Research Article

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On the removability of isolated singular points for elliptic equations involving variable exponent

Abstract: In this paper, we study the problem of removable isolated singularities for elliptic equations with variable exponents. We give a sufficient condition for removability of the isolated singular point for the equations in $W^{1,p(x)}(\Omega)$.

Keywords: Variable exponent, isolated singularity, removable singularity

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1 Introduction

In the studies of electrorheological fluid, nonlinear elasticity and image restoration in practical applications, the classical Lebesgue and Sobolev spaces are not applicable, see [1, 5, 18]. Problems with variable exponential growth conditions are inhomogeneous and nonlinear. So we need to study the problems based on the theory of variable exponent Lebesgue and Sobolev spaces. Since Kováčik and Rákosník first studied the $L^{p(x)}$ spaces and $W^{k,p(x)}$ spaces in [12], many results have been obtained concerning these kinds of variable exponent spaces, see examples in [4, 7, 8, 10, 11, 16, 17, 21].

This paper is devoted to the study of conditions guaranteeing the removability of isolated singularities for solutions of the elliptic equations with nonstandard growth

$$-\sum_{j=1}^{n}\frac{\partial}{\partial x_j}[a_j(x,u,\nabla u)]+g(x,u)=0. \tag{1.1}$$

We assume that for $j=1,2,\dots,n$, the functions $a_j(x,\xi,\eta)$ are measurable and satisfy the following conditions: There exists a number $\mu\in(0,1]$ such that

$$\sum_{j=1}^{n}a_j(x,\xi,\eta)\eta_j\ge\mu|\eta|^{p(x)}, \tag{1.2}$$

$$|a_j(x,\xi,\eta)|\le\mu^{-1}|\eta|^{p(x)-1}, \tag{1.3}$$

$$a_j(x,\xi,-\eta)=-a_j(x,\xi,\eta) \tag{1.4}$$

for almost all $x\in\overline{\Omega}$, $\xi\in\mathbb{R}$, $\eta\in\mathbb{R}^n$, where $p(x)$ is continuous on $\overline{\Omega}$ and satisfies $1<p^-\le p(x)\le p^+<n$ with

$$p^-=\inf_{x\in\overline{\Omega}}p(x)\quad\text{and}\quad p^+=\sup_{x\in\overline{\Omega}}p(x).$$

In addition, it is assumed that the function $g(x,\xi)$ with the conditions of measurability, local integrability in $x\in\overline{\Omega}$ and local boundedness in $\xi\in\mathbb{R}$ satisfies the following condition: There exists a function $q(x)\in C(\overline{\Omega})$

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and $q(x) \gg p(x) - 1$ such that for almost all $x \in \overline{\Omega}$, $\xi \in \mathbb{R}$,

$$g(x, \xi) \operatorname{sgn} \xi \geq |\xi|^{q(x)}, \tag{1.5}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with smooth boundary. Here by $q(x) \gg p(x) - 1$ we denote the fact that $\inf_{x \in \overline{\Omega}}(q(x) - p(x) + 1) > 0$.

The removability of singularities for the equations with standard growth has been considered by many authors. Serrin [19] considered the conditions of removability of an isolated singular point for equation (1.1) in the case of $g(x, u) \equiv 0$ and the condition for this case is

$$u(x) = o(|x - x_0|^{\frac{p-n}{p-1}}), \quad 1 < p < n.$$

Brezis and Veron [2] studied an equation of the form (1.1) with a Laplace operator in the principal part. They proved the removability of isolated singularities for solutions under condition (1.5) with $q \geq \frac{n}{n-2}$ and $n \geq 3$. For equations with weighted functions v, w, Mamedov and Harman [15] proved that an isolated singular point x_0 is removable for solutions of equation (1.1) if the condition of weighted functions

$$v(B(x_0, \varepsilon))\Big(\frac{w(B(x_0, \varepsilon))}{\varepsilon^p v(B(x_0, \varepsilon))}\Big)^{\frac{q}{q-p+1}} = o(1), \quad \varepsilon \to 0$$

with $p > 1$ and $q > p - 1$ is fulfilled. The removability of singularities for solutions of elliptic equations with absorption term was studied in [13, 20]. Skrypnik [20] considered general nonlinear equations and proved that if the conditions

$$1 < p < n, \quad q \geq \frac{n(p-1)}{n-p}$$

are fulfilled, then the singular point is removable. Liskevich and Skrypnik [13] studied the equation (1.1) with a weighted function $x^{-\alpha}$ in the subordinate part $g(x, u)$, and proved that the singular point is removable if the conditions

$$q \geq \frac{(p-1)(n-\alpha)}{n-p}, \quad \alpha < p, \quad 1 < p < n$$

are fulfilled.

Recently, there have been a few papers on the study of the removability of singularities for the equations with nonstandard growth. Lukkari [14] investigated the removability of a compact set for the equation $-\operatorname{div}(|Du|^{p(x)-2}Du) = 0$. For the anisotropic elliptic equation, the removability of a compact set was proved by Cianci [6]. Cataldo and Cianci [3] considered the conditions of removability of an isolated singular point for equation (1.1) in the case of $g(x, u) = |u|^{q-2}u$.

In this paper, we prove that an isolated singular point x_0 is removable for solutions of equation (1.1) if the condition

$$1 < \frac{p(x)q(x)}{q(x) - p(x) + 1} \ll n \tag{1.6}$$

is fulfilled for almost all $x \in \overline{\Omega}$.

2 Preliminaries

We first recall some facts on spaces $L^{p(x)}$ and $W^{k,p(x)}$. For the details see [10, 12].

Let $\mathbf{P}(\Omega)$ be the set of all Lebesgue measurable functions $p : \Omega \to [1, \infty]$, we denote

$$\rho_{p(x)}(u) = \int\limits_{\Omega \setminus \Omega_\infty} |u|^{p(x)}\,dx + \sup_{x \in \Omega_\infty} |u(x)|,$$

where $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$.

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is the class of all functions u such that $\rho_{p(x)}(tu) < \infty$, for some $t > 0$. $L^{p(x)}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{L^{p(x)}} = \inf\Big\{\lambda > 0 : \rho_{p(x)}\Big(\frac{u}{\lambda}\Big) \le 1\Big\}.$$

For any $p \in \mathbf{P}(\Omega)$, we define the conjugate function $p'(x)$ as

$$p'(x) = \begin{cases} \infty, & x \in \Omega_1 = \{x \in \Omega : p(x) = 1\}, \\ 1, & x \in \Omega_\infty, \\ \frac{p(x)}{p(x)-1}, & x \in \Omega \setminus (\Omega_1 \cup \Omega_\infty). \end{cases}$$

Theorem 2.1. *Let $p \in \mathbf{P}(\Omega)$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$,*

$$\int_\Omega |uv|\,dx \le 2\|u\|_{L^{p(x)}}\|v\|_{L^{p'(x)}}.$$

Theorem 2.2. *Let $p \in \mathbf{P}(\Omega)$ with $p^+ < \infty$. For any $u \in L^{p(x)}(\Omega)$, we have*
(1) *if $\|u\|_{L^{p(x)}} \ge 1$, then $\|u\|^{p^-}_{L^{p(x)}} \le \int_\Omega |u|^{p(x)}\,dx \le \|u\|^{p^+}_{L^{p(x)}}$,*
(2) *if $\|u\|_{L^{p(x)}} < 1$, then $\|u\|^{p^+}_{L^{p(x)}} \le \int_\Omega |u|^{p(x)}\,dx \le \|u\|^{p^-}_{L^{p(x)}}$.*

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is the class of all functions $u \in L^{p(x)}(\Omega)$ which have the property $|\nabla u| \in L^{p(x)}(\Omega)$. The space $W^{1,p(x)}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{W^{1,p(x)}} = \|u\|_{L^{p(x)}} + \|\nabla u\|_{L^{p(x)}}.$$

We say that the function $u(x)$ belongs to the space $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$ if $u(x)$ belongs to $W^{1,p(x)}(G)$ in any subdomain G, $\overline{G} \subset \Omega$.

