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Research Article

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Solvable subclasses of a class of nonlinear second-order difference equations

Abstract: We present some solvable subclasses of the class of nonlinear second-order difference equations of the form

$$Ax_{n+1} + Bx_{n+1}x_n + Cx_nx_{n-1} + Gx_{n+1}x_{n-1} + Dx_n + Ex_{n-1} + F = 0, \quad n \in \mathbb{N}_0,$$

where the parameters A, B, C, D, E, F, G and the initial values x_{-1}, x_0 are real numbers. This difference equation is a natural extension of the nonhomogeneous linear second-order difference equation with constant coefficients as well as of the bilinear difference equation with constant coefficients.

Keywords: Second-order difference equation, nonlinear solvable difference equation, closed form solution

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1 Introduction

Nonlinear difference equations and systems of difference equations not closely related to differential ones have attracted some recent attention (see, for example, [1–12, 14, 16–23, 28–30, 33–62] and the references therein). Among other problems, such as boundedness, periodicity and stability, the classical problem of solving such equations and systems also re-attracted some attention recently (see, for example, [1–5, 11, 30, 34, 35, 39, 40, 42–51, 53, 55–62] and the related references therein), especially after the author gave in [34] a theoretical explanation for the formula presented in [14]. For some classical methods for solving difference equations and systems see, for example, [25, 27]. For some other classes of difference equations and systems, and their applications, see, for example [15, 16, 25, 26, 31, 32, 35]. Some papers dealing with nonlinear difference equations and systems which are not solvable, use the behavior of solutions of some solvable ones in order to describe the behavior of their solutions (see, for example, [8, 9, 33, 36, 38, 52, 54]), which shows their importance.

Continuing our line of investigations on solvable difference equations and systems (see, for example, [11, 34, 35, 39, 40, 42–51, 53, 55–61]), we consider here a nonlinear second-order difference equation with constant coefficients which, among others, is a natural extension of the nonhomogeneous linear second-order difference equation with constant coefficients as well as of the bilinear difference equation with constant coefficients. Namely, we consider the difference equation

$$Ax_{n+1} + Bx_{n+1}x_n + Cx_nx_{n-1} + Gx_{n+1}x_{n-1} + Dx_n + Ex_{n-1} + F = 0, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where the coefficients A, B, C, D, E, F, G and the initial values x_{-1}, x_0 (or one of them, if (1.1) is reduced by the vanishing of some of the coefficients to a first-order difference equation) are real numbers.

Our motivation stems from the fact that special cases of (1.1) appear often in the literature but there is no unified treatment of the problem of solvability for the case of the general equation (1.1). Apart from this, it seems that many experts are not aware of the fact that, albeit it is quite "rare", there are numerous special cases of (1.1) which can be solved in closed form.

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Our main aim is to present several subclasses of the difference equation (1.1) which are solvable in closed form and to give formulas for their general solutions. In doing this, for the benefit of the reader and for completeness, we will also recall several known methods for solving difference equations. We will mostly present direct, clear and constructive ways for getting the formulas for the general solutions, although they will be sometimes technically complicated. Here and there, we will use the method of induction just to justify some practically "obvious" relations.

Another aim of ours is to also point out and to remind the importance of the *linear first-order difference equation*, that is, of the equation

$$x_n = p_n x_{n-1} + q_n, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $(p_n)_{n\in\mathbb{N}_0}$, $(q_n)_{n\in\mathbb{N}_0}$ are arbitrary real sequences and $x_{-1} \in \mathbb{R}$.

Equation (1.2) can be solved explicitly in several ways. For example, writing down the first $n+1$ equalities obtained from (1.2) (in the order of decreasing indices), then multiplying the kth equality

$$x_{n-k+1} = p_{n-k+1}x_{n-k} + q_{n-k+1}$$

by

$$\prod_{j=n-k+2}^{n} p_j$$

and finally summing up the resulting $n+1$ equalities, gives that the general solution of (1.2) is

$$x_n = x_{-1}\prod_{j=0}^{n} p_j + \sum_{i=0}^{n} q_i \prod_{j=i+1}^{n} p_j, \quad n \geq -1, \tag{1.3}$$

where, as usual, we use the conventions

$$\prod_{j=s+1}^{s} p_j = 1, \quad \sum_{j=l+1}^{l} q_j = 0, \quad s, l \in \mathbb{Z}.$$

For some recent applications of this equation, see, for example, [5, 9, 11, 30, 34, 35, 39, 40, 42–44, 46–48, 50, 56, 58–60].

Remark 1.1. Although the coefficients $(p_n)_{n\in\mathbb{N}_0}$ and $(q_n)_{n\in\mathbb{N}_0}$ in (1.2) as well as the initial value x_{-1} can also be complex numbers, and many considerations in this and in related papers also hold for the complex case, we will restrict our considerations to the real case for practical reasons.

The solution $(x_n)_{n\geq -s}$ of the difference equation

$$x_n = f(x_{n-1}, \dots, x_{n-s}), \quad n \in \mathbb{N}_0,$$

where $f : \mathbb{R}^s \to \mathbb{R}$, $s \in \mathbb{N}$, is called *periodic* with period p if there is an $n_1 \geq -s$ such that

$$x_{n+p} = x_n, \quad n \geq n_1.$$

It is frequently said that such a solution is *eventually periodic* and that it is periodic with period p only if $n_1 = -s$. For some results in this area, see, for example, [6, 10, 16–19, 22–24, 29, 37, 41] and the related references therein.

2 Solvable subclasses of equation (1.1) for the case $G = 0$

In this section, we will present several subclasses of the difference equation

$$Ax_{n+1} + Bx_{n+1}x_n + Cx_nx_{n-1} + Dx_n + Ex_{n-1} + F = 0, \quad n \in \mathbb{N}_0, \tag{2.1}$$

that is, of (1.1) for the case $G = 0$, which can be solved in closed form. Many of them are already known but we will present them for the benefit of the reader and for completeness.

2.1 The case $A = B = C = D = 0$ for $E \neq 0$

In this case, (2.1) is equivalent to the equation

$$x_{n-1} = -\frac{F}{E}, \quad n \in \mathbb{N}_0,$$

that is, the set of solutions consists of a constant sequence.

2.2 The case $A = B = C = E = 0$ for $D \neq 0$

In this case, (2.1) is equivalent to the equation

$$x_n = -\frac{F}{D}, \quad n \in \mathbb{N}_0,$$

that is, the set of solutions also consists of a constant sequence and the main difference with the previous case is that the domain of indices is not $\mathbb{N}_0 \cup \{-1\}$ but $\mathbb{N}_0$.

2.3 The case $B = C = D = E = 0$ for $A \neq 0$

In this case, (2.1) is equivalent to the equation

$$x_{n+1} = -\frac{F}{A}, \quad n \in \mathbb{N}_0,$$

that is, the set of solutions also consists of a constant sequence and the main difference with the previous two cases is that the domain of indices is $\mathbb{N}$.

