Research Article

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Maximum principles, Liouville-type theorems and symmetry results for a general class of quasilinear anisotropic equations

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Abstract: This paper is concerned with a general class of quasilinear anisotropic equations. We first derive some maximum principles for two appropriate P-functions, in the sense of Payne (see the book of Sperb [18]). These maximum principles are then employed to obtain a Liouville-type result and a Serrin–Weinberger-type symmetry result.

Keywords: Maximum principles, anisotropic equations, Liouville theorems, symmetry, Wulff shapes

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1 Introduction

Let $F\colon \mathbb{R}^N \to [0,\infty)$, $N \geq 2$, be a positive homogeneous function of degree 1, with the following properties:

$$\begin{cases} F \in C^{3,\alpha}_{\mathrm{loc}}(\mathbb{R}^N \setminus \{\mathbf{0}\}),\ \alpha \in (0,1), \\ F(\xi) > 0 \quad \text{for any } \xi \in \mathbb{R}^N \setminus \{\mathbf{0}\}. \end{cases} \tag{1.1}$$

We also have $F(\mathbf{0}) = 0$, since F is homogeneous and defined at the origin. Let $G \in C^{3,\alpha}_{\mathrm{loc}}(0,\infty) \cap C^1[0,\infty)$ be a function such that

$$\begin{cases} G(0) = 0, \quad G'(0) = 0, \\ G(t) > 0, \quad G'(t) > 0, \quad G''(t) > 0 \quad \text{for any } t > 0. \end{cases} \tag{1.2}$$

Let us also assume that either of the following conditions is satisfied (see [2]):

(I) There exists $p > 1$, $k \in [0,1)$, $\gamma > 0$ and $\Gamma > 0$ such that, for any $\xi \in \mathbb{R}^N \setminus \{0\}$, $\zeta \in \mathbb{R}^N$,

$$[\mathrm{Hess}(G \circ F)(\xi)]_{ij}\zeta_i\zeta_j \geq \gamma(k + |\xi|)^{p-2}|\zeta|^2,$$

and

$$\sum_{i,j=1}^{N} \big|[\mathrm{Hess}(G \circ F)(\xi)]_{ij}\big| \leq \Gamma(k + |\xi|)^{p-2}.$$

(II) The composition $G \circ F$ is of class $C^{3,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ and, for any $R > 0$, there exists a positive constant ρ such that for any $\xi, \zeta \in \mathbb{R}^N$ with $|\xi| \leq R$, we have

$$[\mathrm{Hess}(G \circ F)(\xi)]_{ij}\zeta_i\zeta_j \geq \rho|\zeta|^2.$$

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Given a domain $\Omega \subset \mathbb{R}^N$, $\widetilde{f} \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R})$, we now consider the Wolf-type energy functional

$$W_\Omega(u) := \int_\Omega [G(F(\nabla(\mathbf{x}))) - \widetilde{f}(u(\mathbf{x}))]\,dx$$

and the quasilinear anisotropic operator

$$Q(u) := \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(G'(F(\nabla u))F_{\xi_i}(\nabla u)).$$

It is known that the critical points of W_Ω weakly satisfy the Euler–Lagrange equation

$$Qu + f(u) = 0 \quad \text{on } \Omega,$$

where $f := \widetilde{f}'$.

We immediately notice that when $G(t) = t^p/p$, Q is the anisotropic p-Laplace operator, while when $G(t) = \sqrt{1+t^2} - 1$, Q is the anisotropic mean curvature operator.

The first main result of this paper is a Liouville-type (see the survey paper of Farina [5] or the papers [3, 15, 16]), theorem for the solutions to the equation

$$Qu + f(u) = 0 \quad \text{on } \mathbb{R}^N, \tag{1.3}$$

which states the following.

Theorem 1.1. *Assume that F, G verify* (1.1)–(1.2), $f \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R})$ *and one of the following conditions is satisfied:*
(1) *Assumption* (I) *holds and $u \in L^\infty(\mathbb{R}^N) \cap W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ is a weak solution of* (1.3).
(2) *Assumption* (II) *holds and $u \in W^{1,\infty}(\mathbb{R}^N)$ weakly solves* (1.3).
Moreover, assume that there exists a C^2 function $\psi: (0,\infty) \to (0,\infty)$ such that

$$\psi''\psi - (\psi')^2\Big[2 - \frac{N}{N-1}\Big(1 - \frac{G}{tG'}\Big)\Big] \geq 0, \tag{1.4}$$

$$f' \leq -\frac{\psi'}{\psi}\frac{N}{N-1}\Big(1 - \frac{G}{tG'}\Big)f \quad \text{on } (0,\infty). \tag{1.5}$$

If $\inf_{\mathbb{R}^N} u(\mathbf{x}) > 0$, then $u(\mathbf{x})$ must be identically constant.

