Research Article

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Existence and multiplicity of solutions for a class of superlinear elliptic systems

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Abstract: In this paper, we establish the existence and multiplicity of solutions for a class of superlinear elliptic systems without Ambrosetti and Rabinowitz growth condition. Our results are based on minimax methods in critical point theory.

Keywords: Elliptic systems, critical points, variational methods

MSC 2010: 35J50, 35B38, 47J30

1 Introduction and main results

We consider the following elliptic system:

$$\begin{cases} -\Delta u = v^{p-1} & \text{in } \Omega, \\ -\Delta v = f(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < p \le 2$, $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ is a smooth bounded domain and $f \in C(\mathbb{R}, \mathbb{R})$. Here and in the following, we denote $s^\alpha = \operatorname{sgn}(s)|s|^\alpha$.

If $f(s) = s^{q-1}$, system (1.1) is known as the Lane–Emden system (see [11, 15, 20]). Recently, the existence and multiplicity of solutions for system (1.1) has been studied by many authors, see [4, 9, 14, 16, 17, 19–22, 26–29, 31]. Especially, de Figueiredo and Ruf [17] proved the existence of nontrivial solutions for system (1.1), where the function f is superlinear and with no growth restriction. By using variational and perturbative methods, Salvatore [29] established the existence of infinitely many solutions both in the symmetric and in the non-symmetric cases. The following result is stated in [29].

Theorem 1.1 ([29, Theorem 1.1]). *Let $1 < p \le 2$ if $N = 2, 3$ or $1 < p < \frac{N}{N-2}$ if $N \ge 4$. Assume that f verifies following conditions:*

(f1) *there exist constants $\mu > q$ and $s_0 \ge 0$ such that*

$$0 < \mu F(s) \le f(s)s \quad \text{for all } |s| \ge s_0,$$

where $F(s) = \int_0^s f(t)\,dt$ and $q = \frac{p}{p-1}$,

(f2) *$f(s) = o(s^{q-1})$ for s near 0.*

Then, problem (1.1) has a nontrivial solution. If, further, f is odd, then problem (1.1) has infinitely many pairs of solutions. Moreover, if f is Hölder continuous, the solutions are classical.

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Here, condition (f1) is the classical Ambrosetti–Rabinowitz condition, which was first used in [2]. Over the last few decades, the Ambrosetti–Rabinowitz condition has appeared in various kinds of studies for superlinear problems, e.g., elliptic equations, Hamiltonian systems and wave equations, see [12, 17, 20, 23–25, 27, 29, 32] and references therein. Some researchers have been successful in replacing the Ambrosetti–Rabinowitz condition by other superquadratic conditions. In [9], Chen and Zou introduced a class of new condition which is weaker than the Ambrosetti–Rabinowitz condition. They improved Theorem 1.1 and showed that the conclusion of Theorem 1.1 remains valid if (f1) is replaced by the following condition (see [9, Theorem 1.1]):

(f1') There exist a constant $s_0 \geq 0$ and a decreasing function $\theta(s) \in C(\mathbb{R} \setminus (-s_0, s_0))$, such that

$$0 < (q + \theta(s))F(s) \leq f(s)s \quad \text{for all } |s| \geq s_0,$$

where $\theta(s) > 0$ and $\lim_{|s|\to+\infty} |s|\theta(s) = \lim_{|s|\to+\infty} \int_{s_0}^{|s|} \frac{\theta(t)}{t} dt = +\infty$.

In this paper, motivated by the works [9, 17, 27, 29], we consider the existence and multiplicity of solutions for problem (1.1). We state our main results in the following theorems.

Theorem 1.2. *Let $1 < p \leq 2$ if $N = 2, 3$ or $1 < p < \frac{N}{N-2}$ if $N \geq 4$. Suppose that f satisfies the following conditions:*

(f3) *there exists a positive constant $s_0 > 0$ such that*

$$f(s)s - qF(s) \geq \frac{F(s)}{|s|^q} \quad \text{for all } |s| \geq s_0, \quad \text{where } q = \frac{p}{p-1},$$

(f4) $\lim_{|s|\to+\infty} \frac{F(s)}{|s|^q} = +\infty$,

(f5) $\lim_{|s|\to 0} \frac{F(s)}{|s|^q} = 0$.

Then, problem (1.1) *has a nontrivial solution. If, further, f is odd, then problem* (1.1) *has infinitely many pairs of solutions. Moreover, if f is Hölder continuous, then the solutions are classical.*

Remark 1.3. For Hamiltonian systems, the corresponding condition (f3) is due to Tang and Wu [30].

Remark 1.4. Obviously, we deduce from condition (f1') that there is a positive constant M such that

$$F(s) \geq M|s|^q e^{\int_{s_0}^{|s|} \frac{\theta(t)}{t} dt} = M|s|^q G(|s|) \quad \text{for all } |s| \geq s_0,$$

where $G(|s|) = e^{\int_{s_0}^{|s|} \frac{\theta(t)}{t} dt}$. Then, one has

$$\frac{F(s)}{|s|^q} \geq MG(|s|) \to +\infty \quad \text{as } |s| \to \infty.$$

Since $\lim_{|s|\to+\infty} |s|\theta(s) = +\infty$, there exists some positive constant $s_0 > 1$ such that

$$\theta(s) > |s|^{-1} > |s|^{-q} \quad \text{for all } |s| \geq s_0.$$

Thus, condition (f1') implies (f3) and (f4). Hence, our Theorem 1.2 generalizes [29, Theorem 1.1] and [9, Theorem 1.1]. There are functions F satisfying the assumptions of our Theorem 1.2 but not satisfying the corresponding assumptions in [9, 29]. For example,

$$F(s) = |s|^q \ln(1 + |s|^q) + \sin|s|^q - \ln(1 + |s|^q) \quad \text{for all } s \in \mathbb{R}.$$

Theorem 1.5. *Let $N \geq 4$, $\frac{N}{N-2} \leq p \leq 2$ and $q = \frac{p}{p-1}$. Suppose that* (f4), (f5) *hold and*

