Research Article

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A new perspective on the Riesz potential

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Abstract: This paper offers a new perspective to look at the Riesz potential. On the one hand, it is shown that not only $\mathfrak{L}^{q,qp^{-1}(n-\alpha p)}\cap\mathfrak{L}^{p,\kappa-\alpha p}$ contains $I_\alpha L^{p,\kappa}$ under the conditions $1<p<\infty$, $1\le q<\infty$, $q(\kappa/p-\alpha)\le\kappa\le n$, $0<\alpha<\min\{n,1+\kappa/p\}$, but also $\mathfrak{L}^{q,\lambda}$ exists as an associate space under the condition $-q<\lambda<n$, where $I_\alpha L^{p,\kappa}$ and $\mathfrak{L}^{q,\lambda}$ are the Morrey–Sobolev and Campanato spaces on $\mathbb{R}^n$ respectively. On the other hand, a nonnegative Radon measure μ is completely characterized to produce a continuous map $I_\alpha : L_{p,1}\to L^q_\mu$ under the condition $1<p<\min\{q,n/\alpha\}$ or $1<q\le p<\min\{q(n-\alpha p)/(n-\alpha q(q-1)^{-1}),n/\alpha\}$, where $L_{p,1}$ and L^q_μ are the $(p,1)$-Lorentz and (q,μ)-Lebesgue spaces on $\mathbb{R}^n$ respectively.

Keywords: Riesz potential, Radon measure, Morrey and Campanato spaces

MSC 2010: 31B15, 46E30

Introduction

For $n\in\mathbb{N}$ and $(\alpha,p)\in(0,n)\times(1,\infty)$ let ν and L^p_{loc} be the n-dimensional Lebesgue measure and its induced locally p-Lebesgue integrable space respectively. We are interested in the Riesz potentials

$$I_\alpha f(x)=\int_{\mathbb{R}^n}|x-y|^{\alpha-n}f(y)\,d\nu(y)\quad\text{for all } f\in L^p_{\mathrm{loc}}.$$

Roughly speaking, $I_\alpha L^p_{\mathrm{loc}}$ is regarded as the space of tempered distributions having derivatives of order α in L^p_{loc}.

Below is the classical Lebesgue–Sobolev space inclusion which plays a very important role in analysis and partial differential equations (cf., e.g., [4, 19, 21]):

$$I_\alpha L_{p,1}\subset I_\alpha L^p\subset\begin{cases} L^{\frac{n}{\frac{n}{p}-\alpha}} & \text{as } \alpha<\frac{n}{p},\\ \mathrm{BMO} & \text{as } \alpha=\frac{n}{p},\\ C^{\alpha-\frac{n}{p}} & \text{as } 1+\frac{n}{p}>\alpha>\frac{n}{p}.\end{cases}$$

Here and henceforth, $L_{p,1}$ (as a subspace of L^p), BMO and C^β are respectively the $(p,1)$-Lorentz space, the John–Nirenberg space of functions with bounded mean oscillation and the Lipschitz space of order β – more precisely –

$$\begin{cases} f\in L_{p,1} & \Longleftrightarrow\quad \|f\|_{L_{p,1}}=\displaystyle\int_0^\infty\big(\nu(\{x\in\mathbb{R}^n:|f(x)|\ge t\})\big)^{\frac{1}{p}}\,dt<\infty,\\ f\in \mathrm{BMO} & \Longleftrightarrow\quad \|f\|_{\mathrm{BMO}}=\displaystyle\sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} r^{-n}\int_{B(x,r)}|f-f_{B(x,r)}|\,d\nu<\infty,\\ f\in C^\beta & \Longleftrightarrow\quad \|f\|_{C^\beta}=\displaystyle\sup_{x\neq y\text{ in }\mathbb{R}^n}|f(x)-f(y)||x-y|^{-\beta}<\infty\quad\text{for }\beta\in(0,1),\end{cases}$$

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and

$$f_{B(x,r)} = \big(\nu(B(x,r))\big)^{-1} \int\limits_{B(x,r)} f\,d\nu$$

is the ν-integral mean value of f over $B(x,r)$ – the x-centred Euclidean ball with radius r.

As a matter of fact, the above inclusion for BMO and C^β can be obtained from an elementary computation: since

$$\int\limits_{B(x,r)} \int\limits_{B(x,r)} \big||w-y|^{\alpha-n} - |w-z|^{\alpha-n}\big|\,d\nu(y)\,d\nu(z) \lesssim \frac{r^{2n+1}}{(r+|w|)^{n+1-\alpha}}$$

holds for all $(x,r,w) \in \mathbb{R}^n \times (0,\infty) \times \mathbb{R}^n$ (where $\mathsf{A} \lesssim \mathsf{B}$ means that there is a constant $c>0$ such that $\mathsf{A} \le c\mathsf{B}$), it follows that

$$\int\limits_{B(x,r)} |(I_\alpha f) - (I_\alpha f)_{B(x,r)}|\,d\nu \lesssim r^\alpha \|f\|_{L^1}.$$

