

## Research Article

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# Fractional elliptic equations with critical exponential nonlinearity

**Abstract:** We study the existence of positive solutions for fractional elliptic equations of the type

$$(-\Delta)^{\frac{1}{2}} u = h(u), \quad u > 0 \quad \text{in } (-1, 1), \quad u = 0 \quad \text{in } \mathbb{R} \setminus (-1, 1),$$

where  $h$  is a real valued function that behaves like  $e^{u^2}$  as  $u \rightarrow \infty$ . Here  $(-\Delta)^{\frac{1}{2}}$  is the fractional Laplacian operator. We show the existence of mountain-pass solution when the nonlinearity is superlinear near  $t = 0$ . In case  $h$  is concave near  $t = 0$ , we show the existence of multiple solutions for suitable range of  $\lambda$  by analyzing the fibering maps and the corresponding Nehari manifold.

**Keywords:** Trudinger–Moser inequality, square root of laplacian, Nehari manifold

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## 1 Introduction

In this article, we study the existence of positive solutions for fractional elliptic equations of the type

$$(-\Delta)^{\frac{1}{2}} u = h(u), \quad u > 0 \quad \text{in } (-1, 1), \quad u = 0 \quad \text{in } \mathbb{R} \setminus (-1, 1), \quad (P)$$

where the nonlinearity  $h(u)$  satisfies critical growth of exponential type which will be stated later. We also study the following problem for existence and multiplicity of solutions:

$$(-\Delta)^{\frac{1}{2}} u = \lambda f(x) |u|^{q-2} u + g(u), \quad u > 0 \quad \text{in } (-1, 1), \quad u = 0 \quad \text{in } \mathbb{R} \setminus (-1, 1). \quad (Q_\lambda)$$

where  $g(t) = t^p e^{t^\beta}$ ,  $1 < q < 2$ ,  $p > 1$ ,  $1 < \beta \leq 2$ ,  $\lambda > 0$  and the function  $f(x) \in L^{\frac{p+\beta+q}{p+q+\beta-1}}(-1, 1)$ . Here  $(-\Delta)^{\frac{1}{2}}$  is the  $\frac{1}{2}$ -Laplacian operator defined as

$$(-\Delta)^{\frac{1}{2}} u = \int_{\mathbb{R}} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^2} dy \quad \text{for all } x \in \mathbb{R}.$$

The fractional Laplacian operator has been a classical topic in Fourier analysis and nonlinear partial differential equations for a long time. Fractional operators are involved in financial mathematics, where Lévy processes with jumps appear in modeling the asset prices (see [5]). Recently the fractional Laplacian has attracted many researchers. In particular, concerning nonlinear elliptic equations involving fractional operators, the issues of existence and properties of solutions (regularity, multiplicity, a priori bounds and bifurcation phenomena, asymptotic behavior, symmetry, etc.) have been discussed in detail (see for instance [6, 7, 10, 13, 14, 17, 19–24, 29, 30]). The critical exponent problems for the square root of Laplacian have been studied in [8, 28].

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In [30], authors have studied the following problem with convex-concave type nonlinearity

$$(-\Delta)^s u = \lambda f(x)|u|^{q-2}u + |u|^{2_s^*-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \partial\Omega,$$

via Nehari manifold using harmonic extension in the half cylinder  $\Omega \times (0, \infty)$ . The idea of these harmonic extensions was initially introduced and studied in the beautiful work of Caffarelli and Silvestre [7]. In [29], authors studied the square root of Laplacian for exponential type nonlinearities in  $\mathbb{R}^2$ . Here the exponential type nonlinearity is motivated by fractional Moser–Trudinger embedding due to Ozawa [26] (see also [1] for other forms of Trudinger–Moser inequalities). Problems of this type in local settings were studied in [2, 4, 9, 11, 12, 16].

In [17], authors considered the problem in the limiting case  $n = 2$ ,  $s = 1$ ,

$$(-\Delta)^{1/2}u + u = K(x)g(u) \quad \text{in } \mathbb{R},$$

where  $K$  is a real valued positive function and  $g$  has exponential growth and is super-quadratic near 0. Here authors proved the existence of solutions in the subcritical and critical case by studying the corresponding harmonic extension problem under suitable conditions on  $K$  and  $g$ . In the critical case, the existence of non-trivial solutions is obtained in this paper under the condition that the critical exponential growth nonlinearity is globally large enough. In [13], this condition is replaced by the assumption that there exists a Palais–Smale sequence to the associated energy whose level is strictly below the first critical level. In Sections 2 and 3, we improve these results in the critical case by understanding more accurately the first critical level under which the Palais–Smale condition holds. Precisely, in the present paper, Theorems 2.2 and 3.1 state the existence of positive solutions only under condition (h4) (see Section 2) on the asymptotic behavior of the exponential growth nonlinearity. A consequence of these results is that problem (P) admits solutions for nonlinearities  $h$  which behave at infinity as  $ct^p e^{\gamma t^2}$  with  $c, \gamma > 0$  and  $p > -1$ . In the local setting, Adimurthi and Prashanth [3] proved by investigating the radial symmetry case that condition (h4) is in fact sharp to guarantee the existence of nontrivial solutions. To prove Theorems 2.2 and 3.1, we use the Moser functions concentrating on the boundary  $\mathbb{R} \times \{0\}$  in the spirit of [4]. In this regard, assumption (h4) plays an important role to prove the existence of a Palais–Smale sequence whose energy level is strictly below the first critical level where Palais–Smale condition may fail.

Elliptic equations with exponential growth nonlinearities are motivated by the Moser–Trudinger inequality [25], namely

$$\sup_{\|u\|_{H_0^1(\Omega)} \leq 1} \int_{\Omega} e^{\alpha u^2} dx < \infty \quad \text{if and only if} \quad \alpha \leq 4\pi,$$

where  $\Omega \subset \mathbb{R}^2$  bounded domain. The existence of solutions for critical exponent problems was initiated and studied in [2, 4, 9]. Subsequently, these results for non-homogeneous problems with convex-concave nonlinearities have been studied in [11, 12, 27].

The space  $H^{\frac{1}{2}}(\mathbb{R})$  is the Hilbert space with the norm defined as

$$\|u\|_H^2 = \|u\|_{L^2(\mathbb{R})}^2 + \int_{\mathbb{R}} |(-\Delta)^{\frac{1}{4}} u|^2 dx.$$

The space  $H_0^{\frac{1}{2}}(\mathbb{R})$  is the completion of  $C_0^\infty(\mathbb{R})$  under  $[u] = \left(\int_{\mathbb{R}} |(-\Delta)^{\frac{1}{4}} u|^2 dx\right)^{\frac{1}{2}}$ . In case of bounded intervals, say  $I = (-1, 1)$ , the function spaces are defined as

$$X = \{u \in H^{\frac{1}{2}}(\mathbb{R}) : u = 0 \text{ in } \mathbb{R} \setminus (-1, 1)\},$$

equipped with the norm

$$\|u\|_X = [u]_{H^{1/2}(\mathbb{R})} = \sqrt{2\pi} \|(-\Delta)^{\frac{1}{4}} u\|_{L^2(\mathbb{R})}.$$

The problems of type (P) with exponential growth nonlinearities are motivated from the fractional Trudinger–Moser inequality [18], which gives the optimal constant and improves the former results of Ozawa [26], and Kozono, Sato and Wadade [15].

**Theorem 1.1.** *If  $u \in H^{\frac{1}{2}}((-1, 1))$ , then  $e^{au^2} \in L^1((-1, 1))$  for any  $\alpha > 0$ . Moreover, there exists a constant  $C > 0$  such that*

$$\sup_{\|u\|_X \leq 1} \int_0^1 e^{au^2} dx \leq C \quad \text{for all } \alpha \leq \pi.$$

Our approach in the present paper is based on the Caffarelli–Silvestre approach to fractional Laplacian in [7]. In [7], it was shown that for any  $v \in H^{\frac{1}{2}}(\mathbb{R})$ , the unique function  $w(x, y)$  that minimizes the weighted integral

$$\mathcal{E}_{\frac{1}{2}}(w) = \int_0^\infty \int_{\mathbb{R}} |\nabla w(x, y)|^2 dx dy$$

over the set  $\{w(x, y) : \mathcal{E}_{\frac{1}{2}}(w) < \infty, w|_{y=0} = v\}$  satisfies  $\int_{\mathbb{R}} |(-\Delta)^{\frac{1}{2}} v|^2 dx = \mathcal{E}_{\frac{1}{2}}(w)$ . Moreover,  $w(x, y)$  solves the boundary value problem

$$-\operatorname{div}(\nabla w) = 0 \text{ in } \mathbb{R} \times \mathbb{R}_+, \quad w|_{y=0} = v, \quad \frac{\partial w}{\partial \nu} = (-\Delta)^{\frac{1}{2}} v(x),$$

where

$$\frac{\partial w}{\partial \nu} = \lim_{y \rightarrow 0^+} \frac{\partial w}{\partial y}(x, y).$$

In case of bounded domains, in [6], it was observed that the harmonic extension problem is

$$\begin{cases} -\Delta w = 0, \\ w > 0 & \text{in } \mathbb{C} = (-1, 1) \times (0, \infty), \\ w = 0 & \text{on } \{-1, 1\} \times (0, \infty), \\ \frac{\partial w}{\partial \nu} = h(w) & \text{on } (-1, 1) \times \{0\}. \end{cases} \quad (P_E)$$

This can be solved on the space  $H_{0,L}^1(\mathbb{C})$  which is defined as

$$H_{0,L}^1(\mathbb{C}) = \{w \in H^1(\mathbb{C}) : w = 0 \text{ a.e. in } \{-1, 1\} \times (0, \infty)\},$$

equipped with the norm

$$\|w\| = \left( \int_{\mathbb{C}} |\nabla w|^2 dx dy \right)^{\frac{1}{2}}.$$

We have the following “trace embedding” lemma.