Theorem 2.3. *For any $u \in W^{1,p(x)}(\Omega)$, we have*
(1) *if $\|u\|_{W^{1,p(x)}} \ge 1$, then $\|u\|^{p^-}_{W^{1,p(x)}} \le \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\,dx \le \|u\|^{p^+}_{W^{1,p(x)}}$,*
(2) *if $\|u\|_{W^{1,p(x)}} < 1$, then $\|u\|^{p^+}_{W^{1,p(x)}} \le \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\,dx \le \|u\|^{p^-}_{W^{1,p(x)}}$.*

From Zhikov [21, 22], we know that smooth functions are not dense in $W^{1,p(x)}(\Omega)$ without additional assumptions on the exponent $p(x)$. To study the Lavrentiev phenomenon, he considered the following log-Hölder continuous condition:

$$|p(x) - p(y)| \le \frac{C}{-\log(|x-y|)} \tag{2.1}$$

for all $x, y \in \overline{\Omega}$ such that $|x - y| \le \frac{1}{2}$. If the log-Hölder continuous condition holds, then smooth functions are dense in $W^{1,p(x)}(\Omega)$ and we can define the Sobolev spaces with zero boundary values $W^{1,p(x)}_0(\Omega)$, as the closure of $C_0^\infty(\Omega)$ with the norm of $\|\cdot\|_{W^{1,p(x)}(\Omega)}$.

Theorem 2.4. *If $u \in W_0^{1,p}(B(a,R))$, $1 \le p < n$, then for any $1 \le q \le p^*$, the inequality*

$$\Big(\int_{B(a,R)} |u|^q\,dx\Big)^{\frac{1}{q}} \le C(n,p)R^{1+\frac{n}{q}-\frac{n}{p}}\Big(\int_{B(a,R)} |Du|^p\,dx\Big)^{\frac{1}{p}} \tag{2.2}$$

is valid, where $B(a,R)$ is a ball.

Lemma 2.5. *Let Ω be an open bounded subset of $\mathbb{R}^n$, E be a measurable subset of Ω and $p_E^- = \inf_{x\in E} p(x)$, $p_\Omega^- = \inf_{x\in\Omega} p(x)$, then all nonnegative measurable functions f and g defined on E satisfy the inequality*

$$\int_E fg^{p_\Omega^-}\,dx \le \int_E f\,dx + \int_E fg^{p(x)}\,dx.$$

Proof. Since $\frac{p(x)}{p(x)-p_\Omega^-} > 1$, using Young's inequality we have

$$\int_E fg^{p_\Omega^-}\,dx = \int_E f^{\frac{p(x)-p_\Omega^-}{p(x)}}\Big[f^{\frac{p_\Omega^-}{p(x)}} g^{p_\Omega^-}\Big]\,dx \le \int_E f\,dx + \int_E fg^{p(x)}\,dx. \qquad \square$$

Consider a solution $u(x)$ of equation (1.1) with an isolated singularity. Assume that $x_0 \in \Omega$ and x_0 is a singular point of the solution $u(x)$. We say that $u(x)$ is a solution of equation (1.1) in $\Omega \setminus \{x_0\}$ if $u \in W^{1,p(x)}_{\mathrm{loc}}(\Omega \setminus \{x_0\})$ and for any test function $\varphi \in W^{1,p(x)}_{\mathrm{loc}}(\Omega \setminus \{x_0\}) \cap L^\infty_{\mathrm{loc}}(\Omega \setminus \{x_0\})$ with compact support in $\Omega \setminus \{x_0\}$, the following equality is true:

$$\int_\Omega \Big\{\sum_{j=1}^n a_j(x,u,\nabla u)\frac{\partial\varphi}{\partial x_j} + g(x,u)\varphi\Big\}\,dx = 0. \tag{2.3}$$

We say that the solution $u(x)$ of equation (1.1) has a removable singularity at the point x_0 if the function $u(x)$ is a solution in $\Omega \setminus \{x_0\}$ and $u \in W^{1,p(x)}_{\mathrm{loc}}(\Omega \setminus \{x_0\}) \cap L^\infty_{\mathrm{loc}}(\Omega)$ implies that it belongs to the space $W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ and equality (2.3) with any test function $\varphi \in W^{1,p(x)}_0(\Omega) \cap L^\infty(\Omega)$ is fulfilled for it.

In the proof of the main theorem, we use the following lemma.

Lemma 2.6 ([9]). *Let $0 < \theta < 1$, $\sigma > 0$, $\xi(h)$ be a nonnegative function on the interval $[\frac{1}{2}, 1]$, and let*

$$\xi(k) \le A(h-k)^{-\sigma}(\xi(h))^\theta, \quad \frac{1}{2} \le k < h \le 1.$$

Then, there exists $C_{\sigma,\theta} > 0$ such that

$$\xi\Big(\frac{1}{2}\Big) \le C_{\sigma,\theta} A^{\frac{1}{1-\theta}}.$$

3 The behavior of solutions near the isolated singular points

In this section we state and prove the following theorems.

Theorem 3.1. *Let conditions* (1.2)–(1.5) *and* (2.1) *be fulfilled. There exists a constant $0 < \delta < 1$ such that for $u \in W^{1,p(x)}(B(a,r)) \cap L^\infty(B(a,r))$ which is a solution of equation* (1.1) *in $B(a,r)$ with $r < \delta$, $B(a,r) \subset \overline{\Omega}$, the following inequality is valid:*

$$\sup_{x\in B(a,\frac{r}{2})} |u(x)| \le Cr^{-\tau},$$

where $\tau = \tau(p_\delta^-, p_\delta^+, q_\delta^-, \varepsilon) > \frac{p_\delta^+}{q_\delta^- - p_\delta^+ + 1} > 0$, $\varepsilon \in (0, \frac{q_\delta^- - p_\delta^+ + 1}{q_\delta^- - p_\delta^- + 1})$ is a constant and $C = C(n, \mu, p_\delta^-, p_\delta^+, q_\delta^+, q_\delta^-)$. Moreover,

$$p_\delta^+ = \sup_{y\in \overline{B(a,\delta)\cap\Omega}} p(y), \quad p_\delta^- = \inf_{y\in \overline{B(a,\delta)\cap\Omega}} p(y), \quad q_\delta^+ = \sup_{y\in \overline{B(a,\delta)\cap\Omega}} q(y), \quad q_\delta^- = \inf_{y\in \overline{B(a,\delta)\cap\Omega}} q(y).$$

Proof. As $q(x), p(x)$ are continuous on $\overline{\Omega}$, for any $\varepsilon_1 \in (0,1)$ and any $x \in \Omega$, there exists $\delta > 0$ such that $|q(x) - q(y)| < \varepsilon_1$ and $|p(x) - p(y)| < \varepsilon_1$ whenever $|x - y| < \delta$. Take $x \in \overline{\Omega}$. For any $y \in B_\delta(x) \cap \overline{\Omega}$, we have

$$p(y) - 1 < p(x) - 1 + \varepsilon_1 \quad \text{and} \quad q(y) > q(x) - \varepsilon_1.$$

