Research Article

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A convex-valued selection theorem with a non-separable Banach space

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Abstract: In the spirit of Michael's selection theorem [6, Theorem 3.1"'], we consider a nonempty convex-valued lower semicontinuous correspondence $\varphi : X \to 2^Y$. We prove that if φ has either closed or finite-dimensional images, then there admits a continuous single-valued selection, where X is a metric space and Y is a Banach space. We provide a geometric and constructive proof of our main result based on the concept of peeling introduced in this paper.

Keywords: Barycentric coordinates, continuous selections, lower semicontinuous correspondence, closed-valued correspondence, finite-dimensional convex values, separable Banach spaces

MSC 2010: 54C60, 54C65, 54D65

Dedicated to the memory of Monique Florenzano.

1 Introduction

The area of continuous selections is closely associated with the publication by Ernest Michael of two fundamental papers [10]. It is important to notice that the axiom of choice ensures the existence of a selection for any nonempty family of subsets of X (see [9]). Yet, the axiom of choice does not guarantee the continuity of the selection. Michael's studies are more concerned about continuous selections for correspondences $\varphi : X \to 2^Y$. He guarantees the continuity under specific structures on X (paracompact spaces, perfectly normal spaces, collectionwise normal spaces, etc.) and on Y (Banach spaces, separable Banach spaces, Fréchet spaces, etc.) [5]. Without any doubt, the most known selection theorems are: closed-convex valued, compact-valued, zero-dimensional and finite-dimensional theorems [6–8].

The closed-convex valued theorem is considered as one of the most famous of Michael's contributions in the continuous selection theory for correspondences. This theorem gives sufficient conditions for the existence of a continuous selection with the paracompact domain: paracompactness of the domain is a necessary condition for the existence of continuous selections of lower semicontinuous correspondences into Banach spaces with convex closed values [9].

However, despite their importance, all the theorems mentioned above were obtained for closed-valued correspondences. One of the selection theorems obtained by Michael in order to relax the closeness restriction is the convex-valued selection theorem [6, Theorem 3.1"']. The result was obtained by an alternative assumption on X (Perfect normality), a separability assumption on Y and an additional assumption involving three alternative conditions on the images. Besides, Michael shows that when $Y = \mathbb{R}$, then perfect normality is a necessary and sufficient condition in order to get a continuous selection of any convex-valued lower

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semicontinuous correspondence. The proof of the convex-valued selection theorem is based on the existence of a dense family of selections. The technique is quite involved and exploits the characterization of perfect normality of X and separability of Y.

An interesting question is the following: is it possible to relax the separability of Y? To answer this question, Michael provided in his paper a counterexample [6, Example 6.3] showing that the separability of Y can not be omitted. Even though, the correspondence satisfies one of the three conditions, Michael established an overall conclusion. One question arises naturally: is it possible to omit the separability of Y when the images satisfy one of the two remaining conditions? This study aims to prove that the answer is affirmative.

The paper is organized as follows: in Section 2, we begin with some definitions and results which will be very useful in the sequel. Section 3 is dedicated to recall the two Michael's selection theorems that will be used later: the closed-convex valued and the convex-valued theorems. In Section 4, we first state a partial result when the dimension of the images is finite and constant. Then, we introduce and motivate the concept of peeling. Finally, we state the general case and Sections 5 and 6 provide the proofs of our results.

2 Preliminaries and notations

We start by introducing some notations which will be useful throughout this paper.

2.1 Notations

Let Y be a normed space and $C \subset Y$. We shall denote
(i) by $\overline{C}$ the closure of C in Y,
(ii) by $\mathrm{co}(C)$ the convex hull of C and by $\mathrm{aff}(C)$ the affine space of C,
(iii) by $\dim_a(C) = \dim \mathrm{aff}(C)$ the dimension of C which is by definition the dimension of $\mathrm{aff}(C)$,
(iv) if C is finite-dimensional[1], then $\mathrm{ri}(C)$ the relative interior of C in $\mathrm{aff}(C)$ is given by

$$\mathrm{ri}(C) = \{x \in C \mid \text{there exists a neighborhood } V_x \text{ of } x \text{ such that } V_x \cap \mathrm{aff}(C) \subset C\},$$

(v) by $B_C(a, r) := B(a, r) \cap \mathrm{aff}(C)$, where $B(a, r)$ is the open ball of radius $r > 0$ centered at a point $a \in X$ and $\overline{B}_C(a, r) := \overline{B}(a, r) \cap \mathrm{aff}(C)$, where $\overline{B}(a, r)$ is the closed ball of radius $r > 0$ centered at a point $a \in X$,
(vi) by $S_{i-1}(0, 1) := \{x = (x_1, \ldots, x_i) \in \mathbb{R}^i \mid \|x\| = 1\}$ the unit $(i-1)$-sphere of $\mathbb{R}^i$ embedded with the euclidean norm,
(vii) by $(Y^p_{ai})_{p\in\mathbb{N}}$ the set of affinely independent families of $(Y^p)_{p\in\mathbb{N}}$.

We recall that if $\{x^0, x^1, \ldots, x^i\}$ is a set of $i + 1$ affinely independent points of Y. We call an i-simplex the convex hull of $\{x^0, x^1, \ldots, x^i\}$ given by

$$S_i = \left\{ z \in Y \;\middle|\; z = \sum_{k=0}^{i} \alpha_k x^k,\ \alpha_k \geq 0,\ \sum_{k=0}^{i} \alpha_k = 1 \right\} = \mathrm{co}(x^0, \ldots, x^i).$$

2.2 Classical definitions

We go on with formal definitions and related terms of correspondences. Let us consider nonempty topological spaces X and Y.

Definition 2.1. Let $\varphi : X \to 2^Y$ be a correspondence, $B \subset Y$. We define by

$$\varphi^+(B) = \{x \in X \mid \varphi(x) \subset B\}, \quad \varphi^-(B) = \{x \in X \mid \varphi(x) \cap B \neq \emptyset\}.$$

1 The space C is said to be finite-dimensional if C is contained in a finite-dimensional subspace of Y.

Definition 2.2. Let $\varphi : X \to 2^Y$ be any correspondence. A correspondence $\psi : X \to 2^Y$ satisfying $\psi(x) \subset \varphi(x)$, for each $x \in X$, is called a selection of φ. In particular, if ψ is single-valued (associated to some function $f : X \to Y$), then f is a single-valued selection when $f(x) \in \varphi(x)$, for each $x \in X$.

We recall some alternative characteristics of lower semicontinuous correspondences.

Definition 2.3 ([3]). Let $\varphi : X \to 2^Y$ be a correspondence. We say that φ is lower semicontinuous (abbreviated to lsc) if one of the equivalent conditions is satisfied.
(i) For all open sets $V \subset Y$, we have that $\varphi^-(V)$ is open.
(ii) For all closed sets $V \subset Y$, we have that $\varphi^+(V)$ is closed.

In the case of metric spaces, an alternative characterization is given by the following proposition.

Proposition 2.4 ([3]). *Let X and Y be metric spaces and $\varphi : X \to 2^Y$ a correspondence. We have that φ is lsc on X if and only if for all $x \in X$, for all convergent sequences $(x_n)_{n\in\mathbb{N}}$ to x, and for all $y \in \varphi(x)$, there exists a sequence $(y_n)_{n\geq n_0}$ in Y such that $y_n \to y$ and for all $n \geq n_0$, we have $y_n \in \varphi(x_n)$.*

3 Michael's selection theorems (1956)

Let us first recall one of the main selection theorems: the closed-convex valued selection theorem.