2.4 The case $A = B = D = E = 0$ for $C \neq 0$

In this case, (2.1) is equivalent to the equation

$$x_n x_{n-1} = -\frac{F}{C}, \quad n \in \mathbb{N}_0. \tag{2.2}$$

If $F = 0$, then we have that $x_n x_{n-1} = 0$, $n \in \mathbb{N}_0$. This means that among two consecutive members of the sequence $(x_n)_{n\geq -1}$, at least one is zero. There are many such sequences. We do not have a closed form formula for the solutions of (2.2) in this case but they can be described in the following way. Each solution $(x_n)_{n\geq -1}$ consists either of all zeros or of a string of zeros followed by a nonzero number (except if $x_{-1} \neq 0$, when it is followed by a string of zeros which is followed by a nonzero number), which is followed by a string of zeros followed by a nonzero number and so on.

If $F \neq 0$, then applying (2.2) twice, it follows that

$$x_n = \frac{-\frac{F}{C}}{x_{n-1}} = x_{n-2}, \quad n \in \mathbb{N},$$

which means that $(x_n)_{n\geq -1}$ is two-periodic, that is,

$$x_{2n-1} = x_{-1}, \quad x_{2n} = -\frac{F}{Cx_{-1}}, \quad n \in \mathbb{N}_0.$$

2.5 The case $C = D = E = F = 0$ for $B \neq 0$

In this case, (2.1) is equivalent to the equation

$$x_{n+1}(A + Bx_n) = 0, \quad n \in \mathbb{N}_0, \tag{2.3}$$

so we do not have a closed form formula for its solutions and we will give a description of them just for completeness.

If $A = 0$, then we have that $x_n x_{n-1} = 0$, $n \in \mathbb{N}$, that is, the sequences which are essentially described in the previous case. If $A \neq 0$, then each solution $(x_n)_{n\in\mathbb{N}_0}$ of (2.3) in this case consists either of zeros and $-A/B$'s or of a string of zeros and $-A/B$'s followed by a number from the set $\mathbb{R} \setminus \{0, -A/B\}$ (except if $x_0 \notin \{-A/B, 0\}$, when it is followed by a string of zeros and $-A/B$'s, which is followed by a number from the set $\mathbb{R} \setminus \{0, -A/B\}$), which is followed by a string of zeros and $-A/B$'s followed by a number from the set $\mathbb{R} \setminus \{0, -A/B\}$, and so on.

2.6 The case $A = C = D = E = 0$ for $B \neq 0$

In this case, (2.1) is equivalent to the equation

$$x_{n+1}x_n = -\frac{F}{B}, \quad n \in \mathbb{N}_0. \tag{2.4}$$

If $F = 0$, then we have that $x_{n+1}x_n = 0$, $n \in \mathbb{N}_0$, and we have a situation similar to the one in the previous two cases.

If $F \neq 0$, then applying (2.4) twice, it follows that

$$x_{n+1} = \frac{-\frac{F}{B}}{x_n} = x_{n-1}, \quad n \in \mathbb{N},$$

which means that $(x_n)_{n\in\mathbb{N}_0}$ is two-periodic, that is,

$$x_{2n+1} = -\frac{F}{Bx_0}, \quad x_{2n} = x_0, \quad n \in \mathbb{N}_0.$$

2.7 The case $A = B = C = 0$ for $D \neq 0$

In this case, (2.1) is equivalent to the equation

$$x_n = -\frac{E}{D}x_{n-1} - \frac{F}{D}, \quad n \in \mathbb{N}_0, \tag{2.5}$$

which is a nonhomogeneous linear first-order difference equation with constant coefficients. It is a special case of (1.2) with

$$p_n = -\frac{E}{D}, \quad q_n = -\frac{F}{D},$$

which is solvable and whose general solution is given by (1.3). Recall here that the equation can also be solved by using the change of variables $x_n = y_n + c$ for a suitable chosen c. Namely, by using this change of variables, (2.5) becomes

$$y_n = -\frac{E}{D}y_{n-1} - c\left(1 + \frac{E}{D}\right) - \frac{F}{D}, \quad n \in \mathbb{N}_0,$$

which for

$$c = -\frac{F}{D+E},$$

when $D \neq -E$, is reduced to the equation

$$y_n = -\frac{E}{D}y_{n-1}, \quad n \in \mathbb{N}_0,$$

whose general solution is

$$y_n = \left(-\frac{E}{D}\right)^{n+1} y_{-1}, \quad n \geq -1.$$

From this and by using the relation

$$y_n = x_n + \frac{F}{D+E},$$

we get

$$x_n = \left(-\frac{E}{D}\right)^{n+1}\left(x_{-1} + \frac{F}{D+E}\right) - \frac{F}{D+E}, \quad n \ge -1,$$

when $D \ne -E$.

When $D = -E$, (2.5) becomes

$$x_n = x_{n-1} - \frac{F}{D},$$

from which it immediately follows that

$$x_n = -\frac{F}{D}(n+1) + x_{-1}, \quad n \ge -1.$$

2.8 The case $B = C = E = 0$ for $A \ne 0$

In this case, (2.1) is equivalent to the equation

$$x_{n+1} = -\frac{D}{A}x_n - \frac{F}{A}, \quad n \in \mathbb{N}_0, \tag{2.6}$$

which is also a nonhomogeneous linear first-order difference equation with constant coefficients. It can be solved either by using (1.3), but with the initial value x_0, or by using the change of variables in the previous case, which gives that

$$x_n = \left(-\frac{D}{A}\right)^{n}\left(x_0 + \frac{F}{D+A}\right) - \frac{F}{D+A}, \quad n \in \mathbb{N}_0,$$

when $D \ne -A$.

When $D = -A$, (2.6) becomes

$$x_{n+1} = x_n - \frac{F}{A}, \quad n \in \mathbb{N}_0,$$

from which it immediately follows that

$$x_n = -\frac{F}{A}n + x_0, \quad n \in \mathbb{N}_0.$$

2.9 The case $B = C = D = 0$ for $A \ne 0$

In this case, (2.1) is equivalent to the equation

$$x_{n+1} = -\frac{E}{A}x_{n-1} - \frac{F}{A}, \quad n \in \mathbb{N}_0.$$

This equation is a nonhomogeneous linear second-order difference equation with constant coefficients, but of a special type. Namely, it is easy to see that the subsequences $(x_{2m+i})_{m\in\mathbb{N}_0}$, $i \in \{-1, 0\}$, are two independent solutions of the nonhomogeneous linear first-order difference equation with constant coefficients

$$z_{m+1} = -\frac{E}{A}z_m - \frac{F}{A}, \quad m \in \mathbb{N}_0. \tag{2.7}$$

Since (2.7) can be solved either by using (1.3) or by using the change of variables $z_m = y_m + c$, $m \in \mathbb{N}_0$, we obtain that

$$x_{2m+i} = \left(-\frac{E}{A}\right)^{m}\left(x_i + \frac{F}{E+A}\right) - \frac{F}{E+A}, \quad m \in \mathbb{N}_0,\ i \in \{-1, 0\},$$

when $E \ne -A$.

