Research Article

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# A concrete realization of the slow-fast alternative for a semilinear heat equation with homogeneous Neumann boundary conditions

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**Abstract:** We investigate the asymptotic behavior of solutions to a semilinear heat equation with homogeneous Neumann boundary conditions. It was recently shown that the nontrivial kernel of the linear part leads to the coexistence of fast solutions decaying to 0 exponentially (as time goes to infinity), and slow solutions decaying to 0 as negative powers of $t$. Here we provide a characterization of slow/fast solutions in terms of their sign, and we show that the set of initial data giving rise to fast solutions is a graph of codimension one in the phase space.

**Keywords:** Semilinear parabolic equation, decay rates, slow solutions, exponentially decaying solutions, subsolutions and supersolutions

**MSC 2010:** 35K58, 35K90, 35B40

## 1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded connected domain and $p > 0$. In this paper we consider the semilinear heat equation

$$u_t - \Delta u + |u|^p u = 0 \tag{1.1}$$

with homogeneous Neumann boundary conditions on $\partial\Omega$.

The asymptotic behavior of solutions, and in particular their decay rate and asymptotic profile as $t \to +\infty$, has been investigated in the last decade by the third author and collaborators. The starting observation is that the Neumann Laplacian, namely the linear operator associated to (1.1), has a nontrivial kernel consisting of all constant functions. This leads to the coexistence of solutions with different decay rates. In particular, all nonzero solutions to (1.1) are either *fast solutions* that decay exponentially, or *slow solutions* with a decay rate proportional to $t^{-1/p}$. This is the so-called null-slow-fast alternative, and was observed for the first time in [1] (see also [2]). Similar results have been obtained in [3, 8] for solutions to semilinear heat equations such as

$$u_t - \Delta u + |u|^p u - \lambda_1 u = 0$$

with homogeneous Dirichlet boundary conditions. In the concrete model (1.1), as well as in the Dirichlet case, it has been shown earlier in [1, 3, 8] that all positive solutions are slow. In this paper we limit ourselves to the

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model (1.1), and we investigate more completely the sets of initial data giving rise to slow/fast solutions. We provide two results.

(1) In Theorem 3.1 we characterize fast solutions as those solutions which assume both positive and negative values for every $t \geq 0$. What we actually prove is the contrapositive, namely that slow solutions are either eventually positive or eventually negative. This motivates us to introduce the notion of *positive-slow* and *negative-slow* solutions.
(2) In Theorem 3.2 we describe the set of initial data giving rise to slow/fast solutions. We show that in the phase space $L^2(\Omega)$ there are two nonempty open sets of initial data originating positive-slow and negative slow-solutions, respectively. These two open sets are separated by the graph of a continuous function, and all initial data in this graph give rise to a fast (or null) solution.

Our characterization of slow/fast solutions in terms of their sign follows from comparison principles and the construction of suitable subsolutions and supersolutions.

The characterization of Theorem 3.1 is the fundamental tool in the proof of Theorem 3.2. Indeed it implies that "being the initial datum of a positive/negative slow solution" is an open condition. At this point a simple connectedness argument implies that something different should exist in between, and by the null-slow-fast alternative the only remaining option is a fast or null solution.

This paper is organized as follows. In order to make the presentation as self-contained as possible, in Section 2 we collect all we need concerning existence, regularity and decay for solutions to (1.1). In Section 3 we state our main results. In Section 4 we provide the proofs. The final Section 5 is devoted to some comments on possible extensions of the results.

## 2 Basic tools and previous results

Equation (1.1) has been investigated deeply in mathematical literature. For the convenience of the reader, we collect in this section the results that are needed in the sequel.