Theorem 3.1 (Closed-convex valued selection theorem). *Let X be a paracompact space, Y a Banach space and $\varphi : X \to 2^Y$ an lsc correspondence with nonempty closed convex values. Then φ admits a continuous single-valued selection.*

Before stating the next theorem, we recall[2] that a topological space is perfectly normal if it is normal and every closed subset is a G-delta subset (G_δ).

The following Michael selection theorem dedicated for non-closed valued correspondences is much more difficult to prove. The assumption on Y is reinforced by adding the separability. We recall that perfect normality does neither imply paracompactness nor the converse.

Theorem 3.2 (Convex-valued selection theorem). *Let X be a perfectly normal space, Y a separable Banach space and $\varphi : X \to Y$ an lsc correspondence with nonempty convex values. If for any $x \in X$, $\varphi(x)$ is either finite-dimensional, or closed, or has an interior point, then φ admits a continuous single-valued selection.*

Note that as explained before, in his paper [6, Example 6.3], Michael provided the following counterexample showing that the assumption of separability of Y can not be omitted in Theorem 3.2.

Example 3.3. There exists an lsc correspondence φ from the closed unit interval X to the nonempty, open, convex subsets of a Banach space Y for which there exists no selection.

Indeed, let X be the closed unit interval $[0, 1]$ and let

$$Y = \ell_1(X) = \Big\{y : X \to \mathbb{R} \;\Big|\; \sum_{x\in X} |y(x)| < +\infty\Big\}.$$

Michael showed that the correspondence $\varphi : X \to 2^Y$ given by $\varphi(x) = \{y \in Y \mid y(x) > 0\}$ has open values, consequently, images have an interior point but Michael proved that there does not exist a continuous selection. The case where the correspondence is either finite-dimensional or closed valued still remains to be dealt with. In order to provide an answer, we now state the main results of this paper.

2 A subset of a topological space is termed a G_δ subset if it is expressible as a countable intersection of open subsets.

4 The results

We start by recalling that if X is a metric space, then it is both paracompact and perfectly normal. In many applications, both paracompactness and perfect normality aspects are ensured by the metric character. Therefore, throughout this section, we assume that (X, d) is a metric space. In addition, let $(Y, \|\cdot\|)$ be a Banach space. We recall that the relative interior of a convex set C is a convex set of same dimension and that $\mathrm{ri}(\overline{C}) = \mathrm{ri}(C)$ and $\overline{\mathrm{ri}(C)} = \overline{C}$. In the first instance, in order to prove the main result, we focus on the case of constant (finite-)dimensional images. In addition, compared with Theorem 3.2 of Michael, we suppose first that X is a metric space and omit the separability of Y. We denote by D_i the set $D_i := \{x \in X \mid \dim_a \varphi(x) = i\}$. Then we state the following theorem.

Theorem 4.1. *Let $\varphi : X \to 2^Y$ be an lsc correspondence with nonempty convex values. Then, for any $i \in \mathbb{N}$, the restriction of* $\mathrm{ri}(\varphi)$ *to D_i admits a continuous single-valued selection $h_i : D_i \to Y$. In addition, if $i > 0$, then there exists a continuous function $\beta_i : D_i \to]0, +\infty[$ such that for any $x \in D_i$, we have $\overline{B}_{\varphi(x)}(h_i(x), \beta_i(x)) \subset \mathrm{ri}(\varphi(x))$.*

Once we have Theorem 4.1, we will be able to prove our main result given by

Theorem 4.2. *Let X be a metric space and Y a Banach space. Let $\varphi : X \to 2^Y$ be an lsc correspondence with nonempty convex values. If for any $x \in X$, $\varphi(x)$ is either finite-dimensional or closed, then φ admits a continuous single-valued selection.*

Note that the property of φ being either closed or finite-dimensional valued is not inherited by $\mathrm{co}(\varphi)$. Consequently, we can not directly convert Theorem 4.2 in terms of the convex hull. Yet, first, a direct consequence of both Theorem 4.2 and Theorem 3.1 is the following.

Corollary 4.3. *Let X be a metric space and Y a Banach space. Let $\varphi : X \to 2^Y$ be an lsc correspondence with nonempty values. Then $\overline{\mathrm{co}}(\varphi(x))$ admits a continuous single-valued selection.*

Secondly, we can also deduce from Theorem 4.2 the following result.

Corollary 4.4. *Let X be a metric space and Y a Banach space. Let $\varphi : X \to 2^Y$ be an lsc correspondence with nonempty values. If for any $x \in X$, $\varphi(x)$ is either finite-dimensional or closed convex, then* $\mathrm{co}(\varphi)$ *admits a continuous single-valued selection.*

It is also worth noting that under the conditions of Theorem 4.2, we may have an lsc correspondence with both closed and finite-dimensional values. This is made clear in the following example.

Example 4.5. Let Y be an infinite-dimensional Banach space and $X = [0, 1]$. Consider $(e_0, \ldots, e_n, \ldots)$ some linearly independent normed family of Y. Let $\varphi : \mathbb{R} \to 2^Y$ defined by

$$\varphi(x) = \begin{cases} \{0\} & \text{if } x \in \{0, 1\}, \\ \mathrm{ri}(\mathrm{co}(0, e_0)) & \text{if } x \in [\frac{1}{2}, 1), \\ \mathrm{ri}(\mathrm{co}(0, e_0, e_1)) & \text{if } x \in [\frac{1}{3}, \frac{1}{2}), \\ \vdots & \\ Y & \text{if } x \in X^c. \end{cases}$$

Remark that φ is lsc on $]0, 1[$ since it is locally increasing. In other terms, for any $x \in X$, there exists V_x such that for all $x' \in V_x$, we have $\varphi(x) \subset \varphi(x')$.

Besides, φ is lsc at the point $x = 0$. It suffices to remark that for $\varepsilon > 0$, we have $\varphi^-(B(0, \varepsilon)) = \mathbb{R}$. Indeed, since for any $x \in \mathbb{R}$, $0 \in \overline{\varphi}(x)$, then $B(0, \varepsilon) \cap \varphi(x) \neq \emptyset$. The same argument is used for $x = 1$. Therefore, we conclude that φ is lsc.

By Theorem 4.2, we can conclude that φ admits a continuous selection. It should also be noted that we can even build an explicit selection.

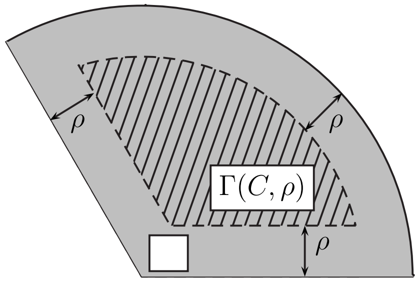

Figure 1. Peeling concept.

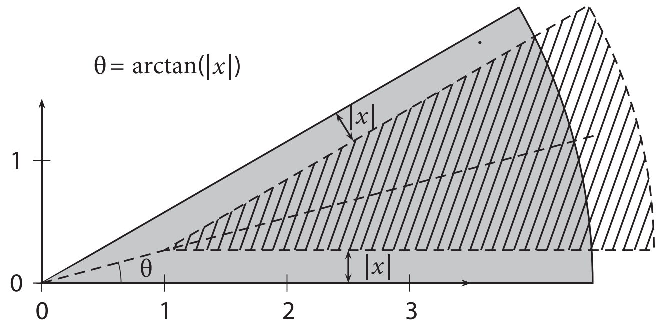

Figure 2. Graphic illustration.