When $E = -A$, (2.7) becomes

$$z_{m+1} = z_m - \frac{F}{A}, \quad m \in \mathbb{N}_0,$$

from which it immediately follows that

$$x_{2m+i} = -\frac{F}{A}m + x_i, \quad m \in \mathbb{N}_0,\ i \in \{-1, 0\}.$$

2.10 The case $C = E = F = 0$ for $B \neq 0 \neq A$

In this case, every well-defined solution to (2.1) satisfies the equation

$$x_{n+1} = -\frac{Dx_n}{A + Bx_n}, \quad n \in \mathbb{N}_0. \tag{2.8}$$

Since the case $D = 0$ is trivial, from now on we will also assume that $D \neq 0$. Now, first note that if $x_0 = 0$, then from (2.8) we get $x_n = 0$ for every $n \in \mathbb{N}$. Also, if n_0 is the smallest natural number such that $x_{n_0} = 0$, then from (2.8) it follows that $x_{n_0-1} = 0$, which implies that $n_0 = 1$. Thus, for a well-defined solution $(x_n)_{n\in\mathbb{N}_0}$ of equation (2.8), we have that $x_0 = 0$ is equivalent to $x_n = 0$ for $n \in \mathbb{N}_0$.

Hence, for any well-defined solution $(x_n)_{n\in\mathbb{N}_0}$ of (2.8), we may use the change of variables

$$y_n = \frac{1}{x_n}, \quad n \in \mathbb{N}_0,$$

and transform (2.8) into the equation

$$y_{n+1} = -\frac{A}{D}y_n - \frac{B}{D}, \quad n \in \mathbb{N}_0. \tag{2.9}$$

Now, note that (2.9) is a linear first-order difference equation with constant coefficients. Hence, by using (1.3) we obtain

$$y_n = \left(-\frac{A}{D}\right)^n \left(y_0 + \frac{B}{A + D}\right) - \frac{B}{A + D}, \quad n \in \mathbb{N}_0,$$

when $D \neq -A$, and

$$y_n = -\frac{B}{D}n + y_0, \quad n \in \mathbb{N}_0,$$

when $D = -A$.

From the last two formulas we obtain

$$x_n = \frac{(A + D)x_0}{(-\frac{A}{D})^n(A + D + Bx_0) - x_0B}, \quad n \in \mathbb{N}_0, \tag{2.10}$$

when $D \neq -A$, and

$$x_n = \frac{Dx_0}{D - Bx_0n}, \quad n \in \mathbb{N}_0, \tag{2.11}$$

when $D = -A$.

Remark 2.1. From (2.10) and (2.11) we see that a solution of (2.8) in the case $D \neq -A$ is well defined if and only if

$$x_0 \neq \frac{(-\frac{A}{D})^n(A + D)}{B(1 - (-\frac{A}{D})^n)}, \quad n \in \mathbb{N},$$

while in the case $D = -A$ if and only if

$$x_0 \neq \frac{D}{Bn}, \quad n \in \mathbb{N}.$$

2.11 The case $A = B = F = 0$ for $C \neq 0 \neq D$

In this case, any well-defined solution of (2.1) satisfies the equation

$$x_n = -\frac{Ex_{n-1}}{D + Cx_{n-1}}, \quad n \in \mathbb{N}_0. \tag{2.12}$$

Since the case $E = 0$ is trivial, from now on we will also assume that $E \neq 0$. Similarly as in the previous case, it can be proved that for a well-defined solution $(x_n)_{n\geq-1}$ of (2.12), we have that the condition $x_{-1} = 0$ is equivalent to $x_n = 0$ for $n \geq -1$.

Hence, for any well-defined solution $(x_n)_{n\ge-1}$ of (2.12), we may use the change of variables

$$y_n = \frac{1}{x_n}, \quad n \ge -1,$$

to transform (2.12) into the equation

$$y_n = -\frac{D}{E}y_{n-1} - \frac{C}{E}, \quad n \in \mathbb{N}_0. \tag{2.13}$$

Now, note that (2.13) is a linear first-order difference equation with constant coefficients. Hence, by using (1.3) we obtain

$$y_n = \left(-\frac{D}{E}\right)^{n+1}\left(y_{-1} + \frac{C}{D+E}\right) - \frac{C}{D+E}, \quad n \ge -1,$$

when $D \ne -E$, and

$$y_n = -\frac{C}{E}(n+1) + y_{-1}, \quad n \ge -1,$$

when $D = -E$.

From the last two formulas we obtain

$$x_n = \frac{(D+E)x_{-1}}{(-\frac{D}{E})^{n+1}(D+E+Cx_{-1}) - x_{-1}C}, \quad n \ge -1, \tag{2.14}$$

when $D \ne -E$, and

$$x_n = \frac{Ex_{-1}}{E - Cx_{-1}(n+1)}, \quad n \ge -1, \tag{2.15}$$

when $D = -E$.

Remark 2.2. From (2.14) and (2.15) we see that a solution of (2.12) in the case $D \ne -E$ is well defined if and only if

$$x_{-1} \ne \frac{(-\frac{D}{E})^{n+1}(D+E)}{C(1-(-\frac{D}{E})^{n+1})}, \quad n \in \mathbb{N}_0,$$

while in the case $D = -E$ if and only if

$$x_{-1} \ne \frac{E}{C(n+1)}, \quad n \in \mathbb{N}_0.$$

2.12 The case $B = C = 0$ for $A, D, E \in \mathbb{R} \setminus \{0\}$

In this case, (2.1) is equivalent to the equation

$$x_{n+1} + \frac{D}{A}x_n + \frac{E}{A}x_{n-1} = -\frac{F}{A}, \quad n \in \mathbb{N}_0, \tag{2.16}$$

which is a nonhomogeneous linear second-order difference equation with constant coefficients. There is a well-known standard method for solving (2.16) (see, for example, [27]). Here, we present a direct method for solving (2.16), that is, a method which does not use the theory of linear difference equations, and we make some comments which could be little known to the experts.

Recall that if λ_1 and λ_2 are the zeros of the characteristic polynomial

$$P_2(\lambda) = \lambda^2 + \frac{D}{A}\lambda + \frac{E}{A},$$

that is,

$$\lambda_{1,2} = \frac{-\frac{D}{A} \pm \sqrt{\frac{D^2-4AE}{A^2}}}{2},$$

then the solution of the homogeneous linear second-order difference equation

$$y_{n+1} + \frac{D}{A}y_n + \frac{E}{A}y_{n-1} = 0, \quad n \in \mathbb{N}_0, \tag{2.17}$$

with initial values y_{-1}, y_0 is

$$y_n = \frac{\lambda_2 y_{-1} - y_0}{\lambda_2 - \lambda_1}\lambda_1^{n+1} + \frac{y_0 - \lambda_1 y_{-1}}{\lambda_2 - \lambda_1}\lambda_2^{n+1}, \quad n \ge -1, \tag{2.18}$$

when $\lambda_1 \ne \lambda_2$, while if $\lambda_1 = \lambda_2$, the solution is

$$y_n = (y_0(n+1) - n\lambda_1 y_{-1})\lambda_1^n, \quad n \ge -1. \tag{2.19}$$

To see how (2.18) and (2.19) can be obtained in a direct way, we refer to, for example, [51] (this is an old idea, see, for example, [27]).