To begin with, we observe that (1.1) can be interpreted as the gradient flow in $L^2(\Omega)$ of the convex functional defined by

$$F_p(u) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u(x)|^2\,dx + \dfrac{1}{p+2}\displaystyle\int_\Omega |u(x)|^{p+2}\,dx & \text{if } u \in H^1(\Omega)\cap L^{p+2}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Assuming that the boundary of $\Omega$ is $C^2$, the subdifferential $B$ of $F_p$ is the operator defined by

$$Bu := -\Delta u + |u|^p u$$

in the domain

$$D(B) := \{u \in H^2(\Omega) \cap L^{2p+2}(\Omega) : \partial u/\partial n = 0 \text{ on } \partial\Omega\}.$$

As a consequence of [4], the operator $B$ generates a contraction semigroup on $L^2(\Omega)$. This provides existence, uniqueness, and continuous dependence on initial conditions of a weak solution $u \in C^0([0,+\infty), L^2(\Omega))$ of (1.1) for any initial condition $u_0 \in L^2(\Omega)$. In the next statements we collect some well-known properties which shall be used in the proofs of our main results.

**Theorem A** (Regularity). *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with boundary of class $C^2$, and let $p$ be a positive real number. Let $u(t,x)$ be the unique solution to equation* (1.1) *with homogeneous Neumann boundary conditions and initial datum $u_0 \in L^2(\Omega)$, as defined previously. Then it turns out that*

$$u \in W^{1,\infty}([\delta,+\infty), L^2(\Omega)) \cap L^\infty([\delta,+\infty), H^2(\Omega)\cap C(\overline{\Omega})) \quad \textit{for all } \delta > 0.$$

*Moreover, the solution satisfies the homogeneous Neumann boundary conditions in the classical sense for every $t > 0$.*

The second result concerns the comparison between two solutions with different initial data. From the general semigroup theory we know that solutions depend continuously on initial data in $L^2(\Omega)$. Here we need more, namely that the semigroup preserves the order, and that one can estimate the norm in $L^\infty(\Omega)$ of the difference between two solutions at positive times in terms of the norm in $L^2(\Omega)$ of the difference between initial data.

**Theorem B** (Comparison between two solutions). *Let $\Omega$ and $p$ be as in Theorem A. Let $u(t,x)$ and $v(t,x)$ be two solutions to* (1.1) *with homogeneous Neumann boundary conditions and initial data $u_0$ and $v_0$, respectively. Then the following statements hold true.*

(1) (Order preservation) *If $u_0(x) \geq v_0(x)$ for almost every $x \in \Omega$, then $u(t,x) \geq v(t,x)$ for every $t > 0$ and every $x \in \overline{\Omega}$.*
(2) (Continuous dependence $L^2(\Omega) \to L^\infty(\Omega)$) *There exists a function $M_1 : (0,+\infty) \to (0,+\infty)$ such that*

$$|u(t,x) - v(t,x)| \leq M_1(t)\|u_0 - v_0\|_{L^2(\Omega)} \quad \textit{for all } t > 0, \textit{ for all } x \in \overline{\Omega}. \tag{2.1}$$

Although estimate (2.1) is rather classical and variants have been used in various contexts, for the sake of completeness we provide a sketch of proof under the sole assumption that a Sobolev-like imbedding $H^1(\Omega) \subseteq L^q(\Omega)$ is satisfied for some $q > 2$.

**Lemma 2.1.** *Let us assume that there exists $q > 2$ such that $H^1(\Omega) \subseteq L^q(\Omega)$ with continuous embedding, namely there exists a constant $K_0$ such that*

$$\|w\|_{L^q(\Omega)} \leq K_0\|w\|_{H^1(\Omega)} \quad \textit{for all } w \in H^1(\Omega). \tag{2.2}$$

*Let $T \in (0,1)$, let $c \in L^\infty((0,T)\times\Omega)$ be a nonnegative function, and let*

$$z \in W^{1,\infty}((0,T), L^2(\Omega)) \cap L^\infty((0,T), H^2(\Omega) \cap L^\infty(\Omega))$$

*be a solution of*

$$z_t - \Delta z + c(t,x)z = 0 \tag{2.3}$$

*with homogeneous Neumann boundary conditions. Then, setting $\beta := q/(2q-4)$, it turns out that*

$$\|z(t,x)\|_{L^\infty(\Omega)} \leq \frac{4^{\beta^2} \cdot K_0^{2\beta}}{t^\beta}\|z(0,x)\|_{L^2(\Omega)} \quad \textit{for all } t \in (0,T]. \tag{2.4}$$