The proofs of Theorem 4.1 and Theorem 4.2 are postponed until Sections 5 and 6 respectively. The proof of Theorem 4.1 is based on the concept of "peeling" that we will introduce and motivate here.

Definition 4.6. Let C be a nonempty finite-dimensional subset of Y. We say that C' is a peeling of C of parameter $\rho \geq 0$ if

$$C' = \Gamma(C, \rho) := \{y \in C \text{ such that } \overline{B}_C(y, \rho) \subset \operatorname{ri}(C)\}.$$

In order to gain some geometric intuition, the concept is illustrated by Figure 1.

Definition 4.7. Let η be a non-negative real-valued function defined on X, and φ a correspondence from X to Y. We will say that the correspondence $\varphi_\eta : X \to 2^Y$ is a peeling of φ of parameter η if for each $x \in X$, we have $\varphi_\eta(x) = \Gamma(\varphi(x), \eta(x))$.

The motivation of the peeling concept is given by the next proposition (whose proof is postponed until the next section) where we show that when the dimension is constant, continuous peeling of an lsc correspondence is also a (possibly empty) lsc correspondence. This proposition is a key argument for the proof of Theorem 4.1.

Proposition 4.8. *Let $\varphi : X \to 2^Y$ be an lsc correspondence with nonempty convex values. If there exists an $i \in \mathbb{N}^*$ such that for any $x \in X$, $\dim_a \varphi(x) = i$, then the continuity of the function η implies the lower semicontinuity of φ_η.*

Remark 1. The following simple example shows that the above proposition is no more valid if the dimension of φ is equal to zero. Let $\varphi : \mathbb{R} \to 2^{\mathbb{R}}$, defined by $\varphi(x) = \{0\}$ and $\eta(x) = |x|$. Obviously φ is lsc and η is continuous, but $\varphi_\eta : \mathbb{R} \to 2^{\mathbb{R}}$ is not lsc since $\varphi_\eta(0) = \{0\}$ while for $x \neq 0$, we have $\varphi_\eta(x) = \emptyset$.

Remark 2. Modifying slightly the previous example, we also show that the above proposition does not hold true if the dimension of φ is not constant. Let us consider the case where $X = \mathbb{R}$, $Y = \mathbb{R}^2$, and $\varphi : X \to 2^Y$, is the lsc correspondence defined by

$$\varphi(x) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq 0 \text{ and } y_2 \geq \tan(2\arctan(|x|))y_1\}, \quad \text{if } x \neq 0 \text{ and } \varphi(0) = [0, 1/2] \times \{0\}.$$

Using the same $\eta(x) = |x|$, we obtain that when $x \neq 0$, $\varphi_\eta(x)$ is a translation of $\varphi(x)$. More precisely,

$$\varphi_\eta(x) = \{(1, |x|)\} + \varphi(x)$$

(see Figure 2). In particular, $d(\varphi(0), \varphi_\eta(x)) \geq 1/2$, which allows us to conclude that φ_η is not lsc.

5 Proof of Theorem 4.1

In Section 5.1, we first present elementary results about the "peeling" of a set. Section 5.2 is dedicated to prove some affine geometry results used to prove Proposition 4.8 in Section 5.3. Finally, we deduce Theorem 4.1 from this proposition in the last subsection.

5.1 Elementary results on a set “peeling”

In this subsection, C is a finite-dimensional set.

Definition 5.1. We define[3] the internal radius of a finite-dimensional set C by

$$\alpha(C) := \sup\{\rho \in \mathbb{R}_+ \mid \text{there exists } y \in C \text{ such that } \overline{B}_C(y,\rho) \subset \mathrm{ri}(C)\}.$$

Lemma 5.2. *Let C be a nonempty convex set. Then one has $\Gamma(C,0) = \mathrm{ri}(C)$. Yet, if $\alpha(C)$ is finite, then we have $\Gamma(C,\alpha(C)) = \emptyset$.*

Proof. The equality on $\Gamma(C,0)$ is a simple consequence of the definition. Let us prove by contradiction that $\Gamma(C,\alpha(C)) = \emptyset$. Indeed, if $y \in \Gamma(C,\alpha(C))$, we have $\overline{B}_C(y,\alpha(C)) \subset \mathrm{ri}(C)$. By a compactness argument on the circle of center y and radius $\alpha(C)$ and the openness of $ri(C)$ in $\mathrm{aff}(C)$, we can prove the existence of some $\varepsilon > 0$ such that $\overline{B}_C(y,\alpha(C)+\varepsilon) \subset \mathrm{ri}(C)$. □

We first establish that a peeling of a convex set remains convex. In addition, we can characterize the nonemptiness.

Lemma 5.3. *Let C be a nonempty convex set and $\rho \in [0,+\infty[$. Then, the set $\Gamma(C,\rho)$ is convex. In addition, the set $\Gamma(C,\rho)$ is nonempty if and only if $\rho < \alpha(C)$.*

Proof. First, we have that $\Gamma(C,\rho)$ is convex. Let $x_1, x_2 \in \Gamma(C,\rho)$ and $\lambda \in [0,1]$. We claim that

$$\overline{B}_C((\lambda x_1 + (1-\lambda)x_2),\rho) \subset \mathrm{ri}(C).$$

By triangle inequality, it is easy to see that

$$\overline{B}(\lambda x_1 + (1-\lambda)x_2,\rho) = \lambda\overline{B}(x_1,\rho) + (1-\lambda)\overline{B}(x_2,\rho).$$

Yet, since $\mathrm{ri}(C)$ is convex, then we have

$$(\lambda\overline{B}(x_1,\rho) + (1-\lambda)\overline{B}(x_2,\rho)) \cap \mathrm{aff}(C) \subset \lambda\,\mathrm{ri}(C) + (1-\lambda)\,\mathrm{ri}(C) = \mathrm{ri}(C),$$

which establishes the result. Now, remark that if $\Gamma(C,\rho) \neq \emptyset$, then by definition of Γ, we have $\rho \le \alpha(C)$ and in view of Lemma 5.2, $\rho < \alpha(C)$. It remains to prove the converse. Using the definition of α, there exists $y_\rho \in C$ such that $\overline{B}_C(y_\rho,\rho) \subset \mathrm{ri}(C)$. Therefore, $y_\rho \in \Gamma(C,\rho)$. □

Lemma 5.4. *Let C be a convex set and ρ_1, ρ_2 non-negative real numbers such that $\rho_1 < \rho_2$. Then we have $\overline{\Gamma}(C,\rho_2) \subset \Gamma(C,\rho_1) \subset \mathrm{ri}(C)$.*

Proof. Let $\overline{y} \in \overline{\Gamma}(C,\rho_2)$. Since $\varepsilon = \rho_2 - \rho_1 > 0$, there exists $y \in \Gamma(C,\rho_2) \cap B(\overline{y},\varepsilon)$. Consequently, by triangle inequality, $\overline{B}_C(\overline{y},\rho_1) \subset \overline{B}_C(y,\rho_2)$, and therefore $y \in \Gamma(C,\rho_1)$. Finally, it comes from the definition that $\Gamma(C,\rho_1) \subset \mathrm{ri}(C)$. □

5.2 Affine geometry

Next, we will use known results about linear independence in order to raise a series of results about affine independence and barycentric coordinates.

