi)_x \varphi\, dx - \int_0^1 (b\psi_{xx} - k(\varphi_x + \psi) - \delta\theta_x - \alpha(t)h(\psi_t))\psi\, dx.$$

Integrating by parts and using the boundary conditions (1.8e), we have

$$K_1'(t) = -\rho_1 \int_0^1 \varphi_t^2\, dx - \rho_2 \int_0^1 \psi_t^2\, dx + b \int_0^1 \psi_x^2\, dx + \int_0^1 k(\varphi_x + \psi)^2\, dx - \delta \int_0^1 \theta\psi_x\, dx + \int_0^1 \alpha(t)h(\psi_t)\psi\, dx.$$

Applying Young's inequality, we obtain (3.5). □

Lemma 3.4. *Let $(\varphi, \psi, \theta, q)$ be a solution of system* (1.8). *Then, the functional*

$$K_2(t) := \rho_2 \int_0^1 \psi\psi_t dx - \rho_2 \int_0^1 \varphi_t w dx - \delta\tau \int_0^1 \psi q\, dx \tag{3.6}$$

satisfies, for any $\varepsilon > 0$,

$$K_2'(t) \le -(b - 2c\varepsilon) \int_0^1 \psi_x^2\, dx + c\left(\int_0^1 \psi_t^2\, dx + \int_0^1 q^2\, dx + \int_0^1 h^2(\psi_t)\, dx \right) + \rho_1 \varepsilon \int_0^1 \varphi_t^2\, dx, \tag{3.7}$$

where w is the solution of the problem

$$\begin{cases} -w_{xx} = \psi_x, \\ w(0) = w(1) = 0. \end{cases} \tag{3.8}$$

Proof. By differentiation of (3.6) and the use of the first, second and fourth equation of (1.8), we get

$$\begin{aligned} K_2'(t) = {} & \rho_2 \int_0^1 \psi_t^2\, dx + b \int_0^1 \psi_{xx}\psi\, dx - k \int_0^1 (\varphi_x + \psi)\psi\, dx - \delta \int_0^1 \theta_x \psi\, dx - \alpha(t) \int_0^1 \psi h(\psi_t)\, dx \\ & + k \int_0^1 (\varphi_x + \psi)_x w\, dx + \rho_1 \int_0^1 \varphi_t w_t\, dx - \tau\delta \int_0^1 \psi_t q\, dx + \delta\beta \int_0^1 \psi q\, dx + \delta \int_0^1 \theta_x \psi\, dx. \end{aligned}$$

Integrating by parts the last equality, using (3.8) and the boundary conditions (1.8e), we have

$$K_2'(t) = \rho_2 \int_0^1 \psi_t^2\,dx - b\int_0^1 \psi_x^2\,dx - k\int_0^1 \psi^2\,dx + k\int_0^1 w_x^2\,dx - \alpha(t)\int_0^1 \psi h(\psi_t)\,dx$$
$$+ \rho_1 \int_0^1 \varphi_t w_t\,dx - \tau\delta\int_0^1 \psi_t q\,dx + \delta\beta\int_0^1 \psi q\,dx.$$

By a simple calculation, we easily deduce that the function w satisfies the following estimates:

$$\int_0^1 w_x^2\,dx \le \int_0^1 \psi^2\,dx, \tag{3.9}$$
$$\int_0^1 w_t^2\,dx \le c\int_0^1 \psi_t^2\,dx. \tag{3.10}$$

Thanks to Young's and Poincaré's inequalities and (3.9)–(3.10), we conclude that

$$K_2'(t) \le \rho_2 \int_0^1 \psi_t^2\,dx - b\int_0^1 \psi_x^2\,dx + \frac{\rho_1}{4\varepsilon}\int_0^1 w_t^2\,dx + \rho_1\varepsilon\int_0^1 \varphi_t^2\,dx + \tau\delta\varepsilon\int_0^1 \psi_t^2\,dx$$
$$+ \frac{\tau\delta}{4\varepsilon}\int_0^1 q^2\,dx + c_p\varepsilon\int_0^1 \psi_x^2\,dx + \frac{(\delta\beta)^2}{4\varepsilon}\int_0^1 q^2\,dx + \varepsilon c_p\int_0^1 \psi_x^2\,dx + \frac{c^2}{4\varepsilon}\int_0^1 h^2(\psi_t)\,dx.$$

Therefore, we obtain (3.7). □

Lemma 3.5. *Let $(\varphi, \psi, \theta, q)$ be a solution of system* (1.8). *Then, the functional*

$$K_3(t) := -\tau\rho_3 \int_0^1 q\bigg(\int_0^x \theta(t,y)\,dy\bigg)\,dx \tag{3.11}$$

satisfies

$$K_3'(t) \le -\frac{\rho_3}{2}\int_0^1 \theta^2\,dx + c\bigg(\int_0^1 q^2\,dx + \int_0^1 \psi_t^2\,dx\bigg). \tag{3.12}$$