Using the Fubini theorem and the Hölder inequality yields that if $p' = p/(p-1)$ and $0 < \alpha < \min\{n, 1 + n/p\}$, then

$$\begin{aligned}
\int\limits_{B(x,r)} |(I_\alpha f) - (I_\alpha f)_{B(x,r)}|\,d\nu &\le \big(\nu(B(x,r))\big)^{-1} \int\limits_{B(x,r)}\int\limits_{B(x,r)} |(I_\alpha f)(y) - (I_\alpha f)(z)|\,d\nu(y)\,d\nu(z) \\
&\lesssim r^{-n} \int\limits_{\mathbb{R}^n} \Bigg(|f(w)| \int\limits_{B(x,r)}\int\limits_{B(x,r)} \big||y-w|^{\alpha-n} - |z-w|^{\alpha-n}\big|\,d\nu(y)\,d\nu(z)\Bigg)\,d\nu(w) \\
&\lesssim r^{-n}\|f\|_{L^p} \Bigg(\int\limits_{\mathbb{R}^n}\Bigg(\int\limits_{B(x,r)}\int\limits_{B(x,r)} \big||y-w|^{\alpha-n} - |z-w|^{\alpha-n}\big|\,d\nu(y)\,d\nu(z)\Bigg)^{p'} d\nu(w)\Bigg)^{\frac{1}{p'}} \\
&\lesssim r^{n+1}\|f\|_{L^p}\Bigg(\int\limits_{\mathbb{R}^n} (r+|w|)^{p'(\alpha-n-1)}\,d\nu(w)\Bigg)^{\frac{1}{p'}} \\
&\lesssim r^{n+\alpha-\frac{n}{p}}\|f\|_{L^p}.
\end{aligned}$$

Interestingly, the preceding calculation of mean oscillation actually tells us that the Lebesgue–Sobolev space $I_\alpha L^p$ is contained in the Campanato space $\mathfrak{L}^{1,n/p-\alpha}$ (cf. Theorem 1 for a detailed information), and moreover suggests exploiting not only the associate space $\mathfrak{H}^{\infty,n/p-\alpha}$ of $\mathfrak{L}^{1,n/p-\alpha}$ (cf. Theorem 2 for a complete account) but also the $\mathfrak{H}^{q',\lambda}$-Sobolev space $I_\alpha\mathfrak{H}^{q',\lambda}$ whose precise description is presented in Theorem 3 which may be regarded as an associate form of Theorem 1. Even more interestingly, the oscillatory nature of $I_\alpha L^p$ motivates us to characterize a nonnegative Radon measure μ on $\mathbb{R}^n$ such that I_α continuously maps $L_{p,1}$ into the (q,μ)-Lebesgue space L^q_μ; see Theorem 4.

1 Morrey–Sobolev spaces in Campanato spaces

Significantly, the foregoing Lebesgue–Sobolev space inclusion can be extended to the following Morrey–Sobolev space inclusion:

$$I_\alpha L^{p,\kappa} \subset \begin{cases} L^{\frac{\kappa}{\frac{\kappa}{p}-\alpha},\kappa} \cap L^{p,\kappa-\alpha p} & \text{as } \alpha < \frac{\kappa}{p}, \\ \mathrm{BMO} & \text{as } \alpha = \frac{\kappa}{p}, \\ C^{\alpha-\frac{\kappa}{p}} & \text{as } 1 + \frac{\kappa}{p} > \alpha > \frac{\kappa}{p}, \end{cases}$$

where $L^{p,\gamma}$ is the Morrey space of order (p,γ) – more precisely –

$$f \in L^{p,\gamma} \iff \|f\|_{L^{p,\gamma}} = \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} \Bigg(r^{\gamma-n}\int\limits_{B(x,r)} |f|^p\,d\nu\Bigg)^{\frac{1}{p}} < \infty \quad \text{under } \gamma \in (-\infty,\infty).$$

The cases $\alpha \le \kappa/p$ of the last inclusion are due to Adams (cf. [2, 4]). To check the case $1 + \kappa/p > \alpha > \kappa/p$, set $\beta = \alpha - \kappa/p \in (0, 1)$. Because I_βBMO is contained in C^β according to Strichartz's papers [22, 23], it follows that (cf. [6, Remark 9])

$$I_\alpha L^{p,\kappa} = I_\beta I_{\kappa/p} L^{p,\kappa} \subset I_\beta \mathrm{BMO} \subset C^\beta = C^{\alpha-\frac{\kappa}{p}}.$$

Observe that BMO can be treated as the limit space of both $L^{\kappa/(\kappa/p-\alpha),\kappa}$ as $\alpha \to (\kappa/p)^-$ and $C^{\alpha-\kappa/p}$ as $\alpha \to (\kappa/p)^+$. So the following new inclusion seems to be a natural matter upon unifying three spaces $L^{\kappa/(\kappa/p-\alpha)}$, BMO and $C^{\alpha-\kappa/p}$ or embedding the Morrey potentials $I_\alpha L^{p,\kappa}$ into the so-called Campanato spaces.