Lemma 5.5. *The set Y^{i+1}_{ai} is an open subset of Y^{i+1}.*

Proof. Let $(z_0,\dots,z_i) \in Y^{i+1}_{\mathrm{ai}}$. In order to get a contradiction, we suppose that for all $r > 0$, and for all $k \in \{0,\dots,i\}$, there exists a $z'_k \in B(z_k,r)$ such that $(z'_0,\dots,z'_i)$ are affinely dependent. In particular, for

3 Note that α may value $+\infty$. The proofs need to distinguish whether α is finite or infinite. Such a difficulty can be avoided by considering on Y the metric given by $\delta(x,y) = \min(1,\|x-y\|)$.

all $p \in \mathbb{N}^*$ and for all $k \in \{0, \dots, i\}$, there exists $z'_{k,p} \in B(z_k, 1/p)$ such that $(z'_{0,p}, \dots, z'_{i,p})$ are affinely dependent. Thus, the family of vectors

$$(v_1^p, \dots, v_i^p) = (z'_{1,p} - z'_{0,p}, \dots, z'_{i,p} - z'_{0,p})$$

is linearly dependent. Hence, there exists a $\lambda^p = (\lambda_1^p, \dots, \lambda_i^p) \in \mathbb{R}^i \setminus \{0\}$ such that $\sum_{k=1}^i \lambda_k^p v_k^p = 0$. We can normalize by letting $\mu^p = \lambda^p / \|\lambda^p\| \in S_{i-1}(0, 1)$. By a compactness argument, the sequence μ^p admits a convergent subsequence $\mu^{\varphi(p)}$ to $\overline{\mu} \in S_{i-1}(0, 1)$. Since for all $k \in \{0, \dots, i\}$, there exists a $z'_{k,p} \in B(z_k, 1/p)$, then the sequence $(z'_{k,p})_{p\in\mathbb{N}^*}$ converges to z_k. Therefore, we have

$$\sum_{k=1}^{i} \mu^{\varphi(p)} v_k^{\varphi(p)} = \sum_{k=1}^{i} (\lambda_k^{\varphi(p)} / \|\lambda^{\varphi(p)}\|) v_k^{\varphi(p)} = 0 \to \sum_{k=1}^{i} \overline{\mu}_k (z_k - z_0).$$

That is, $\sum_{k=1}^i \overline{\mu}_k (z_k - z_0) = 0$. Since, by the starting assumption, $(z_1 - z_0, \dots, z_i - z_0)$ is linearly independent, then we obtain $\overline{\mu} = 0$, which is absurd. □

We recall that if $y^n = (y_0^n, \dots, y_i^n) \in Y_{ai}^{i+1}$, then every point z_n of $\operatorname{aff}(y^n)$ has a unique representation

$$z^n = \sum_{k=0}^{i} \lambda_k^n y_k^n, \quad \lambda^n = (\lambda_0^n, \dots, \lambda_i^n) \in \mathbb{R}^{i+1}, \quad \sum_{k=0}^{i} \lambda_k^n = 1,$$

where $\lambda_0^n, \dots, \lambda_i^n$ are the barycentric coordinates of the point z^n relative to $(y_0^n, \dots, y_i^n)$. Using the previous notations and adopting obvious ones for the limits, our next result is formulated in the following way, where the assumption $\overline{y} \in Y_{ai}^{i+1}$ can not be omitted.

Lemma 5.6. *Let $y^n \in Y_{ai}^{i+1}$ tending to $\overline{y} \in Y_{ai}^{i+1}$. Let $z^n \in \operatorname{aff}(y^n)$. Hence, we have:*
(1) *If z^n is bounded, then λ^n is bounded and z^n has a cluster point in* $\operatorname{aff}(\overline{y})$.
(2) *If z^n converges to $\overline{z}$, then λ^n converges to $\overline{\lambda}$.*

Proof. We start by proving assertion (1). We denote by $w^n = z^n - y_0^n$. Since $\sum_{k=0}^i \lambda_k^n = 1$, it follows that $w^n = \sum_{k=0}^i \lambda_k^n (y_k^n - y_0^n)$. Hence, $w^n = \sum_{k=1}^i \lambda_k^n (y_k^n - y_0^n)$. We denote by $\tilde{\lambda}^n = (\lambda_1^n, \dots, \lambda_i^n) \in \mathbb{R}^i$, where the first component is omitted. First, we will prove by contradiction that $\tilde{\lambda}^n$ is bounded. Assume the contrary. Then there exists a subsequence $\tilde{\lambda}^{\psi(n)}$ of $\tilde{\lambda}^n$ such that $\|\tilde{\lambda}^{\psi(n)}\|$ diverges to infinity. Since z^n is bounded, it follows that $w^{\psi(n)} / \|\tilde{\lambda}^{\psi(n)}\| \to 0$. Moreover, by normalizing, the sequence $\mu^{\psi(n)} = \tilde{\lambda}^{\psi(n)} / \|\tilde{\lambda}^{\psi(n)}\|$ belongs to the compact set $S_{i-1}(0, 1)$. Therefore, the sequence $\mu^{\psi(n)}$ admits a convergent subsequence $\mu^{\psi(\varphi(n))}$ converging to $\overline{\mu}$ in $S_{i-1}(0, 1)$. Thus, we obtain that

$$\frac{w^{\psi(\varphi(n))}}{\|\tilde{\lambda}^{\psi(\varphi(n))}\|} \to \sum_{k=1}^{i} \overline{\mu}_k (\overline{y_k} - \overline{y_0}).$$

By uniqueness of the limit, we deduce that $\sum_{k=1}^i \overline{\mu}_k (\overline{y_k} - \overline{y_0}) = 0$. Since $(\overline{y_1} - \overline{y_0}, \dots, \overline{y_i} - \overline{y_0})$ is independent, we obtain $\overline{\mu} = 0$, which is impossible. Now it suffices to remark that since $\lambda_0^n = 1 - \sum_{k=1}^i \lambda_k^n$, the boundedness of $\tilde{\lambda}^n$ implies the one of λ_0^n and therefore of the whole vector λ^n.

It remains to check that z^n has a cluster point. We have already established that λ^n is bounded. Therefore, there admits a convergent subsequence $\lambda^{\psi(n)}$ converging to $\overline{\lambda}$. Hence,

$$z^{\psi(n)} = \sum_{k=0}^{i} \lambda_k^{\psi(n)} y_k^{\psi(n)} \to \overline{z} = \sum_{k=0}^{i} \overline{\lambda}_k \overline{y}_k.$$

That is, $\overline{z}$ is a cluster point of z^n. Moreover, observe that since $\sum_{k=0}^i \lambda_k^{\psi(n)} = 1$ converges to $\sum_{k=0}^i \overline{\lambda}_k$, we have $\overline{z} \in \operatorname{aff}(\overline{y})$, which establishes the result.