**Lemma 1.2.** *Let  $1 \leq q < \infty$ . Then, there exists  $C_q > 0$  such that for all  $v \in H_{0,L}^1(\mathbb{C})$ ,*

$$\left( \int_{-1}^1 |w(x, 0)|^q dx \right)^{\frac{1}{q}} \leq C_q \|w\|.$$

By the compact trace embedding of  $H_{0,L}^1(\mathbb{C})$  into  $L^2((-1, 1))$ , we can show that the minimization problem

$$\lambda_1 = \min_{w \in H_{0,L}^1(\mathbb{C})} \left\{ \int_{\mathbb{C}} |\nabla w|^2 dx dy : \int_{-1}^1 |w(x, 0)|^2 dx = 1 \right\}$$

admits a non-negative minimizer and  $\lambda_1 > 0$ . The Moser–Trudinger trace inequality for  $H_{0,L}^1(\mathbb{C})$  follows from Theorem 1.1: For any  $\alpha \in (0, \pi]$  there exists  $C_\alpha > 0$  such that

$$\sup_{\|w\| \leq 1} \int_{-1}^1 e^{\alpha w(x, 0)^2} dx \leq C_\alpha. \quad (1.1)$$

In this paper, first we discuss the Adimurthi-type existence result [2] for the fractional Laplacian equation in  $(P)$  with nonlinearity  $h(u)$  that has superlinear growth near zero and exponential growth near  $\infty$  (see Theorem 2.2). To prove our result, we analyze the first critical level using the Moser functions which are

dilations and truncations of fundamental solutions in  $\mathbb{R}^2$  and study the compactness of Palais–Smale sequences below this level. In the second part, we discuss the fractional Laplacian equation in  $(Q_\lambda)$  with sign-changing and exponential type nonlinearity to obtain the multiplicity of solutions with respect to the parameter  $\lambda$ . We show the multiplicity result by extracting Palais–Smale sequences in the Nehari manifold. In the critical case, we translate the functional and show the existence of mountain-pass solution (see Theorem 3.1).

The paper is organized as follows. In Section 2, we consider the critical exponent problem with positive nonlinearity and prove Adimurthi-type existence result. In Section 3, we study the problem with convex-concave and sign-changing nonlinearity by the Nehari manifold approach and show the existence of two solutions that arise from the nature of the Nehari manifold.

## 2 Critical exponent problem

In this section, we consider existence of solution for problem (P), where  $h(u)$  satisfies the following critical growth conditions:

(h1)  $h \in C^1(\mathbb{R})$ ,  $h(t) = 0$  for  $t \leq 0$ ,  $h(t) > 0$  for  $t > 0$  and for any  $\epsilon > 0$ ,  $\lim_{t \rightarrow \infty} h(t)e^{-(1+\epsilon)t^2} = 0$ .

(h2) There exists  $\mu > 2$  such that for all  $u > 0$ ,

$$0 \leq \mu H(u) \equiv \mu \int_0^u h(s) ds \leq uh(u).$$

(h3) There exist positive constants  $t_0$  and  $M$  such that

$$H(t) \leq Mh(t) \quad \text{for all } t \in [t_0, +\infty).$$

(h4)  $\lim_{t \rightarrow \infty} th(t)e^{-t^2} = \infty$ .

(h5)  $\limsup_{u \rightarrow 0} \frac{2H(u)}{u^2} < \lambda_1$ .

**Remark 2.1.** The prototype examples of  $h$  satisfying (h1)–(h5) are  $t^p e^{t^2}$  with  $p > 1$  and  $t^p(e^{t^\beta} - 1)e^{t^2}$  with  $p > 1$  and  $\beta \in (0, 2)$ . Nonlinearities of the form  $t^p e^{\beta t^2}$  ( $\beta > 0$ ,  $p > 1$ ) can be also dealt with suitable modifications in the assumptions and minor changes in the proofs. Note that, in this case the first critical level of the energy functional  $I$  is  $\frac{\pi}{2\beta}$ .

The variational functional associated to problem (P) is given as

$$J(u) = \frac{1}{2} \int_{-1}^1 |(-\Delta)^{\frac{1}{4}} u|^2 dx - \int_{-1}^1 G(u) dx.$$

As discussed in the introduction, problem (P) is equivalent to solving  $(P_E)$  on  $H_{0,L}^1(\mathbb{C})$ . The variational functional  $I: H_{0,L}^1(\mathbb{C}) \rightarrow \mathbb{R}$  related to problem  $(P_E)$  is given as

$$I(w) = \frac{1}{2} \int_{\mathbb{C}} |\nabla w|^2 dx dy - \int_{-1}^1 H(w(x, 0)) dx.$$

Any function  $w \in H_{0,L}^1(\mathbb{C})$  is called the weak solution of problem  $(P_E)$  if for any  $\phi \in H_{0,L}^1(\mathbb{C})$ ,

$$\int_{\mathbb{C}} \nabla w \cdot \nabla \phi dx dy - \int_{-1}^1 h(w(x, 0)) \phi(x, 0) dx = 0. \quad (2.1)$$

It is clear that critical points of  $I$  in  $H_{0,L}^1(\mathbb{C})$  corresponds to the critical points of  $J$  in  $H_0^{\frac{1}{2}}(\mathbb{R})$ . Thus, if  $w$  solves  $(P_E)$ , then  $u = \text{trace}(w) = w(x, 0)$  is the solution of problem (P) and vice versa.

With this introduction we state the main result of this section.

**Theorem 2.2.** *Suppose (h1)–(h5) are satisfied. Then, problem (P) has a solution  $u \in C^2((-1, 1))$ .*

## 2.1 Mountain-pass solution

We will use of mountain-pass structure to show the existence of a solution in the critical case.

**Lemma 2.3.** *Assume that conditions (h1)–(h5) hold. Then,  $I$  satisfies the mountain-pass geometry around 0.*

*Proof.* Using assumption (h2), we get

$$H(s) \geq C_1 |s|^\mu - C_2$$

for some  $C_1, C_2 > 0$  and  $\mu > 2$ . Hence, for function  $w \in H_{0,L}^1(\mathbb{C}) \cap C^\infty$  with support in  $[-1, 1] \times (0, 1)$ , we get

$$I(tw) \leq \frac{1}{2} t^2 \|w\|^2 - C_1 t^\mu \int_{-1}^1 |w(x, 0)|^\mu dx + C_3.$$

Hence,  $I(tw) \rightarrow -\infty$  as  $t \rightarrow \infty$ . Next we will show that there exists  $\alpha, \rho > 0$  such that  $I(w) > \alpha$  for all  $\|w\| < \rho$ . From (h1) and (h5), for  $\epsilon > 0$  and  $r > 2$ , there exists  $C_1 > 0$  such that

$$|H(s)| \leq \frac{\lambda_1 - \epsilon}{2} s^2 + C_1 |s|^r e^{(1+\epsilon)s^2}.$$

Hence, using Hölder's inequality, we get for  $t > 1$  and  $t' = \frac{t}{t-1}$ ,

$$\begin{aligned} \int_{-1}^1 |H(w(x, 0))|^2 dx &\leq \frac{\lambda_1 - \epsilon}{2} \int_{-1}^1 |w(x, 0)|^2 dx + C_1 \int_{-1}^1 |w(x, 0)|^r e^{(1+\epsilon)w(x, 0)^2} dx \\ &\leq \frac{\lambda_1 - \epsilon}{2} \|w(x, 0)\|_{L^2(-1, 1)}^2 + C_2 \|w(x, 0)\|_{L^{t'}(-1, 1)}^r \left( \int_{-1}^1 e^{t(1+\epsilon)w(x, 0)^2} dx \right)^{\frac{1}{t}} \\ &\leq \frac{\lambda_1 - \epsilon}{2} \|w\|_{L^2(-1, 1)}^2 + C_2 \|w\|_{L^{t'}(-1, 1)}^r \left( \int_{-1}^1 e^{t(1+\epsilon)\|w\|^2(w(x, 0)/\|w(x, 0)\|)^2} dx \right)^{\frac{1}{t}}. \end{aligned}$$

Now choosing  $\|w\| \leq \rho$  for sufficiently small  $\rho$  and  $t$  close to 1 such that  $t(1+\epsilon)\|w\|^2 < \pi$ , then using the Moser–Trudinger inequality (1.1), we get

$$\begin{aligned} I(w) &\geq \frac{1}{2} \|w\|^2 - \frac{\lambda_1 - \epsilon}{2} \|w\|_{L^2(-1, 1)}^2 - C_3 \|w\|_{L^{t'}(-1, 1)}^r \\ &\geq \frac{1}{2} \left( 1 - \frac{\lambda_1 - \epsilon}{\lambda_1} \right) \|w\|^2 - C_3 \|w\|_{L^{t'}(-1, 1)}^r. \end{aligned}$$

Hence, we get  $\alpha > 0$  such that  $I(w) > \alpha$  for all  $\|w\| \leq \rho$  for sufficiently small  $\rho$ . □

Next we show the boundedness of Palais–Smale sequences.

**Lemma 2.4.** *Every Palais–Smale sequence of  $I$  is bounded in  $H_{0,L}^1(\mathbb{C})$ .*

*Proof.* Let  $\{w_k\}$  be a  $(PS)_c$  sequence. Then,

$$I(w_k) = c + o(1) \quad \text{and} \quad I'(w_k) = o(1) \quad \text{as } k \rightarrow \infty. \quad (2.2)$$

That is, as  $k \rightarrow \infty$ ,

$$\frac{1}{2} \|w_k\|^2 - \int_{-1}^1 H(w_k(x, 0)) dx = c + o(1) \quad \text{and} \quad \|w_k\|^2 - \int_{-1}^1 h(w_k(x, 0)) w_k(x, 0) dx = o(\|w_k\|).$$

Therefore,

$$\left(\frac{1}{2} - \frac{1}{\mu}\right)\|w_k\|^2 - \frac{1}{\mu} \int_{-1}^1 (\mu H(w_k(x, 0)) - h(w_k(x, 0))w_k(x, 0)) dx = c + o(\|w_k\|).$$

Using assumption (h2), we get  $\|w_k\| \leq C$  for some  $C > 0$ . □

We have the following version of compactness lemma based on Vitali's convergence theorem.