The second main result of our paper is a Serrin-type symmetry result. More precisely, let us consider the following overdetermined anisotropic boundary value problem:

$$\begin{cases} Qu = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ F(\nabla u) = c & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbb{R}^N$ is a connected bounded open set and $c > 0$ is a given constant. Problem (1.6) is overdetermined, so that in general it has no solution. On the other hand, we note that it is not difficult to verify that, under appropriate conditions for F and G, if Ω is a Wulff shape (see Theorem 1.3 below), then problem (1.6) has a unique solution. The delicate task now remains to determine if the overdetermined condition in (1.6) is also necessary for the existence of a solution. The investigation of such problems has raised a good deal of attention in the last decades. The first fundamental contribution is due to Serrin [17] (see, also, the paper of Weinberger [21]) , who consider the case $F(s) = |s|$. His result states that if problem (1.6) has a C^2 solution, then Ω is a ball. This result was later extended by many authors to more general equations, see, for instance, [6, 8–10] (when Q is a quasilinear elliptic operator) or [14, 19] (when Q is an anisotropic operator). Our result is more general and states the following.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a connected bounded open set in $\mathbb{R}^N$, with sufficiently smooth boundary $\partial\Omega$. Assume that the functions F and G satisfy assumptions* (1.1)–(1.2) *and also that one of the conditions* (I) *or* (II) *is satisfied. If problem* (1.6) *has a weak solution $u \in C^1_0(\overline{\Omega})$ then, up to a translation, $\partial\Omega$ is of Wulff shape.*

The converse of the above theorem is also satisfied. More precisely, let us denote

$$F^{\circ}(\mathbf{x}) = \sup_{\xi \neq 0} \frac{\langle \mathbf{x}, \xi \rangle}{F(\xi)},$$

the polar of F (see [8] for more details). We have the following theorem.

Theorem 1.3. *Assume that F and G satisfy the assumptions of Theorem 1.2 and*

$$\Omega := \{\mathbf{x} \in \mathbb{R}^N : F^{\circ}(\mathbf{x}) < NG'(c)\}.$$

Then, problem (1.6) *admits a unique solution of the form*

$$u(\mathbf{x}) = \int_{F^{\circ}(x)}^{NG'(c)} \mathcal{G}(s/N)\, ds,$$

where $\mathcal{G}$ is the inverse of the map $t \to G'(t)$.

The outline of the paper is as follows. The proofs of Theorems 1.1 and 1.2 rely on some key ingredients involving P-functions, which are presented in Section 2. In Section 3 we make use of some maximum principle for P-functions, previously established in Section 2, to prove the Liouville-type result stated in Theorem 1.1. In Section 4 we derive a Pohozaev-type integral identity and give the proofs of Theorems 1.2 and 1.3.

For convenience, notice that throughout the paper the prime is used to indicate differentiation, the comma is used to indicate partial derivatives and the summation from 1 to N is understood on repeated indices. Moreover, when appropriate we use the following notations:

$$\begin{cases} F := F(\nabla u), \quad F_i := F_{\xi_i} = \dfrac{\partial F}{\partial \xi_i}, \quad G' := G'(F(\nabla u)), \quad G'' := G''(F(\nabla u)), \\ a_{ij}(\nabla u)(\mathbf{x}) := \dfrac{\partial^2}{\partial \xi_i \partial \xi_j}(G \circ F)(\nabla u)(\mathbf{x}) = G' F_{ij} + G'' F_i F_j, \\ a_{ijk}(\nabla u)(\mathbf{x}) := \dfrac{\partial^3}{\partial \xi_i \partial \xi_j \partial \xi_k}(G \circ F)(\nabla u)(\mathbf{x}), \end{cases} \tag{1.7}$$

where $i, j, k \in \{1, \dots, N\}$.

2 Maximum principles for appropriate *P*-functions

Let F, G, u and ψ be as in the statement of Theorem 1.1. We introduce the following P-function:

$$P(u, \mathbf{x}) = [G'(F(\nabla u(\mathbf{x})))F - G(F(\nabla u(\mathbf{x})))]\psi(u(\mathbf{x})). \tag{2.1}$$

We then have the following maximum principle.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a connected, bounded open set and let $u(\mathbf{x})$ be as in Theorem 1.1. Assume that* $\inf_{\overline{\Omega}} |\nabla u(\mathbf{x})| > 0$. *If there exists $\mathbf{x}_0 \in \Omega$ such that*

$$P(u, \mathbf{x}_0) = \max_{x \in \overline{\Omega}} P(u, \mathbf{x}),$$

then $P(u, \cdot)$ is identically constant in Ω.

For the proof of Theorem 2.1, the following two lemmas will play an important role.

Lemma 2.2. *If $F \in C^3(\mathbb{R}^N \setminus \{\mathbf{0}\})$ is a positive homogeneous function of degree 1, then we have*

$$F_i(\xi)\xi_i = F(\xi), \quad F_{ij}(\xi)\xi_i = 0, \quad F_{ijk}(\xi)\xi_i = -F_{jk}(\xi)$$

for any $\xi \in \mathbb{R}^N \setminus \{\mathbf{0}\}$ and $i, j, k \in \{1, \dots, N\}$.

For the proof of Lemma 2.2, we refer the reader to [7, Appendix, Lemma 3].

Lemma 2.3. *Let $u(\mathbf{x})$ be as in Theorem 1.1. Then, we have*

$$a_{ij}a_{kl}u_{ik}u_{jl} \geq \frac{(a_{ij}u_{ij})^2}{N} + \frac{N}{N-1}\left(\frac{a_{ij}u_{ij}}{N} - G''F_iF_ju_{ij}\right)^2. \tag{2.2}$$

For a proof of Lemma 2.3, in the particular case $G(t) = t^2$, we refer the reader to [20]. Using similar arguments one may easily prove inequality (2.2) for a general function $G(t)$.