(f6) *there exist two positive constants M_0 and η such that $|f(s)| \leq M_0(|s|^{\eta-1} + 1)$, where $\eta \in (q, \frac{Nq}{N-2q})$ if $2 \leq q < \frac{N}{2}$, or $\eta \in (q, +\infty)$ if $q = \frac{N}{2}$,*

(f7) *there exists a constant $\beta > N(\eta - q)/2q$ such that*

$$\liminf_{|s|\to\infty} \frac{f(s)s - qF(s)}{|s|^\beta} > 0.$$

Then, problem (1.1) *has a nontrivial solution. If, further, f is odd, then problem* (1.1) *has infinitely many pairs of solutions. Moreover, if f is Hölder continuous, then the solutions are classical.*

Remark 1.6. Theorem 1.5 generalizes [29, Theorem 1.2]. There are functions satisfying the assumptions of our Theorem 1.5 and not satisfying the assumptions in [29]. For example,

$$F(s) = |s|^q \ln(1 + |s|^q)$$

satisfies our Theorem 1.5, but does not satisfy (f1), and hence the corresponding results in [17, 27, 29] are not applicable.

Theorem 1.7. *The conclusion of Theorem 1.5 remains valid if we replace* (f5), (f6) *and* (f7) *by* (f2) *and the following condition*

(f8) *there exist three positive constants* s_1 *and* m_1, m_2 *such that*

(j1) $f(s)s - qF(s) \geq m_1|s|^q$, *if* $|s| \geq s_1$,

(j2) $|f(s)|^\sigma \leq m_2(f(s)s - qF(s))|s|^{(q-1)\sigma}$, *if* $|s| \geq s_1$, *where* $\sigma > N/(2q)$.

Remark 1.8. For Schrödinger equations, the corresponding condition (f8) is due to Ding and Luan [13]. The condition (f8) is weaker than the usual Ambrosetti–Rabinowitz-type condition (f1) (see [13]). Thus, Theorem 1.7 extends [29, Theorem 1.2].

We shall also study the following nonhomogeneous system:

$$\begin{cases} -\Delta u = v^{p-1} & \text{in } \Omega, \\ -\Delta v = f(u) + h(x) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

By using perturbative methods, we shall establish the following multiplicity results.

Theorem 1.9. *Let* $1 < p \leq 2$ *and* $q = \frac{p}{p-1}$. *Suppose that* (f4), (f5), (f6) *and* (f7) *hold, where* $\eta \in (q + \frac{2q}{N}, \frac{Nq}{N-2q})$ *if* $2 \leq q < \frac{N}{2}$, *or* $\eta \in (q + \frac{2q}{N}, +\infty)$ *if* $q = \frac{N}{2}$. *If* f *is odd, then for any* $h \in L^{\beta/(\beta-1)}$ *system* (1.2) *has infinitely many solutions, where*

$$\frac{\beta}{\beta - 1} < \frac{2q\eta}{N(\eta - q)} - 1.$$

Furthermore, if f *is Hölder continuous and* $h \in C^1(\overline{\Omega})$, *then the solutions are classical.*

Theorem 1.10. *Suppose that* (f2), (f4) *and* (f8) *hold. If* f *is odd, then for any* $h \in L^p$ *with* $p + 1 < 2q\sigma/N$, *system* (1.2) *has infinitely many solutions. Furthermore, if* f *is Hölder continuous and* $h \in C^1(\overline{\Omega})$, *then the solutions are classical.*

2 Proof of main results

For $1 < p \leq 2$, system (1.2) can be written as

$$\begin{cases} (-\Delta u)^{\frac{1}{p-1}} = v & \text{in } \Omega, \\ -\Delta v = f(u) + h(x) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, system (1.2) is equivalent to the following fourth-order equation:

$$\begin{cases} -\Delta(-\Delta u)^{\frac{1}{p-1}} = f(u) + h(x) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

To set our problem variationally, we consider the space $E = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ endowed with the norm

$$\|u\|_E = \Big(\int_\Omega |\Delta u|^q \Big)^{1/q},$$

which is equivalent to usual intersection norm (see [18, Lemma 9.17])

$$\|u\|_E = \max\left\{\|u\|_{W^{2,q}}, \|u\|_{W_0^{1,q}}\right\}.$$

From [29, Proposition 2.1], it is easy to see that the weak solutions of problem (1.2) are the critical points of the energy functional

$$\Phi(u) = \frac{1}{q}\int_\Omega |\Delta u|^q\,dx - \int_\Omega F(u)\,dx - \int_\Omega hu\,dx, \quad u \in E.$$

Hence, the problem of finding weak solutions of problem (1.2) is equal to the one of seeking the critical points of Φ.

Proposition 2.1 ([29, Proposition 2.2]). *Let $1 < p \le 2$, let f be Hölder continuous and let $h \in C^1(\overline{\Omega})$. Let $u \in E$ be a weak solution of (2.1) and $v = -(\Delta u)^{\frac{1}{p-1}}$. Then (u, v) is a classical solution of (1.2).*

The following embedding results are well known in the literature (see [1]):

(i) $1 < p \le 2$ if $N = 2, 3$ or $1 < p < \frac{N}{N-2}$ if $N \ge 4$, one has $q > \frac{N}{2}$, then the imbedding $W^{2,q}(\Omega) \hookrightarrow C(\Omega)$ is compact, where $C(\Omega)$ is the space of the continuous bounded functions on Ω with the norm

$$\|u\|_\infty = \sup_{x\in\Omega} |u(x)|.$$

Hence, there exists a positive constant K_C such that

$$\|u\|_\infty \le K_C\|u\| \quad \text{for all } u \in E. \tag{2.2}$$

(ii) $p = \frac{N}{N-2}$, $N \ge 4$, i.e. $q = \frac{N}{2}$, then the imbedding $W^{2,q}(\Omega) \hookrightarrow L^r$, for $2 < r < +\infty$ is compact. Hence, there exists a positive constant K_r such that

$$\|u\|_r \le K_r\|u\|, \quad 2 < r < +\infty \quad \text{for all } u \in E. \tag{2.3}$$