**Lemma 2.5.** *For any  $(PS)_c$  sequence  $\{w_k\}$  of  $I$ , there exists  $w_0 \in H_{0,L}^1(\mathbb{C})$  such that, up to subsequence,*

$$h(w_k(x, 0)) \rightarrow h(w_0(x, 0)) \quad \text{and} \quad H(w_k(x, 0)) \rightarrow H(w_0(x, 0)) \quad \text{in } L^1(-1, 1).$$

*Proof.* From Lemma 2.4, we get that the sequence  $\{w_k\}$  is bounded in  $H_{0,L}^1(\mathbb{C})$ . Therefore, up to subsequence,  $w_k \rightharpoonup w_0$  weakly in  $H_{0,L}^1(\mathbb{C})$  for some  $w_0 \in H_{0,L}^1(\mathbb{C})$ . Also from (2.2), there exists  $C > 0$  such that

$$\int_{-1}^1 h(w_k(x, 0))w_k(x, 0) dx \leq C \quad \text{and} \quad \int_{-1}^1 H(w_k(x, 0)) dx \leq C.$$

Now using Vitali's convergence theorem, we get

$$h(w_k(x, 0)) \rightarrow h(w_0(x, 0)) \quad \text{in } L^1(-1, 1).$$

Now to show second part of the above lemma, we use (h3) and the generalized Lebesgue dominated convergence theorem to get

$$H(w_k(x, 0)) \rightarrow H(w_0(x, 0)) \quad \text{in } L^1(-1, 1). \quad \square$$

We need the following version of the “sequence of Moser functions concentrated on the boundary”, see [4].

**Lemma 2.6.** *There exists a sequence  $\{\phi_k\} \subset H_{0,L}^1(\mathbb{C})$  satisfying the following:*

- (i)  $\phi_k \geq 0$ ,  $\text{supp}(\phi_k) \subset B(0, 1) \cap \mathbb{R}_+^2$ ,
- (ii)  $\|\phi_k\| = 1$ ,
- (iii)  $\phi_k$  is constant on  $x \in B(0, \frac{1}{k}) \cap \mathbb{R}_+^2$ , and  $\phi_k^2 = \frac{1}{\pi} \log k$  for  $x \in B(0, \frac{1}{k}) \cap \mathbb{R}_+^2$ .

*Proof.* Let

$$\psi_k(x, y) = \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log k} & 0 \leq \sqrt{x^2 + y^2} \leq 1/k, \\ \frac{\log \frac{1}{\sqrt{x^2 + y^2}}}{\sqrt{\log k}} & 1/k \leq \sqrt{x^2 + y^2} \leq 1, \\ 0 & \sqrt{x^2 + y^2} \geq 1. \end{cases}$$

Then,

$$\int_{\mathbb{R}^2} |\nabla \psi_k|^2 dx dy = 1 \quad \text{and} \quad \int_{\mathbb{R}^2} |\psi_k|^2 dx dy = O\left(\frac{1}{\log k}\right).$$

Let  $\bar{\psi}_k = \psi_k|_{\mathbb{C}}$  and  $\phi_k = \bar{\psi}_k / \|\bar{\psi}_k\|$ . Then,  $\phi_k \geq 0$  and  $\|\phi_k\| = 1$ . Also,

$$\int_{\mathbb{C}} |\nabla \bar{\psi}_k|^2 dx dy = \frac{1}{2} \quad \text{and} \quad \int_{\mathbb{C}} |\bar{\psi}_k|^2 dx dy = O\left(\frac{1}{\log k}\right).$$

Therefore,  $\phi_k^2 = \frac{1}{\pi} \log k$ . □

Define

$$\Gamma = \{\gamma \in C([0, 1]; H_{0,L}^1(\mathbb{C})) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0\}$$

and the corresponding mountain pass level as

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I(\gamma(t)).$$

**Lemma 2.7.** *We have that  $c < \pi/2$ .*

*Proof.* We prove by contradiction. Suppose  $c \geq \pi/2$ . Then, we have

$$\sup_{t \geq 0} I(t\phi_k) = I(t_k\phi_k) \geq \frac{\pi}{2}, \quad (2.3)$$

where the functions  $\phi_k$  are given in Lemma 2.6. From equation (2.3) and since  $h \geq 0$  on  $\mathbb{R}^+$ , we get

$$t_k^2 \geq \pi.$$

Now as  $t_k$  is point of maximum we get  $\frac{d}{dt}I(t\phi_k)|_{t=t_k} = 0$ . Therefore,

$$t_k^2 = t_k^2 \|\phi_k\|^2 = \int_{-1}^1 h(t_k\phi_k(x, 0))t_k\phi_k(x, 0) dx. \quad (2.4)$$

Now we estimate the right-hand side of equation (2.4) using assumption (h4) as follows:

$$\begin{aligned} \int_{-1}^1 h(t_k\phi_k(x, 0))t_k\phi_k(x, 0) dx &\geq \int_{-1/k}^{1/k} h(t_k\phi_k(x, 0))t_k\phi_k(x, 0) dx \\ &\geq \frac{2}{k} h(t_k\phi_k(0, 0))t_k\phi_k(0, 0) \\ &\geq C e^{\frac{1}{\pi}(t_k^2 - \pi) \log k} \frac{h(t_k\phi_k(0, 0))t_k\phi_k(0, 0)}{e^{t_k^2 \phi_k^2(0, 0)}}. \end{aligned}$$

From which it follows that  $t_k^2 \rightarrow \pi$ . Using assumption (h4), the right-hand side of the above expression tends to  $\infty$  as  $k \rightarrow \infty$  which contradicts the boundedness of the left-hand side of (2.4).  $\square$

Next we prove Theorem 2.2 using the above lemmas.

*Proof of Theorem 2.2.* Using Lemma 2.4, there exists a bounded  $(PS)_c$  sequence. So there exists  $w_0 \in H_{0,L}^1(\mathbb{C})$  such that, up to subsequence,  $w_k \rightharpoonup w_0$  in  $H_{0,L}^1(\mathbb{C})$  and  $w_k(x, 0) \rightarrow w_0(x, 0)$  pointwise. We first prove that  $w_0$  solves the problem, then we show that  $w_0$  is nonzero. From Lemma 2.4 and equation (2.2), we get

$$\int_{-1}^1 h(w_k(x, 0))w_k(x, 0) dx < C \quad \text{and} \quad \int_{-1}^1 H(w_k(x, 0)) dx < C.$$

Now from Lemma 2.5, we get  $h(w_k) \rightarrow h(w_0)$  in  $L^1(-1, 1)$ . So for  $\psi \in C_c^\infty$  equation (2.1) holds. Hence, from the density of  $C_c^\infty$  in  $H_{0,L}^1(\mathbb{C})$ ,  $w_0$  is weak solution of  $(P_E)$ .

Next we claim that  $w_0 \neq 0$ . Suppose not. Then, from Lemma 2.5, we get  $H(w_k(x, 0)) \rightarrow 0$  in  $L^1(-1, 1)$ . Hence, from equation (2.2), we get  $\frac{1}{2}\|w_k\|^2 \rightarrow c$  as  $k \rightarrow \infty$  or  $\|w_k\|^2 \leq \pi - \epsilon$  for some  $\epsilon > 0$ . Let  $0 < \delta < \frac{\epsilon}{\pi}$  and  $q = \frac{\pi}{(1+\delta)(\pi-\epsilon)} > 1$ . Using the Moser–Trudinger inequality (1.1), (h1) and Lemma 1.2, we get

$$\begin{aligned} \int_{-1}^1 |h(w_k)w_k|^q dx &\leq A \int_{-1}^1 |w_k(x, 0)|^{2q} dx + C \int_{-1}^1 e^{q(1+\delta)w_k(x, 0)^2} dx \\ &\leq C_1 \|w_k\|^{2q} + C \int_{-1}^1 e^{q(1+\delta)\|w_k\|^2 w_k^2 / \|w_k\|^2} dx < \infty. \end{aligned}$$

Therefore, by

$$\int_{-1}^1 h(w_k(x, 0))w_k(x, 0) dx \rightarrow 0$$

and from equation (2.2), we get  $\lim_k \|w_k\|^2 = 0$ , which is a contradiction. Hence,  $w_0$  is a nontrivial solution of problem  $(P_E)$ . Now by [6, Theorem 5.2], we get  $w_0 \in L^\infty$ .