Proof of Theorem 2.1. First of all we remark that u is in $C^3(\mathbb{R}^N \setminus \mathcal{C})$, where $\mathcal{C} := \{\mathbf{x} \in \mathbb{R}^N : \nabla u(\mathbf{x}) = 0\}$ (see [2, Proposition 3.1 and Proposition 3.2]), therefore $u \in C^3(\overline{\Omega})$.

The proof is mainly based on the construction of an elliptic differential inequality for the P-function defined in (2.1). The conclusion of the theorem will then follow immediately, as a direct consequence of Hopf's first maximum principle (see [18]). To this end, we compute successively

$$\begin{aligned} P_i &= G''FF_ku_{ki}\psi + H\psi'u_i, &(2.3)\\ P_{ij} &= G^{(3)}FF_lF_ku_{ki}u_{lj}\psi + G''F_lF_ku_{lj}u_{ki}\psi + G''FF_{kl}u_{lj}u_{ki}\psi + G''FF_ku_{kij}\psi \\ &\quad + G''FF_ku_{ki}u_j\psi' + G''FF_lu_{lj}\psi'u_i + H\psi''u_iu_j + H\psi'u_{ij}, &(2.4) \end{aligned}$$

where, for simplicity, we denote $H := G'F - G$. Next, making use of notation (1.7), we can see that equation (1.3) may be rewritten as follows:

$$a_{ij}u_{ij} = [G'F_{ij} + G''F_iF_j]u_{ij} = -f. \tag{2.5}$$

From Lemma 2.2, we also have

$$F_iu_i = F, \quad F_{ij}u_j = 0, \quad F_{ijk}u_i = -F_{jk} \tag{2.6}$$

for any $i, j, k \in \{1, \dots, N\}$. Therefore, making use of (2.4), (2.5) and (2.6), we evaluate

$$\begin{aligned} a_{ij}P_{ij} &= G^{(3)}G'FF_lF_kF_{ij}u_{ki}u_{lj}\psi + G^{(3)}G''FF_lF_kF_iF_ju_{ki}u_{lj}\psi + G''G'F_{ij}F_lF_ku_{lj}u_{ki}\psi \\ &\quad + (G'')^2F_kF_lF_iF_ju_{lj}u_{ki}\psi + G''G'FF_{ij}F_{lk}u_{lj}u_{ki}\psi + (G'')^2FF_iF_jF_{lk}u_{lj}u_{ik}\psi \\ &\quad + G''FF_ka_{ij}u_{kij}\psi + 2(G'')^2FF_kF_iF_ju_{ik}u_j\psi' + 2G''G'FF_kF_{ij}u_{ik}u_j\psi' \\ &\quad + G'HF_{ij}u_iu_j\psi'' + G''HF_iF_ju_iu_j\psi'' - Hf\psi'. \end{aligned} \tag{2.7}$$

We are now going to compute each term of (2.7). First, we note that from (2.3) and (2.6) we get

$$F_ku_{ki} = -\frac{H\psi'u_i}{G''F\psi} + (\text{terms containing } P_s), \tag{2.8}$$

$$F_iF_ku_{ki} = -\frac{H\psi'}{G''\psi} + (\text{terms containing } P_s). \tag{2.9}$$

Moreover, making use of the third equation in (2.6) and (2.9) in (2.5), we obtain

$$G'F_{ij}u_{ij} = -G'F_{ijl}u_{li}u_j = -f + \frac{H\psi'}{\psi} + (\text{terms containing } P_s). \tag{2.10}$$

Next, differentiating (2.5), we find

$$2G''F_{il}F_ju_{lk}u_{ij} + G''F_lF_{ij}u_{lk}u_{ij} + G'F_{ijl}u_{lk}u_{ij} + G^{(3)}F_iF_lF_ju_{lk}u_{ij} + a_{ij}u_{ijk} = -f'u_k. \tag{2.11}$$

Now, substituting $a_{ij}u_{ijk}$ from (2.11) into (2.7), we get

$$\begin{aligned} a_{ij}P_{ij} &= G^{(3)}G'FF_lF_kF_{ij}u_{ki}u_{lj}\psi + G^{(3)}G''FF_lF_kF_iF_ju_{ki}u_{lj}\psi + G''G'F_{ij}F_lF_ku_{lj}u_{ki}\psi \\ &\quad + (G'')^2F_kF_lF_iF_ju_{lj}u_{ki}\psi + G''G'FF_{ij}F_{lk}u_{lj}u_{ki}\psi + (G'')^2FF_iF_jF_{lk}u_{lj}u_{ik}\psi \\ &\quad + 2(G'')^2FF_kF_iF_ju_{ik}u_j\psi' + G''HF^2\psi'' - Hf\psi' - G''F^2f'\psi - (G'')^2FF_kF_lF_{ij}u_{lk}u_{ij}\psi \\ &\quad - G''G'FF_kF_{ijl}u_{lk}u_{ij}\psi - G^{(3)}G''FF_kF_lF_iF_ju_{lk}u_{ij}\psi - 2(G'')^2FF_kF_jF_{il}u_{lk}u_{ij}\psi, \end{aligned}$$

where we have used the first two identities from (2.6). Next, from (2.8)–(2.10), we infer