(iii) $\frac{N}{N-2} < p \le 2$, $N \ge 4$, one has $q < \frac{N}{2}$, then the imbedding $W^{2,q}(\Omega) \hookrightarrow L^{\frac{Nq}{N-2q}}$ is continuous. The embedding $W^{2,q}(\Omega) \hookrightarrow L^r$, for $q < r < \frac{Nq}{N-2q}$ is compact. Hence, there exists a positive constant K_r such that

$$\|u\|_r \le K_r\|u\|, \quad q < r \le \frac{Nq}{N-2q} = q^{**} \quad \text{for all } u \in E. \tag{2.4}$$

For reader's convenience, we recall an abstract result introduced by Bolle in [6, 7] for dealing problems with symmetry breaking. Following [10], here we state a version of this theorem which involves functionals which are of class C^1.

Let E be a Banach space with $E = E^- \oplus E^+$, where $\dim(E^-) < +\infty$, and let $\{e_k\}_{k\ge1}$ be a basis of E^+. Set

$$E_0 = E^-, \quad E_{k+1} = E_k \oplus \mathbb{R}e_{k+1}, \quad k \in \mathbb{N},$$

so that

$$E_0 \subset E_1 \subset \cdots \subset E_k \subset \cdots \subset E.$$

Let

$$J_\theta = J(\theta, \cdot) : [0, 1] \times E \to \mathbb{R}, \quad \theta \in [0, 1]$$

be a C^1 functional and $J'_\theta(u) = \frac{\partial J}{\partial u}(\theta, u)$. Then J has the following four properties:

(I1) J satisfies the following variant of the classical condition (C): any sequence $\{(\theta_n, u_n)\}_n \subset [0, 1] \times E$ such that $\{J(\theta_n, u_n)\}_n$ is bounded and $\lim_{n\to\infty}(1 + \|u_n\|)\|J'_{\theta_n}(u_n)\| = 0$ converges up to subsequences,

(I2) for any $b > 0$ there exists $M_b > 0$ such that

$$|J_\theta(u)| \le b \implies \left|\frac{\partial J}{\partial\theta}(\theta, u)\right| \le M_b(\|J'_\theta(u)\| + 1)(\|u\| + 1), \quad (\theta, u) \in [0, 1] \times E,$$

(I3) there exist two continuous maps $g_1, g_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$, Lipschitz continuous with respect to the second variable, such that $g_1(\theta,\cdot) \le g_2(\theta,\cdot)$ and if $(\theta, u) \in [0,1] \times E$ then

$$J'_\theta(u) = 0 \implies g_1(\theta, J_\theta(u)) \le \frac{\partial J}{\partial \theta}(\theta, u) \le g_2(\theta, J_\theta(u)),$$

(I4) J_0 is even and for each finite-dimensional subspace W of E it follows that

$$\lim_{u\in W, \|u\|\to\infty} \sup_{\theta\in[0,1]} J(\theta, u) = -\infty.$$

Property (I4) implies that there is a positive sequence $\{R_k\}$ such that

$$J_\theta(u) < 0, \quad \|u\| \ge R_k \quad \text{for all } u \in E_k.$$

Thus we can define

$$\Gamma_k = \{\gamma \in C(E_k \cap \bar{B}_{R_k}(0), E) : \gamma \text{ odd}, \gamma(u) = u \text{ if } u \in E_k \cap \partial B_{R_k}(0)\},$$
$$c_k = \inf_{\gamma\in\Gamma_k} \sup_{u\in E_k\cap B_{R_k}(0)} J_0(\gamma(u)).$$

Let $\psi_i : [0,1] \times \mathbb{R} \to \mathbb{R}$ be the flow associated to g_i, i.e. the solution of the problem

$$\begin{cases} \frac{\partial \psi_i}{\partial \theta}(\theta, s) = g_i(\theta, \psi_i(\theta, s)), \\ \psi_i(0, s) = s. \end{cases}$$

Here, $\psi_i(\theta,\cdot)$ is continuous, non-decreasing on $\mathbb{R}$ and $\psi_1(\theta,\cdot) \le \psi_2(\theta,\cdot)$. Let

$$\bar{g}_1(s) = \sup_{\theta\in[0,1]} |g_1(\theta, s)|, \quad \bar{g}_2(s) = \sup_{\theta\in[0,1]} |g_2(\theta, s)|.$$

Now, we are in the position to state the following Theorem (for a proof, see [6, Theorem 3] and [7, Theorem 3]).

Theorem 2.2. *There exists a constant $C \in \mathbb{R}$ such that if $k \in \mathbb{N}$, then*
(i1) *either J_1 has a critical level $\tilde{c}_k$ with $\psi_2(1, c_k) < \psi_1(1, c_{k+1}) \le \tilde{c}_k$,*
(i2) *or $c_{k+1} - c_k \le C(\bar{g}_1(c_{k+1}) + \bar{g}_1(c_k) + 1)$.*

Remark 2.3. If $g_2 \ge 0$ in $[0,1] \times \mathbb{R}$, the function $\psi_2(\cdot, s)$ is non-decreasing on $[0,1]$. Hence, $c_k \le \tilde{c}_k$ for all c_k verifying case (i1).

Lemma 2.4 ([29, Lemma 5.1]). *Let Ω be a smooth bounded domain. There exists a Schauder basis $\{e_k\}_k$ in $W^{2,q}(\Omega)$ such that for $r \in [q, \frac{Nq}{N-2q})$ there is $C_1 > 0$ such that*

$$C_1 k^{\frac{2}{N}+\frac{1}{r}-\frac{1}{q}} \|u\|_r \le \|u\|_{W^{2,q}} \quad \textit{for all } u \in \overline{\operatorname{span}\{e_k, e_{k+1}, \ldots\}}.$$

We shall now prove our results.