Now, we are able to prove the second assertion. By assertion (1), we know that λ^n is bounded. By [1], in order to prove that λ^n converges to $\overline{\lambda}$, we claim that the bounded sequence λ^n has a unique cluster point. Using the same notations, let $a \in F$, where F is the set of cluster points of λ^n. Then there exists a subsequence $\lambda^{\varphi(n)}$ of λ^n such that $\lambda^{\varphi(n)} \to a$. Consequently,

$$w^{\varphi(n)} = \sum_{k=1}^{i} \lambda_k^{\varphi(n)} (y_k^{\varphi(n)} - y_0^{\varphi(n)}) \to \sum_{k=1}^{i} a_k (\overline{y}_k - \overline{y}_0).$$

Alternatively, we have that $w^{\varphi(n)} = z^{\varphi(n)} - y_0^{\varphi(n)}$ converges to

$$\overline{z} - \overline{y}_0 = \sum_{k=0}^{i} \lambda_k \overline{y}_k - \overline{y}_0 = \sum_{k=1}^{i} \lambda_k (\overline{y}_k - \overline{y}_0),$$

since $\sum_{k=0}^{i} \lambda_k = 1$. By uniqueness of the limit and given that $(\overline{y_1} - \overline{y_0}, \dots, \overline{y_i} - \overline{y_0})$ is independent, we obtain that $a_k = \overline{\lambda}_k$, for all $k \in \{1, \dots, i\}$. Therefore, we have a unique cluster point. Consequently, $\lambda_k^n \to \overline{\lambda}_k$, for all $k \in \{1, \dots, i\}$. In addition, for the first component, we obtain $\lambda_0^n = 1 - \sum_{k=1}^{n} \lambda_k^n \to 1 - \sum_{k=1}^{n} \overline{\lambda}_k = \overline{\lambda}_0$. Thus, for all $k \in \{0, \dots, i\}$, we have $\lambda_k^n \to \overline{\lambda}_k$. This completes the proof. □

5.3 Proof of Proposition 4.8

The key argument we will use is based on the following lemma. Note that Tan and Yuan [11] have a similar but weaker result, since they only treated the case of a finite-dimensional set Y when the images have a nonempty interior which avoids to introduce the relative interior.

Lemma 5.7 (Fundamental Lemma). *Under the assumptions of Theorem 4.1, let $\overline{x} \in D_i$, $\gamma > 0$ and $\overline{y} \in \Gamma(\varphi(\overline{x}), \gamma)$. Then, for any $\varepsilon \in]0, \gamma[$, there exists a neighborhood V of $\overline{x}$ such that for every $x \in V \cap D_i$, we have that $\Gamma(\varphi(x), \gamma - \varepsilon) \cap B(\overline{y}, \varepsilon) \neq \emptyset$.*

Proof. First, let us fix some $\varepsilon \in]0, \gamma[$. In order to simplify the notations, we will assume within this proof that $X = D_i$. We will start by proving the following claim:

Claim 1. *There exists a neighborhood V_1 of $\overline{x}$, such that for all $x \in V_1$, we have $\overline{B}_{\varphi(x)}(\overline{y}, \gamma - \varepsilon/3) \subset \varphi(x)$.*

To prove the claim, let us denote by r the positive quantity $r = \gamma - \varepsilon/3$. In order to get a contradiction, assume that for all $n > 0$, there exists an $x_n \in B(\overline{x}, 1/n)$, and there exists a $z^n \in \overline{B}_{\varphi(x_n)}(\overline{y}, r)$ such that $z^n \notin \varphi(x_n)$. Since for all $\overline{x} \in X$, we have $\dim_a \varphi(\overline{x}) = i$, there exists $(\hat{y}_0, \dots, \hat{y}_i) \in Y_{ai}^{i+1} \cap \varphi(\overline{x})$. Moreover, we have $x_n \to \overline{x}$ and φ is lsc. Therefore, for n sufficiently large, there exists $y_k^n \to \hat{y}_k$ such that $y_k^n \in \varphi(x_n)$, for all $k \in \{0, \dots, i\}$. Using Lemma 5.5, for n large enough, we obtain that $\dim_a (y_0^n, \dots, y_i^n) = i$.

Besides, we have $z^n \in \operatorname{aff}(\varphi(x_n)) = \operatorname{aff}(y_0^n, \dots, y_i^n)$. Since $z_n \in \overline{B}(\overline{y}, r)$, applying the first assertion of Lemma 5.6, we conclude that z^n has a cluster point $\overline{z}$ in $\operatorname{aff}(\varphi(\overline{x})) = \operatorname{aff}(\hat{y}_0, \dots, \hat{y}_i)$. Furthermore, since $z^n \in \overline{B}(\overline{y}, r)$, we have $\overline{z} \in \overline{B}(\overline{y}, r) \subset B(\overline{y}, \gamma)$. It follows that $\overline{z}$ belongs to $\operatorname{ri}(\overline{B}_{\varphi(\overline{x})}(\overline{y}, \gamma))$.

Hence, in view of the dimension of $\overline{B}_{\varphi(\overline{x})}(\overline{y}, \gamma)$, there exists an i-simplex $S_i = \operatorname{co}(\overline{U}_0, \dots, \overline{U}_i)$ contained in $\overline{B}_{\varphi(\overline{x})}(\overline{y}, \gamma)$ such that $\overline{z} \in \operatorname{ri}(S_i)$. We can write the affine decomposition $\overline{z} = \sum_{k=0}^{i} \overline{\mu}_k \overline{U}_k$, for some $\mu \in \mathbb{R}^{i+1}$ such that $\sum_{k=0}^{i} \overline{\mu}_k = 1$ and $\overline{\mu}_k > 0$. In particular, in view of the assumption of the lemma, $\overline{U}_k \subset \overline{B}_{\varphi(\overline{x})}(\overline{y}, \gamma) \subset \operatorname{ri}(\varphi(\overline{x}))$. Using again that $x_n \to \overline{x}$ and the lower semicontinuity of φ, we obtain that there exists a sequence $U_k^n \to \overline{U}_k$ such that $U_k^n \in \varphi(x_n)$, for all $k \in \{0, \dots, i\}$ and n large enough. Then we consider $\mu^n = (\mu_0^n, \dots, \mu_i^n)$, the affine coordinates of z^n in $(U_0^n, \dots, U_i^n)$. Since z^n has a cluster point $\overline{z}$, there exists a subsequence $z^{\psi(n)}$ of z^n converging to $\overline{z}$. By assertion (2) of Lemma 5.6, we have $\mu_k^{\psi(n)} \to \overline{\mu}_k$, for all $k \in \{0, \dots, i\}$. However, $\overline{\mu}_k > 0$, then $\mu_k^{\psi(n)} \geq 0$, for n large enough. Consequently, $z^{\psi(n)}$ is a convex combination of $U_k^{\psi(n)}$. By the convexity of $\varphi(x_{\psi(n)})$, we conclude that $z^{\psi(n)} \in \varphi(x_{\psi(n)})$, which is a contradiction. This proves Claim 1.

Note that an obvious consequence of Claim 1 is that for all $x \in V_1$, we have

$$\overline{B}_{\varphi(x)}\Big(\overline{y}, \gamma - \frac{2\varepsilon}{3}\Big) \subset \operatorname{ri}(\varphi(x)).$$

Now, since $\varepsilon > 0$, we have $\overline{y} \in \varphi(\overline{x}) \cap B(\overline{y}, \varepsilon/3)$. From the lower semicontinuity of φ, there exists a neighborhood V_2 of $\overline{x}$, such that for all $x \in V_2$, we have the existence of some $y \in \varphi(x) \cap B(\overline{y}, \varepsilon/3)$. Therefore, for any $x \in V = V_1 \cap V_2$, we have

$$\overline{B}_{\varphi(x)}(y, \gamma - \varepsilon) \subset \overline{B}_{\varphi(x)}\Big(\overline{y}, \gamma - \frac{2\varepsilon}{3}\Big) \subset \operatorname{ri}(\varphi(x)).$$

Thus, we finish the proof. □

Note that Proposition 4.8 was already stated without proof in Section 4. We are now ready to prove it.

Proof of Proposition 4.8. Let us consider an open set O and $\overline{x} \in \varphi^-(O)$. This means that $\varphi_\eta(\overline{x}) \cap O$ is nonempty and contains some $\overline{y}$. Since O is an open set containing $\overline{y}$, we can first remark that there exists $r > 0$ such that $\overline{B}(\overline{y}, r) \subset O$.