Proof. By differentiation of (3.11) and the use of the third and fourth equations of (1.8), we get

$$K_3'(t) = \rho_3\beta\int_0^1 q\bigg(\int_0^x \theta(t,y)\,dy\bigg)\,dx + \rho_3\int_0^1 \theta_x\bigg(\int_0^x \theta(t,y)\,dy\bigg)\,dx$$
$$+ \tau\int_0^1 q\bigg(\int_0^x q_x(t,y)\,dy\bigg)\,dx + \tau\delta\int_0^1 q\bigg(\int_0^x \psi_{tx}(t,y)\,dy\bigg)\,dx.$$

By integrating the above equality over the integral (0, 1) and using the boundary conditions (1.8e) (note also that $\int_0^1 \theta\,dx = 0$), we have

$$K_3'(t) = \rho_3\beta\int_0^1 q\bigg(\int_0^x \theta(t,y)\,dy\bigg)\,dx - \rho_3\int_0^1 \theta^2\,dx + \tau\int_0^1 q^2\,dx + \tau\delta\int_0^1 q\psi_t\,dx.$$

Applying again Young's inequality and the fact that

$$\int_0^1 \bigg(\int_0^x \theta(t,y)\,dy\bigg)^2\,dx \le c\int_0^1 \theta^2\,dx,$$

we arrive at (3.12). □

Lemma 3.6. *Let $(\varphi, \psi, \theta, q)$ be a solution of system (1.8). Then, the functional*

$$
\begin{aligned}
K_4(t) := {} & \frac{\tau\rho_2}{k}\int_0^1 \psi_t(\varphi_x+\psi)\,dx + \frac{b\tau\rho_1}{k^2}\int_0^1 \varphi_t\psi_x\,dx \\
& - \frac{b\tau\rho_3}{\delta k}\Big(\frac{\rho_2}{b}-\frac{\rho_1}{k}\Big)\int_0^1 \theta\varphi_t\,dx + \frac{b\tau}{\delta k}\Big(\frac{\rho_2}{b}-\frac{\rho_1}{k}\Big)\int_0^1 q(\varphi_x+\psi)\,dx
\end{aligned}
\tag{3.13}
$$

satisfies

$$
\begin{aligned}
K_4'(t) \le {} & -(\tau-2\varepsilon_1)\int_0^1(\varphi_x+\psi)^2\,dx + C\Big(\int_0^1\psi_t^2\,dx + \int_0^1 q^2\,dx + \int_0^1 h^2(\psi_t)\,dx\Big) \\
& + \frac{b\rho_3}{\delta\rho_1}\Big[\Big(\tau-\frac{\rho_1}{k\rho_3}\Big)\Big(\frac{\rho_2}{b}-\frac{\rho_1}{k}\Big) - \frac{\tau\delta^2\rho_1}{bk\rho_3}\Big]\int_0^1\theta_x(\varphi_x+\psi)\,dx
\end{aligned}
\tag{3.14}
$$

with

$$
C = 2\max\Big[\frac{\tau\rho_2}{k}+\frac{1}{2}, \Big(\frac{b}{\tau k}\Big(\frac{\rho_2}{b}-\frac{\rho_1}{k}\Big)\Big)^2\Big(\frac{\beta^2}{4\varepsilon_1}+\frac{\tau^2}{2}\Big), \frac{c^2\tau^2}{4k^2\varepsilon_1}\Big] \quad \textit{and} \quad \varepsilon_1 > 0.
$$

Proof. By differentiation of (3.13), using (1.8) and integration over $(0,1)$, we get

$$
\begin{aligned}
K_4'(t) = {} & \frac{\tau}{2}\int_0^1(b\psi_{xx} - k(\varphi_x+\psi) - \delta\theta_x - \alpha(t)h(\psi_t))(\varphi_x+\psi)\,dx + \frac{\tau\rho_2}{k}\int_0^1\psi_t(\varphi_x+\psi)_t\,dx \\
& + \frac{b\tau}{k^2}\int_0^1(\varphi_x+\psi)_x\varphi_x + \varphi_t\psi_{tx}\,dx - \frac{b\tau}{\delta k}\Big(\frac{\rho_2}{b}-\frac{\rho_1}{k}\Big)\int_0^1(-(q_x+\delta\psi_{xt})\varphi_t + \theta(\varphi_x+\psi)_x)\,dx \\
& + \frac{b}{\delta k}\Big(\frac{\rho_2}{b}-\frac{\rho_1}{k}\Big)\int_0^1(-(\beta q+\theta_x)(\varphi_x+\psi) + q(\varphi_x+\psi)_t)\,dx.
\end{aligned}
$$

By integration over $(0,1)$ and using the boundary conditions (1.8e), we have

$$
\begin{aligned}
K_4'(t) = {} & -\tau\int_0^1(\varphi_x+\psi)^2\,dx + \frac{\tau\rho_2}{k}\int_0^1\psi_t^2\,dx + \frac{b\tau}{\delta k}\Big(\frac{\rho_2}{b}-\frac{\rho_1}{k}\Big)\int_0^1 q\psi_t\,dx \\
& - \frac{b\beta}{\delta k}\Big(\frac{\rho_2}{b}-\frac{\rho_1}{k}\Big)\int_0^1 q(\varphi_x+\psi)\,dx - \frac{\tau}{k}\int_0^1\alpha(t)h(\psi_t)(\varphi_x+\psi)\,dx \\
& + \frac{b\rho_3}{\delta\rho_1}\Big[\Big(\tau-\frac{\rho_1}{k\rho_3}\Big)\Big(\frac{\rho_2}{b}-\frac{\rho_1}{k}\Big) - \frac{\tau\delta^2\rho_1}{bk\rho_3}\Big]\int_0^1\theta_x(\varphi_x+\psi)\,dx.
\end{aligned}
$$

Applying Young's inequality, we obtain (3.14). □

Next, we define a Lyapunov functional K and show that it is equivalent to the energy functional E.