2.1 The homogeneous case $h = 0$

Proof of Theorem 1.2. As shown in [5], a deformation lemma can be proved with the weaker condition (C) which is due to Cerami [8] replacing the usual Palais–Smale condition, and it turns out that the Mountain Pass Theorem and the symmetric version of Mountain Pass Theorem (see [2, 24]) hold true under condition (C).

First, we will prove that Φ satisfies condition (C), i.e. for every sequence $\{u_n\} \subset E$, $\{u_n\}$ has a convergent subsequence if

$$\{\Phi(u_n)\} \text{ is bounded and } (1 + \|u_n\|)\|\Phi'(u_n)\| \to 0 \text{ as } n \to \infty.$$

Let L_0 be a positive constant such that

$$|\Phi(u_n)| \le L_0, \quad (1 + \|u_n\|)\|\Phi'(u_n)\| \le L_0. \tag{2.5}$$

By using a standard argument, we shall prove that $\{u_n\}$ is a bounded sequence in E. If not, we can assume that $\|u_n\| \to +\infty$ as $n \to \infty$. Let $w_n = \frac{u_n}{\|u_n\|}$, so that $\|w_n\| = 1$. If necessary considering a subsequence, we can suppose that

$$\begin{aligned} &w_n \rightharpoonup w \quad \text{weakly in } E, \\ &w_n \to w \quad \text{strongly in } C(\Omega) \end{aligned} \tag{2.6}$$

as $n \to \infty$.

From (2.5) and (2.6), we have

$$\Big| \int_\Omega \frac{F(u_n)}{\|u_n\|^q}\, dx - \frac{1}{q} \Big| \le \frac{L_0}{\|u_n\|^q}. \tag{2.7}$$

By (f4), there exist two constants $s_2 > s_0$ and L_1 such that

$$F(s) \ge 0 \quad \text{for all } |s| \ge s_2, \tag{2.8}$$

and

$$|f(s)| \le L_1, \quad |F(s)| \le L_1 \quad \text{for all } |s| \le s_2. \tag{2.9}$$

Hence, it follows that

$$F(s) \ge -L_1, \quad s \in \mathbb{R}. \tag{2.10}$$

If $w \not\equiv 0$, letting $\Omega_1^+ = \{x \in \Omega : |w(x)| > 0\}$ and $\Omega_1^- = \Omega \setminus \Omega_1^+$, one has $|\Omega_1^+| > 0$, where $|\cdot|$ is the Lebesgue measure. Since $\|u_n\| \to +\infty$, we get $|u_n| \to \infty$ as $n \to \infty$ for $x \in \Omega_1^+$. From (f4), we obtain

$$\lim_{n\to+\infty} \frac{F(u_n)}{|u_n|^q} = +\infty \quad \text{on } \Omega_1^+.$$

It follows from (2.10) and Fatou's lemma that

$$\liminf_{n\to\infty} \int_\Omega \frac{F(u_n)}{\|u_n\|^q}\, dx \ge \liminf_{n\to\infty} \Big(\int_{\Omega_1^+} \frac{F(u_n)}{|u_n|^q} |w_n|^q\, dx - \|u_n\|^{-q} L_1 |\Omega_1^-| \Big) = +\infty,$$

which contradicts (2.7). Thus, $\|u_n\|$ is bounded.

If $w \equiv 0$, by (2.7), we have

$$\lim_{n\to\infty} \int_\Omega \frac{F(u_n)}{\|u_n\|^q}\, dt = \frac{1}{q}. \tag{2.11}$$

By (f5), there exists a constant s_3 with $s_3 < s_0$ such that

$$|F(s)| \le |s|^q \quad \text{for all } |s| \le s_3. \tag{2.12}$$

It follows from (f3), (2.8), (2.9) and (2.12) that

$$\frac{|F(s)|}{|s|^q} \le f(s)s - qF(s) + 1 + s_3^{-q} L_1 = f(s)s - qF(s) + L_2 \quad \text{for all } s \in \mathbb{R},$$

where $L_2 = 1 + s_3^{-q} L_1$. Thus, it follows that

$$\begin{aligned} \int_\Omega \frac{|F(u_n)|}{|u_n|^q}\, dx &\le \int_\Omega (f(u_n)u_n - qF(u_n))\, dx + L_2|\Omega| \\ &= q\Phi(u_n) - \langle \Phi'(u_n), u_n \rangle + L_2|\Omega| \le (q+1)L_0 + L_2|\Omega|. \end{aligned}$$

Hence, we find

$$\left| \int_\Omega \frac{F(u_n)}{\|u_n\|^q}\,dx \right| = \left| \int_\Omega \frac{F(u_n)}{|u_n|^q}|w_n|^q\,dx \right| \le \|w_n\|_\infty^q((q+1)L_0 + L_2|\Omega|) \to 0$$

as $n \to \infty$, which is a contradiction to (2.11). So, $\|u_n\|$ is bounded. And, condition (C) holds.

Next, we will show that Φ satisfies the conditions of the Mountain Pass Theorem. Let $S = \{v \in E : \|v\| = 1\}$. Fix some $v \in S$, so that

$$\|v\|_q = \Big(\int_\Omega |v|^q\,dx \Big)^{1/q} > 0.$$

We find from condition (f4) that for $L_v = \frac{2}{q\|v\|_q^q}$ there exists a positive constant L_3 such that

$$F(s) \ge L_v|s|^q - L_3, \quad s \in \mathbb{R}.$$

Thus, we find

$$\Phi(\lambda v) = \frac{1}{q}\int_\Omega |\Delta(\lambda v)|^q\,dx - \int_\Omega F(\lambda v)\,dx \le \frac{|\lambda|^q}{q} - L_v|\lambda|^q \int_\Omega |v|^q\,dx + L_3|\Omega| = -\frac{|\lambda|^q}{q} + L_3|\Omega|,$$

and hence

$$\lim_{|\lambda|\to+\infty} \Phi(\lambda v) = -\infty. \tag{2.13}$$

From (f5), there exists a positive constant $\delta > 0$ such that

$$\frac{|F(s)|}{|s|^q} \le \frac{1}{2qK_C^q|\Omega|} \quad \text{for all } |s| \le \delta. \tag{2.14}$$