To show the positivity and regularity of the solution, we proceed as in [6]. By taking odd extension to the whole cylinder  $(-1, 1) \times \mathbb{R}$  and noting that  $w_0, h(w_0) \in L^p(\mathbb{C})$  for any  $p > 1$ , we get, from [6, Proposition 3.1], that  $w_0 \in C^2(\bar{\mathbb{C}})$  for some  $\alpha$ . Therefore,  $u(x) = w_0(x, 0) \in C^2((-1, 1))$ . The positivity of the solution follows from [6, Lemma 4.2].  $\square$

### 3 Convex-concave nonlinearities

In this section, we study the following problem for existence and multiplicity of solutions

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u = \lambda f(x)|u|^{q-2}u + g(u), \\ u > 0 \quad \text{in } (-1, 1), \\ u = 0 \quad \text{in } \mathbb{R} \setminus (-1, 1), \end{cases} \quad (P_\lambda)$$

where  $g(t) = t^p e^{t^\beta}$ ,  $1 < q < 2$ ,  $p > 1$ ,  $1 < \beta \leq 2$ ,  $\lambda > 0$ . The weight function  $f$  satisfies the assumption

$$|f(x)|^{\frac{p+\beta+1}{p+\beta+q-1}} \in L^1((-1, 1)) \quad \text{and} \quad f^\pm(x) \neq 0. \quad (3.1)$$

As discussed in the introduction, this problem is equivalent to the study of

$$\begin{cases} -\operatorname{div}(\nabla w) = 0, \\ w > 0 \quad \text{in } \mathbb{C} = (-1, 1) \times (0, \infty), \\ w = 0 \quad \text{on } \{-1, 1\} \times (0, \infty), \\ \frac{\partial w}{\partial \nu} = \lambda f(x)|w|^{q-2}w + g(w) \quad \text{on } (-1, 1) \times \{0\}, \end{cases} \quad (P_{E\lambda})$$

where  $\frac{\partial w}{\partial \nu} = \lim_{y \rightarrow 0^+} \frac{\partial w}{\partial y}(x, y)$ . The variational functional associated to problem  $(P_\lambda)$  is given as

$$J_\lambda(u) = \frac{1}{2} \int_{-1}^1 |(-\Delta)^{\frac{1}{4}} u|^2 dx - \frac{\lambda}{q} \int_{-1}^1 f(x)|u|^q dx - \int_{-1}^1 G(u) dx.$$

The variational functional  $I_\lambda: H_{0,L}^1(\mathbb{C}) \rightarrow \mathbb{R}$  associated to  $(P_{E\lambda})$  is defined as

$$I_\lambda(w) = \frac{1}{2} \int_{\mathbb{C}} |\nabla w|^2 dx dy - \frac{\lambda}{q} \int_{-1}^1 f(x)|w(x, 0)|^q dx - \int_{-1}^1 G(w(x, 0)) dx.$$

Any function  $w \in H_{0,L}^1(\mathbb{C})$  is called the weak solution of problem  $(P_{E\lambda})$  if for any  $\phi \in H_{0,L}^1(\mathbb{C})$ ,

$$\int_{\mathbb{C}} \nabla w \cdot \nabla \phi dx dy - \lambda \int_{-1}^1 f(x)|w(x, 0)|^{q-2}w(x, 0)\phi(x, 0) dx - \int_{-1}^1 g(w(x, 0))\phi(x, 0) dx = 0.$$

It is clear that the critical points of  $I_\lambda$  in  $H_{0,L}^1(\mathbb{C})$  corresponds to the critical points of  $J_\lambda$  in  $X$ . So as noted in the previous section, we will look for solutions  $w$  of  $(P_{E\lambda})$ .

Existence and multiplicity of solutions for semilinear elliptic equations with sign changing convex-concave type nonlinearities can be investigated using fibering maps and Nehari manifold. We refer to [31], where the Laplacian equation is studied, and to [29], where the  $\frac{1}{2}$ -Laplacian equation is considered. In [29], the author studied the problem with polynomial type nonlinearity. Here we use the same approach as in [12, 27] to study the Nehari manifold for exponential sign changing nonlinearity.

With this introduction we state our main result.

**Theorem 3.1.** *Let  $1 < \beta \leq 2$ . Then, there exists  $\lambda_0 > 0$  such that  $(P_\lambda)$  has at least two solutions for  $\lambda \in (0, \lambda_0)$ .*

### 3.1 The Nehari manifold and fibering maps

Now we consider the Nehari manifold associated to problem  $(P_{E\lambda})$  as

$$N_\lambda = \{w \in H_{0,L}^1(\mathbb{C}) : \langle I'_\lambda(w), w \rangle = 0\}.$$

Thus,  $w \in N_\lambda$  if and only if

$$\|w\|^2 - \lambda \int_{-1}^1 f(x)|w(x,0)|^q dx - \int_{-1}^1 g(w(x,0))w(x,0) dx = 0. \quad (3.2)$$

Now we define fiber map  $\Phi_w: \mathbb{R}^+ \rightarrow \mathbb{R}$  as  $\Phi_w(t) = I_\lambda(tw)$ . Thus,  $tw \in N_\lambda$  if and only if

$$\Phi'_w(t) = t\|w\|^2 - \lambda t^{q-1} \int_{-1}^1 f(x)|w(x,0)|^q dx - \int_{-1}^1 g(tw(x,0))w(x,0) dx = 0.$$

In particular,  $w \in N_\lambda$  if and only if

$$\Phi'_w(1) = \|w\|^2 - \lambda \int_{-1}^1 f(x)|w(x,0)|^q dx - \int_{-1}^1 g(w(x,0))w(x,0) dx = 0.$$

Also,

$$\Phi''_w(1) = \|w\|^2 - (q-1)\lambda \int_{-1}^1 f(x)|w(x,0)|^q dx - \int_{-1}^1 g'(w(x,0))w(x,0)^2 dx. \quad (3.3)$$

We split  $N_\lambda$  into three parts as follows:

$$N_\lambda^\pm = \{w \in N_\lambda : \Phi''_w(1) \gtrless 0\} \quad \text{and} \quad N_\lambda^0 = \{w \in N_\lambda : \Phi''_w(1) = 0\}. \quad (3.4)$$

In order to prove Theorem 3.1, we need following version of [27, Lemma 3.1].

**Lemma 3.2.** *Let*

$$\Gamma = \left\{ w \in H_{0,L}^1(\mathbb{C}) : \|w\|^2 \leq \frac{1}{2-q} \int_{-1}^1 g'(w(x,0))w(x,0)^2 dx \right\}.$$

*Then, there exists a positive  $\lambda_0$  such that*

$$\Gamma_0 := \inf_{w \in \Gamma \setminus \{0\}} \left\{ \int_{-1}^1 (\beta w(x,0)^\beta + p-1)w(x,0)^{p+1} e^{w(x,0)^\beta} dx - (2-q)\lambda \int_{-1}^1 f(x)|w(x,0)|^q dx \right\} > 0 \quad (3.5)$$

*for every  $\lambda \in (0, \lambda_0)$ .*

*Proof.* We divide the proof into three steps.

**Step 1.** We will prove that  $\inf_{w \in \Gamma \setminus \{0\}} \|w\| > 0$ . Suppose that this is not the case. Then, there exists  $\{w_k\}$  in  $H_{0,L}^1(\mathbb{C}) \setminus \{0\}$  such that

$$\|w_k\| \rightarrow 0 \quad \text{and} \quad \|w_k\|^2 \leq \frac{1}{2-q} \int_{-1}^1 g'(w_k(x,0))w_k(x,0)^2 dx. \quad (3.6)$$

Now for  $l > 1$  and from Lemma 1.2,

$$\begin{aligned} \int_{-1}^1 g'(w_k(x,0))w_k(x,0)^2 dx &= \int_{-1}^1 (p + \beta w_k(x,0)^\beta)w_k(x,0)^{p+1} e^{w_k(x,0)^\beta} dx \\ &= p \int_{-1}^1 w_k(x,0)^{p+1} e^{w_k(x,0)^\beta} dx + \beta \int_{-1}^1 w_k(x,0)^{p+1+\beta} e^{w_k(x,0)^\beta} dx \end{aligned}$$

$$\begin{aligned} &\leq C_1 \|w_k\|^{p+1} \left( \int_{-1}^1 e^{l w_k(x,0)^\beta} dx \right)^{\frac{1}{l}} + C_2 \|w_k\|^{p+1+\beta} \left( \int_{-1}^1 e^{l w_k(x,0)^\beta} dx \right)^{\frac{1}{l}} \\ &\leq C_1 \|w_k\|^{p+1} \left( \sup_{\|v_k\| \leq 1} \int_{-1}^1 e^{l \|w_k\|^\beta v_k^\beta} dx \right)^{\frac{1}{l}} + C_2 \|w_k\|^{p+1+\beta} \left( \sup_{\|v_k\| \leq 1} \int_{-1}^1 e^{l \|w_k\|^\beta v_k^\beta} dx \right)^{\frac{1}{l}}. \end{aligned}$$

As  $\|w_k\| \rightarrow 0$ , we can make  $l\|w_k\|^\beta < \pi$ . Hence, from equation (3.6) and the last inequality, we get that  $\|w_k\|^{p-1} \geq C > 0$ , which is a contradiction as  $p > 1$ . Hence, the claim is proved.

**Step 2.** Let

$$C_* = \inf_{w \in \Gamma \setminus \{0\}} \int_{-1}^1 (\beta w^\beta + p - 1) w^{p+1} e^{w^\beta} dx.$$

We will prove that  $C_* > 0$ . Using step 1 and the definition of  $\Gamma$ , we get

$$\inf_{w \in \Gamma \setminus \{0\}} \int_{-1}^1 g'(w_k(x, 0)) w_k(x, 0)^2 dx = \inf_{w \in \Gamma \setminus \{0\}} \int_{-1}^1 (\beta w(x, 0)^\beta + p) w(x, 0)^{p+1} e^{w^\beta} dx > 0.$$

Now multiplying the above integrand by  $\frac{p-1}{p}$ , it is easy to conclude that

$$\inf_{w \in \Gamma \setminus \{0\}} \int_{-1}^1 (\beta w(x, 0)^\beta + p - 1) w(x, 0)^{p+1} e^{w^\beta} dx > 0.$$

**Step 3.** Let  $\lambda < \lambda_0$ . We will prove that equation (3.5) holds. Take  $m > 1$  such that  $mq = p + \beta + 1$ . Then,

$$\begin{aligned} \int_{-1}^1 f(x) |w(x, 0)|^q dx &\leq \left( \int_{-1}^1 |f(x)|^{m'} dx \right)^{\frac{1}{m'}} \left( \int_{-1}^1 |w(x, 0)|^{mq} dx \right)^{\frac{1}{m}} \\ &\leq \frac{C_f^{\frac{m-1}{m}}}{\beta^{\frac{1}{m}}} \left( \int_{-1}^1 (\beta |w(x, 0)|^\beta + p - 1) w(x, 0)^{p+1} e^{w(x, 0)^\beta} dx \right)^{\frac{1}{m}} \\ &\leq \frac{C_f^{\frac{m-1}{m}}}{C_*^{\frac{m-1}{m}} \beta^{\frac{1}{m}}} \left( \int_{-1}^1 (\beta |w(x, 0)|^\beta + p - 1) w(x, 0)^{p+1} e^{w(x, 0)^\beta} dx \right), \end{aligned}$$

where

$$C_f = \|f\|_{L^{\frac{p+\beta+1}{p+\beta+q-1}}((-1,1))}.$$

Thus, if

$$\lambda < \frac{C_f^{\frac{m-1}{m}}}{C_*^{\frac{m-1}{m}} \beta^{\frac{1}{m}}} := \lambda_0,$$

then equation (3.5) holds. □

Now we discuss the behavior of fibering maps  $\Phi_w$  with respect to the sign of  $\int_{-1}^1 f(x) |w(x, 0)|^q dx$ .