As $q(x) \gg p(x) - 1$, take $\varepsilon_1 = \frac{1}{4}\inf_{x\in\overline{\Omega}}(q(x) - p(x) + 1)$. Then,

$$q(x) - \varepsilon_1 - (p(x) - 1 + \varepsilon_1) \ge \frac{1}{2}\inf_{x\in\overline{\Omega}}(q(x) - p(x) + 1) > 0$$

and therefore

$$p(y) - 1 < p(x) - 1 + \varepsilon_1 < q(x) - \varepsilon_1 < q(y).$$

Furthermore,

$$p_\delta^+ - 1 = \sup_{y\in \overline{B(x,\delta)\cap\Omega}} (p(y)-1) < q_\delta^- = \inf_{y\in \overline{B(x,\delta)\cap\Omega}} q(y)$$

and since $a \in \Omega$, we have

$$p_\delta^+ - 1 < q_\delta^-. \tag{3.1}$$

Let $u = FW$, where $F > 1$ is a positive number that will be chosen later. Let $\Omega' = \{x \in B(a,r) : W(x) > 0\}$, $\Omega'' = B(a,r) \setminus \overline{\Omega'}$. Assume that $W(x') > 0$ at some point $x' \in B(a, \frac{r}{2}) \cap \Omega'$ (if $W(x) \le 0$ at some point $B(a, \frac{r}{2})$, then we will consider, in addition, the function $-W(x)$ in $B(a, \frac{r}{2}) \cap \Omega'$). Take

$$M_t = \sup\{W(x) : x \in B(a,tr) \cap \Omega'\}, \quad \frac{1}{2} \le t \le 1.$$

Let $\frac{1}{2} \le s < t \le 1$, $z = W - M_t\xi$ and

$$z_k = (W - M_t\xi - k)_+ = \max(W - M_t\xi - k, 0) \quad \text{in the ball } B(a,tr),$$

where $0 \le k \le \sup_{\Omega'} z$. The function $\xi \in \mathrm{Lip}(B(a,r))$ satisfies: $\xi \equiv 0$ for $x \in B(a,sr)$, $\xi \equiv 1$ for $x \notin B(a, \frac{s+t}{2}r)$,

$$0 \le \xi(x) \le 1 \quad \text{and} \quad |\nabla\xi| \le \frac{C}{r(t-s)} \quad \text{for } x \in B(a,r),$$

where C is a constant. Denote $\Omega_k = \{x \in B(a,tr) : z_k > 0\}$. It is obvious that $z_k \in W_0^{1,p(x)}(B(a,tr))$. It is assumed that $M_{\frac{1}{2}} \ge 1$. The conclusion is obviously right for the case of $0 < M_{\frac{1}{2}} < 1$. Let $\varphi = z_k$ in (2.3), we obtain

$$\sum_{j=1}^n \int_{B(a,tr)} a_j(x,u,\nabla u)\frac{\partial z_k}{\partial x_j}\,dx + \int_{B(a,tr)} g(x,u) z_k\,dx = 0.$$

Then,

$$\sum_{j=1}^n \int_{\Omega_k} a_j(x, FW, F\nabla W)\Big(\frac{\partial W}{\partial x_j} - M_t\frac{\partial \xi}{\partial x_j}\Big)\,dx + \int_{\Omega_k} |FW|^{q(x)} z_k\,dx \le 0.$$

By virtue of conditions (1.2)–(1.5), we have

$$\mu\int_{\Omega_k} F^{p(x)-1}|\nabla W|^{p(x)}\,dx + \int_{\Omega_k} |F|^{q(x)}|k|^{q(x)} z_k\,dx \le \frac{CM_t}{\mu r(t-s)}\int_{\Omega_k} |F|^{p(x)-1}|\nabla W|^{p(x)-1}\,dx \tag{3.2}$$

and using Young's inequality in the right-hand side of (3.2), it does not exceed

$$C(\mu,\varepsilon_2,n)\int_{\Omega_k} \frac{M_t^{p(x)}}{r^{p(x)}(t-s)^{p(x)}}|F|^{p(x)-1}\,dx + \frac{C\varepsilon_2}{\mu}\int_{\Omega_k} |\nabla W|^{p(x)}|F|^{p(x)-1}\,dx.$$

Take $\varepsilon_2 = \frac{\mu^2}{2C}$. Then, from inequality (3.2) we have

$$\frac{\mu}{2}\int_{\Omega_k} F^{p(x)-1}|\nabla W|^{p(x)}\,dx + \int_{\Omega_k} |F|^{q(x)}|k|^{q(x)} z_k\,dx \le C(\mu,n)\int_{\Omega_k} \frac{M_t^{p(x)}}{r^{p(x)}(t-s)^{p(x)}}|F|^{p(x)-1}\,dx.$$

Note that $\nabla z_k = \nabla W - M_t\nabla\xi$ in Ω_k, therefore

$$\frac{\mu}{2}\int_{\Omega_k} F^{p(x)-1}\Big(\frac{1}{2^{p(x)-1}}|\nabla z_k|^{p(x)} - M_t^{p(x)}|\nabla\xi|^{p(x)}\Big)\,dx + \int_{\Omega_k} |F|^{q(x)}|k|^{q(x)} z_k\,dx \le C(\mu,n)\int_{\Omega_k} \frac{M_t^{p(x)}}{r^{p(x)}(t-s)^{p(x)}}|F|^{p(x)-1}\,dx.$$

By Lemma 2.5, we have

$$\int_{\Omega_k} F^{p(x)-1}|\nabla z_k|^{p_\delta^-}\,dx \le \int_{\Omega_k} F^{p(x)-1}\,dx + \int_{\Omega_k} F^{p(x)-1}|\nabla z_k|^{p(x)}\,dx$$

and therefore

$$\frac{\mu}{2^{p_\delta^+}}\int_{\Omega_k} F^{p(x)-1}(|\nabla z_k|^{p_\delta^-}-1)\,dx+\int_{\Omega_k}|F|^{q(x)}|k|^{q(x)}z_k\,dx\le C(\mu,n)\int_{\Omega_k}\frac{M_t^{p(x)}}{r^{p(x)}(t-s)^{p(x)}}|F|^{p(x)-1}\,dx. \tag{3.3}$$

Since $z_k\in W_0^{1,p(x)}(B(a,tr))$, so $z_k\in W_0^{1,p_\delta^-}(B(a,tr))$. By (2.2), we have

$$\Bigg(\int_{B(a,tr)}|z_k|^{\frac{np_\delta^-}{n-1}}\,dx\Bigg)^{\frac{n-1}{np_\delta^-}}\le C(n,p_\delta^-)r^{\frac{p_\delta^--1}{p_\delta^-}}\Bigg(\int_{B(a,tr)}|\nabla z_k|^{p_\delta^-}\,dx\Bigg)^{\frac{1}{p_\delta^-}}.$$

Using Hölder's inequality, we get

$$\int_{B(a,tr)} z_k\,dx\le 2\Bigg[\int_{B(a,tr)}|z_k|^{\frac{np_\delta^-}{n-1}}\,dx\Bigg]^{\frac{n-1}{np_\delta^-}}|\Omega_k|^{\frac{np_\delta^--n+1}{np_\delta^-}},$$

where $|\Omega_k|$ is the Lebesgue measure of Ω_k.