For $u \in E$ with $\|u\| \le \delta/K_C$, we have $\|u\|_\infty \le \delta$. We conclude from (2.14) that

$$\begin{aligned}\Phi(u) &= \frac{1}{q}\int_\Omega |\Delta u|^q\,dx - \int_\Omega F(u)\,dx \ge \frac{1}{q}\int_\Omega |\Delta u|^q\,dx - \frac{1}{2qK_C^q|\Omega|}\int_\Omega |u|^q\,dx \\ &\ge \frac{1}{q}\|u\|^q - \frac{1}{2qK_C^q}\|u\|_\infty^q \ge \frac{1}{2q}\|u\|^q.\end{aligned}$$

Choose $\rho \in (0, \delta/K_C)$; then there exists $\alpha > 0$ such that

$$\inf_{\|u\|=\rho} \Phi(u) \ge \alpha > 0. \tag{2.15}$$

Obviously, we have $\Phi(0) = 0$. Thus, it follows from (2.13) and (2.15) that Φ satisfies the conditions of the Mountain Pass Theorem (see [2, 24]) with condition (C). Hence, there exists a nontrivial critical point, which implies that the problem (1.1) has a nontrivial solution.

Further, if f is odd, then Φ is even. Following the above arguments, it is easy to find that Φ satisfies all the conditions of the symmetric version of Mountain Pass Theorem (see [2, 24]) with condition (C). Thus, Theorem 1.2 holds. □

Proof of Theorem 1.5. First, we claim that Φ satisfies condition (C), i.e. for every sequence $\{u_n\} \subset E$, $\{u_n\}$ has a convergent subsequence if $\{\Phi(u_n)\}$ is bounded and $(1 + \|u_n\|)\|\Phi'(u_n)\| \to 0$ as $n \to \infty$.

Let $M_1 > 0$ be a constant such that

$$|\Phi(u_n)| \le M_1, \quad (1 + \|u_n\|)\|\Phi'(u_n)\| \le M_1. \tag{2.16}$$

From a standard argument, we only need to show that $\{u_n\}$ is a bounded sequence in E.

From (f7), we find that there exist two constants $M_2, M_3 > 0$ such that

$$f(s)s - qF(s) \ge M_2|s|^\beta - M_3 \quad \text{for all } s \in \mathbb{R}. \tag{2.17}$$

Thus, by (2.16) and (2.17), we have

$$(q+1)M_1 \geq q\Phi(u_n) - \langle \Phi'(u_n), u_n\rangle = \int_\Omega (f(u_n)u_n - qF(u_n))\,dx \geq M_2 \int_\Omega |u_n|^\beta\,dx - M_3|\Omega|.$$

Hence, there exists a positive constant M_4 such that

$$\int_\Omega |u_n|^\beta\,dx < M_4. \tag{2.18}$$

By (f6) and (f7), it is clear that $\eta \geq \beta$. If $\eta = \beta$, then $\{\|u_n\|\}$ is obviously bounded in E. Thus, it suffices to consider the case $\eta > \beta$.

Starting with $2 \leq q < \frac{N}{2}$, we have

$$\frac{N(\eta - q)}{2q} < \beta < \eta < \frac{Nq}{N-2q}.$$

Choosing $\alpha = \frac{Nq(\eta-\beta)}{Nq+2q\beta-\beta N}$, we get $0 < \alpha < q$. Set

$$\tau = \frac{\beta}{\eta - \alpha} = \frac{Nq + 2q\beta - \beta N}{Nq + 2q\eta - \eta N} > 1.$$

We conclude from Hölder's inequality, (2.4) and (2.18) that

$$\begin{aligned}\int_\Omega |u_n|^\eta\,dx &= \int_\Omega |u_n|^{\eta-\alpha}|u_n|^\alpha\,dx \\ &= \int_\Omega |u_n|^{\beta/\tau}|u_n|^\alpha\,dx \\ &\leq \Big(\int_\Omega \big(|u_n|^{\beta/\tau}\big)^\tau\,dx\Big)^{1/\tau}\Big(\int_\Omega |u_n|^{\tau'\alpha}\,dx\Big)^{1/\tau'} \leq M_4^{1/\tau}K_{q^{**}}^\alpha \|u_n\|^\alpha\end{aligned} \tag{2.19}$$

for all n, where $\tau' = \tau/(\tau-1) = q^{**}/\alpha$.

From (f6), (2.4) and (2.19), there exists a positive constant M_5 such that

$$\begin{aligned}\langle \Phi'(u_n), u_n\rangle &= \|u_n\|^q - \int_\Omega f(u_n)u_n\,dx \\ &\geq \|u_n\|^q - \int_\Omega |f(u_n)||u_n|\,dx \geq \|u_n\|^q - M_0\int_\Omega (|u_n|^\eta + |u_n|)\,dx \\ &= \|u_n\|^q - M_0\int_\Omega |u_n|^\eta\,dx - M_0\int_\Omega |u_n|\,dx \geq \|u_n\|^q - M_0M_4^{1/\tau}K_{q^{**}}^\alpha\|u_n\|^\alpha - M_5\|u_n\|\end{aligned}$$

for all n. Since $q > \alpha$, by (2.16), $\{\|u_n\|\}$ is bounded in E.