On one hand, by the condition on η, letting $\gamma = \frac{\alpha(\overline{x})-\eta(\overline{x})}{3} > 0$ and $\eta_i = \eta(\overline{x}) + i\gamma$, for any $i = \{0, \dots, 3\}$, we have

$$\eta(\overline{x}) = \eta_0 < \eta_1 < \eta_2 < \eta_3 = \alpha(\overline{x}).$$

By the definition of α, there exists $\overline{z} \in \Gamma(\varphi(\overline{x}), \eta_2)$. Applying the Fundamental Lemma, we obtain that there exists a neighborhood V_1 of $\overline{x}$ such that for any $x \in V_1$, there exists $z \in \Gamma(\varphi(x), \eta_1) \cap B(\overline{z}, \gamma)$.

On the other hand, we will distinguish two cases depending whether at a point x, there is or is not a significant peeling. Let us denote by $M = 1 + \frac{2}{\gamma}(\|\overline{y} - \overline{z}\| + \gamma)$; we will consider the positive number $\overline{\varepsilon} = \min(r/M, \gamma)$.

First case: $\eta(\overline{x}) > 0$. Let us denote $\varepsilon = \min(\overline{\varepsilon}, \eta(\overline{x})/2)$. Since $\overline{y} \in \varphi_\eta(\overline{x}) = \Gamma(\varphi(\overline{x}), \eta(\overline{x}))$, then once again, by the Fundamental Lemma, we have the existence of a neighborhood V_2 of $\overline{x}$ such that for any $x \in V_2$, there exists $y \in \Gamma(\varphi(x), \eta(\overline{x}) - \varepsilon) \cap B(\overline{y}, \varepsilon)$. Now, let us consider $\lambda = \frac{2\varepsilon}{\gamma+\varepsilon} \in]0, 1]$ and $y_\lambda = (1-\lambda)y + \lambda z$.

Second case: $\eta(\overline{x}) = 0$. In this case, $\varphi_\eta(\overline{x}) = \mathrm{ri}(\varphi(\overline{x}))$. Let $\varepsilon = \overline{\varepsilon}$. Since φ is lsc, there exists a neighborhood V_2 of $\overline{x}$ such that for any $x \in V_2$, there exists $y \in \varphi(x) \cap B(\overline{y}, \varepsilon)$ satisfying

$$\overline{B}_{\varphi(x)}(y, 0) \subset \mathrm{ri}(\varphi(x)).$$

Now, let us consider $\lambda = \frac{\varepsilon}{\gamma} \in]0, 1]$ and $y_\lambda = (1-\lambda)y + \lambda z$.

In both cases, since the set $\varphi(x)$ is convex, in view of our choice of λ, by a simple computation, we can prove that if $x \in V_1 \cap V_2$, then $y_\lambda \in \Gamma(\varphi(x), \eta(\overline{x}) + \varepsilon)$.

Note that in both cases $\lambda \le 2\varepsilon/\gamma$. The following computation will show that our choice of ε implies $y_\lambda \in O$:

$$\begin{aligned}
\|y_\lambda - \overline{y}\| &\le (1-\lambda)\|y - \overline{y}\| + \lambda\|z - \overline{y}\| \\
&\le \|y - \overline{y}\| + \lambda(\|z - \overline{z}\| + \|\overline{z} - \overline{y}\|) \\
&\le \varepsilon + \frac{2\varepsilon}{\gamma}(\|z - \overline{z}\| + \|\overline{z} - \overline{y}\|) \\
&\le \varepsilon + \frac{2\varepsilon}{\gamma}(\gamma + \|\overline{y} - \overline{z}\|) = \varepsilon M \le r.
\end{aligned}$$

Finally, by the continuity of η, there exists a neighborhood V_3 of $\overline{x}$ such that for any $x \in V_3$, we have $\eta(\overline{x}) - \varepsilon < \eta(x) < \eta(\overline{x}) + \varepsilon$. Summarizing the previous results, for all $x \in V_1 \cap V_2 \cap V_3$, there exists $y_\lambda \in O$ such that $\overline{B}_{\varphi(x)}(y_\lambda, \eta(x)) \subset \overline{B}_{\varphi(x)}(y_\lambda, \eta(\overline{x}) + \varepsilon) \subset \mathrm{ri}(\varphi(x))$, which means that $y_\lambda \in \varphi_\eta(x) \cap O$ and establishes the result. □

5.4 Proof of Theorem 4.1

Using the above lemma 5.7, we are able to prove the following result on the regularity of the internal radius of an lsc correspondence.

Lemma 5.8. *Let us consider $\varphi : X \to 2^Y$ such that there exists some positive integer i such that for all $x \in X$, $\dim_a \varphi(x) = i$. Then $\alpha \circ \varphi : X \to \mathbb{R}^+ \cap \{+\infty\}$ is a lower semicontinuous function.*

Proof. Let us first recall that

$$\alpha(\varphi(x)) := \sup\{\rho \in \mathbb{R}^+ \mid \text{there exists } y \in \varphi(x) \text{ such that } \overline{B}_{\varphi(x)}(y, \rho) \subset \mathrm{ri}(\varphi(x))\}.$$

Let us fix $\overline{x}$ in X, and $\rho < \alpha(\varphi(\overline{x}))$. We can consider γ such that $\rho < \gamma < \alpha(\varphi(\overline{x}))$. By the definition of α, there exists $y \in \Gamma(\varphi(\overline{x}), \gamma)$. Now, by the Fundamental Lemma, there exists a neighborhood V of $\overline{x}$ such that for all $x \in V$, $\Gamma(\varphi(x), \rho) \ne \emptyset$. Using Lemma 5.3, we obtain that for all x in V, $\alpha(\varphi(x)) > \rho$, which establishes the result. □

An immediate consequence is the following result.

Corollary 5.9. *For all positive integers i, there exists $\eta_i : D_i \to \mathbb{R}^+$ continuous such that* $0 < \eta_i(x) < \alpha(\varphi(x))$.

Proof. Since on D_i, $\alpha \circ \varphi$ is a lower semicontinuous function, the correspondence given by

$$\Lambda : D_i \to 2^{\mathbb{R}}, \quad x \mapsto]0, \alpha(\varphi(x))[$$

is lsc. Applying Michael's Selection Theorem 3.2, we deduce that there exists a single-valued continuous selection η_i of Λ. □

Proof of Theorem 4.1. If $i = 0$, then φ is reduced to a singleton on D_0. Moreover, on D_0, the correspondences φ and $\mathrm{ri}(\varphi)$ coincide. Therefore, there exists a mapping $h_0 : D_0 \to Y$ such that for all $x \in D_0$, $\varphi(x) = \{h_0(x)\}$. Since φ is lsc on D_0, it is well known that h_0 is a continuous function.