From (3.3), we have

$$\begin{aligned}&\frac{\mu}{2^{p_\delta^+}}F^{p_\delta^--1}\Bigg[C|\Omega_k|^{-\frac{np_\delta^--n+1}{np_\delta^-}}r^{-\frac{p_\delta^--1}{p_\delta^-}}\int_{B(a,tr)} z_k\,dx\Bigg]^{p_\delta^-}+F^{q_\delta^-}\min\{k^{q_\delta^-},k^{q_\delta^+}\}\int_{B(a,tr)} z_k\,dx\\&\qquad\le C(\mu,n)F^{p_\delta^+-1}\int_{\Omega_k}\frac{M_t^{p(x)}}{r^{p(x)}(t-s)^{p(x)}}\,dx+\frac{\mu}{2^{p_\delta^+}}F^{p_\delta^+-1}|\Omega_k|.\end{aligned}$$

Take $\varepsilon\in(0,\frac{q_\delta^--p_\delta^++1}{q_\delta^--p_\delta^-+1})$. Then,

$$\begin{aligned}&\frac{\varepsilon\mu}{2^{p_\delta^+}}F^{p_\delta^--1}\Bigg[C\int_{B(a,tr)} z_k\,dx\Bigg]^{p_\delta^-}r^{1-p_\delta^-}+(1-\varepsilon)F^{q_\delta^-}\min\{k^{q_\delta^-},k^{q_\delta^+}\}|\Omega_k|^{\frac{np_\delta^--n+1}{n}}\int_{B(a,tr)} z_k\,dx\\&\qquad\le C(\mu,n)F^{p_\delta^+-1}|\Omega_k|^{\frac{np_\delta^--n+1}{n}}\int_{\Omega_k}\frac{M_t^{p(x)}}{r^{p(x)}(t-s)^{p(x)}}\,dx+\frac{\mu}{2^{p_\delta^+}}F^{p_\delta^+-1}|\Omega_k|^{1+\frac{np_\delta^--n+1}{n}}.\end{aligned} \tag{3.4}$$

Applying Young's inequality $a^\varepsilon b^{1-\varepsilon}\le\varepsilon a+(1-\varepsilon)b$ in the left-hand side of (3.4), we obtain

$$\begin{aligned}&\min\{k^{q_\delta^-(1-\varepsilon)},k^{q_\delta^+(1-\varepsilon)}\}F^{q_\delta^-(1-\varepsilon)+(p_\delta^--1)\varepsilon}r^{(1-p_\delta^-)\varepsilon}\Bigg[\int_{B(a,tr)} z_k\,dx\Bigg]^{p_\delta^-\varepsilon+1-\varepsilon}\\&\qquad\le CF^{p_\delta^+-1}\Bigg[|\Omega_k|^{\frac{np_\delta^--n+1}{n}\varepsilon}\int_{\Omega_k}\frac{M_t^{p(x)}}{r^{p(x)}(t-s)^{p(x)}}\,dx+|\Omega_k|^{\frac{np_\delta^--n+1}{n}\varepsilon+1}\Bigg]\end{aligned}$$

and

$$\begin{aligned}&\min\{k^{q_\delta^-(1-\varepsilon)},k^{q_\delta^+(1-\varepsilon)}\}F^{q_\delta^--p_\delta^++1-(q_\delta^--p_\delta^-+1)\varepsilon}\\&\qquad\le C\Bigg[\int_{B(a,tr)} z_k\,dx\Bigg]^{-(p_\delta^-\varepsilon+1-\varepsilon)}r^{(p_\delta^--1)\varepsilon}\Bigg[\frac{M_t^{p_\delta^+}}{r^{p_\delta^+}(t-s)^{p_\delta^+}}+1\Bigg]|\Omega_k|^{\frac{np_\delta^--n+1}{n}\varepsilon+1},\end{aligned} \tag{3.5}$$

where $C=C(n,\mu,p_\delta^-,p_\delta^+)$.

Denote $\alpha=\frac{np_\delta^--n+1}{n}\varepsilon+1$. Integrate (3.5) with respect to k and take the equality $\frac{d}{dk}(\int_{\Omega_k}z_k dx)=-|\Omega_k|$ into account, then we have

$$F^{\frac{q_\delta^--p_\delta^++1-(q_\delta^--p_\delta^-+1)\varepsilon}{\alpha}}\int_0^{\sup_{\Omega'}z}\min\Big\{k^{\frac{q_\delta^-(1-\varepsilon)}{\alpha}},k^{\frac{q_\delta^+(1-\varepsilon)}{\alpha}}\Big\}\,dk\le C\Bigg[\frac{M_t^{p_\delta^+}}{r^{p_\delta^+}(t-s)^{p_\delta^+}}+1\Bigg]^{\frac{1}{\alpha}}r^{\frac{(p_\delta^--1)\varepsilon}{\alpha}}\int_0^{\sup_{\Omega'}z}\Bigg[\int_{\Omega_k} z_k\,dx\Bigg]^{-\frac{p_\delta^-\varepsilon+1-\varepsilon}{\alpha}}|\Omega_k|\,dk.$$

Since $1 - \frac{p_\delta^-\varepsilon+1-\varepsilon}{\alpha} = 1 - \frac{np_\delta^-\varepsilon+n-n\varepsilon}{np_\delta^-\varepsilon+n-n\varepsilon+\varepsilon} > 0$, $\sup_{\Omega'} z > 1$, so

$$\Big(\sup_{\Omega'} z\Big)^{\frac{q_\delta^-(1-\varepsilon)}{\alpha}+1} F^{\frac{q_\delta^- - p_\delta^+ + 1 - (q_\delta^- - p_\delta^- + 1)\varepsilon}{\alpha}} \le C\Bigg[\int_{\Omega_0} z_0\,dx\Bigg]^{1-\frac{p_\delta^-\varepsilon+1-\varepsilon}{\alpha}} \Bigg[\frac{M_t^{p_\delta^+}}{r^{p_\delta^+}(t-s)^{p_\delta^+}}+1\Bigg]^{\frac{1}{\alpha}} r^{\frac{(p_\delta^- -1)\varepsilon}{\alpha}}.$$

Apply the estimate $\int_{\Omega_0} z_0\,dx \le M_t|B(a,r)|$ and note that $\sup_{\Omega'} z \ge \sup_{\Omega'\cap B(a,sr)} z = M_s$, then we have

$$M_s^{\frac{q_\delta^-(1-\varepsilon)}{\alpha}+1} F^{\frac{q_\delta^- - p_\delta^+ + 1 - (q_\delta^- - p_\delta^- + 1)\varepsilon}{\alpha}} \le CM_t^{1-\frac{p_\delta^-\varepsilon+1-\varepsilon}{\alpha}} \Bigg[\frac{M_t^{p_\delta^+}}{r^{p_\delta^+}(t-s)^{p_\delta^+}}\Bigg]^{\frac{1}{\alpha}} r^{\frac{(p_\delta^- -1)\varepsilon}{\alpha}} r^{n\left(1-\frac{p_\delta^-\varepsilon+1-\varepsilon}{\alpha}\right)},$$

where $C = C(n, \mu, p_\delta^-, p_\delta^+, q_\delta^+)$.