**Case 1:**  $\int_{-1}^1 f(x) |w(x, 0)|^q dx \leq 0$ . We define

$$\Psi_w(t) = t^{2-q} \|w\|^2 - t^{1-q} \int_{-1}^1 g(tw(x, 0)) w(x, 0) dx. \quad (3.7)$$

It is clear that  $tw \in N_\lambda$  if and only if  $\Psi_w(t) = \lambda \int_{-1}^1 f(x) |w(x, 0)|^q dx$ . Observe that  $\lim_{t \rightarrow \infty} \Psi_w(t) \rightarrow -\infty$ ,  $\lim_{t \rightarrow \infty} \Psi'_w(t) \rightarrow -\infty$  and  $\lim_{t \rightarrow 0^+} \Psi'_w(t) > 0$ . Hence, there exists a unique  $t_*(w) > 0$  such that  $\Psi_w(t)$  is increasing in  $(0, t_*)$  and decreasing in  $(t_*, \infty)$ . Hence, for all  $\lambda > 0$ , there exists a unique  $t^-(w) > t_*(w)$  such that  $\Psi_w(t^-) = \lambda \int_{-1}^1 f(x) |w(x, 0)|^q dx \leq 0$ . Since  $\Psi'_w(t) < 0$  for  $t > t_*$ , using the relation  $\Phi''_{tw}(1) = t^{q+1} \Psi'_w(t)$ , we get  $t^-w \in N_\lambda^-$ .

Case 2:  $\int_{-1}^1 f(x)|w(x, 0)|^q dx > 0$ . As discussed in case 1,  $\Psi'_w(t_*) = 0$ , we get

$$(2 - q)\|t_* w\|^2 - (1 - q) \int_{-1}^1 g(t_* w(x, 0))t_* w(x, 0) dx - \int_{-1}^1 g'(t_* w(x, 0))(t_* w(x, 0))^2 dx = 0. \quad (3.8)$$

From equation (3.8), it is easy to show that  $t_* w \in \Gamma \setminus \{0\}$ . Now from equation (3.7), we obtain

$$\begin{aligned} \Psi_w(t_*) &= \frac{1}{(2 - q)t_*^q} \left( \int_{-1}^1 g'(t_* w(x, 0))(t_* w(x, 0))^2 - g(t_* w(x, 0))t_* w(x, 0) dx \right) \\ &= \frac{1}{(2 - q)t_*^q} \int_{-1}^1 (\beta(t_* w(x, 0))^\beta + p - 1)(t_* w(x, 0))^{p+1} e^{(t_* w(x, 0))^\beta} dx. \end{aligned}$$

Now from Lemma 3.2,

$$\begin{aligned} \Psi_w(t_*) - \lambda \int_{-1}^1 f(x)|w(x, 0)|^q dx &= \frac{1}{(2 - q)t_*^q} \left( \int_{-1}^1 (\beta(t_* w(x, 0))^\beta + p - 1)(t_* w(x, 0))^{p+1} e^{(t_* w(x, 0))^\beta} dx \right. \\ &\quad \left. - (2 - q) \int_{-1}^1 f(x)|t_* w(x, 0)|^q dx \right) > 0. \end{aligned}$$

Therefore, there exists unique  $t^+(w) < t_*(w) < t^-(w)$  such that  $\Psi'_w(t^+) > 0$ ,  $\Psi'_w(t^-) < 0$  and

$$\Psi_w(t^+) = \Psi_w(t^-) = \lambda \int_{-1}^1 f(x)|w(x, 0)|^q dx.$$

As a consequence  $t^+ w \in N_\lambda^+$  and  $t^- w \in N_\lambda^-$ . Moreover, since  $\Phi'_w(t^-) = \Phi'_w(t^+) = 0$ ,  $\Phi'_w(t) < 0$  on  $(0, t^+)$  and  $\Phi'_w(t) > 0$  on  $(t^+, t^-)$ , we have that  $I_\lambda(t^+ w) = \min_{0 < t < t^-} I_\lambda(tw)$  and  $I_\lambda(t^- w) = \max_{t > t^+} I_\lambda(tw)$ . From the discussion above we have the following lemma.

**Lemma 3.3.** For any  $w \in H_{0,L}^1(\mathbb{C})$ , the following hold:

- (i) There exists a unique  $t^-(w) > 0$  such that  $t^- w \in N_\lambda^-$  for every  $\lambda > 0$ .
- (ii) If  $\int_{-1}^1 f(x)|w(x, 0)|^q dx > 0$ , then there exist unique  $t^+(w) < t_*(w) < t^-(w)$  such that  $t^+ w \in N_\lambda^+$  and  $t^- w \in N_\lambda^-$  for every  $\lambda$  satisfying Lemma 3.2. Moreover,  $I_\lambda(t^+ w) < I_\lambda(tw)$  for any  $t \in [0, t^-]$  such that  $t \neq t^+$  and  $I_\lambda(t^- w) = \max_{t > t^+} I_\lambda(tw)$ .

Now, the next lemma shows that  $N_\lambda$  is a manifold.

**Lemma 3.4.** Let  $\lambda < \lambda_0$  as in Lemma 3.2. Then  $N_\lambda^0 = \{0\}$ .

*Proof.* We prove this lemma by contradiction. Let  $w \in N_\lambda^0$  such that  $w \neq 0$ . Then, from equations (3.3) and (3.4), we get

$$\|w\|^2 - \lambda(q - 1) \int_{-1}^1 f(x)|w(x, 0)|^q dx - \int_{-1}^1 g'(w(x, 0))w(x, 0)^2 dx = 0. \quad (3.9)$$

From equations (3.9) and (3.2), it is easy to show that  $w \in \Gamma$ . Now from equation (3.2) and (3.9), we get

$$\lambda(2 - q) \int_{-1}^1 f(x)|w(x, 0)|^q dx = \int_{-1}^1 (\beta w(x, 0)^\beta + p - 1)w(x, 0)^{p+1} e^{w(x, 0)^\beta} dx.$$

Hence, if  $\lambda \in (0, \lambda_0)$ , we get a contradiction. □

**Lemma 3.5.** The variational functional  $I_\lambda$  is coercive and bounded below on  $N_\lambda$ .

*Proof.* Let  $w \in N_\lambda$ . Using equations (3.2), and the fact that  $g$  satisfies  $(p+1)G(s) < sg(s)$  for all  $s > 0$ , we get

$$\begin{aligned} I_\lambda(w) &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \|w\|^2 - \lambda \left(\frac{1}{q} - \frac{1}{p+1}\right) \int_{-1}^1 f(x) |w(x, 0)|^q dx \\ &\quad - \int_{-1}^1 \left( G(w(x, 0)) - \frac{1}{p+1} g(w(x, 0)) w(x, 0) \right) dx \\ &\geq \left(\frac{1}{2} - \frac{1}{p+1}\right) \|w\|^2 - \lambda \left(\frac{1}{q} - \frac{1}{p+1}\right) \int_{-1}^1 f(x) |w(x, 0)|^q dx. \end{aligned} \quad (3.10)$$

So from Hölder's inequality, Lemma 1.2 and the fact that  $q < 2$ , the lemma is proved.  $\square$

The following lemma shows that the minimizers of  $I_\lambda$  on  $N_\lambda$  are critical points of  $I_\lambda$ .