$$\begin{aligned}
a_{ij}P_{ij} &= G''G'FF_{ij}F_{lk}u_{lj}u_{ki}\psi + \frac{H^2(\psi')^2}{\psi} + HG''F^2\psi'' - 2\frac{G''F^2(\psi')^2H}{\psi} - H\psi' f - G''F^2 f'\psi \\
&\quad + \frac{G''FH\psi'}{G'}\Big(\frac{H\psi'}{\psi} - f\Big) + HG'F_{ijl}u_{lk}u_l\psi' + (\text{terms containing } P_s) \\
&= G''G'FF_{ij}F_{lk}u_{lj}u_{ki}\psi + HG''F^2\psi'' - 2\frac{G''F^2(\psi')^2H}{\psi} \\
&\quad + FH^2\frac{G''}{G'}\frac{(\psi')^2}{\psi} - \frac{G''}{G'}HFf\psi' - G''F^2f'\psi' + (\text{terms containing } P_s),
\end{aligned}$$

so that

$$\begin{aligned}
a_{ij}P_{ij} =& G''G'FF_{ij}F_{kl}u_{lj}u_{ki}\psi + G''FH\Big(F\psi'' - 2F\frac{(\psi')^2}{\psi} + \frac{F}{G'}\frac{(\psi')^2}{\psi}\Big) \\
& - G''F\Big(\frac{H}{G'}\psi' f + Ff'\psi\Big) + (\text{terms containing } P_s).
\end{aligned} \tag{2.12}$$

Now, the term $F_{ij}F_{kl}u_{lj}u_{ki}$ can be computed as follows:

$$\begin{aligned}
F_{ij}F_{kl}u_{lj}u_{ki} &= \Big(\frac{a_{ij}}{G'} - \frac{G''}{G'}F_iF_j\Big)\Big(\frac{a_{kl}}{G'} - \frac{G''}{G'}F_kF_l\Big)u_{lj}u_{ki} \\
&= \frac{1}{G'^2}a_{ij}a_{kl}u_{lj}u_{ki} + \frac{(G'')^2}{G'^2}\Big[-\frac{H\psi'}{G''\psi} + (\text{terms containing } P_s)\Big]^2 \\
&\quad - 2\frac{G''}{G'}\Big[F_{kl} + \frac{G''}{G'}F_kF_l\Big]\Big[\frac{H^2\psi'^2}{(G'')^2F^2\psi^2}u_lu_k + (\text{terms containing } P_s)\Big] \\
&= \frac{a_{ij}a_{kl}u_{lj}u_{ki}}{G'^2} - \frac{G^2\psi'^2}{G'^2\psi^2} + (\text{terms containing } P_s),
\end{aligned} \tag{2.13}$$

where we have used (2.6), (2.8) and (2.9). Thus, from (2.13) and (2.12) it follows that

$$a_{ij}P_{ij} = \frac{G''}{G'}F\psi a_{ij}a_{kl}u_{lj}u_{ki} + G''F^2H\Big(\psi'' - 2\frac{(\psi')^2}{\psi}\Big) - G''F\Big(\frac{H}{G'}\psi' f + Ff'\psi\Big) + (\text{terms containing } P_s). \tag{2.14}$$

Now, Lemma 2.3, equation (2.5) and (2.9) lead to

$$\begin{aligned}
a_{ij}a_{kl}u_{lj}u_{ki} &\geq \frac{f^2}{N} + \frac{N}{N-1}\Big(\frac{f}{N} + \frac{H\psi'}{\psi} + (\text{terms containing } P_s)\Big)^2 \\
&= \frac{f^2}{N-1} + \frac{2Hf\psi'}{(N-1)\psi} + \frac{N}{N-1}\frac{H^2\psi'^2}{\psi^2} + (\text{terms containing } P_s).
\end{aligned} \tag{2.15}$$

Consequently, making use of (2.15) in (2.14), we obtain

$$\begin{aligned}
a_{ij}P_{ij} \geq f^2\frac{G''F\psi}{G'(N-1)} &+ G''F^2H\Big(\psi'' - 2\frac{(\psi')^2}{\psi} + \frac{NH}{(N-1)FG'}\frac{(\psi')^2}{\psi}\Big) \\
&- G''F^2\Big(\frac{H}{FG'}\frac{N+1}{N-1}f\psi' + f'\psi\Big) + (\text{terms containing } P_s).
\end{aligned}$$

Thus, under assumptions (1.4)–(1.5), we have the inequality

$$a_{ij}P_{ij} + (\text{terms containing } P_s) \geq 0 \quad \text{in } \Omega,$$

and Theorem 2.1 follows as a direct consequence of Hopf's first maximum principle (see [18]). □

In order to prove Theorem 1.2, let us consider another functional defined as follows:

$$\widetilde{P}(u, \mathbf{x}) := G'(F(\nabla u(\mathbf{x})))F - G(F(\nabla u(\mathbf{x}))) + \frac{1}{N}u(\mathbf{x}). \tag{2.16}$$

We have the following theorem.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^N$ be a connected bounded open set with sufficiently smooth boundary and let $u \in C_0^1(\overline{\Omega})$ be a weak solution of problem* (1.6). *Then, $\widetilde{P}(u, \mathbf{x})$ takes its maximum value on $\partial\Omega$. Moreover, if $\widetilde{P}$ is not constant in Ω, then any maximum point for $\widetilde{P}$ in Ω is necessarily a critical point for u.*

Proof. Under the assumptions concerning the functions F and G, we know that $\widetilde{P} \in C^3(\Omega \setminus \mathcal{C})$, where $\mathcal{C} := \{\mathbf{x} \in \Omega : \nabla u(\mathbf{x}) = \mathbf{0}\}$. The following computations are all considered in $\Omega \setminus \mathcal{C}$.