Lemma 3.7. *Let $(\varphi, \psi, \theta, q)$ be a solution of system (1.8). Then, the functional*

$$
K(t) := NE(t) + K_1 + N_2K_2 + N_3K_3 + N_4K_4, \tag{3.15}
$$

where N is sufficiently large and N_1, N_2 are positive real numbers to be chosen properly, satisfies

$$
c_1E(t) \le K(t) \le c_2E(t), \tag{3.16}
$$

for c_1 and c_2 two positive constants and

$$\begin{aligned}K'(t) \le &-(\rho_1 - N_2\rho_1\varepsilon)\int_0^1 \varphi_t^2\,dx - \rho_2\int_0^1 \psi_t^2\,dx - (N_2(b-2c\varepsilon)-c)\int_0^1 \psi_x^2\,dx\\ &-\int_0^1 (N_4(\tau-2\varepsilon_1)-k)(\varphi_x+\psi)^2\,dx - \Big(\frac{N_3\rho_3}{2}-\frac{\delta}{2}\Big)\int_0^1 \theta^2\,dx - (N\beta - cN_2 - cN_3 - cN_4)\int_0^1 q^2\,dx\\ &+c\int_0^1 (\psi_t^2 + h^2(\psi_t))\,dx + N_4\frac{b\rho_3}{\delta\rho_1}\Big[\Big(\tau-\frac{\rho_1}{k\rho_3}\Big)\Big(\frac{\rho_2}{b}-\frac{\rho_1}{k}\Big)-\frac{\tau\delta^2\rho_1}{bk\rho_3}\Big]\int_0^1 \theta_x(\varphi_x+\psi)\,dx.\end{aligned} \tag{3.17}$$

Proof. From Lemmas 3.3–3.6, we find

$$\begin{aligned}|K(t)-NE(t)| \le \rho_1\int_0^1 |\varphi\varphi_t|\,dx + (\rho_2+N_2)\int_0^1 |\psi\psi_t|\,dx + N_2\rho_1\int_0^1 |\varphi_t w|\,dx\\ + N_2\tau\delta\int_0^1 |\psi q|\,dx + \tau\rho_3\int_0^1 \Big|q\Big(\int_0^x \theta(t,y)\,dy\Big)\Big|\,dx.\end{aligned}$$

Applying the Young, Poincaré and Cauchy–Schwarz inequalities and the fact that

$$\varphi_x^2 \le 2(\varphi_x+\psi)^2 + 2\psi^2 \le 2(\varphi_x+\psi)^2 + 2c\psi_x^2,$$

we obtain (3.16), and therefore we get

$$K(t) \sim E(t).$$

To prove (3.17), it suffices to differentiate (3.15) and use Lemmas 3.2–3.6. □

Theorem 3.8. *Let us suppose that*

$$\mu = \Big(\tau - \frac{\rho_1}{k\rho_3}\Big)\Big(\frac{\rho_2}{b}-\frac{\rho_1}{k}\Big) - \frac{\tau\delta^2\rho_1}{bk\rho_3} = 0.$$

Then, there exist positive constants k_1, k_2, k_3 and ε_0 such that the energy $E(t)$ associated with (1.8) *satisfies*

$$E(t) \le k_3 H_1^{-1}\Big(k_1\int_0^t \alpha(s)\,ds + k_2\Big) \quad \textit{for all } t \ge 0, \tag{3.18}$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)}\,ds, \quad H_2(t) = tH'(\varepsilon_0 t).$$

Here H_1 is a strictly decreasing and convex function on $(0,1]$ *with* $\lim_{t\to 0} H_1(t) = +\infty$.

Proof. Estimate (3.17), with $\mu = 0$, takes the form

$$\begin{aligned}K'(t) \le &-(\rho_1 - N_2\rho_1\varepsilon)\int_0^1 \varphi_t^2\,dx - \rho_2\int_0^1 \psi_t^2\,dx - (N_2(b-2c\varepsilon)-c)\int_0^1 \psi_x^2\,dx - \int_0^1 (N_4(\tau-2\varepsilon_1)-k)(\varphi_x+\psi)^2\,dx\\ &-\Big(\frac{N_3\rho_3}{2}-\frac{\delta}{2}\Big)\int_0^1 \theta^2\,dx - (N\beta - cN_2 - cN_3 - cN_4)\int_0^1 q^2\,dx + c\int_0^1 (\psi_t^2 + h^2(\psi_t))\,dx.\end{aligned}$$

Now, we choose the constants in the above estimate as follows: first, ε and ε_1 are such that

$$\varepsilon = \frac{1}{2N_2} \quad \text{and} \quad \varepsilon_1 < \frac{\tau}{2}.$$