If $i > 0$, let us first apply Corollary 5.9 in order to get a continuous function η_i. Consequently, by Proposition 4.8, we have for any $i \in \mathbb{N}^*$, that φ_{η_i} is lsc on D_i. By a classical result, this implies that $\overline{\varphi}_{\eta_i}$ is also lsc on D_i. Moreover, in view of the double inequality satisfied by η_i, we can apply Lemma 5.3, in order to state that $\overline{\varphi}_{\eta_i}$ is a correspondence with nonempty convex-closed values. Therefore, applying Theorem 3.1 of Michael gives a continuous single-valued selection h_i of $\overline{\varphi}_{\eta_i}$. That is for any $x \in D_i$, $h_i(x) \in \overline{\Gamma}(\varphi(x), \eta_i(x))$. Since $\eta_i(x) > 0$, we can apply Lemma 5.4 in order to show that $h_i(x) \in \Gamma(\varphi(x), \eta_i(x)/2)$. Now, for all $x \in D_i$, let $\beta_i(x) := \eta_i(x)/2$; then the previous condition can be rewritten as $\overline{B}_{\varphi(x)}(h_i(x), \beta_i(x)) \subset \mathrm{ri}(\varphi(x))$, which proves the result. □

It is worth noting that the idea of peeling should be distinguished from the approximation method introduced by Cellina [2]. Indeed, mainly Cellina's method consists of approximating an upper semicontinuous correspondence φ by a lower semicontinuous one.[4] In addition, unlike the peeling concept which can be seen as an "inside approximation" (the approximated set is a subset of the original one), the approximation of Cellina is an "outside one" (the original set is a subset of the approximated one). As it is well known (see [2]), applying selection theorems to Cellina approximations is used to deduce Kakutani's fixed point theorem.

We are now ready to prove the main result of this paper.

6 Proof of Theorem 4.2

6.1 Notations and preliminaries

We denote
(i) by $D_\infty = \{x \in X \mid \varphi(x) \text{ has infinite dimension}\}$,
(ii) by $D_{\le i} = D_0 \cup D_1 \cup \cdots \cup D_i$,
(iii) by $D_{\ge i} = (D_i \cup D_{i+1} \cup \cdots) \cup D_\infty$.
As in the previous section, we begin by listing a series of parallel results which will be used in order to prove Theorem 4.2. The first one is a classical result (see for example [4]).

Lemma 6.1. *Let C be a convex set such that* $\mathrm{ri}(C)$ *is nonempty. Then, for any α such that* $0 < \alpha < 1$, *we have* $(1-\alpha)\overline{A} + \alpha\,\mathrm{ri}(A) \subset \mathrm{ri}(A)$.

Lemma 6.2. *The set $D_{\ge i}$ is an open subset of X.*

Proof. Let $x \in D_{\ge i}$. Then there exists $j \ge i$ such that $x \in D_j$. Therefore, since $\dim_a \varphi(x) = j$, then we have $Y^{j+1}_{ai} \cap \mathrm{aff}(\varphi^{j+1}(x)) \neq \emptyset$ where $\varphi^{j+1}(x)$ is the cartesian product $\varphi^{j+1}(x) = \varphi(x) \times \cdots \times \varphi(x)$. Since the lower semicontinuity of φ implies that φ^{j+1} is also lsc, in view of the openness of Y^{j+1}_{ai}, there exists a neighborhood V_x of x such that for any $x' \in V_x$, $Y^{j+1}_{ai} \cap \mathrm{aff}(\varphi^{j+1}(x')) \neq \emptyset$. This implies that $\dim_a \varphi(x') \ge j$. That is, $V_x \subset D_{\ge j}$. Yet, since $D_{\ge j} \subset D_{\ge i}$, we finish the proof. □

Lemma 6.3. *The set $D_{\le(i-1)}$ is closed in $D_{\le i}$.*

4 The approximation is given by $\varphi_\varepsilon(x) = \mathrm{co}(\bigcup_{z \in (B(x,\varepsilon)\cap X)} \varphi(z))$.

Proof. In view of the partition $D_{\leq i} = D_{\leq i-1} \cup D_i$, in order to prove the result, it suffices to prove that D_i is an open set in $D_{\leq i}$. Using the previous remark, we already know that $D_{\geq i}$ is an open set of X. Yet, since $D_i = D_{\geq i} \cap D_{\leq i}$, the result is established. □

Lemma 6.4. *Let X and Y be two topological spaces and F a closed subset of X. Suppose that $\varphi : X \to 2^Y$ is an lsc correspondence and $f : F \to Y$ is a continuous single-valued selection of $\varphi_{|F}$. Then, the correspondence ψ given by*

$$\psi(x) = \begin{cases} \{f(x)\} & \text{if } x \in F, \\ \varphi(x) & \text{if } x \notin F, \end{cases}$$

is also lsc.

Proof. Let V be a closed subset of Y. We have

$$\psi^+(V) = \{x \in F \mid \psi(x) \subset V\} \cup \{x \in X \setminus F \mid \psi(x) \subset V\} = \{x \in F \mid f(x) \in V\} \cup \{x \in X \setminus F \mid \varphi(x) \subset V\}.$$

Since f is a selection of $\varphi_{|F}$, we deduce that

$$\begin{aligned} \{x \in X \mid \varphi(x) \subset V\} &= \{x \in X \setminus F \mid \varphi(x) \subset V\} \cup \{x \in F \mid \varphi(x) \subset V\} \\ &= \{x \in X \setminus F \mid \varphi(x) \subset V\} \cup \{x \in F \mid f(x) \in \varphi(x) \subset V\}. \end{aligned}$$

Therefore,

$$\psi^+(V) = \{x \in F \mid f(x) \in V\} \cup \{x \in X \mid \varphi(x) \subset V\} = f^{-1}(V) \cup \varphi^+(V).$$

Since φ is lsc, the set $\varphi^+(V)$ is closed. Moreover, since V is a closed subset of Y and f is continuous, it follows that $f^{-1}(V)$ is a closed subset of F. Now, since F is closed in X, the set $f^{-1}(V)$ is closed in X. Hence, $\psi^+(V)$ is closed, as required. □

6.2 Proof of Theorem 4.2

The proof of Theorem 4.2 is outlined in three steps as follows:

(Step 1) For any $k \in \mathbb{N}$, we have that $\mathrm{ri}(\varphi)$ admits a continuous selection $h_{\leq k}$ on $D_{\leq k}$.

(Step 2) For any $k \in \mathbb{N}$, there exists $j^k : X \to Y$ such that
- j^k is a continuous selection of $\overline{\varphi}$ on X.
- j^k is a continuous selection of $\mathrm{ri}(\varphi)$ on $D_{\leq k}$.

(Step 3) There exists a continuous selection f of φ on X.

Proof of Step 1. Let us apply for any $k \in \mathbb{N}$, Theorem 4.1 in order to get the existence of a continuous single-valued function h_k defined on D_k such that for all $x \in D_k$, $h_k(x) \in \mathrm{ri}(\varphi_k(x))$. Let P_n be the following heredity property: the restriction of $\mathrm{ri}(\varphi)$ to $D_{\leq n}$ admits a continuous selection $h_{\leq n}$.

For $n = 0$, it suffices to notice that $D_{\leq 0} = D_0$. Therefore, we can let $h_{\leq 0} := h_0$ which is a continuous selection of $\mathrm{ri}(\varphi)$. Thus, P_0 is true.