By using similar arguments and computations as in the proof of Theorem 2.1, one may see that

$$a_{ij}\widetilde{P}_{ij} = G''G'FF_{ij}F_{kl}u_{lj}u_{ki} - \frac{1}{N}\frac{G''}{G'} + \frac{1}{N^2}\frac{G''}{G'}F + (\text{terms containing } P_s). \tag{2.17}$$

Moreover, using the fact that

$$F_{ij}F_{kl}u_{lj}u_{ki} = \frac{1}{G'^2}\left(a_{ij}a_{kl}u_{lj}u_{ki} - \frac{1}{N^2}\right) + (\text{terms containing } P_s), \tag{2.18}$$

and substituting (2.18) into (2.17), we get

$$a_{ij}\widetilde{P}_{ij} = \frac{G''F}{G'}\left(a_{ij}a_{kl}u_{lj}u_{ki} - \frac{1}{N}\right) + (\text{terms containing } P_s). \tag{2.19}$$

Now, since the matrix (a_{ij}) is symmetric and positive define in $\mathbb{R}^N \setminus \{\mathbf{0}\}$ for a fixed point $\mathbf{x}$, we can choose coordinates around $\mathbf{x}$ such that $a_{ij}(\mathbf{x}) = \lambda_i\delta_{ij}$ with $\lambda_i > 0$ for any i. Obviously, the first equation in (1.6) is rewritten as $\lambda_i u_{ii} = -1$, and therefore (2.19) becomes

$$\begin{aligned} a_{ij}\widetilde{P}_{ij} + (\text{terms containing } P_s) &= \frac{G''F}{G'}\left(\lambda_i\lambda_j u_{ij}^2 - \frac{1}{N}\right) \\ &\geq \frac{G''F}{G'}\left(\lambda_i^2 u_{ii}^2 - \frac{1}{N}\right) \\ &\geq \frac{G''F}{NG'}[(\lambda_i u_{ii})^2 - 1] = 0. \end{aligned}$$

According to the maximum principle we have that $\max_{\overline{\Omega\setminus\mathcal{C}}} \widetilde{P} = \max_{\partial\Omega} \widetilde{P}$. Indeed, if we suppose that there exists a point $x_0 \in \Omega$ such that $\widetilde{P}(x_0) > \max_{\partial\Omega} \widetilde{P}$, then x_0 belongs to the interior of $\mathcal{C}$. From the first equation in (1.6), the interior of $\mathcal{C}$ is empty, thus $\widetilde{P}$ achieves its maximum over $\overline{\Omega}$ on $\partial\Omega$. Moreover, if $\widetilde{P}$ is not constant in Ω, then any maximum point for $\widetilde{P}$ in Ω belongs necessarily to $\mathcal{C}$. This completes the proof. □

3 Proof of Theorem 1.1

For the proof we adapt an idea employed by Caffarelli, Garofalo and Segala [1] to obtain a different Liouville-type theorem (see also the recent works [2, 7]).

First of all, we notice that if u satisfies the statement of Theorem 1.1, then $u \in C^3(\mathbb{R}^N)$ (for clear proofs of these statements, see [2, Proposition 3.1 and Proposition 3.2]). Now, let us define the following set:

$$\mathcal{S}_u := \left\{v \text{ satisfies (1.3)} : \inf_{\mathbb{R}^N} u \leq v(\mathbf{x}) \leq \sup_{\mathbb{R}^N} u \text{ for all } \mathbf{x} \in \mathbb{R}^N\right\},$$

which is compact in the topology of $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ (see [11, 13]). Let us now define

$$P_0 := \sup\{P(v; \mathbf{x}) : v \in \mathcal{S}_u,\ \mathbf{x} \in \mathbb{R}^N\}. \tag{3.1}$$

We claim that $P_0 \equiv 0$. From this, Theorem 1.1 follows immediately. To this end, we argue by contradiction and assume contrariwise that

$$P_0 > 0.$$

By (3.1), there exist two sequences $(v_k)_{k\in\mathbb{N}} \subset \mathcal{S}_u$ and $(\mathbf{x}_k)_{k\in\mathbb{N}} \subset \mathbb{R}^N$ such that

$$\lim_{k\to\infty} P(v_k, \mathbf{x}_k) = P_0 > 0.$$

Consider now the functions $w_k(\mathbf{x}) := v_k(\mathbf{x} + \mathbf{x}_k)$. One can easily checks that $w_k \in \mathcal{S}_u$, $P(v_k, \mathbf{x}_k) = P(w_k, \mathbf{0})$ and

$$\lim_{k\to\infty} P(v_k, \mathbf{x}_k) = \lim_{k\to\infty} P(w_k, \mathbf{0}) = P_0 > 0.$$

Moreover, up to a subsequence, we can suppose that there exists $w \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N) \cap \mathcal{S}_u$ such that

$$\lim_{k\to\infty} w_k = w \quad \text{and} \quad \lim_{k\to\infty} P(w_k, \mathbf{0}) = P(w, \mathbf{0}) = P_0 > 0.$$