*Proof.* Let $r$ be any nonnegative real number. Let us multiply (2.3) by $|z|^{2r}z$ and let us integrate over $\Omega$. After integrating by parts the term $|z|^{2r}z\Delta z$, and recalling that $c(t,x)$ is nonnegative, we obtain that

$$\frac{1}{2r+2}\frac{d}{dt}\int_\Omega |z|^{2r+2}\,dx + (2r+1)\int_\Omega |z|^{2r}|\nabla z|^2\,dx = -\int_\Omega c(t,x)|z|^{2r+2}\,dx \leq 0. \tag{2.5}$$

This implies in particular that

$$\text{the function } t \to \|z(t,x)\|_{L^\alpha(\Omega)} \text{ is nonincreasing for every } \alpha \geq 2. \tag{2.6}$$

Now we introduce the function $\psi_r(\sigma) := |\sigma|^r\sigma$, and we observe that

$$\nabla[\psi_r(z)] = (r+1)|z|^r\nabla z.$$

As a consequence, (2.5) can be rewritten as

$$\frac{1}{2r+2}\frac{d}{dt}\left(\|z\|_{L^{2r+2}(\Omega)}^{2r+2}\right) + \frac{2r+1}{(r+1)^2}\|\nabla[\psi_r(z)]\|_{L^2(\Omega)}^2 \leq 0.$$

Given any $\tau \in (0,T]$, integrating in $[0,\tau]$ we deduce that

$$\int_0^\tau \|\nabla[\psi_r(z(s,x))]\|_{L^2(\Omega)}^2\,ds \leq \frac{1}{2}\|z(0,x)\|_{L^{2r+2}(\Omega)}^{2r+2}. \tag{2.7}$$

On the other hand, from (2.6) with $\alpha := 2r + 2$, we obtain also that

$$\int_0^\tau \|\psi_r(z(s,x))\|^2_{L^2(\Omega)}\,ds \le \tau \|z(0,x)\|^{2r+2}_{L^{2r+2}(\Omega)}. \tag{2.8}$$

Adding (2.7) and (2.8) we conclude that

$$\int_0^\tau \|\psi_r(z(s,x))\|^2_{H^1(\Omega)}\,ds \le \Big(\tau + \frac{1}{2}\Big)\|z(0,x)\|^{2r+2}_{L^{2r+2}(\Omega)} \quad \text{for all } \tau \in (0,T].$$

Now we exploit the continuous imbedding (2.2). From (2.6) with $\alpha := (r+1)q$ we obtain that

$$\begin{aligned}
\tau\|\psi_r(z(\tau,x))\|^2_{L^q(\Omega)} &\le \int_0^\tau \|\psi_r(z(s,x))\|^2_{L^q(\Omega)}\,ds \\
&\le K_0^2 \int_0^\tau \|\psi_r(z(s,x))\|^2_{H^1(\Omega)}\,ds \\
&\le K_0^2\Big(\tau + \frac{1}{2}\Big)\|z(0,x)\|^{2r+2}_{L^{2r+2}(\Omega)},
\end{aligned}$$

and hence

$$\|\psi_r(z(\tau,x))\|^2_{L^q(\Omega)} \le K_0^2\Big(1 + \frac{1}{2\tau}\Big)\|z(0,x)\|^{2r+2}_{L^{2r+2}(\Omega)} \quad \text{for all } \tau \in (0,T].$$

Setting $\alpha := 2r + 2$ and $\lambda := q/2$, this can be written in the more suggestive form

$$\|z(\tau,x)\|_{L^{\lambda\alpha}(\Omega)} \le \Big[K_0^2\Big(1 + \frac{1}{2\tau}\Big)\Big]^{1/\alpha}\|z(0,x)\|_{L^\alpha(\Omega)} \quad \text{for all } \tau \in (0,T].$$

Due to the time-translation invariance, this implies also that

$$\|z(\theta+\tau,x)\|_{L^{\lambda\alpha}(\Omega)} \le \Big[K_0^2\Big(1 + \frac{1}{2\tau}\Big)\Big]^{1/\alpha}\|z(\theta,x)\|_{L^\alpha(\Omega)} \tag{2.9}$$

whenever $0 \le \theta < \theta + \tau \le T$.