Theorem 1. *For $(q, \lambda) \in [1, \infty) \times (-\infty, \infty)$ let $\mathfrak{L}^{q,\lambda}$ be the Campanato space of all ν-measurable functions f on $\mathbb{R}^n$ obeying*

$$\|f\|_{\mathfrak{L}^{q,\lambda}} = \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} \Big(r^{\lambda-n} \int_{B(x,r)} |f - f_{B(x,r)}|^q \, d\nu \Big)^{\frac{1}{q}} < \infty.$$

If

$$1 < p < \infty, \quad 0 < \kappa \le n, \quad qp^{-1}(\kappa - \alpha p) \le \kappa, \quad 0 < \alpha < \min\Big\{n, 1 + \frac{\kappa}{p}\Big\},$$

then

$$I_\alpha L^{p,\kappa} \subset \mathfrak{L}^{q,qp^{-1}(\kappa-\alpha p)} \cap \mathfrak{L}^{p,\kappa-\alpha p}.$$

Proof. The inclusion

$$I_\alpha L^{p,\kappa} \subset \mathfrak{L}^{p,\kappa-\alpha p}$$

follows from the inclusion stated above since (cf. [13, p. 70])

$$L^{p,\kappa-\alpha p} \subset \mathfrak{L}^{p,\kappa-\alpha p} = \begin{cases} \mathrm{BMO} & \text{as } \kappa = \alpha p, \\ C^{\alpha-\frac{\kappa}{p}} & \text{as } p + \kappa > \alpha p > \kappa. \end{cases}$$

In order to verify the inclusion

$$I_\alpha L^{p,\kappa} \subset \mathfrak{L}^{q,qp^{-1}(\kappa-\alpha p)}$$

let us handle two cases below.

Case $\kappa/p > \alpha$. Using the Hölder inequality we obtain that if $p_1 > p_2$, then

$$\int_{B(x,r)} |f|^{p_2} \, d\nu \lesssim \Big(\int_{B(x,r)} |f|^{p_1} \, d\nu \Big)^{\frac{p_2}{p_1}} r^{n(1-\frac{p_1}{p_2})} \lesssim \|f\|^{p_2}_{L^{p_1,\kappa}} r^{n-\frac{\kappa p_1}{p_2}} \quad \text{for all } (x, r) \in \mathbb{R}^n \times (0, \infty),$$

in other words,

$$L^{p_1,\kappa} \subset L^{p_2,\frac{\kappa p_1}{p_2}} \subset \mathfrak{L}^{p_2,\frac{\kappa p_1}{p_2}}.$$

As a consequence, we get

$$L^{\frac{\kappa}{\frac{\kappa}{p}-\alpha},\kappa} \subset L^{\frac{\lambda}{\frac{\kappa}{p}-\alpha},\lambda} \quad \text{as } \lambda = q\Big(\frac{\kappa}{p} - \alpha\Big) \le \kappa \le n.$$

Now, via the Adams inclusion

$$I_\alpha L^{p,\kappa} \subset L^{\frac{\kappa}{\frac{\kappa}{p}-\alpha},\kappa}$$

we have

$$I_\alpha L^{p,\kappa} \subset L^{q,q(\frac{\kappa}{p}-\alpha)} \subset \mathfrak{L}^{q,q(\frac{\kappa}{p}-\alpha)} \quad \text{under } q\Big(\frac{\kappa}{p} - \alpha\Big) \le \kappa.$$

Case $\kappa/p \le \alpha < 1 + \kappa/p$. Again, according to [13, p. 70] we have

$$\mathfrak{L}^{q,q(\frac{\kappa}{p}-\alpha)} = \begin{cases} \mathrm{BMO} & \text{as } \frac{\kappa}{p} = \alpha, \\ C^{\alpha-\frac{\kappa}{p}} & \text{as } \frac{\kappa}{p} < \alpha < 1 + \frac{\kappa}{p}. \end{cases}$$

So the desired inclusion under this situation is valid. □

2 Pre-associate spaces of Campanato spaces

To get a better understanding of Theorem 1, it is appropriate to point out that $L^{p,\kappa}$ and $\mathfrak{L}^{q,\lambda}$ actually exist as associate spaces. For $\kappa \in [0,n)$, $p \in [1,\infty)$ and $p' = p/(p-1) \in (1,\infty]$ denote by $H^{p',\kappa}$ the space of all functions $f \in L^{p'}_{\mathrm{loc}}$ obeying

$$\|f\|_{H^{p',\kappa}} = \inf\Big\{\sum_j |c_j| : f = \sum_{j=1}^{\infty} c_j a_j\Big\} < \infty,$$

where $\{a_j\}$ are $(p', (1-\kappa/n)(1-1/p'))$-atoms. Recall that such an atom a is defined to be a function enjoying that $\operatorname{supp}(a)$, the support of a, is contained in a ball B and

$$\|a\|_{L^{p'}} \le (\nu(B))^{-(1-\frac{\kappa}{n})(1-\frac{1}{p'})}.$$

According to [5, Theorems 2.3–3.3] and their proofs, we have that $H^{p',\kappa}$ is the associate space of $L^{p,\kappa}$ – in other words (cf. [20, Theorem 4.1]) – if

$$[H^{p',\kappa}]^\star := \Big\{f \in L^p_{\mathrm{loc}} : \|f\|_{[H^{p',\kappa}]^\star} = \sup_{\|g\|_{H^{p',\kappa}}\le 1} \Big|\int_{\mathbb{R}^n} fg\,d\nu\Big| < \infty\Big\},$$

then $[H^{p',\kappa}]^\star = L^{p,\kappa}$. Since $L^{p,\kappa} = \{0\}$ under $\kappa < 0$, this last identity can be trivially extended to $\kappa < 0$ via setting $H^{p',\kappa} = \{0\}$. However, this triviality will not appear in discovering the pre-associate space of a Campanato space as seen below.