In particular, we find that $F(\nabla w(\mathbf{0})) \neq \mathbf{0}$ and therefore $\nabla w(\mathbf{0}) \neq 0$. By continuity, there exists $\rho > 0$ such that

$$\inf_{B_\rho(\mathbf{0})} |\nabla u| > 0.$$

On the other hand, by the definition of P_0, we know that

$$P(w; \mathbf{x}) \leq P_0 = P(w, \mathbf{0}) \quad \text{for all } \mathbf{x} \in B_\rho(\mathbf{0}),$$

so that $\mathbf{0}$ is a local maximum for $P(w, \cdot)$ in $B_\rho(\mathbf{0})$. Then, Theorem 2.4 implies that $P(w; \cdot)$ is constant in $B_\rho(\mathbf{0})$. By the continuity of $P(w; \cdot)$ and connectedness arguments one can see that $P(w; \cdot)$ is constant on the whole $\mathbb{R}^N$. More exactly, we have

$$P(w; \mathbf{x}) := (G'F - G)(F(\nabla w(\mathbf{x})))\psi(w(\mathbf{x})) = P_0 > 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}^N. \tag{3.2}$$

Next, since w is bounded on $\mathbb{R}^N$, we must have $\inf_{\mathbb{R}^N} |\nabla w| = 0$.

Let $(\mathbf{y}_k)_{k\in\mathbb{N}} \subset \mathbb{R}^N$ be a sequence such that $\lim_{k\to\infty} |\nabla w(\mathbf{y}_k)| = 0$. From (3.2) we get

$$(G'F - G)(F(\nabla w(\mathbf{y}_k)))\psi(w(\mathbf{y}_k)) = P_0 > 0. \tag{3.3}$$

Letting $k \to \infty$ in (3.3) and taking into account that have $\inf_{\mathbb{R}^N} w(\mathbf{x}) = 0$, we get $P_0 = 0$, which contradicts our assumption. Therefore, we have $P_0 \equiv 0$ on $\mathbb{R}^N$. On the other hand, from (1.2) and (1.4) one can easily see that $(G'(t)t - G(t))\psi(t) > 0$ for any $t > 0$, therefore $\nabla u \equiv \mathbf{0}$ on $\mathbb{R}^N$, and thus the proof of Theorem 1.1 is achieved. □

In what follows we give some examples of functions ψ and f which verify assumptions (1.4)–(1.5). More precisely, we look for functions of the form $\psi(u) = u^{-\beta_N}$, where β_N is a positive constant. Let us define

$$\theta(t) := 1 - \frac{G}{tG'}, \quad \theta_0 := \inf_{t>0} \theta(t).$$

From these definitions one may immediately notice that $\theta, \theta_0 \in [0, 1)$, so that $k_N := (2 - \frac{N}{N-1}\theta_0) > 0$ for $N > 1$. Obviously, if the following inequalities hold:

$$\psi''\psi \geq k_N(\psi')^2, \tag{3.4}$$

$$f' \leq -\theta_0 \frac{\psi'}{\psi} \frac{N+1}{N} f \quad \text{for any } t > 0, \tag{3.5}$$

then conditions (1.4)–(1.5) are satisfied. Replacing $\psi = u^{-\beta_N}$ into (3.4) and (3.5), we obtain

$$1 \geq \beta_N(1 - k_N),$$

respectively,

$$f' \leq \theta_0 \beta_N \frac{f}{t} \frac{N+1}{N} \quad \text{for any } t > 0. \tag{3.6}$$

Thus, as $k_N > 1 \Leftrightarrow \theta_0 < \frac{N-1}{N}$, one can choose

$$\beta_N := \frac{1}{k_N - 1} \quad \text{for } \theta_0 < \frac{N-1}{N},$$
$$\beta_N \geq 0 \quad \text{for } \theta_0 \geq \frac{N-1}{N},$$

and see that (3.6) can be rewritten as

$$f' \leq \begin{cases} \dfrac{\theta_0(N+1)}{N(1-\theta_0)-1}\dfrac{f}{t} & \text{if } \theta_0 < \dfrac{N-1}{N}, \\ L\dfrac{f}{t} & \text{if } \theta_0 \geq \dfrac{N-1}{N}, \end{cases} \tag{3.7}$$

for any $t > 0$, where $L \geq 0$ is an arbitrary constant.

Summarizing, we obtain that the conclusion of Theorem 1.1 holds if f satisfies (3.7).

Let us now consider some particular examples for G. If $G := t^p/p,\ p > 1$, then $\theta = \theta_0 = 1 - \frac{1}{p}$ and (3.7) becomes

$$f'(t) \leq \begin{cases} (p-1)\dfrac{N+1}{N-p}\dfrac{f(t)}{t} & \text{if } 1 < p < N, \\ L\dfrac{f(t)}{t} & \text{if } p \geq N, \end{cases}$$

for all $t > 0$. On the other hand, if $G(t) := \sqrt{1+t^2} - 1$, we have $\theta = 1/(\sqrt{1+t^2}+1)$ and $\theta_0 = 0$. In this case, (3.7) implies that f must be a non-decreasing function.

4 Proof of Theorems 1.2 and 1.3

4.1 A Pohozaev-type identity

In what follows we establish a Pohozaev-type integral identity for $\widetilde{P}$. More precisely, we have the following lemma.

Lemma 4.1. *If u is given as in Theorem 1.2, then the auxiliary function $\widetilde{P}$, defined in (2.16), satisfies the Pohozaev-type identity*

$$\int_\Omega \widetilde{P}(u(\mathbf{x}); \mathbf{x})\, d\mathbf{x} = (cG'(c) - G(c))|\Omega|, \tag{4.1}$$

where $|\Omega|$ is the n-dimensional volume of Ω. Moreover, $\widetilde{P}$ is identically constant in Ω.