After that, we choose N, N_2, N_3 and N_4 sufficiently large such that

$$N_2 > \frac{2c}{b}, \quad N_3 > \frac{\delta}{\rho_3}, \quad N_4 > \frac{k}{\tau - 2\varepsilon_1} \quad \text{and} \quad N > \frac{c}{\beta}\Big(\frac{2c}{b} + \frac{\delta}{\rho_3} + \frac{k}{\tau - 2\varepsilon_1}\Big).$$

Then, we deduce that

$$K'(t) \le -dE(t) + c\int_0^1 (\psi_t^2 + h^2(\psi_t))\,dx, \tag{3.19}$$

where

$$d = \min\Big(\rho_1 - N_2\rho_1\varepsilon, \rho_2, N_2(b - 2c\varepsilon) - c, N_4(\tau - 2\varepsilon_1) - k, \frac{N_3\rho_3}{2} - \frac{\delta}{2}, N\beta - cN_2 - cN_3 - cN_4\Big).$$

First case: Let h_0 be a linear function over $[0, \varepsilon]$. The hypothesis (A_2^*) implies that

$$c_1'|s| \le |h(s)| \le c_2'|s| \quad \text{for all } s \in \mathbb{R}.$$

Consequently, by multiplying inequality (3.19) by $\alpha(t)$, we obtain

$$\begin{aligned}
\alpha(t)K'(t) &\le -d\alpha(t)E(t) + c\alpha(t)\int_0^1 (\psi_t^2 + h^2(\psi_t))\,dx \\
&\le -d\alpha(t)E(t) + c\alpha(t)\int_0^1 \Big(\frac{1}{c_1'}|\psi_t h(\psi_t)| + c_2'|\psi_t h(\psi_t)|\Big)\,dx \\
&\le -d\alpha(t)E(t) + c_0\alpha(t)\int_0^1 \psi_t h(\psi_t)dx = -d\alpha(t)E(t) - c_0E'(t),
\end{aligned} \tag{3.20}$$

where $c_0 = c(\frac{1}{c_1'} + c_2')$. Using now hypothesis (A_1), this yields

$$(\alpha K + c_0E)'(t) \le \alpha(t)K'(t) + c_0E'(t) \le -d\alpha(t)E(t). \tag{3.21}$$

We integrate inequality (3.21) and use the fact that $\alpha K + c_0E \sim E$, we obtain for some $k, c > 0$,

$$E(t) \le k\exp\Big(-dc\int_0^t \alpha(s)\,ds\Big). \tag{3.22}$$

Finally, by a simple computation we get (3.18).

Second case: Let h_0 be a nonlinear function over $[0, \varepsilon]$. We assume that $\max(r, h_0(r)) < \varepsilon$, where r is defined in hypothesis (A_2^*). Let $\varepsilon_1 = \min(r, h_0(r))$, we deduce from hypothesis (A_2^*) that

$$\frac{h_0(\varepsilon_1)}{\varepsilon}|s| \le \frac{h_0(|s|)}{|s|}|s| \le |h(s)| \le \frac{h_0^{-1}(|s|)}{|s|}|s| \le \frac{h_0(\varepsilon)}{\varepsilon_1}|s|$$

for all s satisfying $\varepsilon_1 \le |s| \le \varepsilon$. Then, the estimates in hypothesis (A_2^*) become

$$\begin{cases} h_0(|s|) \le |h(s)| \le h_0^{-1}(|s|) & \text{for all } |s| \le \varepsilon_1, \\ c_1'|s| \le |h(s)| \le c_2'|s| & \text{for all } |s| \ge \varepsilon_1, \end{cases} \tag{3.23}$$

and we have

$$s^2 + h^2(s) \le 2H^{-1}(sh(s)). \tag{3.24}$$

To estimate the last term of (3.19), we consider the following partition of (0.1):

$$\Omega_1 = \{x \in (0,1) : |\psi_t| \le \varepsilon_1\}, \quad \Omega_2 = \{x \in (0,1) : |\psi_t| > \varepsilon_1\}.$$

Then, we obtain

$$\psi_t h(\psi_t) \le H(r^2) \quad \text{and} \quad \psi_t h(\psi_t) \le r^2 \quad \text{on } \Omega_1. \tag{3.25}$$

Now, we apply Jensen's inequality to the term

$$I(t) := \frac{1}{|\Omega_1|} \int_{\Omega_1} \psi_t h(\psi_t)\,dx,$$

and we infer that

$$H^{-1}(I(t)) \ge c \int_{\Omega_1} H^{-1}(\psi_t h(\psi_t))\,dx. \tag{3.26}$$

Using (3.23), (3.24) and (3.26), the right-hand side of (3.19) multiplied by $\alpha(t)$ becomes

$$\begin{aligned}
\alpha(t)\int_0^1 (\psi_t^2 + h^2(\psi_t))\,dx &= \alpha(t)\int_{\Omega_1} (\psi_t^2 + h^2(\psi_t))\,dx + \alpha(t)\int_{\Omega_2} (\psi_t^2 + h^2(\psi_t))\,dx\\
&\le 2\alpha(t)\int_{\Omega_1} H^{-1}(\psi_t h(\psi_t))\,dx + \alpha(t)\int_{\Omega_2}\Big(|\psi_t|\frac{1}{c_1'}|h(\psi_t)| + c_2'|\psi_t||h(\psi_t)|\Big)\,dx\\
&\le c\alpha(t)H^{-1}(I(t)) + \alpha(t)c\int_0^1 \psi_t h(\psi_t)\,dx\\
&\le c\alpha(t)H^{-1}(I(t)) - cE'(t).
\end{aligned}$$