Theorem 2. *Let $q \in [1,\infty)$ and $q' = q/(q-1) \in (1,\infty]$.*

(i) *For $\lambda \in [0,n)$ write $\mathfrak{H}^{q',\lambda}$ as the space of all functions $f \in L^{q'}_{\mathrm{loc}}$ satisfying*

$$\|f\|_{\mathfrak{H}^{q',\lambda}} = \inf\Big\{\sum_j |c_j| : f = \sum_{j=1}^{\infty} c_j a_j\Big\} < \infty,$$

where each a_j is a $(q', (1-\lambda/n)(1-1/q'), 0)$-atom, i.e., a $(q', (1-\lambda/n)(1-1/q'))$-atom with

$$\int_{\mathbb{R}^n} a_j\,d\nu = 0.$$

(ii) *For $\lambda \in (-q,0]$ and $\bar q = n/(n-\lambda/q)$ write $\mathfrak{H}^{q',\lambda}$ as the space of all functions $f \in L^{q'}_{\mathrm{loc}}$ satisfying*

$$\|f\|_{\mathfrak{H}^{q',\lambda}} = \inf\Big\{\Big(\sum_j |c_j|^{\bar q}\Big)^{\frac{1}{\bar q}} : f = \sum_{j=1}^{\infty} c_j a_j\Big\} < \infty,$$

where each a_j is a $(q', (1-\lambda/n)(1-1/q'), -\lambda n/q)$-atom, i.e., a $(q', (1-\lambda/n)(1-1/q'))$-atom with all moments

$$\int_{\mathbb{R}^n} x_1^{\gamma_1}\cdots x_n^{\gamma_n} a_j(x)\,d\nu(x) = 0 \quad \text{for } \sum_{i=1}^n \gamma_i \le -\frac{\lambda n}{q}.$$

Then $\mathfrak{H}^{q',\lambda}$ is the pre-associate space of $\mathfrak{L}^{q,\lambda}$ – namely – if

$$[\mathfrak{H}^{q',\lambda}]^\star := \Big\{f \in L^q_{\mathrm{loc}} : \|f\|_{[\mathfrak{H}^{q',\lambda}]^\star} = \sup_{\|g\|_{\mathfrak{H}^{q',\lambda}}\le 1} \Big|\int_{\mathbb{R}^n} fg\,d\nu\Big| < \infty\Big\},$$

then $[\mathfrak{H}^{q',\lambda}]^\star = \mathfrak{L}^{q,\lambda}$.

Proof. Under (ii), the desired associate relation says essentially that the dual space of the Hardy space $H^{\bar q}$ of Fefferman–Stein [12] is identical to BMO under $\bar q = 1$ and is identical to $C^{-\lambda/q}$ under $\bar q < 1$ (cf. [9–11]). Therefore it is enough to check the associate relation under (i). However, under $(q,\lambda) \in (1,\infty) \times [0,n)$ the associate relation follows from Zorko's duality [27, Proposition 5], so it remains to deal with the case $(q,\lambda) \in \{1\} \times [0,n)$.

On the one hand, if $f \in \mathfrak{L}^{1,\lambda}$ and $g = \sum_j c_j a_j \in \mathfrak{H}^{\infty,\lambda}$ with $\sum_j |c_j| < \infty$ and a_j being an $(\infty, 1-\lambda/n, 0)$-atom (supported in a ball B_j), then

$$\Big|\int_{\mathbb{R}^n} f a_j \, dv\Big| = \Big|\int_{\mathbb{R}^n} (f - f_{B_j}) a_j \, dv\Big| \leq \big(v(B_j)\big)^{\frac{\lambda-n}{n}} \int_{B_j} |f - f_{B_j}| \, dv \lesssim \|f\|_{\mathfrak{L}^{1,\lambda}},$$

and hence

$$\Big|\int_{\mathbb{R}^n} fg \, dv\Big| \leq \sum_j |c_j| \Big|\int_{\mathbb{R}^n} f a_j \, dv\Big| \lesssim \|f\|_{\mathfrak{L}^{1,\lambda}} \sum_j |c_j|,$$

namely, each $f \in \mathfrak{L}^{1,\lambda}$ induces a member of the associate space $[\mathfrak{H}^{\infty,\lambda}]^\star$.