Now, note that from (2.16) it follows that

$$x_{n+2} + \frac{D}{A}x_{n+1} + \frac{E}{A}x_n = -\frac{F}{A} = x_{n+1} + \frac{D}{A}x_n + \frac{E}{A}x_{n-1}, \quad n \in \mathbb{N}_0,$$

which implies

$$x_{n+2} - x_{n+1} + \frac{D}{A}(x_{n+1} - x_n) + \frac{E}{A}(x_n - x_{n-1}) = 0, \quad n \in \mathbb{N}_0,$$

so that the sequence

$$b_n = x_{n+1} - x_n, \quad n \ge -1,$$

is the solution of (2.17) with initial values b_{-1}, b_0.

Hence, if $1 \ne \lambda_1 \ne \lambda_2 \ne 1$, by using (2.18) we have that

$$x_n - x_{n-1} = b_{n-1} = \frac{b_{-1}\lambda_2 - b_0}{\lambda_2 - \lambda_1}\lambda_1^n + \frac{b_0 - \lambda_1 b_{-1}}{\lambda_2 - \lambda_1}\lambda_2^n, \quad n \in \mathbb{N}_0. \tag{2.20}$$

By summing up (2.20) from 0 to n and by some simple calculations, we get

$$x_n = x_{-1} + \frac{(x_{-1}\lambda_2 - x_0)(\lambda_1 - 1) + \frac{F}{A}}{\lambda_2 - \lambda_1}\frac{\lambda_1^{n+1} - 1}{\lambda_1 - 1} + \frac{(x_0 - x_{-1}\lambda_1)(\lambda_2 - 1) - \frac{F}{A}}{\lambda_2 - \lambda_1}\frac{\lambda_2^{n+1} - 1}{\lambda_2 - 1}. \tag{2.21}$$

For the case when one of the numbers λ_1 or λ_2 is equal to 1, say λ_2, then from (2.20) we obtain

$$x_n = x_{-1} + \frac{(x_0 - x_{-1})(\lambda_1 - 1) - \frac{F}{A}}{\lambda_1 - 1}\frac{\lambda_1^{n+1} - 1}{\lambda_1 - 1} + \frac{F}{A(\lambda_1 - 1)}(n+1). \tag{2.22}$$

If $\lambda_1 = \lambda_2 \ne 1$, by using (2.19) we have that

$$x_n - x_{n-1} = b_{n-1} = ((b_0 - \lambda_1 b_{-1})n + \lambda_1 b_{-1})\lambda_1^{n-1}, \quad n \in \mathbb{N}_0. \tag{2.23}$$

By summing up equations (2.23) from 0 to n and by some simple calculations, we get

$$x_n = x_{-1} + \left((x_0 - \lambda_1 x_{-1})(\lambda_1 - 1) - \frac{F}{A}\right)\frac{1 - (n+1)\lambda_1^n + n\lambda_1^{n+1}}{(1 - \lambda_1)^2} + (x_0 - x_{-1})\frac{\lambda_1^{n+1} - 1}{\lambda_1 - 1}. \tag{2.24}$$

If $\lambda_1 = \lambda_2 = 1$, then from (2.23) we have that

$$x_n = x_0 + \left(x_0 - x_{-1} - \frac{F}{2A}\right)n - \frac{Fn^2}{2A}. \tag{2.25}$$

Remark 2.3. The solution in (2.21) for the case $1 \ne \lambda_1 \ne \lambda_2 \ne 1$ can be written in the form

$$x_n = \frac{\lambda_2 x_{-1} - x_0}{\lambda_2 - \lambda_1}\lambda_1^{n+1} + \frac{x_0 - \lambda_1 x_{-1}}{\lambda_2 - \lambda_1}\lambda_2^{n+1} + \frac{\frac{F}{A}}{\lambda_2 - \lambda_1}\left(\frac{\lambda_1^{n+1} - 1}{\lambda_1 - 1} - \frac{\lambda_2^{n+1} - 1}{\lambda_2 - 1}\right), \quad n \ge -1,$$

which is a natural decomposition of the solution as a sum of the solution

$$x_n^h = \frac{\lambda_2 x_{-1} - x_0}{\lambda_2 - \lambda_1}\lambda_1^{n+1} + \frac{x_0 - \lambda_1 x_{-1}}{\lambda_2 - \lambda_1}\lambda_2^{n+1}, \quad n \ge -1,$$

of the homogeneous equation with initial values x_{-1}, x_0 and the solution

$$x_n^p = \frac{\frac{F}{A}}{\lambda_2 - \lambda_1}\left(\frac{\lambda_1^{n+1} - 1}{\lambda_1 - 1} - \frac{\lambda_2^{n+1} - 1}{\lambda_2 - 1}\right), \quad n \geq -1,$$

of the nonhomogeneous equation (2.16) with initial conditions $x_{-1} = x_0 = 0$, that is,

$$x_n = x_n^h + x_n^p, \quad n \geq -1. \tag{2.26}$$

The solution in (2.22) for the case $\lambda_1 \neq 1$, $\lambda_2 = 1$ can be written in the form (2.26), where

$$x_n^h = \frac{x_{-1} - x_0}{1 - \lambda_1}\lambda_1^{n+1} + \frac{x_0 - \lambda_1 x_{-1}}{1 - \lambda_1}, \quad n \geq -1,$$

is the solution of the homogeneous equation with initial values x_{-1}, x_0 and

$$x_n^p = \frac{\frac{F}{A}}{1 - \lambda_1}\left(\frac{\lambda_1^{n+1} - 1}{\lambda_1 - 1} - (n+1)\right), \quad n \geq -1,$$

is the solution of the nonhomogeneous equation (2.16) with initial conditions $x_{-1} = x_0 = 0$.

When $\lambda_1 = \lambda_2 \neq 1$, (2.24) can be also written as a sum of the solution

$$x_n^h = ((x_0 - \lambda_1 x_{-1})n + x_0)\lambda_1^n, \quad n \geq -1,$$

of the homogeneous equation with initial values x_{-1}, x_0 and the solution

$$x_n^p = -\frac{F}{A} \cdot \frac{1 - (n+1)\lambda_1^n + n\lambda_1^{n+1}}{(1 - \lambda_1)^2}, \quad n \geq -1,$$

of the nonhomogeneous equation (2.16) with initial conditions $x_{-1} = x_0 = 0$.

Finally, when $\lambda_1 = \lambda_2 = 1$, (2.25) can be also written as a sum of the solution

$$x_n^h = x_0 + (x_0 - x_{-1})n, \quad n \geq -1,$$

of the homogeneous equation with initial values x_{-1}, x_0 and the solution

$$x_n^p = -\frac{Fn(n+1)}{2A}, \quad n \geq -1,$$

of the nonhomogeneous equation (2.16) with initial conditions $x_{-1} = x_0 = 0$.