Let $n \geq 1$. Suppose that P_{n-1} holds true and let us prove that P_n is true. By the heredity hypothesis, we have that $\mathrm{ri}(\varphi)$ admits a continuous selection $h_{\leq(n-1)}$ on $D_{\leq(n-1)}$. We will introduce an auxiliary mapping $\widetilde{\varphi}_n$ defined on $D_{\leq n}$, which is lsc and such that the graph is contained in the graph of φ, by taking

$$\widetilde{\varphi}_n : D_{\leq n} \to 2^Y \quad \text{defined by} \quad \widetilde{\varphi}_n(x) = \begin{cases} \{h_{\leq(n-1)}(x)\} & \text{if } x \in D_{\leq n-1}, \\ \overline{\varphi}(x) & \text{if } x \in D_n. \end{cases}$$

By Lemma 6.3 and Lemma 6.4, we conclude that $\widetilde{\varphi}_n$ is lsc. Moreover, $\widetilde{\varphi}_n$ has closed-convex values. By Michael's selection Theorem 3.1, there exists g_n continuous such that $g_n(x) \in \widetilde{\varphi}_n(x) \subset \overline{\varphi}(x)$, for every $x \in D_{\leq n}$. In the following, we will construct a continuous single-valued selection $h_{\leq n}$ of $\mathrm{ri}(\varphi)$ on $D_{\leq n}$. We consider the continuous application $\lambda : D_n \to]0, 1[$ given by

$$\lambda(x) = \frac{\min(d(x, D_{\leq n-1}), 1)}{2 + \|h_n(x)\|}.$$

Then we define $h_{\leq n} : D_{\leq n} \to Y$ given by

$$h_{\leq n}(x) = \begin{cases} g_n(x) & \text{if } x \in D_{\leq n-1}, \\ (1 - \lambda(x))g_n(x) + \lambda(x)h_n(x) & \text{if } x \in D_n. \end{cases}$$

Using the heredity property, we know that $h_{\leq(n-1)}$ is a selection of $\mathrm{ri}(\varphi)$ on $D_{\leq(n-1)}$. On the other hand, on D_n, the function $h_{\leq n}$ is a strict convex combination of $g_n \in \overline{\varphi}$ and $h_n \in \mathrm{ri}(\varphi)$, thus by Lemma 6.1, $h_{\leq n} \in \mathrm{ri}(\varphi)$. Therefore, we conclude that $h_{\leq n}$ is a selection of $\mathrm{ri}(\varphi)$ on $D_{\leq n}$. It remains to check the continuity of $h_{\leq n}$. It is clear that the restriction of $h_{\leq n}$ on D_n (respectively on $D_{\leq n-1}$) is continuous. Since D_n is open in $D_{\leq n}$, it suffices to consider the case of a sequence $x_k \in D_n$ such that $x_k \to \overline{x} \in D_{\leq n-1}$. It is obvious that since $d(x_k, D_{\leq n-1})$ tends to zero, $\lambda(x_k)$ tends to zero when x_k tends to $\overline{x}$. Besides, we have that $\|h_n(x)\|/(2 + \|h_n(x)\|)$ is bounded. That is, $\lambda(x_k)h_n(x_k)$ tends to zero when x_k tends to $\overline{x}$. Therefore, combining the previous remarks and the continuity of g_n allows us to conclude that $h_{\leq n}(x_k)$ tends to $g_n(\overline{x}) = h_{\leq n}(\overline{x})$, which establishes the result. □

Proof of Step 2. Using Lemma 6.2, we have already proved that $D_{\geq i}$ is an open set of X. Consequently, $D_{\leq i-1}$ is closed in X. Using again Lemma 6.3, we can define an lsc function $T^k : X \to 2^Y$ given by

$$T^k(x) = \begin{cases} \{h_{\leq k}(x)\} & \text{if } x \in D_{\leq k}, \\ \overline{\varphi}(x) & \text{if } x \in D_{\geq k+1}. \end{cases}$$

By Michael's Selection Theorem 3.1, there exists j^k continuous such that for all $x \in X$, $j^k(x) \in T^k(x) \subset \overline{\varphi}(x)$. That is, j^k is a selection of $\overline{\varphi}$. On the other hand, for any $x \in D_{\leq k}$, we have $j^k(x) \in T^k(x) = \{h_{\leq k}(x)\}$, which finishes the proof of Step 2. □

Proof of Step 3. We can write X as a partition between X_1 and D_∞, where $X_1 = \bigcup_{k\in\mathbb{N}} D_k$. Under the hypothesis of Theorem 4.2, we have that D_∞ is a subset of X_2, where $X_2 = \{x \in X \mid \varphi(x) \text{ has closed values}\}$.

Now, in the spirit of Michael's proof, for any $k \in \mathbb{N}^*$, we define

$$f^k(x) = \lambda_k(x)j^k(x) + (1 - \lambda_k(x))j^0(x), \quad \text{where } \lambda_k(x) = \frac{1}{\max(1, \|j^k(x) - j^0(x)\|)}.$$

It is easy to check in view of Lemma 6.1 that for each $k \in \mathbb{N}^*$, f^k is also a selection of $\overline{\varphi}$ on X satisfying for any $x \in D_{\leq k}$, $f^k(x) \in \mathrm{ri}(\varphi(x))$. In addition, we have that $f^k(x)$ is bounded since $f^k(x)$ can be written as $f^k(x) = j^0(x) + \lambda_k(x)(j^k(x) - j^0(x))$ and our choice of $\lambda_k(x)$ ensures that $\|f^k(x)\| \leq \|j_0(x)\| + 1$.

We can now define for any $k \in \mathbb{N}^*$,

$$\tilde{f}^n(x) = \sum_{k=1}^{n} \frac{1}{2^k} f^k(x), \quad f(x) = \sum_{k\in\mathbb{N}^*} \frac{1}{2^k} f^k(x).$$

First, we claim that f is continuous at any $\overline{x} \in X$. As a consequence of the continuity of j_0, the set

$$W_{\overline{x}} = \{x \in X \mid \|j_0(x)\| \leq \|j_0(\overline{x})\| + 1\}$$

is an open neighborhood of $\overline{x}$. Indeed, each f^k is continuous and the series $\tilde{f}^n$ converges uniformly to f on $W_{\overline{x}}$, since

$$\|f(x) - \tilde{f}^n(x)\| \leq \sum_{k=n+1}^{\infty} \frac{1}{2^k} \|f^k(x)\| \leq (\|j^0(\overline{x})\| + 2) \sum_{k=n+1}^{\infty} \frac{1}{2^k}.$$

Moreover, we have $f(x) \in \overline{\varphi}(x)$ for any $x \in X$, since f is an "infinite convex combination"[5] of elements of $\overline{\varphi}(x)$.

5 We recall that if C is a convex subset of Y, then for any bounded sequence $(c_k)_{k\in\mathbb{N}} \in C$ and for any sequence of non-negative real numbers $(\lambda_k)_{\in\mathbb{N}}$ with $\sum_{k\in\mathbb{N}} \lambda_k = 1$, the series $\sum_{\in\mathbb{N}} \lambda_k c_k$ converges to an element of $\overline{C}$.

Now, we claim that for all $x \in X$, $f(x)$ is an element of $\varphi(x)$. We have to distinguish three cases.

- If $x \in D_0$, then $f(x) = j_0(x)$ since $\varphi(x)$ is a singleton.
- If $x \in X_1 \setminus D_0$, then there exists $k_0(x) > 0$ such that $x \in D_{k_0(x)} \subset D_{\leq k_0(x)}$. Let us remark that $f(x)$ can be written as
$$f(x) = \mu f^{k_0(x)}(x) + (1-\mu) \sum_{k \neq k_0(x)} \frac{f^k(x)}{2^k(1-\mu)},$$
where $\mu = \frac{1}{2^{k_0(x)}}$. Using once again Footnote 5, we easily check that $\sum_{k \neq k_0(x)} \frac{f^k(x)}{2^k(1-\mu)}$ belongs to $\overline{\varphi}(x)$. Therefore, by Lemma 6.1, $f(x)$ is an interior point of $\varphi(x)$, then a selection of φ, which finishes the proof.
- If $x \in D_\infty$, then $x \in X_2$. That is, $\overline{\varphi}(x) = \varphi(x)$. Since we have already established that $f(x) \in \overline{\varphi}(x)$ for all $x \in X$, the result is immediate. □

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