When $q = \frac{N}{2}$, we consider the case $\eta \in (q, +\infty)$ and $\beta > \eta - q$. Let $\alpha_1 = \frac{q+\eta-\beta}{2}$, to have $0 < \alpha_1 < q$. Set

$$\tau_1 = \frac{\beta}{\eta - \alpha_1} = \frac{2\beta}{\eta - q + \beta} > 1.$$

We conclude from Hölder's inequality, (2.3) and (2.18) that

$$\begin{aligned}\int_\Omega |u_n|^\eta\,dx &= \int_\Omega |u_n|^{\eta-\alpha_1}|u_n|^{\alpha_1}\,dx = \int_\Omega |u_n|^{\beta/\tau_1}|u_n|^{\alpha_1}\,dx \\ &\leq \Big(\int_\Omega \big(|u_n|^{\beta/\tau_1}\big)^{\tau_1}\,dx\Big)^{1/\tau_1}\Big(\int_\Omega |u_n|^{\tau_1'\alpha_1}\,dx\Big)^{1/\tau_1'} \leq M_4^{1/\tau_1}K_{\tau_1'\alpha_1}^{\alpha_1}\|u_n\|^{\alpha_1}\end{aligned} \tag{2.20}$$

for all n, where $\tau_1' = \tau_1/(\tau_1 - 1)$.

Now from (f6), (2.3) and (2.20), we find

$$\begin{aligned}\langle \Phi'(u_n), u_n\rangle &= \|u_n\|^q - \int_\Omega f(u_n)u_n\,dx \geq \|u_n\|^q - \int_\Omega |f(u_n)||u_n|\,dx \geq \|u_n\|^q - M_0\int_\Omega (|u_n|^\eta + |u_n|)\,dx\\ &= \|u_n\|^q - M_0\int_\Omega |u_n|^\eta\,dx - M_0\int_\Omega |u_n|\,dx \geq \|u_n\|^q - M_0 M_4^{1/\tau_1} K_{\tau_1'\alpha_1}^{\alpha_1}\|u_n\|^{\alpha_1} - M_6\|u_n\|\end{aligned}$$

for all n and some positive constant $M_6 > 0$. Since $q > \alpha_1$, $\{\|u_n\|\}$ is bounded in E.

Thus it follows that $\{u_n\}$ is bounded in E. From a standard argument, it follows that Φ satisfies condition (C).

Finally, from (f4) and (f5), as in the proof of Theorem 1.2, the conclusion follows immediately. □

Proof of Theorem 1.7. We claim that Φ satisfies condition (C), i.e. for every sequence $\{u_n\} \subset E$, $\{u_n\}$ has a convergent subsequence if $\{\Phi(u_n)\}$ is bounded and

$$(1 + \|u_n\|)\|\Phi'(u_n)\| \to 0 \quad \text{as } n \to \infty.$$

It follows from (f8) that there exist two positive constants m_3 and m_4 such that

$$m_3 \geq q\Phi(u_n) - \langle \Phi'(u_n), u_n\rangle = \int_\Omega (f(u_n)u_n - qF(u_n))\,dx \geq m_1\int_\Omega |u_n|^q\,dx - m_4|\Omega|. \tag{2.21}$$

Thus, we find

$$\int_\Omega |u_n|^q\,dx \leq m_5 \tag{2.22}$$

for all n and some positive constant m_5. Assume by contradiction that $\|u_n\| \to \infty$ as $n \to \infty$. Setting $v_n = \frac{u_n}{\|u_n\|}$, by (2.22) we get

$$\int_\Omega |v_n|^q\,dx = \frac{1}{\|u_n\|^q}\int_\Omega |u_n|^q\,dx \leq \frac{m_5}{\|u_n\|^q} \to 0 \quad \text{as } n \to \infty. \tag{2.23}$$

Since

$$\langle \Phi'(u_n), u_n\rangle = \|u_n\|^q - \int_\Omega f(u_n)u_n\,dx = \|u_n\|^q\Big(1 - \int_\Omega \frac{f(u_n)u_n}{\|u_n\|^q}\,dx\Big),$$

we have

$$1 - \int_\Omega \frac{f(u_n)u_n}{\|u_n\|^q}\,dx = o(1). \tag{2.24}$$

If $2 \leq q < \frac{N}{2}$, by $\|v_n\| = 1$ we get $\|v_n\|_{L^r} \leq K_r$ for all $r \in (q, \frac{Nq}{N-2q}]$. Hence, for $r \in (q, \frac{Nq}{N-2q})$, we deduce from Hölder's inequality and (2.23) that

$$\int_\Omega |v_n|^r\,dx \leq K_{q^{**}}^{q^{**}(r-q)/(q^{**}-q)}\Big(\int_\Omega |v_n|^q\,dx\Big)^{(q^{**}-r)/(q^{**}-q)} \to 0 \quad \text{as } n \to \infty. \tag{2.25}$$

From (f2), (f8), (2.21) and (2.25), we can find a positive constant m_6 such that

$$\begin{aligned}\Big|\int_\Omega \frac{f(u_n)u_n}{\|u_n\|^q}\,dx\Big| &\leq \int_\Omega \frac{|f(u_n)|}{|u_n|^{q-1}}|v_n|^q\,dx \leq \Big(\int_\Omega \Big(\frac{|f(u_n)|}{|u_n|^{q-1}}\Big)^\sigma dx\Big)^{1/\sigma}\Big(\int_\Omega |v_n|^{q\sigma'}\,dx\Big)^{1/\sigma'}\\ &\leq m_6\Big(\int_\Omega |v_n|^{q\sigma'}\,dx\Big)^{1/\sigma'} \to 0\end{aligned} \tag{2.26}$$

as $n \to \infty$, where $\sigma' = \sigma/(\sigma - 1)$. Therefore, from (2.24) and (2.26), we obtain $1 = o(1)$, which is a contradiction. Hence, $\{u_n\}$ is bounded.

If $q = \frac{N}{2}$, then we have $\|v_n\|_{L^r} \le K_r$ for all $r \in (q, +\infty)$. And, for $r \in (q, +\infty)$, it follows from Hölder's inequality and (2.23) that

$$\int_\Omega |v_n|^r \, dx \le \Big(\int_\Omega |v_n|^{p(r-1)} \, dx\Big)^{1/p} \Big(\int_\Omega |v_n|^q \, dx\Big)^{1/q} \to 0 \quad \text{as } n \to \infty. \tag{2.27}$$

Clearly, by applying (2.27) we obtain again (2.26), then the contradiction follows from (2.24) and (2.26). Hence, $\{u_n\}$ is bounded.