2.13 The case $A = B = 0$ for $C \neq 0$

In this case, every well-defined solution to (2.1) satisfies the equation

$$x_n = -\frac{Ex_{n-1} + F}{Cx_{n-1} + D}, \quad n \in \mathbb{N}_0, \tag{2.27}$$

which is the bilinear difference equation with constant coefficients, which is also solvable in closed form. A detailed study of this difference equation can be found, for example, in [13] (see also [51]). We will briefly explain how (2.27) can be solved. First, note that it can be written in the form

$$x_n = -\frac{E}{C} + \frac{1}{C}\frac{ED - CF}{Cx_{n-1} + D}, \quad n \in \mathbb{N}_0,$$

from which it follows that

$$Cx_n + D = D - E + \frac{ED - CF}{Cx_{n-1} + D}, \quad n \in \mathbb{N}_0. \tag{2.28}$$

Since for a well-defined solution of (2.27) it must be

$$Cx_{n-1} + D \neq 0, \quad n \in \mathbb{N}_0,$$

we can use the change of variables

$$b_{n-1} = \frac{1}{Cx_{n-1} + D}, \quad n \in \mathbb{N}_0, \tag{2.29}$$

in (2.28) and obtain

$$b_n = \frac{1}{D - E + (ED - CF)b_{n-1}}, \quad n \in \mathbb{N}_0. \tag{2.30}$$

If we further use the change of variables

$$b_n = \frac{c_n}{c_{n+1}}, \quad n \geq -1, \tag{2.31}$$

in (2.30), we get

$$c_{n+1} - (D - E)c_n + (CF - ED)c_{n-1} = 0, \quad n \in \mathbb{N}_0,$$

which is a homogeneous second-order difference equation with constant coefficients, whose solutions with initial values c_{-1}, c_0 can be calculated by using (2.18) and (2.19).

By using such formulas for $(c_n)_{n\geq -1}$ in the relation

$$x_n = \frac{1}{C}\left(\frac{c_{n+1}}{c_n} - D\right), \quad n \in \mathbb{N}_0,$$

which follows from (2.29) and (2.31), we obtain the general solution of (2.27).

2.14 The case $C = E = 0$ for $B \neq 0$

In this case, every well-defined solution to (2.1) satisfies the equation

$$x_{n+1} = -\frac{Dx_n + F}{Bx_n + A}, \quad n \in \mathbb{N}_0,$$

which is a bilinear difference equation with constant coefficients and can be solved as in the previous case. Hence, we omit the details.

2.15 The case $B = D = E = F = 0$ for $A \neq 0 \neq C$

In this case, (2.1) can be written in the form

$$x_{n+1} = -\frac{Cx_n x_{n-1}}{A}, \quad n \in \mathbb{N}_0. \tag{2.32}$$

By using the change of variables

$$y_n = -\frac{Cx_n}{A}, \quad n \geq -1, \tag{2.33}$$

we transform (2.32) into

$$y_{n+1} = y_n y_{n-1}, \quad n \in \mathbb{N}_0. \tag{2.34}$$

Let

$$a_1 = 1, \quad b_1 = 1 \tag{2.35}$$

and rewrite (2.34) into the form

$$y_{n+1} = y_n^{a_1} y_{n-1}^{b_1}, \quad n \in \mathbb{N}_0. \tag{2.36}$$

Using (2.34) in (2.36) we get

$$y_{n+1} = (y_{n-1}y_{n-2})^{a_1} y_{n-1}^{b_1} = y_{n-1}^{a_1+b_1} y_{n-2}^{a_1} = y_{n-1}^{a_2} y_{n-2}^{b_2}, \quad n \in \mathbb{N},$$

where

$$a_2 := a_1 + b_1, \quad b_2 := a_1.$$

Assume that for some $k \ge 2$ we have

$$y_{n+1} = y_{n-k+1}^{a_k} y_{n-k}^{b_k}, \quad n \ge k-1, \tag{2.37}$$

where

$$a_k = a_{k-1} + b_{k-1}, \quad b_k = a_{k-1} \tag{2.38}$$

and (2.35) holds.

By using (2.34) into (2.37), we have that

$$y_{n+1} = (y_{n-k}y_{n-k-1})^{a_k} y_{n-k}^{b_k} = y_{n-k}^{a_k+b_k} y_{n-k-1}^{a_k} = y_{n-k}^{a_{k+1}} y_{n-k-1}^{b_{k+1}}, \quad 2 \le k \le n,$$

where the sequences $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ satisfy the system of difference equations

$$a_{k+1} = a_k + b_k, \quad b_{k+1} = a_k, \quad k \in \mathbb{N}, \tag{2.39}$$

with the initial conditions (2.35). Hence, we have proved by induction that (2.37) holds for every $k \in \mathbb{N}$ and $n \ge k-1$, where the sequences $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ satisfy (2.38).

From (2.39) it follows that

$$a_{k+1} = a_k + a_{k-1}, \quad k \ge 2,$$

with $a_1 = 1$, $a_2 = 2$, while $b_k = a_{k-1}$, $k \ge 2$, with $b_1 = 1$. Hence, $b_k = f_k$ and $a_k = f_{k+1}$, where $(f_k)_{k\in\mathbb{N}}$ is the Fibonacci sequence, and consequently

$$y_n = y_0^{a_n} y_{-1}^{b_n} = y_0^{f_{n+1}} y_{-1}^{f_n}, \quad n \ge -1,$$

where, as usual, we regard that $f_0 = f_2 - f_1 = 0$ and $f_{-1} = f_1 - f_0 = 1$.

From this and (2.33) it follows that

$$x_n = -\frac{A}{C} y_n = \left(-\frac{C}{A}\right)^{f_{n+1}+f_n-1} x_0^{f_{n+1}} x_{-1}^{f_n} = \left(-\frac{C}{A}\right)^{f_{n+2}-1} x_0^{f_{n+1}} x_{-1}^{f_n}, \quad n \ge -1,$$

which is the general solution to (2.32).

Remark 2.4. It is a common situation that mathematicians, including some experts, try to solve equations of the type related to (2.32) by taking the logarithm and reducing them to linear difference equations with constant coefficients. However, for some values of the parameters and the initial values, the values of the solutions can be negative and so the method for solving the equations essentially does not work. For the case of (2.34), such a situation appears if $y_{-1}y_0 < 0$.

2.16 The case $B = 0$ for $A, C, D, E, F \in \mathbb{R} \setminus \{0\}$

In this case, (2.1) can be written in the form

$$x_{n+1} = -\frac{Cx_n x_{n-1} + Dx_n + Ex_{n-1} + F}{A}, \quad n \in \mathbb{N}_0. \tag{2.40}$$

Equation (2.40) seems not to be solvable in closed form for all values of the parameters A, C, D, E, F. Hence, we will give some sufficient conditions for which a subclass of the equation is solvable.

From (2.40) it follows that

$$\begin{aligned} x_{n+1}+g &= -\frac{Cx_nx_{n-1}+Dx_n+Ex_{n-1}+F-gA}{A} \\ &= -\frac{C}{A}\Big(x_nx_{n-1}+\frac{D}{C}x_n+\frac{E}{C}x_{n-1}+\frac{F-gA}{C}\Big) \\ &= -\frac{C}{A}\Big(x_n\Big(x_{n-1}+\frac{D}{C}\Big)+\frac{E}{C}\Big(x_{n-1}+\frac{F-gA}{E}\Big)\Big) \end{aligned} \tag{2.41}$$

for $n \in \mathbb{N}_0$.