Choosing

$$F^{\frac{q_\delta^- - p_\delta^+ + 1 - (q_\delta^- - p_\delta^- + 1)\varepsilon}{\alpha}} = r^{\frac{(p_\delta^- -1)\varepsilon}{\alpha}} r^{n\left(1-\frac{p_\delta^-\varepsilon+1-\varepsilon}{\alpha}\right)} r^{-\frac{p_\delta^+}{\alpha}},$$

then $F = r^{-\tau}$, where

$$\begin{aligned}\tau &= -\Big[\frac{(p_\delta^- - 1)\varepsilon}{\alpha} + n\Big(1 - \frac{p_\delta^-\varepsilon + 1 - \varepsilon}{\alpha}\Big) - \frac{p_\delta^+}{\alpha}\Big]\Big[\frac{q_\delta^- - p_\delta^+ + 1 - (q_\delta^- - p_\delta^- + 1)\varepsilon}{\alpha}\Big]^{-1}\\ &= -\frac{p_\delta^-\varepsilon - p_\delta^+}{q_\delta^- - p_\delta^+ + 1 - (q_\delta^- - p_\delta^- + 1)\varepsilon} > \frac{p_\delta^+}{q_\delta^- - p_\delta^+ + 1} > 0\end{aligned}$$

when $\varepsilon \in (0, \frac{q_\delta^- - p_\delta^+ + 1}{q_\delta^- - p_\delta^- + 1})$.

On the other hand,

$$M_s \le C(n, \mu, p_\delta^-, p_\delta^+, q_\delta^+)\frac{M_t^\theta}{(t-s)^\sigma},$$

where

$$\begin{aligned}\theta &= \Big[\Big(1 - \frac{p_\delta^-\varepsilon + 1 - \varepsilon}{\alpha}\Big) + \frac{p_\delta^+}{\alpha}\Big]\Big[\frac{q_\delta^-(1-\varepsilon)}{\alpha} + 1\Big]^{-1} = \frac{\alpha - (p_\delta^- - 1)\varepsilon + p_\delta^+ - 1}{\alpha - q_\delta^-\varepsilon + q_\delta^-} < 1,\\ \sigma &= \frac{p_\delta^+}{\alpha}\Big[\frac{q_\delta^-(1-\varepsilon)}{\alpha} + 1\Big]^{-1} > 0.\end{aligned}$$

By virtue of Lemma 2.6, we derive $M_{\frac{1}{2}} \le C(n, \mu, p_\delta^-, p_\delta^+, q_\delta^+, q_\delta^-)$.

From the substitution $u = FW$ we obtain

$$\sup\Big\{u(x) : x \in B\Big(a, \frac{r}{2}\Big) \cap \Omega'\Big\} = FM_{\frac{1}{2}} \le CF. \tag{3.6}$$

If $\sup_{B(a,\frac{r}{2})\cap\Omega'} W \le 0$, then there exists $x'' \in B(a, \frac{r}{2}) \cap \Omega''$ such that $-W(x'') > 0$, in the same way, we can obtain

$$\sup\Big\{-u(x) : x \in B\Big(a, \frac{r}{2}\Big) \cap \Omega''\Big\} = FM_{\frac{1}{2}} \le CF. \tag{3.7}$$

Combining (3.6) and (3.7), we can give the estimate

$$\sup_{B(a,\frac{r}{2})} |u(x)| \le Cr^{-\tau},$$

where $\tau = \tau(p_\delta^-, p_\delta^+, q_\delta^-, \varepsilon) > \frac{p_\delta^+}{q_\delta^- - p_\delta^+ + 1} > 0$, $\varepsilon \in (0, \frac{q_\delta^- - p_\delta^+ + 1}{q_\delta^- - p_\delta^- + 1})$ is a constant and $C = C(n, \mu, p_\delta^-, p_\delta^+, q_\delta^+, q_\delta^-)$. □

From Theorem 3.1, we can easily obtained the following theorem.

Theorem 3.2. *Let conditions* (1.2)–(1.5) *and* (2.1) *be fulfilled. Let* $u \in W_{\mathrm{loc}}^{1,p(x)}(\Omega \setminus \{x_0\}) \cap L_{\mathrm{loc}}^\infty(\Omega \setminus \{x_0\})$ *be a solution of equation* (1.1) *in* $\Omega \setminus \{x_0\}$, *then for any* $0 < |x - x_0| = r < \min\{\frac{1}{2}\operatorname{dist}(x_0, \partial\Omega), \frac{\delta}{2}\}$, *the following inequality is valid:*

$$|u(x)| \le Cr^{-\tau'}, \tag{3.8}$$

where

$$\tau' = \tau'(p^-_{x_0,\delta}, p^+_{x_0,\delta}, q^-_{x_0,\delta}, \varepsilon') > \frac{p^+_{x_0,\delta}}{q^-_{x_0,\delta} - p^+_{x_0\delta} + 1} > 0,$$

$\varepsilon' \in (0, \frac{q^-_{x_0,\delta}-p^+_{x_0,\delta}+1}{q^-_{x_0,\delta}-p^-_{x_0,\delta}+1})$ *is a constant and* $C = C(n, \mu, p^-_{x_0,\delta}, p^+_{x_0,\delta}, q^+_{x_0,\delta}, q^-_{x_0,\delta})$. *Moreover,*

$$p^+_{x_0,\delta} = \sup_{y\in \overline{B(x_0,\delta)\cap\Omega}} p(y), \quad p^-_{x_0,\delta} = \inf_{y\in \overline{B(x_0,\delta)\cap\Omega}} p(y), \quad q^+_{x_0,\delta} = \sup_{y\in \overline{B(x_0,\delta)\cap\Omega}} q(y), \quad q^-_{x_0,\delta} = \inf_{y\in \overline{B(x_0,\delta)\cap\Omega}} q(y).$$

4 The removability of isolated singular points

The following theorem is the main theorem in this paper.

Theorem 4.1. *Let conditions* (1.2)–(1.6) *and* (2.1) *be fulfilled. Let u be a solution of equation* (1.1) *in* $\Omega \setminus \{x_0\}$. *Then, the singularity of* $u(x)$ *at the point* x_0 *is removable.*

Proof. We let $R_0 = \min\{\frac{1}{2}\operatorname{dist}(x_0, \partial\Omega), \frac{\delta}{2}\}$ and for $0 < r < R_0$ we denote $m(r) = \sup\{|u(x)| : r \le |x - x_0| \le R_0\}$. It is assumed that $\lim_{r\to 0} m(r) = \infty$, then there exists $0 < \rho < R_0$ such that $m(\rho) > 1$. For sufficiently small values $r \le \min\{\frac{1}{e^2}, R_0^2\}$, we define the function $\psi_r(x)$ as follows:

$$\begin{aligned} &\psi_r(x) \equiv 0 && \text{for } |x - x_0| < r,\\ &\psi_r(x) \equiv 1 && \text{for } |x - x_0| > \sqrt{r},\\ &\psi_r(x) = \frac{2}{\ln\frac{1}{r}} \ln\frac{|x - x_0|}{r} && \text{for } r < |x - x_0| < \sqrt{r}. \end{aligned}$$

For $r < |x - x_0| < \sqrt{r}$, we have

$$\begin{gathered} \frac{\partial}{\partial x_i}\psi_r(x) = \frac{2}{\ln\frac{1}{r}}\frac{\partial}{\partial x_i}\ln\frac{|x - x_0|}{r} = \frac{2(x_i - x_{0i})}{|x - x_0|^2 \ln\frac{1}{r}},\\ \left|\frac{\partial}{\partial x_i}\psi_r(x)\right| \le \frac{2}{|x - x_0|} \le \frac{2}{r} \end{gathered}$$

and

$$|\nabla\psi_r(x)| = \frac{2}{|x - x_0|^2 \ln\frac{1}{r}}\left[\sum_{i=1}^n (x_i - x_{0i})^2\right]^{\frac{1}{2}} = \frac{2}{|x - x_0| \ln\frac{1}{r}}.$$

We take the test function

$$\varphi(x) = \psi_r^\gamma(x)\left[\ln\frac{u}{m(\rho)}\right]_+ \tag{4.1}$$

for any $x \in \Omega_\rho$, where $\Omega_\rho = \{x \in \Omega : u(x) > m(\rho)\}$, $\gamma = \sup_{x\in\overline{\Omega}} \frac{p(x)q(x)}{q(x)-p(x)+1}$ is a constant and $\varphi(x) \equiv 0$ for $x \notin \Omega_\rho$.