On the other hand, suppose $f \in [\mathfrak{H}^{\infty,\lambda}]^\star$. For a given ball $B(x,r)$ let c be a constant such that

$$B_{f\geq c}(x,r) = \{x \in B(x,r) : f(x) \geq c\}, \quad B_{f<c}(x,r) = B(x,r) \setminus B_{f\geq c}(x,r)$$

obeys

$$v\big(B_{f<c}(x,r)\big) \geq v\big(B_{f\geq c}(x,r)\big).$$

The existence of such a constant c follows from the basic equation

$$v(B(x,r)) = v\big(B_{f<c}(x,r)\big) + v\big(B_{f\geq c}(x,r)\big).$$

Without loss of generality, we may assume

$$\int_{B_{f\geq c}(x,r)} |f - c| \, dv \geq \int_{B_{f<c}(x,r)} |f - c| \, dv.$$

Then, choosing

$$a(y) = \big(v(B(x,r))\big)^{\frac{\lambda-n}{n}} \begin{cases} 1 & \text{as } y \in B_{f\geq c}(x,r), \\ -\Big(\frac{v(B_{f\geq c}(x,r))}{v(B_{f<c}(x,r))}\Big) & \text{as } y \in B_{f<c}(x,r), \end{cases}$$

we find that a is an $(\infty, (1-\lambda/n), 0)$-atom with

$$\|a\|_{\mathfrak{H}^{\infty,\lambda}} \leq \|a\|_{L^\infty} \big(v(B(x,r))\big)^{\frac{n-\lambda}{n}} \leq 1 \quad \text{and} \quad \int_{B_{f<c}(x,r)} (f-c) a \, dv \geq 0.$$

Consequently, we compute

$$\begin{aligned}
\int_{B(x,r)} |f - f_{B(x,r)}| \, dv &\leq 2 \int_{B(x,r)} |f - c| \, dv \\
&= 2\Big(\int_{B_{f\geq c}(x,r)} |f - c| \, dv + \int_{B_{f<c}(x,r)} |f - c| \, dv\Big) \\
&\leq 4 \int_{B_{f\geq c}(x,r)} |f - c| \, dv \\
&= 4\big(v(B(x,r))\big)^{\frac{n-\lambda}{n}} \int_{B_{f\geq c}(x,r)} (f-c) a \, dv \\
&\leq 4\big(v(B(x,r))\big)^{\frac{n-\lambda}{n}} \Big(\int_{B_{f\geq c}(x,r)} (f-c) a \, dv + \int_{B_{f<c}(x,r)} (f-c) a \, dv\Big) \\
&= 4\big(v(B(x,r))\big)^{\frac{n-\lambda}{n}} \int_{B(x,r)} (f-c) a \, dv \\
&\leq 4\big(v(B(x,r))\big)^{\frac{n-\lambda}{n}} \Big|\int_{\mathbb{R}^n} f a \, dv\Big|,
\end{aligned}$$

whence reading off

$$(\nu(B(x,r)))^{\frac{\lambda-n}{n}} \int\limits_{B(x,r)} |f - f_{B(x,r)}|\,d\nu \le 4\left|\int\limits_{\mathbb{R}^n} fa\,d\nu\right| \le 4\|f\|_{[\mathfrak{H}^{\infty,\lambda}]^*}\|a\|_{\mathfrak{H}^{\infty,\lambda}} \le 4\|f\|_{[\mathfrak{H}^{\infty,\lambda}]^*}$$

whence arriving at $f \in \mathfrak{L}^{1,\lambda}$ with $\|f\|_{\mathfrak{L}^{1,\lambda}} \lesssim \|f\|_{[\mathfrak{H}^{\infty,\lambda}]^*}$. □

3 Riesz images of pre-associate Campanato spaces

Given the atomic decomposition of $\mathfrak{H}^{q',\lambda}$ in Theorem 2 and Theorem 1 connecting $I_\alpha L^{p,\kappa}$ and $\mathfrak{L}^{q,\lambda}$, it seems natural that there should be a relation between $I_\alpha\mathfrak{H}^{q',\lambda}$ and $H^{p',\kappa}$. Indeed, we can make the following assertion.

Theorem 3. *Let $q \in (1,\infty)$ and $q' = q/(q-1)$.*
(i) *If $-q < \lambda \le 0$, then*

$$I_\alpha\mathfrak{H}^{q',\lambda} \subset H^{\frac{\kappa q'}{\lambda-(\alpha+\lambda-\kappa)q'},\kappa} \quad \text{as } 0 < \frac{\kappa}{p} \le \alpha < n.$$

(ii) *If $0 < \lambda < n$, then*

$$I_\alpha\mathfrak{H}^{q',\lambda} \subset H^{\frac{\lambda q'}{\lambda-\alpha q'},\lambda} \cap H^{q',\lambda-\alpha q} \quad \text{as } 1 < \frac{\lambda}{\lambda-\alpha} < q' < \frac{\lambda}{\alpha}.$$

Proof. To reach the desired inclusion, recall an equivalent description of $H^{q',\lambda}$ – if $\Lambda^{(\infty)}_{n-\lambda}(E)$ expresses the Hausdorff capacity of $E \subset \mathbb{R}^n$ with order $n-\lambda$, i.e., $\Lambda^{(\infty)}_{n-\kappa}(E) = \inf\sum_j r_j^{n-\lambda}$, where the infimum ranges over all countable coverings of E by open balls of radius r_j, then

$$H^{q',\lambda} = \left\{f \in L^{q'}_{\mathrm{loc}} : \|\|f\|\|_{H^{q',\lambda}} = \inf_{w\in B_1^{(n-\lambda)}}\left(\int\limits_{\mathbb{R}^n} |f|^{q'} w^{1-q'}\,d\nu\right)^{\frac{1}{q'}} < \infty\right\},$$

where $B_1^{(n-\lambda)}$ comprises all nonnegative functions w with

$$\int\limits_{\mathbb{R}^n} w\,d\Lambda^{(\infty)}_{n-\lambda} = \int\limits_0^\infty \Lambda^{(\infty)}_{n-\lambda}(\{y \in \mathbb{R}^n : w(y) > t\})\,dt \le 1.$$

Now, using [5, Theorem 2.3] and modifying its argument, we have that not only

$$(H^{q',\lambda})^* = L^{\frac{q'}{q'-1},\lambda},$$

but also $\mathfrak{H}^{q',\lambda}$ consists of all $H^{q',\lambda}$-functions f with vanishing moment $\int_{\mathbb{R}^n} f\,d\nu = 0$.