This is the starting point of a classical iteration procedure. Given any $t \in (0,T]$, for every $n \in \mathbb{N}$ we set

$$t_n := \Big(1 - \frac{1}{2^n}\Big)t, \qquad \lambda_n := 2\lambda^n,$$

and from (2.9) with $\theta := t_n$, $\tau := t_{n+1} - t_n$, and $\alpha := \lambda_n$ we deduce that

$$\|z(t_{n+1},x)\|_{L^{\lambda_{n+1}}(\Omega)} \le \Big[K_0^2\Big(1 + \frac{2^n}{t}\Big)\Big]^{1/\lambda_n}\|z(t_n,x)\|_{L^{\lambda_n}(\Omega)} \quad \text{for all } n \in \mathbb{N}.$$

Since $t \le 1$, this implies the simpler formula

$$\|z(t_{n+1},x)\|_{L^{\lambda_{n+1}}(\Omega)} \le \Big[\frac{K_0^2 \cdot 2^{n+1}}{t}\Big]^{1/\lambda_n}\|z(t_n,x)\|_{L^{\lambda_n}(\Omega)} \quad \text{for all } n \in \mathbb{N}.$$

At this point an easy induction yields

$$\|z(t_n,x)\|_{L^{\lambda_n}(\Omega)} \le 2^{\gamma_n}\Big[\frac{K_0^2}{t}\Big]^{\beta_n}\|z(0,x)\|_{L^2(\Omega)} \quad \text{for all } n \in \mathbb{N}, \tag{2.10}$$

where

$$\beta_n := \sum_{k=0}^{n-1}\frac{1}{\lambda_k} \le \sum_{k=0}^{\infty}\frac{1}{2\lambda^k} = \frac{1}{2}\,\frac{\lambda}{\lambda - 1} = \frac{q}{2(q-2)}$$

and

$$\gamma_n := \sum_{k=0}^{n-1}\frac{k+1}{\lambda_k} \le \sum_{k=0}^{\infty}\frac{k+1}{2\lambda^k} = \frac{1}{2}\,\frac{\lambda^2}{(\lambda-1)^2} = \frac{q^2}{2(q-2)^2}.$$

Letting $n \to +\infty$ in (2.10), we obtain (2.4). □

We are now ready to prove estimate (2.1). We consider first the case where $u_0$ and $v_0$ are of class $C^2$ with compact support in $\Omega$. In this case $u$, $v$ and $z := u - v$ are bounded on $(0, t)$ with values in $H^2(\Omega) \cap L^\infty(\Omega)$, and $z$ satisfies (2.3) with

$$c(t, x) := \frac{|u|^p u - |v|^p v}{u - v} \geq 0$$

with the convention that the quotient is 0 whenever the denominator vanishes. Moreover, the continuous embedding $H^1(\Omega) \subseteq L^q(\Omega)$ holds true with any $q \geq 2$ if $n \leq 2$, and with $q = 2^* := 2n/(n-2)$ if $n \geq 3$. Therefore, we are in a position to apply Lemma 2.1, from which we obtain (2.1) with a function $M_1(t)$ independent of the initial data. At this point, the result for general initial data follows from a density argument. □

The next statement describes all possible decay rates and asymptotic profiles for solutions to (1.1). We refer to [1, Theorem 1.3] and [7, Theorem 4.4] for further details and proofs.