Hence, by a standard argument, we find that Φ satisfies condition (C). The conclusion follows immediately by (f2) and (f4) arguing as in the proof of Theorem 1.2. □

2.2 The non-homogeneous case $h \neq 0$

Proof of Theorem 1.9. Let

$$\Phi_\theta(u) = \Phi(\theta, u) = \frac{1}{q}\int_\Omega |\Delta u|^q \, dx - \int_\Omega F(u) \, dx - \theta \int_\Omega hu \, dx = \Phi_0(u) - \theta \int_\Omega hu \, dx \quad \text{for all } (\theta, u) \in [0,1] \times E.$$

Obviously, $\Phi_0(u) = \Phi(0, u)$ is an even functional. As in the proof of Theorem 1.5, it is easy to see that Φ satisfies the Cerami-type assumption (I1). Here, we have

$$\frac{\partial \Phi}{\partial \theta}(\theta, u) = -\int_\Omega hu \, dx, \quad \langle \Phi'_\theta(u), v\rangle = \langle \Phi'_0(u), v\rangle - \theta \int_\Omega hv \, dx, \quad u, v \in E.$$

And there exists a positive constant M_7 such that

$$\Big|\frac{\partial \Phi}{\partial \theta}(\theta, u)\Big| \le \int_\Omega |hu| \, dx \le M_7 \|u\| \quad \text{for all } (\theta, u) \in [0,1] \times E. \tag{2.28}$$

Thus, condition (I2) holds. Further, from (2.17), we have

$$\begin{aligned} q\Phi_\theta(u) - \langle \Phi'_\theta(u), u\rangle &= \int_\Omega (f(u)u - qF(u)) \, dx + \theta(1-q)\int_\Omega hu \, dx \\ &\ge M_2 \|u\|_\beta^\beta - M_3|\Omega| - \theta(q-1)\|h\|_{\beta'}\|u\|_\beta. \end{aligned}$$

Since $\beta > 1$, if $\Phi'_\theta(u) = 0$, there exists a positive constant M_8 such that

$$\Big|\frac{\partial \Phi}{\partial \theta}(\theta, u)\Big| \le \|h\|_{\beta'}\|u\|_\beta \le M_8(\Phi_\theta^2(u) + 1)^{\frac{1}{2\beta}}.$$

Then, condition (I3) holds with

$$g_2 = -g_1 = M_8(s^2 + 1)^{\frac{1}{2\beta}}. \tag{2.29}$$

Let $\{e_k\}_k$ be a Schauder basis of $W^{2,q}(\Omega)$. Take

$$E_k = W_0^{1,q}(\Omega) \bigcap \operatorname{span}\{e_1, \dots, e_k\}, \quad E_k^\perp = W_0^{1,q}(\Omega) \bigcap \overline{\operatorname{span}\{e_k, e_{k+1}, \dots\}}.$$

For each $k \in \mathbb{N}$, since $\dim E_k < \infty$, all the norms are equivalent, so there exists a constant $\delta_k > 0$ such that

$$\|u\| \le \delta_k \|u\|_q \quad \text{for all } u \in E_k. \tag{2.30}$$

From (f4), there exists $M_9 > 0$ such that

$$F(s) \ge \frac{2\delta_k^q}{q}|s|^q - M_9 \quad \text{for all } s \in \mathbb{R}. \tag{2.31}$$

For $u \in E_k$, we conclude from (2.28), (2.30) and (2.31) that

$$\begin{aligned}\Phi(u) &= \frac{1}{q}\int_\Omega |\Delta u|^q\,dx - \int_\Omega F(u)\,dx - \theta\int_\Omega hu\,dx \\ &\le \frac{1}{q}\|u\|^q - \frac{2\delta_k^q}{q}\|u\|_q^q - M_9|\Omega| + M_7\|u\| \le -\frac{1}{q}\|u\|^q - M_9|\Omega| + M_7\|u\|\end{aligned}$$

which implies that

$$\Phi(u) \to -\infty, \quad \text{as } \|u\| \to \infty \quad \text{in } E_k.$$

Thus, condition (I4) holds.

Now we have

$$c_k = \inf_{\gamma\in\Gamma_k}\sup_{u\in E_k\cap B_{R_k}(0)} \Phi_0(\gamma(u)),$$

where

$$\Gamma_k = \{\gamma \in C(E_k \cap \bar{B}_{R_k}(0), X) : \gamma \text{ odd},\ \gamma(u) = u \text{ if } u \in E_k \cap \partial B_{R_k}(0)\}.$$

According to Theorem 2.2, if the case (i2) occurs for k large enough, by (2.29), we obtain

$$c_{k+1} - c_k \le C(c_{k+1}^{\frac{1}{\beta}} + c_k^{\frac{1}{\beta}} + 1). \tag{2.32}$$

It follows from [3, Lemma 3.5] that there exists an integer $k_0 \in \mathbb{N}$ such that

$$c_k \le Ck^{\frac{\beta}{\beta-1}}, \quad k \ge k_0. \tag{2.33}$$

Further, by (f6), we get

$$\Phi_0(u) \ge \frac{1}{q}\|u\|^q - \frac{M_0}{\eta}\|u\|_\eta^\eta - M_0\|u\|_1. \tag{2.34}$$

Thus, it follows from Lemma 2.4 that there exist two positive constants M_{10} and M_{11} such that

$$\Phi_0(u) \ge \frac{1}{q}\|u\|^q - M_{10}k^{-\left(\frac{2}{N}+\frac{1}{\eta}-\frac{1}{q}\right)\eta}\|u\|^\eta - M_{11}, \quad u \in E_{k-1}^\perp.$$

By an easy calculation, for k large enough there exists $\rho_k > 0$ such that

$$\Phi_0(u) \ge M_{12}k^{\frac{2q\eta}{N(\eta-q)}-1}, \quad u \in E_{k-1}^\perp \cap \partial B_{\rho_k}(0),$$

where $M_{12} > 0$ is independent of k.