Take $\overline{G} \subset \Omega\backslash\{x_0\}$. Since $u \in W^{1,p(x)}_{\mathrm{loc}}(\Omega\backslash\{x_0\}) \cap L^\infty_{\mathrm{loc}}(\Omega\backslash\{x_0\})$, we have

$$\int_G |\varphi(x)|^{p(x)}\,dx = \int_{G\cap\Omega_\rho} |\psi_r(x)|^{\gamma p(x)}\left(\ln\frac{u}{m(\rho)}\right)^{p(x)} dx \le \int_{G\cap\Omega_\rho}\left(\frac{u}{m(\rho)}\right)^{p(x)} dx < \infty$$

and

$$\begin{aligned} \int_G \left|\frac{\partial\varphi(x)}{\partial x_j}\right|^{p(x)} dx &= \int_{G\cap\Omega_\rho}\left[\gamma\psi_r^{\gamma-1}\frac{\partial\psi_r(x)}{\partial x_j}\left(\ln\frac{u}{m(\rho)}\right) + \psi_r^\gamma\frac{1}{u}\frac{\partial u}{\partial x_j}\right]^{p(x)} dx\\ &\le C\int_{G\cap\Omega_\rho} \gamma^{p(x)}\psi_r^{(\gamma-1)p(x)}\left|\frac{\partial\psi_r(x)}{\partial x_j}\right|^{p(x)}\left(\ln\frac{u}{m(\rho)}\right)^{p(x)} + \psi_r^{\gamma p(x)}\frac{1}{u^{p(x)}}\left|\frac{\partial u}{\partial x_j}\right|^{p(x)} dx\\ &\le C\int_{G\cap\Omega_\rho} \gamma^{p(x)}\left(\frac{2}{r}\right)^{p(x)}\left(\frac{u}{m(\rho)}\right)^{p(x)} + \frac{1}{(m(\rho))^{p(x)}}\left|\frac{\partial u}{\partial x_j}\right|^{p(x)} dx < \infty. \end{aligned}$$

Then, we obtain $\varphi(x) \in W^{1,p(x)}_{\mathrm{loc}}(\Omega\backslash\{x_0\}) \cap L^\infty_{\mathrm{loc}}(\Omega\backslash\{x_0\})$ with compact support in $\Omega\backslash\{x_0\}$.

For some $0 < \rho < R_0$, let the domain Ω_ρ be nonempty. Testing the equality (2.3) by the test function φ, we have

$$\int_{\Omega_\rho} \sum_{j=1}^{n} a_j(x,u,\nabla u)\frac{\partial u}{\partial x_j}\frac{\psi_r^\gamma}{u} + g(x,u)\psi_r^\gamma(x)\ln\Big(\frac{u}{m(\rho)}\Big)\,dx + \int_{\Omega_\rho}\sum_{j=1}^{n} a_j(x,u,\nabla u)\gamma\psi_r^{\gamma-1}(x)\frac{\partial \psi_r}{\partial x_j}\ln\Big(\frac{u}{m(\rho)}\Big)\,dx = 0.$$

By virtue of the conditions (1.2)–(1.5), we have

$$\int_{\Omega_\rho} \mu\frac{|\nabla u|^{p(x)}}{u}\psi_r^\gamma(x)\,dx + \int_{\Omega_\rho} u^{q(x)}\psi_r^\gamma(x)\ln\Big(\frac{u}{m(\rho)}\Big)\,dx \le n\mu^{-1}\gamma\int_{\Omega_\rho}|\nabla u|^{p(x)-1}|\nabla\psi_r|\psi_r^{\gamma-1}(x)\ln\Big(\frac{u}{m(\rho)}\Big)\,dx.$$

As

$$\begin{aligned} &n\mu^{-1}\gamma\int_{\Omega_\rho}|\nabla u|^{p(x)-1}|\nabla\psi_r|\psi_r^{\gamma-1}(x)\ln\Big(\frac{u}{m(\rho)}\Big)\,dx \\ &\quad\le C(n,\mu,\gamma,\varepsilon_3)\int_{\Omega_\rho} u^{p(x)-1}\psi_r^{\gamma-p(x)}|\nabla\psi_r|^{p(x)}\Big(\ln\frac{u}{m(\rho)}\Big)^{p(x)}\,dx + n\mu^{-1}\gamma\varepsilon_3\int_{\Omega_\rho}\psi_r^\gamma u^{-1}|\nabla u|^{p(x)}\,dx, \end{aligned}$$

take $\varepsilon_3 = \frac{\mu^2}{2n\gamma}$. Then,

$$\begin{aligned} &\frac{\mu}{2}\int_{\Omega_\rho}\frac{|\nabla u|^{p(x)}}{u}\psi_r^\gamma(x)\,dx + \int_{\Omega_\rho} u^{q(x)}\psi_r^\gamma(x)\ln\Big(\frac{u}{m(\rho)}\Big)\,dx \\ &\quad\le C(n,\mu,\gamma)\int_{\Omega_\rho} u^{p(x)-1}\psi_r^{\gamma-p(x)}|\nabla\psi_r|^{p(x)}\Big(\ln\frac{u}{m(\rho)}\Big)^{p(x)}\,dx. \end{aligned}$$

Furthermore,

$$\begin{aligned} &\int_{\Omega_\rho} u^{p(x)-1}\psi_r^{\gamma-p(x)}|\nabla\psi_r|^{p(x)}\Big(\ln\frac{u}{m(\rho)}\Big)^{p(x)}\,dx \\ &\quad\le C(\varepsilon_4)\int_{\Omega_\rho}\Big(\ln\frac{u}{m(\rho)}\Big)^{1+\frac{(p(x)-1)q(x)}{q(x)-p(x)+1}}|\nabla\psi_r|^{\frac{p(x)q(x)}{q(x)-p(x)+1}}\,dx + \varepsilon_4\int_{\Omega_\rho}\ln\Big(\frac{u}{m(\rho)}\Big)u^{q(x)}\psi_r^{\frac{(\gamma-p(x))q(x)}{p(x)-1}}\,dx. \end{aligned}$$