**Lemma 3.6.** *Let  $w$  be a local minimizer of  $I_\lambda$  in any decompositions of  $N_\lambda$  such that  $w \notin N_\lambda^0$ . Then,  $w$  is a critical point of  $I_\lambda$ .*

*Proof.* If  $w$  minimizes  $I_\lambda$  in  $N_\lambda$ , then by the Lagrange multiplier theorem we get

$$I'_\lambda(w) = \mu C'_\lambda(w), \quad \text{where } C_\lambda(u) = \langle I'_\lambda(w), w \rangle = 0. \quad (3.11)$$

Now

$$\langle I'_\lambda(w), w \rangle = \mu \langle C'_\lambda(w), w \rangle = \mu \Phi''_w(1) = 0.$$

But  $\Phi''_w(1) \neq 0$  as  $w \notin N_\lambda^0$ . Thus,  $\mu = 0$ . Hence, by (3.11), we get  $I'_\lambda(w) = 0$ .  $\square$

**Lemma 3.7.** *Let  $\lambda < \lambda_0$  such that (3.5) holds. Then, given  $w \in N_\lambda \setminus \{0\}$ , there exist  $\epsilon > 0$  and a differentiable function  $\xi: \mathcal{B}(0, \epsilon) \subset H_{0,L}^1(\mathbb{C}) \rightarrow \mathbb{R}$  such that  $\xi(0) = 1$ , the function  $\xi(v)(w - v) \in N_\lambda$  and, for all  $v \in H_{0,L}^1(\mathbb{C})$ ,*

$$\begin{aligned} \langle \xi'(0), v \rangle &= \frac{\int_{-1}^1 \nabla w \nabla v \, dx \, dy - \int_{-1}^1 f(x) |w(x, 0)|^{q-2} w(x, 0) v(x, 0) \, dx - \int_{-1}^1 g(w(x, 0)) v(x, 0) \, dx}{(2-q) \|w\|^2 + (q-1) \int_{-1}^1 g(w(x, 0)) w(x, 0) \, dx - \int_{-1}^1 g'(w(x, 0)) w(x, 0)^2 \, dx}. \end{aligned} \quad (3.12)$$

*Proof.* For fixed  $w \in N_\lambda \setminus \{0\}$ , define  $\mathcal{F}_w: \mathbb{R} \times H_{0,L}^1(\mathbb{C}) \rightarrow \mathbb{R}$  as follows:

$$\mathcal{F}_w(t, v) = t \|w - v\|^2 - t^{q-1} \lambda \int_{-1}^1 f(x) |w(x, 0) - v(x, 0)|^q \, dx - \int_{-1}^1 g(t(w(x, 0) - v(x, 0))) (w(x, 0) - v(x, 0)) \, dx.$$

Then,  $\mathcal{F}_w(1, 0) = 0$  and  $\frac{\partial}{\partial t} \mathcal{F}_w(1, 0) \neq 0$  as  $N_\lambda^0 = \{0\}$ . So we can apply the implicit function theorem to get a differentiable function  $\xi: \mathcal{B}(0, \epsilon) \subset H_{0,L}^1(\mathbb{C}) \rightarrow \mathbb{R}$  such that  $\xi(0) = 1$  and (3.12) is satisfied. Moreover,  $\mathcal{F}_w(\xi(v), v) = 0$  for all  $v \in \mathcal{B}(0, \epsilon)$ . Therefore,

$$\begin{aligned} \|\xi(v)(w - v)\|^2 - \lambda \int_{-1}^1 f(x) |\xi(v)(w(x, 0) - v(x, 0))|^q \, dx \\ - \int_{-1}^1 g(\xi(v)(w(x, 0) - v(x, 0))) \xi(v)(w(x, 0) - v(x, 0)) \, dx = 0. \end{aligned}$$

Hence,  $\xi(v)(w - v) \in N_\lambda$ .  $\square$

We define  $\theta_\lambda := \inf\{I_\lambda(w) | w \in N_\lambda\}$  and we prove the following lemma.

**Lemma 3.8.** *There exists a constant  $C > 0$  such that  $\theta_\lambda < -\frac{(p-q+1)(p-1)}{2q(p+1)} C$ .*

*Proof.* Let  $v \in H_{0,L}^1(\mathbb{C})$  such that  $\int_{-1}^1 f(x)|v(x, 0)|^q dx > 0$ . Then, from Lemma 3.3 (ii), we get a  $w \in N_\lambda^+$ . Thus,

$$I_\lambda(w) \leq \left(\frac{1+q}{2q}\right) \int_{-1}^1 g(w(x, 0))w(x, 0) dx - \frac{1}{2q} \int_{-1}^1 g'(w(x, 0))w(x, 0)^2 dx - \int_{-1}^1 G(w(x, 0)) dx. \quad (3.13)$$

Take

$$\rho(t) = \left(\frac{1+q}{2q}\right)g(t)t - \frac{1}{2q}g'(t)t^2 - G(t).$$

Now using  $g(t) = t^p e^{t^\beta}$ , we get

$$\begin{aligned} \rho'(t) &= \left(\frac{q-1}{2q}\right)g'(t)t - \frac{q-1}{2q}g(t) - \frac{1}{2q}g''(t)t^2 \\ &= \left(\frac{(q-p-1)(p-1)}{2q}\right)g(t) + \beta\left(\frac{q-2p-\beta}{2q}\right)t^\beta g(t) - \frac{\beta^2}{2q}t^{2\beta}g(t), \end{aligned}$$

which implies that  $\rho(t)$  is a decreasing function with  $\rho(0) = 0$ . Therefore,  $\rho(t) \leq 0$  for all  $t \geq 0$ .

Also,

$$\lim_{t \rightarrow 0} \frac{\rho(t)}{|t|^{p+1}} = -\frac{(p-q+1)(p-1)}{2q(p+1)} \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{\rho(t)}{|t|^{p+1+\beta}e^{t^\beta}} = -\frac{\beta}{2q}.$$

Hence, there exists  $M > 0$  independent of  $t$  such that

$$\rho(t) \leq -M \frac{(p-q+1)}{2q(p+1)}(p-1+\beta|t|^\beta)|t|^{p+1}e^{t^\beta}. \quad (3.14)$$

Now using equation (3.14) in (3.13), we get

$$\begin{aligned} I_\lambda(w) &\leq -M \frac{(p-q+1)(p-1)}{2q(p+1)} \int_{-1}^1 (p-1+\beta|w(x, 0)|^\beta)|w|^{p+1}e^{w(x, 0)^\beta} dx \\ &\leq -M \frac{(p-q+1)(p-1)}{2q(p+1)} \int_{-1}^1 |w(x, 0)|^{p+\beta+1} dx \\ &\leq -\frac{(p-q+1)(p-1)}{2q(p+1)}C, \end{aligned}$$

where  $C = M \int_{-1}^1 |w(x, 0)|^{p+\beta+1} dx$ . □

In the next proposition we show the existence of a Palais–Smale sequence.

**Proposition 3.9.** *Let  $\lambda \in (0, \lambda_0)$ . Then, there exists a minimizing sequence  $\{w_k\} \subset N_\lambda$  such that*

$$I_\lambda(w_k) = \theta_\lambda + o_k(1) \quad \text{and} \quad I'_\lambda(w_k) = o_k(1). \quad (3.15)$$

*Proof.* From Lemma 3.5,  $I_\lambda$  is bounded below on  $N_\lambda$ . So by the Ekeland variational principle, there exists a minimizing sequence  $\{w_k\} \in N_\lambda \setminus \{0\}$  such that

$$I_\lambda(w_k) \leq \theta_\lambda + \frac{1}{k}, \quad (3.16)$$

$$I_\lambda(v) \geq I_\lambda(w_k) - \frac{1}{k}\|v - w_k\| \quad \text{for all } v \in N_\lambda. \quad (3.17)$$

Using equation (3.16) and Lemma 3.8, it is easy to show that  $w_k \neq 0$ . From Lemma 3.5, we have that  $\sup_k \|w_k\| < \infty$ . Next we claim that  $\|I'_\lambda(w_k)\| \rightarrow 0$  as  $k \rightarrow \infty$ . Now, using Lemma 3.7 we get the differentiable functions  $\xi_k: \mathcal{B}(0, \epsilon_k) \rightarrow \mathbb{R}$  for some  $\epsilon_k > 0$ , such that  $\xi_k(v)(w_k - v) \in N_\lambda$  for all  $v \in \mathcal{B}(0, \epsilon_k)$ . For fixed  $k$ , choose  $0 < \rho < \epsilon_k$ . Let  $w \in H_{0,L}^1(\mathbb{C})$  with  $w \neq 0$  and let  $v_\rho = \frac{\rho w}{\|w\|}$ . We set  $\eta_\rho = \xi_k(v_\rho)(w_k - v_\rho)$ . Since  $\eta_\rho \in N_\lambda$ , from equation (3.2), we get

$$I_\lambda(\eta_\rho) - I_\lambda(w_k) \geq -\frac{1}{k}\|\eta_\rho - w_k\|.$$

Now by the mean value theorem, we get

$$\langle I'_\lambda(w_k), \eta_\rho - w_k \rangle + o_k(\|\eta_\rho - w_k\|) \geq -\frac{1}{k}\|\eta_\rho - w_k\|.$$

Hence,

$$\langle I'_\lambda(w_k), -v_\rho \rangle + (\xi_k(v_\rho) - 1)\langle I'_\lambda(w_k), (w_k - v_\rho) \rangle \geq -\frac{1}{k}\|\eta_\rho - w_k\| + o_k(\|\eta_\rho - w_k\|)$$

and since  $\langle I'_\lambda(\eta_\rho), (w_k - v_\rho) \rangle = 0$ ,

$$-\rho \left\langle I'_\lambda(w_k), \frac{w}{\|w\|} \right\rangle + (\xi_k(v_\rho) - 1)\langle I'_\lambda(w_k) - I'_\lambda(\eta_\rho), (w_k - v_\rho) \rangle \geq -\frac{1}{k}\|\eta_\rho - w_k\| + o_k(\|\eta_\rho - w_k\|).$$

Thus,

$$\left\langle I'_\lambda(w_k), \frac{w}{\|w\|} \right\rangle \leq \frac{1}{k\rho}\|\eta_\rho - w_k\| + \frac{o_k(\|\eta_\rho - w_k\|)}{\rho} + \frac{(\xi_k(v_\rho) - 1)}{\rho} \langle I'_\lambda(w_k) - I'_\lambda(\eta_\rho), (w_k - v_\rho) \rangle. \quad (3.18)$$