Proof. Let us multiply the first equation in (1.6) with u and integrate it by parts. Then, we obtain

$$\int_\Omega u\, d\mathbf{x} = \int_\Omega G'(F(\nabla u))F(\nabla u)\, d\mathbf{x}. \tag{4.2}$$

Taking into account the assumptions of Theorem 1.2, we can apply [4, Theorem 2] in order to obtain

$$\begin{aligned} &\int_{\partial\Omega} G(F(\nabla u))\langle \mathbf{x}, \nu\rangle\, d\sigma - \int_{\partial\Omega} G'(F(\nabla u))\nabla_\xi F(\nabla u)\cdot \nabla u \langle \mathbf{x}, \nu\rangle\, d\sigma \\ &\quad = N\int_\Omega G(F(\nabla u)))\, d\mathbf{x} - \int_\Omega G'(F(\nabla u))\nabla u \nabla_\xi F(\nabla u)\, d\mathbf{x} - \int_\Omega \nabla_\xi G(F(\nabla u))\cdot \nabla u\, d\mathbf{x} + \int_\Omega \langle \mathbf{x}, \nabla u\rangle\, d\mathbf{x} + \int_\Omega u\, d\mathbf{x} \\ &\quad = N\int_\Omega G(F(\nabla u))\, d\mathbf{x} - \int_\Omega G'(F(\nabla u))F(\nabla u)\, d\mathbf{x} + \int_\Omega \langle \mathbf{x}, \nabla u\rangle\, d\mathbf{x}, \end{aligned} \tag{4.3}$$

where we have used (2.6) and (4.2). Making use of the known identities

$$\int_{\partial\Omega} \langle \mathbf{x}, \nu\rangle\, d\mathbf{x} = N|\Omega|,$$

$$\int_\Omega \langle \mathbf{x}, \nabla u\rangle\, d\mathbf{x} = -N\int_\Omega u\, d\mathbf{x}$$

together with (4.3), and taking into account the last equation in (1.6), we derive

$$\begin{aligned} N(G(c) - G'(c)c)|\Omega| &= N\int_\Omega (G(F(\nabla u)) - G'(F(\nabla u))F(\nabla u))\,d\mathbf{x} + (N-1)\int_\Omega G'(F(\nabla u))F(\nabla u)d\mathbf{x} + \int_\Omega \langle \mathbf{x}, \nabla u\rangle\,d\mathbf{x} \\ &= N\int_\Omega \Big(G(F(\nabla u)) - G'(F(\nabla u))F(\nabla u) - \frac{1}{N}u\Big)\,d\mathbf{x}, \end{aligned}$$

from which (4.1) follows.

Finally, from Theorem 2.4 and equality (4.1), one can conclude that

$$\widetilde{P} \equiv G'(c)c - G(c) \quad \text{on } \overline{\Omega},$$

and the proof is completed. □

4.2 Proof of Theorem 1.2

If $\widetilde{P} \equiv G'(c)c - G(c)$ on $\overline{\Omega}$, then

$$\Lambda(F(\nabla u(\mathbf{x}))) := G'(F(\nabla u(\mathbf{x})))F(\nabla u(\mathbf{x})) - G(F(\nabla u(\mathbf{x}))) = \Lambda(c) - \frac{1}{N}u(\mathbf{x}) \quad \text{on } \overline{\Omega}, \tag{4.4}$$

where $\Lambda(t) := tG'(t) - G(t)$, $t > 0$. Now, the strict monotonicity of the map $t \to \Lambda(t)$ on $(0, \infty)$ can be used to write $F(\nabla u)$ explicitly as a function of u, i.e.,

$$F(\nabla u(\mathbf{x})) = \Lambda^{-1}\Big(\Lambda(c) - \frac{1}{N}u(\mathbf{x})\Big) := g(u(\mathbf{x})), \tag{4.5}$$

where $g(u)$ is of class C^1, according to assumptions (1.2). Next, we can see that ∇u vanishes only at points where u attains its maximum on Ω and, also, that u is positive in Ω.

Indeed, if $\nabla u(\mathbf{x}_0) = 0$, then by (4.4) we have $N\Lambda(c) = u(\mathbf{x}_0)$. On the other hand,

$$u(\mathbf{x}) = N(\Lambda(c) - \Lambda(F(\nabla u(\mathbf{x})))) \le u(c) \quad \text{in } \Omega,$$

so $u(\mathbf{x}_0) = \max_{\overline{\Omega}} u$. Also, u is positive on Ω. Otherwise, there exists $\mathbf{x}_1 \in \Omega$ with $u(\mathbf{x}_1) = \min_{\overline{\Omega}} u \le 0 < N\Lambda(c)$, hence $\nabla u(\mathbf{x}_1)$ does not vanish, which is a contradiction. Hence, $\nu = \frac{\nabla u}{|\nabla u|}$ is well defined on the open set $U := \{\mathbf{x} \in \Omega : 0 < u(\mathbf{x}) < \max_{\overline{\Omega}} u\}$.