**Theorem C** (Classification of decay rates). *Let $\Omega$ and $p$ be as in Theorem A and assume, in addition, that $\Omega$ is connected. Let $u(t, x)$ be any solution to* (1.1) *with homogeneous Neumann boundary conditions and initial datum in $L^2(\Omega)$. Then one and only one of the following statements apply to $u(t, x)$.*

(1) (Null solution) *The solution is the null solution $u(t, x) \equiv 0$.*

(2) (Slow solutions) *There exist $t_0 > 0$ and $M_2 \geq 0$ such that*

$$\left| |u(t, x)| - \frac{1}{(pt)^{1/p}} \right| \leq \frac{M_2}{t^{1+1/p}} \quad \text{for all } t \geq t_0, \text{ for all } x \in \overline{\Omega}.$$

(3) (Spectral fast solutions) *There exist an eigenvalue $\lambda > 0$ of the Neumann Laplacian, and a corresponding eigenfunction $\varphi_\lambda(x)$, such that*

$$\lim_{t \to +\infty} \|u(t, x) - \varphi_\lambda(x) e^{-\lambda t}\|_{L^2(\Omega)} e^{\gamma t} = 0$$

*for some $\gamma > \lambda$.*

## 3 Statements

In the first result of this paper we characterize slow solutions in terms of sign.

**Theorem 3.1** (Characterization of slow solutions). *Let $n$ be a positive integer, let $\Omega \subseteq \mathbb{R}^n$ be a bounded connected open set with $C^2$ boundary, and let $p$ be a positive real number. Let $u(t, x)$ be a solution to equation* (1.1) *with homogeneous Neumann boundary conditions. Then the following three statements are equivalent.*

(i) *There exist $t_0 > 0$ and $M_3 \geq 0$ such that*

$$\left| |u(t, x)| - \frac{1}{(pt)^{1/p}} \right| \leq \frac{M_3}{t^{1+1/p}} \quad \text{for all } t \geq t_0, \text{ for all } x \in \overline{\Omega}.$$

(ii) *There exist $t_0 > 0$ and $m : [t_0, +\infty) \to (0, +\infty)$ such that*

$$|u(t, x)| \geq m(t) \quad \text{for all } t \geq t_0, \text{ for all } x \in \overline{\Omega}.$$

(iii) *There exists $t_0 \geq 0$ such that either $u(t_0, x) \geq 0$ for almost every $x \in \Omega$ or $u(t_0, x) \leq 0$ for almost every $x \in \Omega$, but $u(t_0, x)$ is not identically 0 in $\Omega$ (in the almost everywhere sense).*

Theorem 3.1 above implies that there are only two types of slow solutions:
- *positive-slow solutions*, which are eventually positive and decay as $(pt)^{-1/p}$,
- *negative-slow solutions*, which are eventually negative and decay as $-(pt)^{-1/p}$.

Moreover, statement (iii) implies that fast solutions are necessarily sign changing functions for every $t \geq 0$.

The main result of this paper concerns the structure of slow/fast solutions. We show that, in the phase space $L^2(\Omega)$, positive-slow and negative-slow solutions are separated by a manifold of codimension one consisting of fast solutions. As a consequence, the set of initial data generating slow solutions is open and dense in $L^2(\Omega)$. The separating manifold is the graph of a continuous function $\Phi$ defined in subspace $N^\perp$ orthogonal to constant functions (which are the kernel of the Neumann Laplacian). The function $\Phi$ turns out to be Lipschitz continuous when restricted to $L^\infty(\Omega)$.

**Theorem 3.2** (Structure of slow/fast solutions). *Let $\Omega$ and $p$ be as in Theorem 3.1. Let us consider the space*

$$N^\perp := \left\{ w \in L^2(\Omega) : \int_\Omega w(x)\,dx = 0 \right\}.$$

*Then there exists a continuous function $\Phi : N^\perp \to \mathbb{R}$ with the following property. For every $w_0 \in N^\perp$, the solution $u(t,x)$ to equation* (1.1) *with homogeneous Neumann boundary conditions and initial datum $u(0,x) = w_0(x)+k$ is*

- *positive-slow if $k > \Phi(w_0)$,*
- *fast if $k = \Phi(w_0)$ (or null if $w_0 = 0$, in which case also $\Phi(w_0) = 0$),*
- *negative-slow if $k < \Phi(w_0)$.*