On the one hand, note that if $\lambda \in (-q, 0]$, then $\mathfrak{H}^{q',\lambda} = H^{\bar q}$, where $0 < \bar q = n/(n-\lambda/q) \le 1$ (see the argument for Theorem 2). So a combination of Kalita's [14, Theorem 2] on $I_\alpha H^{\bar q}$ (which is also investigated in Strichartz's [24, Theorem 5.2] via some suitable atoms) and Adams–Xiao's [5, Theorem 3.3] gives the desired inclusion under $\lambda \in (-q, 0]$.

On the other hand, suppose $\lambda \in (0, n)$. According to [6, Theorem 16 (ii)] we find that if

$$1 < q' < \min\left\{\frac{\lambda}{\alpha}, (q'-1)\frac{\lambda}{\alpha}\right\}$$

i.e.

$$1 < \frac{\lambda}{\lambda-\alpha} < q' < \frac{\lambda}{\alpha},$$

then

$$I_\alpha H^{q',\lambda} \subset H^{\frac{\lambda q'}{\lambda-\alpha q'},\lambda} \cap H^{q',\lambda-\alpha q},$$

and hence

$$I_\alpha\mathfrak{H}^{q',\lambda} \subset H^{\frac{\lambda q'}{\lambda-\alpha q'},\lambda} \cap H^{q',\lambda-\alpha q}.$$

This completes the argument for the case $\lambda \in (0, n)$. □

4 Trace measures for Lorentz–Sobolev spaces

Recall that for a given nonnegative Radon measure μ on $\mathbb{R}^n$, the characteristic function 1_E and the Riesz (α, p)-capacity of a set E

$$C_{\alpha,p}(E) = \inf\{\|f\|_{L^p}^p : f \in L^p \text{ and } I_\alpha f \geq 1_E\},$$

we have that under $1 < p < n/\alpha$ (cf. Adams [1] for $p < q$, Adams and Maz'ya [3, 17, 18] for $p = q$, and Cascante–Ortega–Verbitsky [8] for $p > q$):

$$I_\alpha : L^p \to L^q_\mu \text{ is continuous } \iff \begin{cases} \sup\limits_{(x,r)\in\mathbb{R}^n\times(0,\infty)} \mu(B(x,r))r^{p^{-1}q(\alpha p-n)} < \infty & \text{as } p < q, \\ \sup\limits_{\text{compact } K\subset\mathbb{R}^n} \mu(K)\big(C_{\alpha,p}(K)\big)^{-1} < \infty & \text{as } p = q, \\ \int\limits_{\mathbb{R}^n} \bigg(\int\limits_0^\infty \Big(\frac{\mu(B(x,r))}{r^{n-\alpha p}}\Big)^{\frac{1}{p-1}} \frac{dr}{r}\bigg)^{\frac{q(p-1)}{p-q}} d\nu(x) < \infty & \text{as } p > q, \end{cases}$$

thereby finding that the continuity of $I_\alpha : L^p \to L^q_\mu$ under $p \geq q$ cannot be fully determined by the trace measure condition $\mu(B(x,r)) \lesssim r^{p^{-1}q(n-\alpha p)}$. But nevertheless, upon replacing L^p with $L_{p,1}$, we can establish the following trace measure principle whose case $p = q \in (1,2)$ is obtained via a dyadic partitioning technique in [15].

Theorem 4. *Let μ be a nonnegative Radon measure on $\mathbb{R}^n$. If*

$$1 < p < \min\Big\{q, \frac{n}{\alpha}\Big\} \quad \text{or} \quad 1 < q \leq p < \min\Big\{\frac{q(n-\alpha p)}{n - \alpha q(q-1)^{-1}}, \frac{n}{\alpha}\Big\},$$

then

$$I_\alpha : L_{p,1} \to L^q_\mu \text{ is continuous } \iff |||\mu|||_{p,q,\alpha} = \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} \mu(B(x,r))r^{p^{-1}q(\alpha p-n)} < \infty.$$

Proof. On the one hand, assume that $I_\alpha : L_{p,1} \to L^q_\mu$ is continuous. Choose f as the indicator $1_{B(x,r)}$ of a given ball $B(x,r)$, and compute

$$\|1_{B(x,r)}\|_{L_{p,1}}^p = \nu\big(B(x,r)\big) \approx r^n$$

and

$$I_\alpha 1_{B(x,r)}(y) = \int\limits_{B(x,r)} |y-z|^{\alpha-n}\, d\nu(z) \geq (2r)^{\alpha-n}\nu\big(B(x,r)\big) \gtrsim r^\alpha \quad \text{for all } y \in B(x,r).$$

Consequently,

$$r^\alpha\big(\mu(B(x,r))\big)^{\frac{1}{q}} \lesssim \|I_\alpha 1_{B(x,r)}\|_{L^q_\mu} \lesssim \|1_{B(x,r)}\|_{L_{p,1}} \lesssim r^{\frac{n}{p}},$$

namely, $|||\mu|||_{p,q,\alpha} < \infty$.