As in [23, Lemma 1.44], from the Borsuk–Ulam Theorem, we can deduce that

$$\gamma(E_k \cap B_{R_k}(0)) \cap \partial B_\rho(0) \cap E_{k-1}^\perp \ne \emptyset$$

for all integers k, $\rho < R_k$ and $\gamma \in \Gamma_k$. Thus, we have

$$\max \Phi_0(\gamma(E_k \cap B_{R_k}(0))) \ge \inf \Phi_0(\partial B_{\rho_k}(0) \cap E_{k-1}^\perp)$$

and

$$c_k \ge M_{12}k^{\frac{2q\eta}{N(\eta-q)}-1} \quad \text{for } k \text{ large.} \tag{2.35}$$

From (2.33) and (2.35), if

$$\frac{\beta}{\beta-1} < \frac{2q\eta}{N(\eta-q)} - 1,$$

this leads to a contradiction. Therefore, case (i1) of Theorem 2.2 occurs. From Remark 2.3, Φ has a critical level $\tilde{c}_k \ge c_k$ for k large. Thus, Theorem 1.9 holds. □

Proof of Theorem 1.10. First, following the proof of Theorem 1.7, we can show that Φ satisfies the Cerami-type condition (I1). As in the proof of Theorem 1.9, it also follows that Φ satisfies conditions (I2) and (I4). Further, we deduce from (f8) that there exists a positive constant m_7 such that

$$\begin{aligned} q\Phi_\theta(u) - \langle \Phi'_\theta(u), u\rangle &= \int_\Omega (f(u)u - qF(u))\,dx + \theta(1-q)\int_\Omega hu\,dx \\ &\geq m_1\|u\|_q^q - m_7|\Omega| - \theta(q-1)\|h\|_p\|u\|_q. \end{aligned}$$

Since $q > 1$, if $\Phi'_\theta(u) = 0$, we obtain

$$\left|\frac{\partial \Phi}{\partial \theta}(\theta, u)\right| \leq \|h\|_p\|u\|_q \leq m_8(\Phi_\theta^2(u) + 1)^{\frac{1}{2q}}$$

for some positive constant m_8. Then, condition (I3) holds with

$$g_2 = -g_1 = m_8(s^2+1)^{\frac{1}{2q}}. \tag{2.36}$$

According to Theorem 2.2, if the case (i2) occurs for k large enough, by (2.36), we obtain

$$c_{k+1} - c_k \leq C_1(c_{k+1}^{\frac{1}{q}} + c_k^{\frac{1}{q}} + 1). \tag{2.37}$$

It follows from [3, Lemma 3.5] that there exists an integer $k_1 \in \mathbb{N}$ such that

$$c_k \leq C_1 k^{\frac{q}{q-1}}, \quad k \geq k_1. \tag{2.38}$$

Further, by (f4), there exists a positive constant s_4 such that

$$F(s) \geq 0, \quad |s| \geq s_4. \tag{2.39}$$

From (f8) and (2.39), for $|s| \geq s_5 = \max\{s_1, s_4\}$, one has

$$|f(s)|^\sigma \leq m_2(f(s)s - qF(s))|s|^{(q-1)\sigma} \leq m_2|f(s)||s|^{(q-1)\sigma+1}. \tag{2.40}$$

We conclude from (2.40) that

$$|f(s)| \leq m_2|s|^{\frac{(q-1)\sigma+1}{\sigma-1}} \quad \text{for all } |s| \geq s_5.$$

Then, there exist two positive constants m_9 and m_{10} such that

$$|F(s)| \leq m_9|s|^{\frac{q\sigma}{\sigma-1}} + m_{10}, \quad s \in \mathbb{R}.$$

Thus, we have

$$\Phi_0(u) \geq \frac{1}{q}\|u\|^q - m_9\|u\|^{\frac{q\sigma}{\sigma-1}}_{\frac{q\sigma}{\sigma-1}} - m_{10}\|u\|_1. \tag{2.41}$$

Now it follows from Lemma 2.4 that there exist two positive constants m_{11} and m_{12} such that

$$\Phi_0(u) \geq \frac{1}{q}\|u\|^q - m_{11}k^{-\left(\frac{2}{N}-\frac{1}{q\sigma}\right)\frac{q\sigma}{\sigma-1}}\|u\|^{\frac{q\sigma}{\sigma-1}} - m_{12}.$$

By an easy calculation, for k large enough there exists $\rho_k > 0$ such that

$$\Phi_0(u) \geq m_{13}k^{\frac{2q\sigma}{N}-1}, \quad u \in E_{k-1}^\perp \cap \partial B_{\rho_k}(0),$$

where $m_{13} > 0$ is independent of k.

As in [23, Lemma 1.44], from the Borsuk–Ulam Theorem, we can deduce that

$$\gamma(E_k \cap B_{R_k}(0)) \cap \partial B_\rho(0) \cap E_{k-1}^\perp \neq \emptyset$$

for all integers k, $\rho < R_k$ and $\gamma \in \Gamma_k$. Then, one has

$$\max \Phi_0(\gamma(E_k \cap B_{R_k}(0))) \geq \inf \Phi_0(\partial B_{\rho_k}(0) \cap E_{k-1}^\perp)$$

and

$$c_k \geq m_{13}k^{\frac{2q\sigma}{N}-1} \quad \text{for } k \text{ large.} \tag{2.42}$$

By (2.38) and (2.42), if

$$\frac{q}{q-1} < \frac{2q\sigma}{N} - 1,$$

this leads to a contradiction. Therefore, case (i1) of Theorem 2.2 occurs. From Remark 2.3, Φ has a critical level $\tilde{c}_k \geq c_k$ for k large. In conclusion, Theorem 1.10 holds. □

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