If we choose constants A, C, D, E, F such that

$$g=\frac{D}{C}=\frac{E}{C}=\frac{F-gA}{E},$$

then (2.41) is reduced to

$$x_{n+1}+\frac{D}{C}=-\frac{C}{A}\Big(x_n+\frac{D}{C}\Big)\Big(x_{n-1}+\frac{D}{C}\Big),$$

which by the change of variables

$$y_n=x_n+\frac{D}{C}, \quad n\ge -1, \tag{2.42}$$

becomes

$$y_{n+1}=-\frac{C}{A}y_ny_{n-1}, \quad n\ge -1,$$

which is identical to (2.32). Hence, its general solution is

$$y_n=\Big(-\frac{C}{A}\Big)^{f_{n+2}-1}y_0^{f_{n+1}}y_{-1}^{f_n}, \quad n\ge -1,$$

from which, by using (2.42), it follows that

$$x_n=\Big(-\frac{C}{A}\Big)^{f_{n+2}-1}\Big(x_0+\frac{D}{C}\Big)^{f_{n+1}}\Big(x_{-1}+\frac{D}{C}\Big)^{f_n}-\frac{D}{C}, \quad n\ge -1,$$

is the general solution of (2.40).

2.17 The case $A \neq 0 \neq B$

In this case, every well-defined solution of (2.1) satisfies the equation

$$x_{n+1}=-\frac{Cx_nx_{n-1}+Dx_n+Ex_{n-1}+F}{A+Bx_n}, \quad n\in\mathbb{N}_0, \tag{2.43}$$

which can be rewritten in the form

$$x_{n+1}=\frac{cx_nx_{n-1}+dx_n+ex_{n-1}+f}{x_n+a}, \quad n\in\mathbb{N}_0, \tag{2.44}$$

where

$$a=\frac{A}{B}, \quad c=-\frac{C}{B}, \quad d=-\frac{D}{B}, \quad e=-\frac{E}{B}, \quad f=-\frac{F}{B}. \tag{2.45}$$

Equation (2.45) also seems not to be solvable in closed form for all values of the parameters a, c, d, e, f. Hence, we present some sufficient conditions which guarantee its solvability, and consequently the solvability of the corresponding equation in the class (2.43).

Equation (2.44) is equivalent to the equation

$$x_{n+1}+a=\frac{cx_nx_{n-1}+(d+a)x_n+ex_{n-1}+f+a^2}{x_n+a}, \quad n\in\mathbb{N}_0. \tag{2.46}$$

First, assume that $c > 0$. Then, (2.46) can be written as

$$x_{n+1} + a = \frac{\sqrt{c}x_{n-1}\big(\sqrt{c}x_n + \frac{e}{\sqrt{c}}\big) + \frac{d+a}{\sqrt{c}}\big(\sqrt{c}x_n + \frac{\sqrt{c}(f+a^2)}{d+a}\big)}{x_n + a}, \quad n \in \mathbb{N}_0.$$

Now, assume that

$$e = \frac{c(f + a^2)}{d + a} = d + a. \tag{2.47}$$

Under the conditions (2.47), we have that the sequence

$$u_n = \frac{x_{n+1} + a}{\sqrt{c}x_n + \frac{e}{\sqrt{c}}}, \quad n \ge -1,$$

satisfies the relation

$$u_{n+1} = \frac{1}{u_n}$$

from which it follows that $(u_n)_{n\ge-1}$ is two-periodic. Hence, we have that

$$\frac{x_{2n+i} + a}{\sqrt{c}x_{2n-1+i} + \frac{e}{\sqrt{c}}} = \frac{x_i + a}{\sqrt{c}x_{i-1} + \frac{e}{\sqrt{c}}}$$

for $n \in \mathbb{N}_0$ and $i \in \{0, 1\}$, that is,

$$x_{2n} = \frac{x_0 + a}{x_{-1} + \frac{f+a^2}{d+a}} x_{2n-1} + \frac{x_0 + a}{x_{-1} + \frac{f+a^2}{d+a}} \frac{f + a^2}{d + a} - a, \tag{2.48}$$

$$x_{2n+1} = \frac{x_1 + a}{x_0 + \frac{f+a^2}{d+a}} x_{2n} + \frac{x_1 + a}{x_0 + \frac{f+a^2}{d+a}} \frac{f + a^2}{d + a} - a \tag{2.49}$$

for $n \in \mathbb{N}_0$.

From (2.48) and (2.49) we obtain

$$x_{2n} = \frac{x_0 + a}{x_{-1} + \frac{f+a^2}{d+a}} \frac{x_1 + a}{x_0 + \frac{f+a^2}{d+a}} x_{2n-2} + \frac{x_0 + a}{x_{-1} + \frac{f+a^2}{d+a}} \left(\frac{x_1 + a}{x_0 + \frac{f+a^2}{d+a}} \frac{f + a^2}{d + a} - a \right) + \frac{x_0 + a}{x_{-1} + \frac{f+a^2}{d+a}} \frac{f + a^2}{d + a} - a, \tag{2.50}$$

$$x_{2n-1} = \frac{x_1 + a}{x_0 + \frac{f+a^2}{d+a}} \frac{x_0 + a}{x_{-1} + \frac{f+a^2}{d+a}} x_{2n-3} + \frac{x_1 + a}{x_0 + \frac{f+a^2}{d+a}} \left(\frac{x_0 + a}{x_{-1} + \frac{f+a^2}{d+a}} \frac{f + a^2}{d + a} - a \right) + \frac{x_1 + a}{x_0 + \frac{f+a^2}{d+a}} \frac{f + a^2}{d + a} - a \tag{2.51}$$

for $n \in \mathbb{N}$.

This means that the sequences $(x_{2n})_{n\in\mathbb{N}_0}$ and $(x_{2n-1})_{n\in\mathbb{N}_0}$ are solutions of the linear first-order difference equations

$$z_n = \frac{x_0 + a}{x_{-1} + \frac{f+a^2}{d+a}} \frac{x_1 + a}{x_0 + \frac{f+a^2}{d+a}} z_{n-1} + \frac{x_0 + a}{x_{-1} + \frac{f+a^2}{d+a}} \left(\frac{x_1 + a}{x_0 + \frac{f+a^2}{d+a}} \frac{f + a^2}{d + a} - a \right) + \frac{x_0 + a}{x_{-1} + \frac{f+a^2}{d+a}} \frac{f + a^2}{d + a} - a, \tag{2.52}$$

$$u_n = \frac{x_1 + a}{x_0 + \frac{f+a^2}{d+a}} \frac{x_0 + a}{x_{-1} + \frac{f+a^2}{d+a}} u_{n-1} + \frac{x_1 + a}{x_0 + \frac{f+a^2}{d+a}} \left(\frac{x_0 + a}{x_{-1} + \frac{f+a^2}{d+a}} \frac{f + a^2}{d + a} - a \right) + \frac{x_1 + a}{x_0 + \frac{f+a^2}{d+a}} \frac{f + a^2}{d + a} - a \tag{2.53}$$

for $n \in \mathbb{N}$, respectively.

From (2.52) and (2.53), by using (1.3), (2.44) with $n = 0$, and (2.45), we easily obtain formulas for the general solution of (2.43) in this case.