On the other hand, let $|||\mu|||_{p,q,\alpha} < \infty$. The verification of the continuity of $I_\alpha : L_{p,1} \to L^q_\mu$ is split into two steps.

Step 1. Setting K be any compact subset of $\mathbb{R}^n$ we show

$$\int\limits_{\mathbb{R}^n} (I_\alpha 1_K)^q\, d\mu \lesssim |||\mu|||_{p,q,\alpha}\big(\nu(K)\big)^{\frac{q}{p}}.$$

Here, it is appropriate to mention that our forthcoming approach is partially influenced by the argument for [7, Theorem 3.2 (iii)], but nevertheless essentially different from the method of proving [15, Theorem 0.3].

To proceed, for $1 < p < n/\alpha$ and $1 < q < \infty$ let

$$\gamma = p^{-1}q(n-\alpha p) < q(n-\alpha)$$

and the parameter pair $(\alpha_1, \alpha_2) \in (0,n) \times (0,n)$ obey

$$\alpha = \alpha_1 q^{-1} + \alpha_2(1-q^{-1}).$$

Then an application of the Hölder inequality gives

$$I_\alpha 1_K(y) = \int_K |y-z|^{\alpha-n}\,d\nu(z) \le \Big(\int_K |y-z|^{\alpha_1-n}\,d\nu(z)\Big)^{\frac{1}{q}}\Big(\int_K |y-z|^{\alpha_2-n}\,d\nu(z)\Big)^{1-\frac{1}{q}}.$$

First, according to [16, Lemma 1.30], we have

$$\int_K |y-z|^{\alpha_j-n}\,d\nu(z) \lesssim (\nu(K))^{\frac{\alpha_j}{n}}, \quad j=1,2.$$

Via fixing a point $x_0 \in K$ and taking an x_0-symmetric ball $B(x_0,r_0)$ such that

$$r_0^n = \frac{\nu(K)}{\nu(B(0,1))} \quad \text{and} \quad K \subset B(x_0, 2r_0),$$

we consider the following two cases under the granted condition $1 < p < n/\alpha$.

Case $1 < p < q$. Under this condition α_1 can be chosen to ensure $n-\gamma < \alpha_1 < n$. Utilizing the Fubini theorem and [16, Lemma 1.27] we calculate

$$\begin{aligned}\int_{B(x_0,2r_0)}\Big(\int_K \frac{d\nu(z)}{|y-z|^{n-\alpha_1}}\Big)d\mu(y) &= \int_K\Big(\int_{B(x_0,2r_0)}\frac{d\mu(y)}{|y-z|^{n-\alpha_1}}\Big)d\nu(z)\\
&\lesssim \int_K\Big(\int_0^\infty \mu(B(x_0,2r_0)\cap B(z,s))s^{\alpha_1-n}\,ds\Big)d\nu(z)\\
&\lesssim \int_K\Big(\int_0^{2r_0}+\int_{2r_0}^\infty\Big)\mu(B(x_0,2r_0)\cap B(z,s))s^{\alpha_1-n}\,ds\,d\nu(z)\\
&\lesssim |||\mu|||_{p,q,\alpha}\int_K\Big(\int_0^{2r_0} s^{\gamma+\alpha_1-n-1}ds + \int_{2r_0}^\infty (2r_0)^\gamma s^{\alpha_1-n-1}ds\Big)d\nu(z)\\
&\lesssim |||\mu|||_{p,q,\alpha}(\nu(K))^{\frac{\gamma+\alpha_1-n}{n}}.\end{aligned}$$

The previous analysis guarantees

$$\int_{B(x_0,2r_0)} (I_\alpha 1_K)^q\,d\mu \lesssim \nu(K)|||\mu|||_{p,q,\alpha}(\nu(K))^{\frac{\gamma+\alpha_1-n}{n}}(\nu(K))^{\frac{\alpha_2(q-1)}{n}} \lesssim |||\mu|||_{p,q,\alpha}(\nu(K))^{\frac{q}{p}}.$$

Second, a combination of the Minkowski inequality and [16, Lemma 1.27] or [26, (2.4.7)] produces

$$\begin{aligned}\Big(\int_{\mathbb{R}^n\setminus B(x_0,2r_0)} (I_\alpha 1_K)^q\,d\mu\Big)^{\frac{1}{q}} &\le \int_K\Big(\int_{\mathbb{R}^n\setminus B(x_0,2r_0)} |y-z|^{(\alpha-n)q}\,d\mu(y)\Big)^{\frac{1}{q}} d\nu(z)\\
&\lesssim \int_K\Big(\int_{2r_0}^\infty s^{(\alpha-n)q-1}\mu(B(z,s)\cap(\mathbb{R}^n\setminus B(x,2r_0)))\,ds\Big)^{\frac{1}{q}} d\nu(z)\\
&\lesssim \nu(K)|||\mu|||_{p,q,\alpha}^{\frac{1}{q}}\Big(\int_{2r_0}^\infty s^{\gamma-(n-\alpha)q-1}\,ds\Big)^{\frac{1}{q}}\\
&\lesssim |||\mu|||_{p,q,\alpha}^{\frac{1}{q}}(\nu(K))^{\frac{1}{p}}.\end{aligned}$$