*Moreover, the function $\Phi$ is* 1*-Lipschitz continuous if restricted to $L^\infty(\Omega)$, namely*

$$|\Phi(w_1) - \Phi(w_2)| \leq \|w_1 - w_2\|_{L^\infty(\Omega)} \quad \textit{for all } (w_1, w_2) \in [N^\perp \cap L^\infty(\Omega)]^2.$$

**Remark 3.3.** The projections of a function $u_0 \in L^2(\Omega)$ on the kernel of the Neumann Laplacian and on the orthogonal space $N^\perp$ are given, respectively, by

$$\int_\Omega u_0(x)\,dx, \quad u_0(x) - \int_\Omega u_0(x)\,dx.$$

Therefore, the result of Theorem 3.2 above can be rephrased by saying that the solution to (1.1) with some initial datum $u_0 \neq 0$ is positive-slow, fast, or negative-slow according to the sign of

$$\int_\Omega u_0(x)\,dx - \Phi\left( u_0(x) - \int_\Omega u_0(x)\,dx \right).$$

**Remark 3.4.** A very important sub-product of the two above results is that, at the exception of a manifold of codimension 1 in $L^\infty(\Omega)$, all essentially bounded initial data produce a solution which has a constant sign for $t$ large enough. This property of ultimate constant sign for “most" solutions might be true for very general nonlinearities, independently of asymptotic properties, assuming for instance some conditions implying that all solutions are global. We believe this may be an interesting clue for future research in the field.

# 4 Proofs

To begin with, we recall that for every $p > 0$ there exists a constant $K_p$ such that

$$\big||a+b|^p(a+b) - |a|^p a\big| \leq K_p(|a|^p + |b|^p)|b| \quad \text{for all } (a,b) \in \mathbb{R}^2. \tag{4.1}$$

This inequality follows from the mean value theorem applied to the function $|x|^p x$.

**Lemma 4.1.** *Let $n$, $\Omega$, and $p$ be as in Theorem 3.1. Let $v_0$ and $w_0$ be two functions in $L^2(\Omega)$ such that $v_0(x) \geq w_0(x)$ for almost every $x \in \Omega$, and $v_0(x) > w_0(x)$ on a set of positive measure. Let $v$ and $w$ be the solutions to equation* (1.1) *with homogeneous Neumann boundary conditions and initial data $v_0$ and $w_0$, respectively. Let us assume that $w$ is a fast or null solution in the sense of Theorem C. Then $v$ is a slow solution in the sense of Theorem C.*

*Proof.* Let $z(t,x) := v(t,x) - w(t,x)$ denote the difference, which is a nonnegative function because of statement (1) of Theorem B, and satisfies

$$z_t = \Delta z - \big(|w+z|^p(w+z) - |w|^p w\big).$$

Applying inequality (4.1) with $a := w(t, x)$ and $b := z(t, x)$, we can estimate the nonlinear term in the right-hand side, and obtain that

$$z_t \geq \Delta z - K_p(|z|^p + |w|^p)z.$$

Let us consider now the function $I : [0, +\infty) \to [0, +\infty)$ defined by

$$I(t) := \int_\Omega z(t, x)\, dx \quad \text{for all } t \geq 0.$$

Our assumption on $v_0$ and $w_0$ implies that $I(0) > 0$. Since the function $I(t)$ is continuous in $[0, +\infty)$ (due to the continuity of $v$ and $w$ with values in $L^2(\Omega)$), there exists $\delta > 0$ such that $I(\delta) > 0$.

Now we argue by contradiction. Let us assume that $v$, as well as $w$, is not a slow solution. By Theorem C, this implies that $z$ decays exponentially to 0 in $L^2(\Omega)$, and hence also in $L^\infty(\Omega)$ because of statement (2) of Theorem B, and therefore there exist constants $\nu > 0$ and $C_\delta > 0$ such that

$$z_t \geq \Delta z - C_\delta e^{-\nu t} z \quad \text{for all } (t, x) \in [\delta, +\infty) \times \Omega.$$