Now, assume that $c < 0$. Then, (2.46) can be written as

$$x_{n+1} + a = -\frac{\sqrt{-c}x_{n-1}\big(\sqrt{-c}x_n - \frac{e}{\sqrt{-c}}\big) - \frac{d+a}{\sqrt{-c}}\big(\sqrt{-c}x_n + \frac{\sqrt{-c}(f+a^2)}{d+a}\big)}{x_n + a}, \quad n \in \mathbb{N}_0.$$

Assume that the conditions in (2.47) hold. Then, we have that the sequence

$$v_n = \frac{x_n + a}{\sqrt{-c}x_{n-1} - \frac{e}{\sqrt{-c}}}, \quad n \ge -1,$$

satisfies the relation

$$v_{n+1} = -\frac{1}{v_n}, \quad n \ge -1,$$

from which it follows that it is two-periodic. Hence, we have that

$$\frac{x_{2n+i} + a}{\sqrt{-c}x_{2n-1+i} - \frac{e}{\sqrt{-c}}} = \frac{x_i + a}{\sqrt{-c}x_{i-1} - \frac{e}{\sqrt{-c}}}, \quad n \in \mathbb{N}_0,\ i \in \{0,1\},$$

from which it follows that (2.50) and (2.51) hold, and consequently that the sequences $(x_{2n})_{n\in\mathbb{N}_0}$ and $(x_{2n-1})_{n\in\mathbb{N}_0}$ are solutions of (2.52) and (2.53), respectively. Hence, again from (2.52) and (2.53), by using (1.3), (2.44) with $n = 0$, and (2.45), we easily obtain formulas for the general solution of (2.43) in this case.

Now, we present another subclass of (2.43) which can be solved in closed form. Equation (2.44) is also equivalent to

$$x_{n+1} + g = \frac{cx_nx_{n-1} + (d+g)x_n + ex_{n-1} + f + ag}{x_n + a}, \quad n \in \mathbb{N}_0,$$

which can be written in the form

$$x_{n+1} + g = c\frac{x_n(x_{n-1} + \frac{d+g}{c}) + \frac{e}{c}(x_{n-1} + \frac{f+ag}{e})}{x_n + a}, \quad n \in \mathbb{N}_0. \tag{2.54}$$

Now, assume that the conditions

$$a = \frac{e}{c}, \quad \frac{d+g}{c} = \frac{f+ag}{e} = g \tag{2.55}$$

hold. Then, if $(x_n)_{n\ge-1}$ is a well-defined solution to (2.54), it can be written in the form

$$x_{n+1} + \frac{d}{c-1} = c\Big(x_{n-1} + \frac{d}{c-1}\Big), \quad n \in \mathbb{N}_0,$$

from which it easily follows that

$$x_{2m+i} = c^m\Big(x_i + \frac{d}{c-1}\Big) - \frac{d}{c-1}, \quad m \in \mathbb{N}_0,\ i \in \{-1,0\},$$

are formulas for the general solution of (2.44) in this case. From this, along with (2.45), it follows that

$$x_{2m+i} = \Big(-\frac{C}{B}\Big)^m\Big(x_i + \frac{D}{C+B}\Big) - \frac{D}{C+B}, \quad m \in \mathbb{N}_0,\ i \in \{-1,0\},$$

is the general solution to equation (2.43).

Remark 2.5. Note that from (2.55) it follows that c must not be equal to 1, that is, $C \ne -B$, which implies that the quantity $d/(c-1)$, that is, $D/(C+B)$, is defined.

3 Solvable subclasses of equation (1.1) for the case $G \ne 0$

Here, we present several solvable subclasses of the difference equation (1.1) for the case $G \ne 0$.

3.1 The case $A = B = C = D = E = 0$ for $G \ne 0 \ne F$

In this case, (1.1) is equivalent to the equation

$$x_{n+1}x_{n-1} = -\frac{F}{G}, \quad n \in \mathbb{N}_0. \tag{3.1}$$

Using (3.1) twice, it follows that

$$x_n = \frac{-\frac{F}{G}}{x_{n-2}} = x_{n-4}, \quad n \ge 3,$$

which means that $(x_n)_{n\ge-1}$ is four-periodic, that is,

$$x_{2n-1} = x_{-1}, \quad x_{2n} = x_0, \quad x_{2n+1} = -\frac{F}{Gx_{-1}}, \quad x_{2n+2} = -\frac{F}{Gx_0}, \quad n \in \mathbb{N}_0.$$

3.2 The case $A = B = C = E = F = 0$ for $D \neq 0 \neq G$

In this case, every well-defined solution of (1.1) satisfies the equation

$$x_{n+1} = -\frac{Dx_n}{Gx_{n-1}}, \quad n \in \mathbb{N}_0. \tag{3.2}$$

By using the change of variables

$$x_n = -\frac{D}{G} y_n, \tag{3.3}$$

we transform (3.2) into the equation

$$y_{n+1} = \frac{y_n}{y_{n-1}}, \quad n \in \mathbb{N}_0. \tag{3.4}$$

It is easy to see that a solution of (3.4) is well defined if and only if $y_{-1} \neq 0 \neq y_0$, which is equivalent to $y_n \neq 0$ for every $n \geq -1$.

To find a formula for the general solution of (3.4), we will use the method described in Section 2.15, that is, in the case $B = D = E = F = 0$ for $A \neq 0 \neq C$. Let

$$a_1 = 1, \quad b_1 = -1 \tag{3.5}$$

and rewrite (3.4) into the form

$$y_{n+1} = y_n^{a_1} y_{n-1}^{b_1}, \quad n \in \mathbb{N}_0. \tag{3.6}$$

Using (3.4) in (3.6), we get

$$y_{n+1} = (y_{n-1} y_{n-2}^{-1})^{a_1} y_{n-1}^{b_1} = y_{n-1}^{a_1+b_1} y_{n-2}^{-a_1} = y_{n-1}^{a_2} y_{n-2}^{b_2}, \quad n \in \mathbb{N},$$

where

$$a_2 := a_1 + b_1, \quad b_2 := -a_1.$$

Assume that for some $k \geq 2$ we have

$$y_{n+1} = y_{n-k+1}^{a_k} y_{n-k}^{b_k}, \quad n \geq k-1, \tag{3.7}$$

where

$$a_k = a_{k-1} + b_{k-1}, \quad b_k = -a_{k-1} \tag{3.8}$$

and (3.5) holds.

By using (3.4) into (3.7) we have that

$$y_{n+1} = (y_{n-k} y_{n-k-1}^{-1})^{a_k} y_{n-k}^{b_k} = y_{n-k}^{a_k+b_k} y_{n-k-1}^{-a_k} = y_{n-k}^{a_{k+1}} y_{n-k-1}^{b_{k+1}}, \quad 2 \leq k \leq n,$$

where the sequences $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ satisfy the system of difference equations

$$a_{k+1} = a_k + b_k, \quad b_{k+1} = -a_k, \quad k \in \mathbb{N}, \tag{3.9}$$

with the initial conditions in (3.5). Hence, we have proved by induction that (3.7) holds for every $k \in \mathbb{N}$ and $n \geq k - 1$, where the sequences $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ satisfy (3.8) with the initial conditions in (3.5).