Now, putting the first and second estimates together deduces

$$\int_{\mathbb{R}^n} (I_\alpha 1_K)^q\,d\mu \lesssim |||\mu|||_{p,q,\alpha}(\nu(K))^{\frac{q}{p}}.$$

Case $p \ge q > p(n-\alpha q(q-1)^{-1})(n-\alpha p)^{-1}$. Under this circumstance we easily find $q \le p < q/(q-1)$ and so $q \in (1,2)$. Choosing α_2 as $n-\gamma < \alpha_2 < n$, we use the Hölder inequality and the Fubini theorem to similarly get

$$\begin{aligned}\int_{B(x_0,2r_0)} \bigg(\int_K \frac{d\nu(z)}{|y-z|^{n-\alpha_2}}\bigg)^{q-1} d\mu(y) &\le \bigg(\int_{B(x_0,2r_0)} \bigg(\int_K \frac{d\nu(z)}{|y-z|^{n-\alpha_2}}\bigg) d\mu(y)\bigg)^{q-1} (\mu(B(x_0,2r_0)))^{2-q}\\ &\lesssim \Big(|||\mu|||_{p,q,\alpha}(\nu(K))^{\frac{\gamma+\alpha_2}{n}}\Big)^{q-1} |||\mu|||_{p,q,\alpha}^{2-q}(\nu(K))^{\frac{\gamma(2-q)}{n}}\\ &\lesssim |||\mu|||_{p,q,\alpha}(\nu(K))^{\frac{\gamma+(q-1)\alpha_2}{n}}.\end{aligned}$$

This in turn implies

$$\begin{aligned}\int_{B(x_0,2r_0)} (I_\alpha 1_K)^q\, d\mu &\le \int_{B(x_0,2r_0)} \bigg(\bigg(\int_K |y-z|^{\alpha_1-n}\, d\nu(z)\bigg)\bigg(\int_K |y-z|^{\alpha_2-n}\, d\nu(z)\bigg)^{q-1}\bigg) d\mu(y)\\ &\lesssim (\nu(K))^{\frac{\alpha_1}{n}} |||\mu|||_{p,q,\alpha}(\nu(K))^{\frac{\gamma+(q-1)\alpha_2}{n}}\\ &\lesssim (\nu(K))^{p^{-1}q} |||\mu|||_{p,q,\alpha}.\end{aligned}$$

Meanwhile, just repeating the calculation made in the case $p < q$ we can get

$$\int_{\mathbb{R}^n \setminus B(x_0,2r_0)} (I_\alpha 1_K)^q\, d\mu \lesssim |||\mu|||_{p,q,\alpha}(\nu(K))^{\frac{q}{p}},$$

thereby finding

$$\int_{\mathbb{R}^n} (I_\alpha 1_K)^q\, d\mu \lesssim |||\mu|||_{p,q,\alpha}(\nu(K))^{\frac{q}{p}}.$$

Step 2. Notice that $0 \le g \in L_{p,1}$ can be written as $g = \lim_{j\to\infty} g_j$ (cf. [25, p. 88]) with

$$\begin{cases} g_j = \sum_{i=1}^{j2^j} (j-1)2^{-j} 1_{E_{i,j}} + j1_{E_j},\\ E_{i,j} = \{x \in \mathbb{R}^n : (i-1)2^{-j} \le g(x) < i2^{-j}\},\\ E_j = \{x \in \mathbb{R}^n : g(x) \ge j\}.\end{cases}$$

Thus, an application of the Minkowski inequality, plus the estimate for $\|I_\alpha 1_K\|_{L^q_\mu}$ obtained in Step 1, yields

$$\begin{aligned}\|I_\alpha g_j\|_{L^q_\mu} &\le \sum_{i=1}^{j2^j} (j-1)2^{-j}\|I_\alpha 1_{E_{i,j}}\|_{L^q_\mu} + j\|I_\alpha 1_{E_j}\|_{L^q_\mu}\\ &\lesssim |||\mu|||_{p,q,\alpha}^{\frac{1}{q}}\bigg(\sum_{i=1}^{j2^j} (j-1)2^{-j}(\nu(E_{i,j}))^{\frac{1}{p}} + j(\nu(E_j))^{\frac{1}{p}}\bigg)\\ &\lesssim |||\mu|||_{p,q,\alpha}^{\frac{1}{q}}\|g_j\|_{L_{p,1}}.\end{aligned}$$

Now, for $f \in L_{p,1}$ let $g = |f|$. Then sending j to ∞ in the last estimation derives

$$\|I_\alpha f\|_{L^q_\mu} \le \|I_\alpha g\|_{L^q_\mu} \lesssim |||\mu|||_{p,q,\alpha}^{\frac{1}{q}}\|g\|_{L_{p,1}} \lesssim |||\mu|||_{p,q,\alpha}^{\frac{1}{q}}\|f\|_{L_{p,1}}.$$ □

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