Consequently, estimate (3.19) gives

$$R_0'(t) \le -d\alpha(t)E(t) + c\alpha(t)H^{-1}(I(t)), \tag{3.27}$$

where $R_0 = \alpha K + cE$. On the one hand, for $\varepsilon_0 < r^2$, using (3.27), $H' \ge 0$ and $H'' \ge 0$ over $(0, r^2]$ and $E' \le 0$ the functional R_1 defined by

$$R_1(t) := H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big)R_0(t) + c_0E(t),$$

is equivalent to $E(t)$. On the other hand, using the fact that

$$\varepsilon_0\frac{E'(t)}{E(0)}H''\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big)R_0(t) \le 0$$

and (3.27), we conclude that

$$\begin{aligned}
R_1'(t) &= \varepsilon_0\frac{E'(t)}{E(0)}H''\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big)R_0(t) + H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big)R_0'(t) + c_0E'(t)\\
&\le -d\alpha(t)E(t)H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big) + c\alpha(t)H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big)H^{-1}(I(t)) + c_0E'(t).
\end{aligned} \tag{3.28}$$

Our goal now is to estimate the second term in the right-hand side of (3.28). For that purpose, we introduce the convex conjugate H^* of H defined by

$$H^*(s) = s(H')^{-1}(s) - H((H')^{-1}(s)) \quad \text{for } s \in (0, H'(r^2)) \tag{3.29}$$

and H^* satisfies the following Young inequality:

$$AB \le H^*(A) + H(B) \quad \text{for } A \in (0, H'(r^2)),\ B \in (0, r^2). \tag{3.30}$$

Now, taking $A = H'(\varepsilon_0\frac{E(t)}{E(0)})$ and $B = H^{-1}(I(t))$, we obtain

$$\begin{aligned}
R_1'(t) &\le -d\alpha(t)E(t)H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big) + c\alpha(t)H^*\Big(H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big)\Big) + c\alpha(t)H(H^{-1}(I(t)) + c_0E'(t)\\
&\le -d\alpha(t)E(t)H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big) + c\varepsilon_0\frac{E(t)}{E(0)}\alpha(t)H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big) - c\alpha(t)H\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big) + c\alpha(t)I(t) + c_0E'(t)\\
&\le -d\alpha(t)E(t)H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big) + c\varepsilon_0\frac{E(t)}{E(0)}\alpha(t)H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big) - cE'(t) + c_0E'(t).
\end{aligned}$$

With a suitable choice of ε_0 and c_0, from the last inequality we deduce that

$$R_1'(t) \le -(dE(0) - c\varepsilon_0)\alpha(t)\frac{E(t)}{E(0)}H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big) \le -k\alpha(t)H_2\Big(\frac{E(t)}{E(0)}\Big), \tag{3.31}$$

where $k = dE(0) - c\varepsilon_0 > 0$ and $H_2(s) = sH'(\varepsilon_0 s)$. Since $E(t) \sim R_1(t)$, there exist a_1 and a_2 such that

$$a_1R_1(t) \le E(t) \le a_2R_1(t).$$

We set now

$$R(t) = \frac{a_1R_1(t)}{E(0)}.$$

It is clear that $R(t) \sim E(t)$. We use the fact that $H_2'(t),\ H_2(t) > 0$ over $(0, 1]$ (this is due to the fact that H is strictly convex on $(0, r^2]$) and we deduce from (3.31) that

$$R'(t) \le -k_1\alpha(t)H_2(R(t)) \quad \text{for all } t \in \mathbb{R}_+$$

with $k_1 > 0$. By integrating the last inequality, we obtain

$$H_1(R(t)) \ge H_1(R(0)) + k_1\int_0^t \alpha(s)\,ds.$$

Finally, using the fact that H_1^{-1} is decreasing (because H_1 is also), we have

$$R(t) \le H_1^{-1}\Big(k_1\int_0^t \alpha(s)\,ds + k_2\Big) \quad \text{with } k_2 > 0.$$

Taking into account that $E(t) \sim R(t)$, we deduce (3.18). □

3.1.1 Examples

In the following, we will apply inequality (3.18) on some examples in order to show explicit stability results in term of asymptotic profiles in time. For that, we choose the function H strictly convex near zero.