Since  $\|\eta_\rho - w_k\| \leq \rho|\xi_k(v_\rho)| + |\xi_k(v_\rho) - 1|\|w_k\|$  and  $\lim_{\rho \rightarrow 0^+} |\xi_k(v_\rho) - 1|/\rho \leq \|\xi'_k(0)\|$ , taking the limit  $\rho \rightarrow 0^+$  in (3.18), we get

$$\left\langle I'_\lambda(w_k), \frac{w}{\|w\|} \right\rangle \leq \frac{C}{k}(1 + \|\xi'_k(0)\|) \quad (3.19)$$

for some constant  $C > 0$  independent of  $w$ . So if we can show that  $\|\xi'_k(0)\|$  is bounded then we are done. Now from Lemma 3.7, the boundedness of  $\{w_k\}$  and Hölder's inequality, for some  $M > 0$ , we get

$$\langle \xi'(0), v \rangle = \frac{M\|v\|}{(2-q)\|w_k\|^2 + (q-1) \int_{-1}^1 g(w_k(x, 0))w_k(x, 0) dx - \int_{-1}^1 g'(w_k(x, 0))w_k(x, 0)^2 dx}. \quad (3.20)$$

So to prove the claim we only need to prove that

$$(2-q)\|w_k\|^2 + (q-1) \int_{-1}^1 g(w_k(x, 0))w_k(x, 0) dx - \int_{-1}^1 g'(w_k(x, 0))w_k(x, 0)^2 dx$$

is bounded away from zero. Suppose not. Then, there exists a subsequence still denoted  $\{w_k\}$  such that

$$(2-q)\|w_k\|^2 + (q-1) \int_{-1}^1 g(w_k(x, 0))w_k(x, 0) dx - \int_{-1}^1 g'(w_k(x, 0))w_k(x, 0)^2 dx = o_k(1). \quad (3.21)$$

From equation (3.21) and the fact that  $\|w_k\| \geq C > 0$ , we obtain

$$\liminf_{k \rightarrow \infty} \int_{-1}^1 g(w_k(x, 0))w_k(x, 0) dx > 0.$$

Therefore,  $w_k \in \Gamma \setminus \{0\}$  for all  $k$  large. Now using that  $w_k \in N_\lambda$ , equation (3.21) becomes

$$\int_{-1}^1 (\beta w_k^\beta + p-1)w_k(x, 0)^{p+1} e^{w_k(x, 0)^\beta} dx - (2-q)\lambda \int_{-1}^1 f(x)|w_k(x, 0)|^q dx = o_k(1)$$

which contradicts Lemma 3.2 for  $\lambda \in (0, \lambda_0)$ . Hence, the proof of the lemma is completed.  $\square$

### 3.2 Existence and multiplicity results

In this subsection, we show the existence and multiplicity of solutions by minimizing  $I_\lambda$  on nonempty decompositions of  $N_\lambda$  and the generalized mountain-pass theorem, for both subcritical and critical case for suitable range of  $\lambda$ .

**Theorem 3.10.** *Let  $\lambda < \lambda_0$  be such that Lemma 3.2 holds. Then, there exists a function  $w_0 \in N_\lambda^+$  such that  $I_\lambda(w_0) = \inf_{w \in N_\lambda \setminus \{0\}} I_\lambda(w)$ .*

*Proof.* Using Lemma 3.5 and the Ekeland variational principle, we get a minimizing sequence  $\{w_k\}$  in  $N_\lambda^+$  satisfying equation (3.15). From Lemma 3.5, it follows that  $\{w_k\}$  is bounded in  $H_{0,L}^1(\mathbb{C})$ . So up to subsequence,  $w_k \rightharpoonup w_0$  in  $H_{0,L}^1(\mathbb{C})$  and  $w_k(x, 0) \rightarrow w_0(x, 0)$  pointwise a.e. in  $(-1, 1)$ . Now (3.1) implies that

$$\int_{-1}^1 f(x)|w_k(x, 0)|^q dx \rightarrow \int_{-1}^1 f(x)|w_0(x, 0)|^q dx.$$

Also using the compactness of  $w \mapsto \int_{-1}^1 f(x)|w(x, 0)|^q dx$  and the fact that  $w_k \in N_\lambda$ , we get

$$\int_{-1}^1 g(w_k(x, 0)w_k(x, 0)) dx < \infty$$

and from Vitali's convergence theorem,

$$\int_{-1}^1 g(w_k(x, 0)\phi(x, 0)) dx \rightarrow \int_{-1}^1 g(w_0(x, 0)\phi(x, 0)) dx.$$

Hence,  $w_0$  solves  $(P_{E\lambda})$  and  $w_0 \in N_\lambda$ . Moreover,  $w_0$  is a minimizer for  $I_\lambda$  on  $N_\lambda$ . Next we show that  $w_0 \in N_\lambda^+$ . Using Lemma 3.8, (3.10) and  $(p+1)G(s) \leq g(s)s$ , we get  $\int_{-1}^1 f(x)|w_0(x, 0)|^q dx > 0$ . Hence, by Lemma 3.3 (ii), there exists  $t^+(w_0)$  such that  $t^+w_0 \in N_\lambda^+$ . Our claim is that  $t^+(w_0) = 1$ . If  $t^+(w_0) < 1$ , then  $t^-(w_0) = 1$ . So  $w_0 \in N_\lambda^-$ . Now from Lemma 3.3 (ii),  $I_\lambda(t^+(w_0)w_0) < I_\lambda(w_0) = \theta_\lambda$ , which is impossible as  $t^+(w_0)w_0 \in N_\lambda^+$ . This completes the proof of the theorem.  $\square$

**Theorem 3.11.** *Let  $\lambda$  be such that Lemma 3.2 holds. Then, the function  $w_0 \in N_\lambda^+$  is a local minimum of  $I_\lambda(w)$  in  $H_{0,L}^1(\mathbb{C})$ .*

*Proof.* Since  $w_0 \in N_\lambda^+$ , we have  $t^+(w_0) = 1 < t_*(w_0)$ . Hence, by the continuity of  $w \mapsto t_*(w)$ , given  $\epsilon > 0$ , there exists  $\delta = \delta(\epsilon) > 0$  such that  $1 + \epsilon < t_*(w_0 - w)$  for all  $\|w\| < \delta$ . Also, from Lemma 3.7, for  $\delta > 0$  small enough, we obtain a  $C^1$  map  $t: \mathcal{B}(0, \delta) \rightarrow \mathbb{R}^+$  such that  $t(w)(w_0 - w) \in N_\lambda$ ,  $t(0) = 1$ . Therefore, for  $\delta > 0$  small enough, we have  $t^+(w_0 - w) = t(w) < 1 + \epsilon < t_*(w_0 - w)$  for all  $\|w\| < \delta$ . Since  $t_*(w_0 - w) > 1$ , we obtain  $I_\lambda(w_0) \leq I_\lambda(t_1(w_0 - w)(w_0 - w)) \leq I_\lambda(w_0 - w)$  for all  $\|w\| < \delta$ . This shows that  $w_0$  is a local minimizer for  $I_\lambda$  in  $H_{0,L}^1(\mathbb{C})$ .  $\square$

We will show the existence of second solution in the critical case by the generalized mountain-pass theorem. We use the truncation technique to show the existence of a non negative critical point for the transformed functional. It is easy to check that  $w_1 = v + w_0$  solves problem  $(P_{E\lambda})$  if  $v$  solves

$$\begin{cases} -\operatorname{div}(\nabla v) = 0 \\ v \geq 0 \quad \text{in } \mathbb{C}, \\ v = 0 \quad \text{on } \{-1, 1\} \times (0, \infty), \\ \frac{\partial v}{\partial \nu} = \lambda h(x, v + w_0) + g(v + w_0) - \lambda h(x, w_0) - g(w_0) \quad \text{on } (-1, 1) \times \{0\}, \end{cases} \quad (E_\lambda^t)$$

where  $h(x, s) = f(x)|s|^{q-1}$  and  $g(s) = s^p e^{s^2}$ . In order to show the second solution we cut the nonlinearity from the first solution. Set

$$\begin{aligned} \tilde{h}(x, s) &= \begin{cases} h(x, s + w_0(x, 0)) - h(x, w_0(x, 0)) & \text{for } s \geq 0, \\ 0 & \text{for } s < 0, \end{cases} \\ \tilde{g}(s) &= \begin{cases} g(s + w_0(x, 0)) - g(w_0(x, 0)) & \text{for } s \geq 0, \\ 0 & \text{for } s < 0. \end{cases} \end{aligned}$$

Hence, problem  $(E_\lambda^t)$  will be formulated as

$$\begin{cases} -\operatorname{div}(\nabla v) = 0, \\ v \geq 0 \\ \frac{\partial v}{\partial \nu} = \lambda \tilde{h}(x, v) + \tilde{g}(v) \end{cases} \quad \text{in } \mathbb{C}, \quad \text{on } (-1, 1) \times \{0\}. \quad (\tilde{E}_\lambda^t)$$

Define

$$\tilde{I}_\lambda(v) = \frac{1}{2} \|v\|^2 - \lambda \int_{-1}^1 \tilde{H}(x, v(x, 0)) \, dx - \int_{-1}^1 \tilde{G}(v(x, 0)) \, dx.$$

So, if we could find a non-negative critical point of  $\tilde{I}_\lambda$ , then we are through. For that, we use the following generalized definition of Palais–Smale sequence for  $\tilde{I}_\lambda$  at a level  $\rho$  around  $F$  (in short  $(PS)_{F, \rho}$ ):

$$\lim_{k \rightarrow \infty} \operatorname{dist}(v_k, F) = 0, \quad \lim_{k \rightarrow \infty} \tilde{I}_\lambda(v_k) = \rho, \quad \lim_{k \rightarrow \infty} \|\tilde{I}'_\lambda(v_k)\| = 0.$$

In the next proposition we recall the following compactness result similar to Lemma 2.5.