From (3.9) it follows that

$$a_{k+1} = a_k - a_{k-1}, \quad k \geq 2,$$

with $a_1 = 1$ and $a_2 = 0$, while $b_k = -a_{k-1}$, $k \geq 2$, with $b_1 = -1$. Hence, by using formula (2.18) (with shifted indices) we obtain that

$$a_k = -\frac{\lambda_1^{k-2} - \lambda_2^{k-2}}{i\sqrt{3}}, \quad k \in \mathbb{N},$$

where

$$\lambda_{1,2} = \frac{1 \pm i\sqrt{3}}{2},$$

from which it follows that

$$a_k = \frac{2}{\sqrt{3}} \sin \frac{(k+1)\pi}{3}, \quad b_k = -\frac{2}{\sqrt{3}} \sin \frac{k\pi}{3}, \quad k \in \mathbb{N},$$

and, consequently,

$$y_n = y_0^{a_n} y_{-1}^{-a_{n-1}} = y_0^{\frac{2}{\sqrt{3}} \sin \frac{(n+1)\pi}{3}} y_{-1}^{-\frac{2}{\sqrt{3}} \sin \frac{n\pi}{3}}, \quad n \geq -1, \tag{3.10}$$

where we naturally regard that $a_0 = a_1 - a_2 = 1$, $a_{-1} = a_0 - a_1 = 0$ and $a_{-2} = a_{-1} - a_0 = -1$.

From this and (3.3) it follows that

$$x_n = -\frac{D}{G} y_n = \left(-\frac{G}{D}\right)^{a_n - a_{n-1} - 1} x_0^{a_n} x_{-1}^{-a_{n-1}} = \left(-\frac{G}{D}\right)^{a_{n+1} - 1} x_0^{a_n} x_{-1}^{-a_{n-1}}, \quad n \geq -1,$$

which is the general solution to (3.2).

Remark 3.1. It is a well-known fact that (3.4) is six-periodic. It is one of the standard examples for periodicity. However, (3.10) is less known, although we expect that it can be found somewhere in the literature.

3.3 The case $B = C = D = 0$ for $G \neq 0$

In this case, every well-defined solution to (1.1) satisfies the equation

$$x_{n+1} = -\frac{Ex_{n-1} + F}{Gx_{n-1} + A}, \quad n \in \mathbb{N}_0. \tag{3.11}$$

From (3.11) we see that the sequences $(x_{2m+i})_{m \in \mathbb{N}_0}$, $i \in \{-1, 0\}$, are respectively the solutions of the bilinear difference equation

$$z_{m+1} = -\frac{Ez_m + F}{Gz_m + A}, \quad m \in \mathbb{N}_0, \tag{3.12}$$

with the initial conditions $z_0 = x_{-1}$ and $z_0 = x_0$. Since (3.12) is solvable (see Section 2.13), we can find formulas for $(x_{2m+i})_{m \in \mathbb{N}_0}$, $i \in \{-1, 0\}$, in closed form. We omit the details.

3.4 The case $A = D = E = F = 0$ for $B, C, G \in \mathbb{R} \setminus \{0\}$

In this case, every well-defined solution to (1.1) satisfies the equation

$$x_{n+1} = -\frac{Cx_n x_{n-1}}{Bx_n + Gx_{n-1}}, \quad n \in \mathbb{N}_0. \tag{3.13}$$

It is easy to see that for a well-defined solution $(x_n)_{n \geq -1}$ of (3.13), the condition $x_{-1} \neq 0 \neq x_0$ is equivalent to $x_n \neq 0$ for every $n \geq -1$. Hence, we may use the change of variables

$$x_n = \frac{1}{y_n}, \tag{3.14}$$

which transforms (3.13) into

$$y_{n+1} + \frac{G}{C} y_n + \frac{B}{C} y_{n-1} = 0, \quad n \in \mathbb{N}_0,$$

which is a second-order difference equation with constant coefficients. Hence, we can apply (2.18) with

$$\lambda_{1,2} = \frac{-\frac{G}{C} \pm \sqrt{\frac{G^2 - 4BC}{C^2}}}{2},$$

which along with (3.14) gives

$$x_n = x_{-1} x_0 \left(\frac{\lambda_2 x_0 - x_{-1}}{\lambda_2 - \lambda_1} \lambda_1^{n+1} + \frac{x_{-1} - \lambda_1 x_0}{\lambda_2 - \lambda_1} \lambda_2^{n+1} \right)^{-1}, \quad n \geq -1, \tag{3.15}$$

when $\lambda_1 \neq \lambda_2$, while if $\lambda_1 = \lambda_2$, by using (2.19), we get that the solution is

$$x_n = x_{-1} x_0 ((x_{-1}(n+1) - n\lambda_1 x_0)\lambda_1^n)^{-1}, \quad n \geq -1. \tag{3.16}$$

Remark 3.2. From (3.15) we see that, in the case $G^2 \neq 4BC$, a solution of (3.13) is well defined if and only if $x_0 \neq 0 \neq x_{-1}$ and

$$\frac{\lambda_2 x_0 - x_{-1}}{\lambda_2 - \lambda_1}\lambda_1^{n+1} + \frac{x_{-1} - \lambda_1 x_0}{\lambda_2 - \lambda_1}\lambda_2^{n+1} \neq 0$$

for every $n \in \mathbb{N}$, while from (3.16) we see that, in the case $G^2 = 4BC$, a solution of equation (3.13) is well defined if and only if $x_0 \neq 0 \neq x_{-1}$ and

$$x_{-1}(n+1) \neq n\lambda_1 x_0$$

for every $n \in \mathbb{N}$.

3.5 The case $B = 0$ for $G \neq 0 \neq C$

In this case, every well-defined solution to (1.1) satisfies the equation

$$x_{n+1} = -\frac{Cx_n x_{n-1} + Dx_n + Ex_{n-1} + F}{Gx_{n-1} + A}, \quad n \in \mathbb{N}_0. \tag{3.17}$$

Equation (3.17) is equivalent to

$$x_{n+1} + h = \frac{-Cx_n x_{n-1} - Dx_n + (hG - E)x_{n-1} + Ah - F}{Gx_{n-1} + A}, \quad n \in \mathbb{N}_0,$$

which can be written in the form

$$x_{n+1} + h = \frac{-\frac{C}{G}x_n\left(Gx_{n-1} + \frac{GD}{C}\right) + \frac{(hG-E)}{G}\left(Gx_{n-1} + \frac{G(Ah-F)}{hG-E}\right)}{Gx_{n-1} + A}, \quad n \in \mathbb{N}_0. \tag{3.18}$$

If we assume that

$$A = \frac{GD}{C} = \frac{G(Ah - F)}{hG - E},$$

then, if $(x_n)_{n\geq -1}$ is a well-defined solution to (3.18), it can be written in the form

$$x_{n+1} = -\frac{C}{G}x_n - \frac{E}{G}$$

from which it follows that

$$x_n = \left(-\frac{C}{G}\right)^n\left(x_0 + \frac{E}{C+G}\right) - \frac{E}{C+G}.$$

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