Integrating over $\Omega$, taking account of the Neumann boundary conditions we find

$$I'(t) \geq -C_\delta e^{-\nu t} I(t) \quad \text{for all } t \geq \delta,$$

hence

$$I(t) \geq I(\delta) \exp\biggl(-C_\delta \int_\delta^{+\infty} e^{-\nu s}\, ds\biggr) \geq I(\delta) \exp\Bigl(-\frac{C_\delta}{\nu}\Bigr) \quad \text{for all } t \geq \delta.$$

This contradicts the fact that $z$ tends to 0 in $L^2(\Omega)$. □

## Proof of Theorem 3.1

The implications (i) ⇒ (ii) ⇒ (iii) are almost trivial. As for (iii) ⇒ (i), assuming for instance the positive sign, it is enough to apply Lemma 4.1 with $w(t, x) := 0$ and $v(t, x) := u(t + t_0, x)$.

## Proof of Theorem 3.2

In this proof we deal with many different initial conditions. For this reason we adopt the semigroup notation, namely we write $S_t(v_0)$ or $[S_t(v_0)](x)$ in order to denote the solution at time $t$ which has $v_0$ as initial condition.

### Existence and uniqueness

For every $w_0 \in L^2(\Omega)$, let $\mathcal{K}^+(w_0)$ denote the set of real numbers $k$ for which the solution with initial datum $w_0(x) + k$ is positive-slow, and let $\mathcal{K}^-(w_0)$ denote the set of real numbers $k$ for which the solution is negative-slow.

The main point is proving that $\mathcal{K}^+(w_0)$ and $\mathcal{K}^-(w_0)$ are, respectively, an open right half-line and an open left half-line, and these two half-lines are separated by a unique element. When $w_0 \in N^\perp$, this separator is the value $\Phi(w_0)$ that we are looking for. We prove these claims through several steps.

**Step 1.** We prove that $\mathcal{K}^+(w_0)$ and $\mathcal{K}^-(w_0)$ are nonempty for every $w_0 \in L^2(\Omega)$. To this end, we concentrate on $\mathcal{K}^+(w_0)$, since the argument for $\mathcal{K}^-(w_0)$ is symmetric. Let us choose two positive constants $m_0$ and $t_0$, and let us set

$$m_1 := \frac{m_0}{2(1 + p m_0^p t_0)^{1/p}}, \qquad \delta := \frac{m_1}{M_1(t_0)},$$

where $M_1(t_0)$ is the constant which appears in (2.1). Let us choose $v_0 \in C^0(\overline{\Omega})$ such that $\|w_0 - v_0\|_{L^2(\Omega)} \leq \delta$ (this is possible because $C^0(\overline{\Omega})$ is dense in $L^2(\Omega)$). Due to boundedness of $v_0$, there exists $k_0 \in \mathbb{R}$ such that

$$v_0(x) + k_0 \geq m_0 \quad \text{for all } x \in \overline{\Omega}.$$

Now we claim that

$$[S_t(v_0 + k_0)](x) \geq \frac{m_0}{(1 + pm_0^p t)^{1/p}} \quad \text{for all } t \geq 0, \text{ for all } x \in \overline{\Omega}.$$

This inequality follows from the usual comparison principle because it is true when $t = 0$, and in addition both the left-hand and the right-hand side are solutions to (1.1) with homogeneous Neumann boundary conditions. Setting $t = t_0$, and recalling our definition of $m_1$, we obtain that

$$[S_{t_0}(v_0 + k_0)](x) \geq 2m_1 \quad \text{for all } x \in \overline{\Omega}. \tag{4.2}$$

On the other hand, statement (2) of Theorem B applied to initial data $w_0 + k_0$ and $v_0 + k_0$ implies that

$$|[S_{t_0}(w_0 + k_0)](x) - [S_{t_0}(v_0 + k_0)](x)| \leq M_1(t_0)\|w_0 - v_0\|_{L^2(\Omega)} \leq m_1 \tag{4.3}$$

for every $x \in \overline{\Omega}$. From (4.2) and (4.3) it follows that

$$[S_{t_0}(w_0 + k_0)](x) \geq m_1 \quad \text{for all } x \in \overline{\Omega}.$$

Thanks to Theorem 3.1, this is enough to conclude that the solution with initial condition $w_0(x) + k_0$ is positive-slow, and hence $k_0 \in \mathcal{K}^+(w_0)$.