Example 3.9. Let h be a function that satisfies

$$c_3\min(|s|, |s|^p) \le |h(s)| \le c_4\max(|s|, |s|^{\frac{1}{p}})$$

for some $c_3, c_4 > 0$ and $p \ge 1$. For $h_0(s) = cs^p$, hypothesis (A_2^*) is verified. Then, $H(s) = cs^{\frac{p+1}{2}}$. Therefore, we distinguish the following two cases:

- If $p = 1$, we have that h_0 is linear, $H_2(s) = cs$, $H_1(s) = -\frac{\ln(s)}{c}$ and $H_1^{-1}(t) = \exp(-ct)$. Applying (3.18) of Theorem 3.8, we conclude that

$$E(t) \le k_3\exp\Big(-c\Big(k_1\int_0^t \alpha(s)\,ds + k_2\Big)\Big).$$

- If $p > 1$, then this implies that h_0 is nonlinear and we have $H_2(s) = cs^{\frac{p+1}{2}}$ and

$$H_1(t) = \int_t^1 \frac{1}{\delta}s^{-\frac{p-1}{2}}\,ds = \frac{2}{\delta(1-p)} - \frac{2}{\delta(1-p)}t^{\frac{p-1}{2}} \quad \text{with } \delta = c\frac{p+1}{2}\varepsilon_0^{\frac{p-1}{2}}.$$

  Therefore,

$$H_1^{-1}(t) = \Big(\delta\frac{p-1}{2}t + 1\Big)^{-\frac{2}{p-1}}.$$

  Using again (3.18), we obtain

$$E(t) \le H_1^{-1}\Big(k_1\int_0^t \alpha(s)\,ds + k_2\Big) = \Big(\delta\frac{p-1}{2}\Big(k_1\int_0^t \alpha(s)\,ds + k_2\Big) + 1\Big)^{-\frac{2}{p-1}}.$$

Example 3.10. Let $h_0(s) = \exp(-\frac{1}{s})$. This yields $H(s) = \sqrt{s}\exp(-\frac{1}{\sqrt{s}})$ and

$$H_2(s) = \left(\frac{\sqrt{s}}{2\sqrt{\varepsilon_0}} + \frac{1}{2\varepsilon_0}\right)\exp\left(-\frac{1}{\sqrt{\varepsilon_0 s}}\right).$$

Moreover, we have

$$\begin{aligned} H_1(t) &= \int_t^1 \left(\frac{1}{\frac{\sqrt{s}}{2\sqrt{\varepsilon_0}} + \frac{1}{2\varepsilon_0}}\right)\exp\left(\frac{1}{\sqrt{\varepsilon_0 s}}\right) ds \\ &\le \int_t^1 \frac{2\sqrt{\varepsilon_0}}{\sqrt{s}}\exp\left(\frac{1}{\sqrt{\varepsilon_0 s}}\right) ds \\ &\le c\int_t^1 \frac{1}{2s\sqrt{\varepsilon_0 s}}\exp\left(\frac{1}{\sqrt{\varepsilon_0 s}}\right) ds \\ &= c\exp\left(\frac{1}{\sqrt{\varepsilon_0 t}}\right) - c\exp\left(\frac{1}{\sqrt{\varepsilon_0}}\right). \end{aligned}$$

Then,

$$t \le \varepsilon_0^{-1}\left(\ln\left(\frac{H_1(t) + c\exp(\frac{1}{\sqrt{\varepsilon_0 s}})}{c}\right)\right)^{-2}.$$

Replacing t by $H_1^{-1}(k_1\int_0^t \alpha(s)\,ds + k_2)$ in the last inequality, we find

$$H_1^{-1}\left(k_1\int_0^t \alpha(s)\,ds + k_2\right) \le \varepsilon_0^{-1}\left(\ln\left(\frac{k_1\int_0^t \alpha(s)\,ds + k_2 + c\exp(\frac{1}{\sqrt{\varepsilon_0}})}{c}\right)\right)^{-2}.$$

Therefore,

$$E(t) \le k_3\varepsilon_0^{-1}\left(\ln\left(\frac{k_1\int_0^t \alpha(s)\,ds + k_2 + c\exp(\frac{1}{\sqrt{\varepsilon_0}})}{c}\right)\right)^{-2}.$$

Example 3.11. Let $h_0(s) = \frac{1}{s}\exp(-\frac{1}{s^2})$. Following the same steps as in Example 3.10, we find that the energy of (1.8) satisfies

$$E(t) \le \varepsilon\left(\ln\left(\frac{k_1\int_0^t \alpha(s)\,ds + k_2 + c\exp(\frac{1}{\varepsilon_0})}{c}\right)\right)^{-1}.$$

Example 3.12. Let $h_0(s) = \frac{1}{s}\exp(-\frac{1}{4}(\ln s)^2)$. Then, we have $H(s) = \exp(-\frac{1}{4}(\ln s)^2)$,

$$H_2(s) = -\frac{1}{2}\frac{\ln\varepsilon_0 s}{\varepsilon_0}\exp\left(-\frac{1}{4}(\ln\varepsilon_0 s)^2\right) \quad \text{and} \quad H_1(t) = \int_t^1 -2\frac{\varepsilon_0}{\ln\varepsilon_0 s}\exp\left(\frac{1}{4}(\ln\varepsilon_0 s)^2\right) ds.$$