Take $\varepsilon_4 = \min\{\frac{1}{2C(n,\mu,\gamma)},\frac{1}{2}\}$. Since $\frac{(\gamma-p(x))q(x)}{p(x)-1} > \gamma$, $\psi_r(x)\le 1$ and $\gamma < n$, we have

$$\begin{aligned} &\frac{\mu}{2}\int_{\Omega_\rho}\frac{|\nabla u|^{p(x)}}{u}\psi_r^\gamma(x)dx + \frac{1}{2}\int_{\Omega_\rho} u^{q(x)}\psi_r^\gamma(x)\ln\Big(\frac{u}{m(\rho)}\Big)\,dx \\ &\quad\le C\int_{\Omega_\rho\cap\{x:r\le|x-x_0|<\sqrt{r}\}}\Big(\ln\frac{u}{m(\rho)}\Big)^{1+\frac{(p(x)-1)q(x)}{q(x)-p(x)+1}}|\nabla\psi_r|^{\frac{p(x)q(x)}{q(x)-p(x)+1}}\,dx \\ &\quad= C\int_{\Omega_\rho\cap\{x:r\le|x-x_0|<\sqrt{r}\}}\Big(\ln\frac{u}{m(\rho)}\Big)^{1+\frac{(p(x)-1)q(x)}{q(x)-p(x)+1}}\Big(\frac{2}{|x-x_0|\ln\frac{1}{r}}\Big)^{\frac{p(x)q(x)}{q(x)-p(x)+1}}\,dx \\ &\quad\le C\int_{\Omega_\rho\cap\{x:r\le|x-x_0|<\sqrt{r}\}}\big(\ln|x-x_0|^{-\tau'}\big)^{1+\frac{(p(x)-1)q(x)}{q(x)-p(x)+1}}\Big(\frac{2}{|x-x_0|\ln\frac{1}{r}}\Big)^{\frac{p(x)q(x)}{q(x)-p(x)+1}}\,dx \\ &\quad\le C\Big(\ln\frac{1}{r}\Big)^{-\frac{q^-_{x_0,\delta}p^-_{x_0,\delta}}{q^+_{x_0,\delta}-p^-_{x_0,\delta}+1}}\int_{\Omega_\rho\cap\{x:r\le|x-x_0|<\sqrt{r}\}}\Big(\ln\frac{1}{|x-x_0|}\Big)^{1+\frac{(p(x)-1)q(x)}{q(x)-p(x)+1}}\Big(\frac{1}{|x-x_0|}\Big)^{\frac{p(x)q(x)}{q(x)-p(x)+1}}\,dx \\ &\quad\le C\Big(\ln\frac{1}{r}\Big)^{-\frac{q^-_{x_0,\delta}p^-_{x_0,\delta}}{q^+_{x_0,\delta}-p^-_{x_0,\delta}+1}}\int_{\Omega_\rho\cap\{x:r\le|x-x_0|<\sqrt{r}\}}\Big(\ln\frac{1}{|x-x_0|}\Big)^{1+\frac{(p^+_{x_0,\delta}-1)q^+_{x_0,\delta}}{q^-_{x_0,\delta}-p^+_{x_0,\delta}+1}}\Big(\frac{1}{|x-x_0|}\Big)^{\gamma}\,dx \\ &\quad\le C\Big(\ln\frac{1}{r}\Big)^{-\frac{q^-_{x_0,\delta}p^-_{x_0,\delta}}{q^+_{x_0,\delta}-p^-_{x_0,\delta}+1}}\int_r^{\sqrt{r}}\Big(\frac{1}{t}\Big)^{\gamma}\Big(\ln\frac{1}{t}\Big)^{1+\frac{(p^+_{x_0,\delta}-1)q^+_{x_0,\delta}}{q^-_{x_0,\delta}-p^+_{x_0,\delta}+1}}t^{n-1}\,dt, \end{aligned}$$

where $C = C(n,\mu,\gamma,p^+_{x_0,\delta},p^-_{x_0,\delta},q^-_{x_0,\delta},q^+_{x_0,\delta})$.

Further, by (1.6), if $\gamma < n$, then

$$\begin{aligned}
&\Big(\ln\frac{1}{r}\Big)^{-\frac{q^-_{x_0,\delta}p^-_{x_0,\delta}}{q^+_{x_0,\delta}-p^-_{x_0,\delta}+1}}\int_r^{\sqrt{r}}\Big(\frac{1}{t}\Big)^{\gamma}\Big(\ln\frac{1}{t}\Big)^{1+\frac{(p^+_{x_0,\delta}-1)q^+_{x_0,\delta}}{q^-_{x_0,\delta}-p^+_{x_0,\delta}+1}}t^{n-1}\,dt\\
&\quad\le\Big(\ln\frac{1}{r}\Big)^{-\frac{q^-_{x_0,\delta}p^-_{x_0,\delta}}{q^+_{x_0,\delta}-p^-_{x_0,\delta}+1}}\Big(\ln\frac{1}{r}\Big)^{1+\frac{(p^+_{x_0,\delta}-1)q^+_{x_0,\delta}}{q^-_{x_0,\delta}-p^+_{x_0,\delta}+1}}\int_r^{\sqrt{r}}t^{n-1-\gamma}\,dt\\
&\quad=\Big(\ln\frac{1}{r}\Big)^{-\frac{q^-_{x_0,\delta}p^-_{x_0,\delta}}{q^+_{x_0,\delta}-p^-_{x_0,\delta}+1}}\Big(\ln\frac{1}{r}\Big)^{1+\frac{(p^+_{x_0,\delta}-1)q^+_{x_0,\delta}}{q^-_{x_0,\delta}-p^+_{x_0,\delta}+1}}\frac{1}{n-\gamma}r^{\frac{1}{2}(n-\gamma)}(1-r^{\frac{1}{2}(n-\gamma)})\to 0\quad\text{as } r\to 0.
\end{aligned}$$

Therefore, we obtain

$$\lim_{r\to 0}\frac{\mu}{2}\int_{\Omega_\rho}\frac{|\nabla u|^{p(x)}}{u}\psi_r^{\gamma}(x)\,dx+\frac{1}{2}\int_{\Omega_\rho}u^{q(x)}\psi_r^{\gamma}(x)\ln\Big(\frac{u}{m(\rho)}\Big)\,dx\le 0$$

and then

$$\mu\int_{\Omega_\rho}\frac{|\nabla u|^{p(x)}}{u}\,dx+\int_{\Omega_\rho}u^{q(x)}\ln\frac{u}{m(\rho)}\,dx=0.$$

Hence, $u(x)\equiv m(\rho)$ almost everywhere in Ω_ρ and the Lebesgue measure of Ω_ρ equals zero. Considering further the function $-u(x)$ instead of $u(x)$, we obtain the boundedness of $-u(x)$ in a neighborhood of the point x_0. Thus, we have proved that $u\in L^\infty(\Omega)$.

Next, we take the test function

$$\widetilde{\varphi}=\psi^{p^+}u,$$

where $\psi\equiv 1$ in $B(x_0,2\rho)\backslash B(x_0,\rho)$, $\psi\equiv 0$ outside $B(x_0,\frac{5\rho}{2})\backslash B(x_0,\frac{\rho}{2})$, $0\le\psi(x)\le 1$, $|\nabla\psi|\le\frac{C}{\rho}$ and $0<\rho\le 1$. Testing the equality (2.3) by the test function $\widetilde{\varphi}$, we have

$$\int_\Omega\sum_{j=1}^{n}a_j(x,u,\nabla u)\Big(p^+\psi^{p^+-1}\frac{\partial\psi}{\partial x_j}u+\psi^{p^+}\frac{\partial u}{\partial x_j}\Big)+g(x,u)\psi^{p^+}u\,dx=0.$$