On the other hand, denoting by $H_F(S_t)$ the H-mean curvature of the level set of u, $S_t := \{\mathbf{x} \in \overline{\Omega} : u(\mathbf{x}) = t\}$, we have (see [19, Theorem 3])

$$Q(u(\mathbf{x})) = G(F(\nabla u))H_F(S_t) + G''(F(\nabla u))\frac{\partial^2 u}{\partial \nu_F^2} \tag{4.6}$$

for all $\mathbf{x}$ such that $u(\mathbf{x}) = t$, where ν_F is the normal of S_t with respect to F, namely $\nu_F = F_\xi(\nabla u)$.

Now from (4.6) and (4.5), $H_F(S_t)$ depends only on u if $\frac{\partial^2 u}{\partial \nu_F^2}$ depends only on u. Taking into account the first equation in (2.6) and (4.5), one can see that

$$\begin{aligned} \frac{\partial u}{\partial \nu_F} &= \nabla u \cdot F_\xi(\nabla u) = g(u), \\ \frac{\partial}{\partial \nu_F}\Big(\frac{\partial u}{\partial \nu_F}\Big)^2 &= 2\frac{\partial^2 u}{\partial \nu_F^2}\frac{\partial u}{\partial \nu_F} = \frac{\partial}{\partial \nu_F}(g^2(u)) = 2g(u)g'(u)\frac{\partial u}{\partial \nu_F}. \end{aligned} \tag{4.7}$$

In consequence, as

$$\frac{\partial u}{\partial \nu_F} \ne 0 \quad \text{on } U,$$

we find from (4.7) that

$$\frac{\partial^2 u}{\partial \nu_F^2} = g(u)g'(u) \quad \text{on } U.$$

Therefore, using (4.6) we deduce

$$H_F(S_t) = -\frac{1}{G(g(u))}\left(1 + G''(g(u))g(u)g'(u)\right) \quad \text{for all } 0 < t < \max_{\overline{\Omega}} u.$$

But this equality says that every level set of u at height between 0 and $\max_{\overline{\Omega}} u$ is a set of constant H_F-mean curvature. Therefore, each connected component of it must be a Wulff shape, up to a translation (see [12]). In particular, Ω must be simply connected, because otherwise we have a foliation of $N-1$ submanifolds $\mathcal{T}_t$, $t \in (0, t_0)$, for some small positive constant $t_0 < N\Lambda(c)$ with the property that $\mathcal{T}_t$ is one of the connected component of S_t. If the level set of u has another connected component $\mathcal{T}_t'$, then one of them must be enclosed in the other. Since both of them are of Wulff shape of equal radius, we conclude that $S_t = \mathcal{T}_t$, $t \in (0, t_0)$, has only one connected component of Wulff shape.

Therefore, $\partial\Omega = S_0$ is also a Wulff shape and this concludes the proof. □

It is worth pointing out that there exists at most one solution $u \in C_0^1(\overline{\Omega})$ of problem (1.6). Indeed, it is sufficient to point out that any solution of (1.6) is a critical point of the following functional:

$$\mathcal{W}_\Omega(u) = \int_\Omega \left(G(F(\nabla u(\mathbf{x}))) - u(\mathbf{x})\right) d\mathbf{x}. \tag{4.8}$$

Due to assumption (1.2), (I) or (II), the map $G \circ F$ is strictly convex, therefore u must coincide with its unique minimizer.

4.3 Proof of Theorem 1.3

Taking into account the previous remarks, we know that if problem (1.6) admits a solution, then it is unique and coincides with the minimizer of the functional given in (4.8). According to standard arguments, we are looking for a minimizer of the form $u(\mathbf{x}) = u(F^\circ(\mathbf{x})) := v(r)$, where $r = F^\circ(\mathbf{x})$ is decreasing as function of r.

First let us note that

$$\mathcal{W}_\Omega(u) = N\omega_N \int_0^1 \left(G(F(v'(r)\nabla F^\circ(\mathbf{x}))) - v(r)\right) r^{N-1}\, dr = N\omega_N \int_0^1 \left(G(-v'(r)) - v(r)\right) r^{N-1}\, dr,$$

where we have used the equality $F(\nabla F^\circ(\mathbf{x})) = 1$, the positively homogeneity of F of degree 1 as well as the fact that $v'(r) < 0$.

Next, we remark that the corresponding Euler equation of our problem is

$$\left(G'(-v'(r))r^{N-1}\right)' = r^{N-1} \quad \text{on } [0, R]. \tag{4.9}$$

Integrating (4.9) from 0 to r, we get

$$G'(-v'(r)) = \frac{r}{N}. \tag{4.10}$$

Hence, if $\mathcal{G}$ is the inverse function of $t \to G'(t)$, we have $v'(r) = -\mathcal{G}(r/N)$, which implies

$$u(\mathbf{x}) = \int_{F^\circ(\mathbf{x})}^{R} \mathcal{G}\left(\frac{s}{N}\right) ds \quad \text{for any } \mathbf{x} \in \Omega.$$

Now, writing (4.10) for $r = R$, and making use of the third equality in (1.7), we get $R = NG'(c)$.

Finally, the formula obtained for u implies $u = 0$ and

$$\nabla u(\mathbf{x}) = -\mathcal{G}\left(\frac{F^\circ(\mathbf{x})}{N}\right)\nabla F^\circ(\mathbf{x}) \Rightarrow F(\nabla u(\mathbf{x})) = \mathcal{G}\left(\frac{R}{N}\right) = \mathcal{G}(G'(c)) = c$$

on $\{\mathbf{x} \in \mathbb{R}^N : F^\circ(\mathbf{x}) = NG'(c)\}$. This completes the proof. □

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