As $\lim_{s\to 0}\frac{4\varepsilon_0^2 s}{(\ln(\varepsilon_0 s))^2} = 0$, it follows that the function $s \mapsto \frac{4\varepsilon_0^2 s}{(\ln(\varepsilon_0 s))^2}$ is bounded on $(0, 1]$, and we infer that

$$H_1(t) \le c\int_t^1 -\frac{1}{2}\frac{\ln\varepsilon_0 s}{\varepsilon_0 s}\exp\left(\frac{1}{4}(\ln s)^2\right) ds = \exp\left(\frac{1}{4}(\ln\varepsilon_0 t)^2\right) - \underbrace{\exp\left(\frac{1}{4}(\ln\varepsilon_0)^2\right)}_{c_1}.$$

Hence, we have

$$t \le \frac{1}{\varepsilon_0}\exp\left(-2(\ln(H_1(t)) + c_1)^{\frac{1}{2}}\right).$$

Replacing t by $H_1^{-1}(k_1\int_0^t \alpha(s)\,ds + k_2)$ in the last inequality, we find

$$E(t) \le k_3 H_1^{-1}\left(k_1\int_0^t \alpha(s)\,ds + k_2\right) = \frac{k_3}{\varepsilon_0}\exp\left(-2\left(\ln k_1\int_0^t \alpha(s)\,ds + k_2 + c_1\right)^{\frac{1}{2}}\right).$$

3.2 The case $\mu \neq 0$ and $\alpha(t) = 1$

This subsection is devoted to the statement and the proof of the stability result for system (1.8) when $\mu \neq 0$ and $\alpha(t) = 1$. We suppose here that the derivative of the function h is bounded. We have the following theorem.

Theorem 3.13. *Let us suppose that conditions* (A_1) *and* (A_2^*) *hold. Then for*

$$\mu = \Big(\tau - \frac{\rho_1}{k\rho_3}\Big)\Big(\frac{\rho_2}{b} - \frac{\rho_1}{k}\Big) - \frac{\tau\delta^2\rho_1}{bk\rho_3} \neq 0,$$

the energy solution of (1.8) *satisfies*

$$E(t) \leq H_2^{-1}\Big(\frac{c}{t}\Big), \tag{3.32}$$

where

$$H_2(t) = tH'(\varepsilon_0 t) \quad \text{with} \quad \lim_{t\to 0} H_2(t) = 0.$$

Proof. Let $(\varphi, \psi, \theta, q)$ be a solution of system (1.8). First, we define

$$E(t) := \frac{1}{2}\int_0^1 (\rho_1\varphi_t^2 + \rho_2\psi_t^2 + b\psi_x^2 + k(\varphi_x + \psi)^2 + \rho_3\theta^2 + \tau q^2)\,dx$$

and

$$\tilde{E}(t) := \frac{1}{2}\int_0^1 (\rho_1\varphi_{tt}^2 + \rho_2\psi_{tt}^2 + b\psi_{tx}^2 + k(\varphi_{tx} + \psi_t)^2 + \rho_3\theta_t^2 + \tau q_t^2)\,dx.$$

Then, the functional E satisfies

$$E'(t) = -\beta\int_0^1 q^2\,dx - \int_0^1 \psi_t h(\psi_t)\,dx \leq 0.$$

Analogously, the functional $\tilde{E}$ satisfies

$$\tilde{E}'(t) = -\beta\int_0^1 q_t^2\,dx - \int_0^1 \psi_{tt}^2 h'(\psi_t)\,dx \leq 0.$$

Using the results in Section 3.1 (recall the expressions of the functionals $K_1, \ldots, K_4$), we have the following lemma.

Lemma 3.14. *Let* $(\varphi, \psi, \theta, q)$ *be a solution of system* (1.8). *Then, the functional*

$$L(t) := N(E(t) + \tilde{E}(t)) + K_1 + N_2K_2 + N_3K_3 + N_4K_4 \tag{3.33}$$

satisfies

$$L'(t) \leq -d'(t) + c\int_0^1 (\psi_t^2 + h^2(\psi_t))\,dx \tag{3.34}$$

for N *large enough and* $d' > 0$.

Proof. By differentiation of (3.33), and using (3.17) and Young's inequality, we obtain

$$L'(t) \leq -dE(t) + c\int_0^1 (\psi_t^2 + h^2(\psi_t))\,dx + c\int_0^1 (\theta_x^2 + (\varphi_x + \psi)^2)\,dx - N\beta\int_0^1 q_t^2\,dx - N\int_0^1 \psi_{tt}^2 h'(\psi_t)\,dx.$$

Now, from (1.8d), we deduce that

$$\int_0^1 \theta_x^2\,dx \leq c\Big(\int_0^1 q^2\,dx + \int_0^1 q_t^2\,dx\Big).$$

Consequently, we get

$$L'(t) \le -d'E(t) + c\int_0^1 (\psi_t^2 + h^2(\psi_t))\,dx - (\beta N - c)\int_0^1 q_t^2\,dx - N\int_0^1 \psi_{tt}^2 h'(\psi_t)\,dx.$$

where $d' = d - c > 0$ and d is the same constant that appears in (3.19). Finally, we choose N large enough and using the monotony of the function h we arrive at (3.34). □