**Proposition 3.12.** *Let  $F \subset H_{0,L}^1(\mathbb{C})$  be a closed set and  $\rho \in \mathbb{R}$ . Let  $\{v_k\} \subset H_{0,L}^1(\mathbb{C})$  be a  $(PS)_{F, \rho}$  sequence. Then, up to a subsequence,  $v_k \rightharpoonup v_0$  in  $H_{0,L}^1(\mathbb{C})$ ,  $\tilde{g}(v_k(x, 0)) \rightarrow \tilde{g}(v_0(x, 0))$  in  $L^1(-1, 1)$  and  $\tilde{G}(v_k(x, 0)) \rightarrow \tilde{G}(v_0(x, 0))$  in  $L^1(-1, 1)$ .*

Note that  $\tilde{I}_\lambda(0) = 0$  and  $v = 0$  is local minimum for  $\tilde{I}_\lambda$  and  $\lim_{t \rightarrow \infty} \tilde{I}_\lambda(tv) = -\infty$ . Hence, we can fix  $e \in H_{0,L}^1(\mathbb{C})$  such that  $\tilde{I}_\lambda(e) < 0$ . Let

$$\Gamma = \{\gamma: [0, 1] \rightarrow H_{0,L}^1(\mathbb{C}) : \gamma \text{ continuous on } [0, 1], \gamma(0) = 0, \gamma(1) = e\}.$$

Define the mountain-pass level  $\rho_0 = \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} \tilde{I}_\lambda(\gamma(t))$ . Note that  $\rho \geq 0$ . Let  $R_0 = \|e\|$ . If  $\rho_0 = 0$ , we obtain that  $\inf\{\tilde{I}_\lambda(v) : \|v\| = R\} = 0$  for all  $R \in (0, R_0)$ . We now let  $F = H_{0,L}^1(\mathbb{C})$  if  $\rho > 0$  and  $F = \{v \in H_{0,L}^1(\mathbb{C}) : \|v\| = \frac{R_0}{2}\}$  if  $\rho = 0$ . Now we have the following lemma.

**Lemma 3.13.** *We have that  $\rho_0 < \pi/2$ .*

*Proof.* We prove the lemma by contradiction. Suppose that  $\rho_0 \geq \pi/2$ . Consider the Moser functions as in the proof of Lemma 2.7. Then, for each  $k$ , there exists a  $t_k$  such that

$$\sup_{t > 0} \tilde{I}_\lambda(t\phi_k) = \tilde{I}_\lambda(t_k\phi_k) \geq \frac{\pi}{2}.$$

This implies that

$$\frac{t_k^2}{2} \|\phi_k\|^2 - \lambda \int_{-1}^1 \tilde{H}(x, t_k\phi_k(x, 0)) \, dx - \int_{-1}^1 \tilde{G}(t_k\phi_k(x, 0)) \, dx \geq \frac{\pi}{2}.$$

Since  $f$  is changing sign, we can assume that  $f > 0$  in the support of  $\phi_k$  (large  $k$  if needed). Hence,

$$\frac{t_k^2}{2} \|\phi_k\|^2 \geq \frac{\pi}{2}.$$

Using the relation from Lemma 2.6 we get

$$t_k^2 \geq \pi.$$

Since  $t_k$  is a point of maximum for  $\tilde{I}_\lambda(t\phi_k)$ , we have that  $\frac{d}{dt} \tilde{I}_\lambda(t\phi_k)|_{t=t_k} = 0$  which implies

$$t_k^2 \|\phi_k\|^2 - \lambda \int_{-1}^1 \tilde{h}(x, t_k\phi_k(x, 0)) t_k\phi_k(x, 0) \, dx - \int_{-1}^1 \tilde{g}(t_k\phi_k(x, 0)) t_k\phi_k(x, 0) \, dx = 0.$$

Therefore,

$$t_k^2 = t_k^2 \|\phi_k\|^2 \geq \int_{-1}^1 \tilde{g}(t_k\phi_k(x, 0)) t_k\phi_k(x, 0) \, dx \geq C t_k^{p+1} (\log k)^{\frac{p+1}{2}} e^{\frac{1}{\pi}(t_k^2 - \pi) \log k},$$

from which it follows that  $t_k^2 \rightarrow \pi$ . Then, the right-hand side of the above expression is unbounded and we get a contradiction.  $\square$

We now prove the following.

**Proposition 3.14.** *The functional  $\tilde{I}_\lambda$  has a non-negative critical point  $v_0 \in H_{0,L}^1(\mathbb{C})$ .*

*Proof.* Let  $\{v_k\}$  be a  $(PS)_{F,\rho}$  sequence for  $\tilde{I}_\lambda$ . Then, up to subsequence, we have that  $v_k \rightharpoonup v_0$  in  $H_{0,L}^1(\mathbb{C})$  and  $v_k(x, 0) \rightarrow v_0(x, 0)$  pointwise almost everywhere in  $(-1, 1)$ . Using Proposition 3.12, it is easy to show that  $v_0$  is a solution of  $(\tilde{E}_\lambda^t)$ . Hence, a critical point of  $\tilde{I}_\lambda$ .

Now we show that  $v_0$  is a non-negative critical point of  $\tilde{I}_\lambda$ . We argue by contradiction. Suppose  $v_0 \equiv 0$ . Consider the following two cases.

If  $\rho_0 = 0$ , then we recall that  $F = \{v \in H_{0,L}^1(\mathbb{C}) : \|v\| = \frac{R_0}{2}\}$  and from Proposition 3.12, we get

$$o_k(1) = \tilde{I}_\lambda(v_k) = \frac{1}{2} \|v_k\|^2 + o_k(1),$$

which contradicts the fact that  $\text{dist}(v_k, F) = o(1)$ .

If  $\rho_0 \in (0, \frac{\pi}{2})$ , then using  $\tilde{I}_\lambda(v_k) = \rho_0 + o(1)$ , we get  $\frac{1}{2} \|v_k\|^2 \rightarrow \rho_0$  as  $k \rightarrow \infty$  or  $\|v_k\|^2 \leq \pi - \epsilon$  for some  $\epsilon > 0$ . Let  $0 < \delta < \frac{\epsilon}{\pi}$  and  $q = \frac{\pi}{(1+\delta)(\pi-\epsilon)} > 1$ . Now following the steps in the proof of Theorem 2.2 we get

$$\sup_k \int_{-1}^1 |\tilde{g}(v_k(x, 0))v_k(x, 0)|^q dx \leq C \sup_k \int_{-1}^1 e^{q\|v_k\|^2 v_k(x, 0)^2 / \|v_k\|^2} dx < \infty.$$

Since  $v_k(x, 0) \rightarrow v_0(x, 0) = 0$  almost everywhere in  $(-1, 1)$  and from Vitali's convergence theorem we get  $\int_{-1}^1 \tilde{g}(v_k(x, 0))v_k(x, 0) \rightarrow 0$  as  $k \rightarrow \infty$ . Hence,  $o(1) = \langle \tilde{I}'_\lambda(v_k), v_k \rangle = \frac{1}{2} \|v_k\|^2 + O(1)$ , which contradicts the fact that  $\|v_k\|^2 \rightarrow \rho_0 > 0$ . So  $v_0$  is a non-negative critical point of  $\tilde{I}_\lambda$  and  $w_1 = v_0 + w_0$  is a second solution to problem  $(P_{E\lambda})$ .  $\square$

**Remark 3.15** (Non-existence). In case of  $f(x) \geq 0, f(x) \not\equiv 0$ , it is not difficult to see that problem  $(Q_\lambda)$  has no solution for large  $\lambda$ . Indeed, if  $u_\lambda$  exists for all  $\lambda$ , then using the inequality

$$(\lambda_1 + \epsilon)s \leq \lambda s^q + g(s) \quad \text{for all } s > 0,$$

we see that  $u_\lambda$  is a super solution to the problem

$$(-\Delta)^{1/2}u = (\lambda_1 + \epsilon)u \quad \text{in } (0, 1), \quad u = 0 \quad \text{in } \mathbb{R} \setminus (0, 1).$$

Now by the monotone iteration method and noting that  $t\phi_1 \leq u_\lambda$  for  $t$  small and  $t\phi_1$  is a subsolution, we can show that the above problem has a solution. This contradicts the isolatedness of smallest eigenvalue [10].

**Remark 3.16** (Global multiplicity). In case of  $f(x) \geq 0, f(x) \not\equiv 0$ . We define  $\Lambda = \sup\{\lambda : (Q_\lambda) \text{ admits solution}\}$ . Then, we can show that problem  $(Q_\lambda)$  admits solution in the whole interval  $(0, \Lambda)$ . Indeed, we can use the monotone iterations:  $n \in \mathbb{N}$ ,

$$(-\Delta)^{1/2}u_{n+1} = \lambda u_n^q + g(u_n) \quad \text{in } (0, 1), \quad u = 0 \quad \text{in } \mathbb{R} \setminus (0, 1),$$

starting with  $u_1 = \epsilon\phi_1$ . It is easy to see that the first eigenfunction  $\phi_1$  is a subsolution of  $(Q_\lambda)$ . This sequence  $\{u_n\}$  is a monotone sequence thanks to the weak maximum principle in [6, Lemma 4.1]. Now by standard weak compactness arguments, one can show that this sequence  $\{u_n\}$  converges to a solution  $u_\lambda$  of  $(Q_\lambda)$ . By [13, Theorem 1.1], this solution is a local minimum in  $H^{\frac{1}{2}}(0, 1)$ . As in Section 3, we can translate the problem to origin, in order to find a second solution in the form  $u_\lambda + v_\lambda$ , where  $v_\lambda$  is a solution of

$$(-\Delta)^{1/2}v = \lambda((v + u_\lambda)^q - u_\lambda) + g(v + u_\lambda) - g(u_\lambda) \quad \text{in } (0, 1), \quad v = 0 \quad \text{in } \mathbb{R} \setminus (0, 1).$$

Now using the mountain-pass lemma around the first solution, one can show the existence of a second positive solution  $v_\lambda$  in the whole interval  $(0, \Lambda)$ . Again by harmonic extension arguments, it is not difficult to show that this problem has solution  $v_\lambda$  for all  $\lambda \in (0, \Lambda)$ .

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