By virtue of the conditions (1.2)–(1.5), we have

$$\begin{aligned}
&\int_{B(x_0,\frac{5\rho}{2})}\mu|\nabla u|^{p(x)}\psi^{p^+}+|u|^{q(x)+1}\psi^{p^+}\,dx\\
&\quad\le np^+\mu^{-1}\int_{B(x_0,\frac{5\rho}{2})}|\nabla u|^{p(x)-1}\psi^{p^+-1}|\nabla\psi||u|\,dx\\
&\quad=np^+\mu^{-1}\int_{B(x_0,\frac{5\rho}{2})}\Big[|\nabla\psi||u|\psi^{p^+-1-\frac{p^+}{p'(x)}}\Big]\Big[|\nabla u|^{p(x)-1}\psi^{\frac{p^+}{p'(x)}}\Big]\,dx\\
&\quad\le C(n,\mu,p^+,\varepsilon_5)\int_{B(x_0,\frac{5\rho}{2})}|\nabla\psi|^{p(x)}|u|^{p(x)}\psi^{p^+-p(x)}\,dx+np^+\mu^{-1}\varepsilon_5\int_{B(x_0,\frac{5\rho}{2})}|\nabla u|^{p(x)}\psi^{p^+}\,dx.
\end{aligned}$$

Take $\varepsilon_5=\frac{\mu^2}{2np^+}$. Then, we have

$$\begin{aligned}
\int_{B(x_0,\frac{5\rho}{2})}|\nabla u|^{p(x)}\psi^{p^+}\,dx&\le C(n,\mu,p^+)\int_{B(x_0,\frac{5\rho}{2})}|\nabla\psi|^{p(x)}|u|^{p(x)}\psi^{p^+-p(x)}\,dx\\
&\le C(n,\mu,p^+)\frac{1}{\rho^{p^+}}\max\big\{\|u\|_\infty^{p^+},\|u\|_\infty^{p^-}\big\}\Big|B\Big(x_0,\frac{5\rho}{2}\Big)\Big|\\
&\le C(n,\mu,p^+)\frac{1}{\rho^{p^+}}\omega_n\Big(\frac{5\rho}{2}\Big)^n=C(n,\mu,p^+)\rho^{n-p^+},
\end{aligned}$$

where ω_n is the volume of the unit ball. Further,

$$\int\limits_{B(x_0,2\rho)\setminus B(x_0,\rho)} |\nabla u|^{p(x)}\,dx \leq C(n,\mu,p^+)\rho^{n-p^+}$$

and then we obtain

$$\int\limits_{B(x_0,\rho)} |\nabla u|^{p(x)}\,dx = \sum_{j=1}^{\infty} \int\limits_{B(x_0,2^{1-j}\rho)\setminus B(x_0,2^{-j}\rho)} |\nabla u|^{p(x)}\,dx \leq C\sum_{j=1}^{\infty}(2^{-j}\rho)^{n-p^+} \leq C(n,\mu,p^+)\rho^{n-p^+} \to 0 \quad \text{as } \rho \to 0.$$

So $|\nabla u| \in L^{p(x)}(\Omega)$ and thus we have proved that $u \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$.

Next, we will show that $u(x)$ is a solution of equation (1.1) in the domain Ω. Let $\eta_\rho \in C_0^\infty(R^n)$ be the cutoff function for the ball $B(x_0,\rho)$ with $\eta_\rho \equiv 1$ in $B(x_0,\rho)$, $\eta_\rho \equiv 0$ outside the ball $B(x_0,2\rho)$, $|\nabla\eta_\rho| \leq \frac{C}{\rho}$ and $0 < \rho \leq 1$. Let $\varphi \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$. Testing the equation (2.3) by the test function $(1-\eta_\rho)\varphi$, we have

$$\int\limits_\Omega \sum_{j=1}^n a_j(x,u,\nabla u)\frac{\partial(1-\eta_\rho)\varphi}{\partial x_j}\,dx + \int\limits_\Omega g(x,u)(1-\eta_\rho)\varphi\,dx = 0,$$

that is

$$\int\limits_\Omega \sum_{j=1}^n a_j(x,u,\nabla u)\frac{\partial\varphi}{\partial x_j}(1-\eta_\rho)\,dx - \int\limits_\Omega \sum_{j=1}^n a_j(x,u,\nabla u)\frac{\partial\eta_\rho}{\partial x_j}\varphi\,dx + \int\limits_\Omega g(x,u)(1-\eta_\rho)\varphi\,dx = 0.$$

Indeed,

$$\begin{aligned}\Bigg|\sum_{j=1}^n a_j(x,u,\nabla u)\frac{\partial\varphi}{\partial x_j}(1-\eta_\rho)\Bigg| &\leq n\mu^{-1}|\nabla u|^{p(x)-1}|\nabla\varphi| \\ &\leq n\mu^{-1}\Big(\frac{p(x)-1}{p(x)}|\nabla u|^{p(x)} + \frac{1}{p(x)}|\nabla\varphi|^{p(x)}\Big) \in L^1(\Omega),\end{aligned}$$

therefore we have

$$\lim_{\rho\to 0}\int\limits_\Omega \sum_{j=1}^n a_j(x,u,\nabla u)\frac{\partial\varphi}{\partial x_j}(1-\eta_\rho)\,dx = \int\limits_\Omega \sum_{j=1}^n a_j(x,u,\nabla u)\frac{\partial\varphi}{\partial x_j}\,dx.$$

In the same way,

$$\lim_{\rho\to 0}\int\limits_\Omega g(x,u)(1-\eta_\rho)\varphi\,dx = \int\limits_\Omega g(x,u)\varphi\,dx.$$

Since

$$\begin{aligned}&\Bigg|\int\limits_\Omega \sum_{j=1}^n a_j(x,u,\nabla u)\frac{\partial\eta_\rho}{\partial x_j}\varphi\,dx\Bigg| \\ &\quad\leq \frac{Cn}{\mu\rho}\int\limits_{B(x_0,2\rho)\setminus B(x_0,\rho)} |\nabla u|^{p(x)-1}\varphi\,dx \\ &\quad\leq \frac{C(n,\mu)}{\rho}\int\limits_{B(x_0,2\rho)\setminus B(x_0,\rho)} |\nabla u|^{p(x)-1}\,dx \\ &\quad\leq \frac{C(n,\mu)}{\rho}\big\||\nabla u|^{p(x)-1}\big\|_{L^{\frac{p(x)}{p(x)-1}}(B(x_0,2\rho)\setminus B(x_0,\rho))}\|1\|_{L^{p(x)}(B(x_0,2\rho)\setminus B(x_0,\rho))} \\ &\quad\leq \frac{C(n,\mu)}{\rho}\max\Bigg\{\Bigg[\int\limits_{B(x_0,2\rho)\setminus B(x_0,\rho)} |\nabla u|^{p(x)}\,dx\Bigg]^{\frac{p^--1}{p^+}}, \Bigg[\int\limits_{B(x_0,2\rho)\setminus B(x_0,\rho)} |\nabla u|^{p(x)}\,dx\Bigg]^{\frac{p^+-1}{p^-}}\Bigg\}\cdot|B(x_0,2\rho)\setminus B(x_0,\rho)|^{\frac{1}{p^+}} \\ &\quad\leq \frac{C(n,\mu,p^+)}{\rho}\rho^{\frac{(p^--1)(n-p^+)}{p^+}}(\rho^n)^{\frac{1}{p^+}} = C(n,\mu,p^+)\rho^{\frac{p^-(n-p^+)}{p^+}} \to 0 \quad \text{as } \rho \to 0.\end{aligned}$$

So we have obtained that equality (2.3) is fulfilled for any test function. Therefore, an isolated singular point x_0 is removable for solutions of equation (1.1). $\square$

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