Now, using the following partition of $(0, 1)$ defined in Section 3.1, the right-hand side of (3.34) becomes

$$\int_0^1 (\psi_t^2 + h^2(\psi_t))\,dx = \int_{\Omega_1} (\psi_t^2 + h^2(\psi_t))\,dx + \int_{\Omega_2} (\psi_t^2 + h^2(\psi_t))\,dx.$$

Now, estimates (3.23)–(3.26) imply that

$$\begin{aligned}\int_0^1 (\psi_t^2 + h^2(\psi_t))\,dx &\le 2\int_{\Omega_1} H^{-1}(\psi_t h(\psi_t))\,dx + \int_{\Omega_2} \Big(|\psi_t|\frac{1}{c_1'}|h(\psi_t)| + c_2'|\psi_t||h(\psi_t)|\Big)\,dx \\ &\le cH^{-1}(I(t)) + c\int_0^1 \psi_t h(\psi_t)\,dx.\end{aligned}$$

Consequently,

$$L'(t) \le -d'E(t) + cH^{-1}(I(t)) + c\int_0^1 \psi_t h(\psi_t)\,dx + c\beta\int_0^1 q^2\,dx \le -d'E(t) + cH^{-1}(I(t)) - cE'(t).$$

Hence, we deduce that

$$(L + cE)'(t) \le -d'E(t) + cH^{-1}(I(t)).$$

We then define

$$R_1(t) := H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big)(L + cE)(t) + c_0E(t),$$

which verifies

$$R_1'(t) \le -d_1E(t)H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big) + cH'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big)H^{-1}(I(t)) + \epsilon E'(t),$$

as we have

$$\varepsilon_0\frac{E'(t)}{E(0)}H''\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big)R_0(t) \le 0.$$

We recall the definition of the convex conjugate H^* of H, given by (3.29), which satisfies the following Young inequality:

$$AB \le H^*(A) + H(B) \quad \text{for } A \in (0, H'(r^2)),\ B \in (0, r^2).$$

With the same choice of A and B as in (3.30), we obtain

$$R_1'(t) \le -d_1E(t)H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big) + c\varepsilon_0\frac{E(t)}{E(0)}\alpha(t)H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big) - cE'(t) + \epsilon E'(t).$$

With a suitable choice of ε_0 and ϵ, from the above inequality we deduce that

$$R_1'(t) \le -(dE(0) - c\varepsilon_0)\frac{E(t)}{E(0)}H'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big) \le -k_1H_2\Big(\frac{E(t)}{E(0)}\Big), \tag{3.35}$$

where $k = dE(0) - c\varepsilon_0 > 0$ and $H_2(s) = sH'(\varepsilon_0(s))$.

Finally, we have

$$R_1'(t) \le -k_1H_2\Big(\frac{E(t)}{E(0)}\Big) \quad \text{for all } t \in \mathbb{R}_+$$

with $k_1 > 0$, which yields

$$tH_2\Big(\frac{E(t)}{E(0)}\Big) \le \int_0^t H_2\Big(\frac{E(s)}{E(0)}\Big)\,ds \le -(R_1(t) - R_1(0)) \le R_1(0).$$

Then, we easily deduce that

$$H_2\Big(\frac{E(t)}{E(0)}\Big) \le \frac{R_1(0)}{k_1 t}.$$

Thus,

$$E(t) \le E(0)H_2^{-1}\Big(\frac{R_1(0)}{k_1 t}\Big).$$

This concludes the proof of Theorem 3.13. □

3.2.1 Examples

Example 3.15. Let $h_0(s) = cs^p$. Then $H(s) = cs^{\frac{p+1}{2}}$. Therefore, we distinguish the following two cases:

- If $p = 1$, we have that h_0 is linear and $H_2^{-1}(t) = cs$. Applying (3.32) of Theorem 3.13, we conclude that

$$E(t) \le \frac{c}{t}.$$

- If $p > 1$, then this implies that h_0 is nonlinear and we have $H_2(s) = cs^{\frac{p+1}{2}}$. Therefore,

$$H_2^{-1}(t) = ct^{\frac{2}{p+1}}.$$

  Using (3.32), we obtain

$$E(t) \le ct^{-\frac{2}{p+1}}.$$

Example 3.16. Let h be given by $h(x) = \frac{1}{x^3}\exp(-\frac{1}{x^2})$ and we choose $h_0(x) = \frac{1+x^2}{x^3}\exp(-\frac{1}{x^2})$. We obtain

$$H(x) = \frac{1+x}{x}\exp\Big(-\frac{1}{x}\Big) \quad \text{and} \quad H_2(x) = \frac{\exp(-\frac{1}{\varepsilon_0 x})}{\varepsilon_0^3 x^2}.$$

Then, using the property

$$\lim_{x\to 0^+} \exp\Big(\frac{1}{\varepsilon_0 x}\Big)H_2(x) = +\infty,$$

we deduce that

$$\exp\Big(-\frac{1}{\varepsilon_0 x}\Big) \le H_2(x).$$

We infer that there exists $x_0 > 0$ such that

$$\exp\Big(-\frac{1}{\varepsilon_0 x}\Big) \le H_2(x) \quad \text{on } (0, x_0].$$

Consequently, the energy of the solution of (1.8) satisfies the estimate

$$E(t) \le c(\ln(t))^{-1}.$$

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