**Step 2.** We prove that $\mathcal{K}^+(w_0)$ is an open right half-line, and analogously $\mathcal{K}^-(w_0)$ is an open left half-line. To this end, let us consider $\mathcal{K}^+(w_0)$ (the argument for $\mathcal{K}^-(w_0)$ is symmetric). It is a right half-line because, if $w_0(x) + k_0$ gives rise to a positive-slow solution $u(t, x)$, then every solution with initial datum $w_0(x) + k$ with $k > k_0$ is greater than $u(t, x)$, and hence it is positive-slow as well.

It remains to show that $\mathcal{K}^+(w_0)$ is an open set. Let us assume that $k_0 \in \mathcal{K}^+(w_0)$, so that the solution $u(t, x)$ with initial datum $w_0(x) + k_0$ is positive-slow. Due to Theorem 3.1, it turns out that

$$[S_{t_0}(w_0 + k_0)](x) \geq m_0 \quad \text{for all } x \in \overline{\Omega}$$

for suitable constants $t_0 > 0$ and $m_0 > 0$. Applying statement (2) of Theorem B as in the previous step, we obtain that

$$[S_{t_0}(w_0 + k)](x) \geq \frac{m_0}{2} \quad \text{for all } x \in \overline{\Omega}$$

provided that $k$ is close enough to $k_0$. Applying Theorem 3.1 once again, we can conclude that all these neighboring solutions are positive-slow as well.

**Step 3.** The structure of $\mathcal{K}^+(w_0)$ and $\mathcal{K}^-(w_0)$ implies that

$$\sup \mathcal{K}^-(w_0) \leq \inf \mathcal{K}^+(w_0), \tag{4.4}$$

and any $k$ in between (endpoints included) lies neither in $\mathcal{K}^-(w_0)$ nor in $\mathcal{K}^+(w_0)$. Due to the null-slow-fast alternative of Theorem C, the corresponding solutions are necessarily fast or null. Finally, as a consequence of Lemma 4.1, we do have equality in (4.4), and hence for every $w_0 \in L^2(\Omega)$ there exists a unique $k$ such that $w_0(x) + k$ generates a fast (or null) solution.

### Continuity

We show that the map $w_0 \to \Phi(w_0)$ is continuous with respect to the norm of $L^2(\Omega)$, namely for every $w_0 \in N^\perp$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\Phi(w_0) - \varepsilon \leq \Phi(v_0) \leq \Phi(w_0) + \varepsilon \tag{4.5}$$

for every $v_0 \in N^\perp$ with $\|v_0 - w_0\|_{L^2(\Omega)} \leq \delta$.

Let us consider the solution with initial condition $w_0(x) + (\Phi(w_0) + \varepsilon)$. It is positive-slow, and hence from Theorem 3.1 we know that

$$[S_{t_0}(w_0 + (\Phi(w_0) + \varepsilon))](x) \geq m_0 \quad \text{for all } x \in \overline{\Omega}$$

for suitable constants $t_0 > 0$ and $m_0 > 0$. Applying statement (2) of Theorem B as in the existence part, we deduce that

$$[S_{t_0}(v_0 + (\Phi(w_0) + \varepsilon))](x) \geq \frac{m_0}{2} \quad \text{for all } x \in \overline{\Omega}$$

provided that $\|v_0 - w_0\|_{L^2(\Omega)}$ is small enough. Applying Theorem 3.1 once again, we deduce that the solution with initial condition $v_0(x) + (\Phi(w_0) + \varepsilon)$ is positive-slow as well. It follows that $\Phi(w_0) + \varepsilon \in \mathcal{K}^+(v_0)$, and therefore $\Phi(v_0) \leq \Phi(w_0) + \varepsilon$.

This proves that the inequality on the right in (4.5) holds true for every $v_0 \in N^\perp$ which is close enough to $w_0$ with respect to the norm of $L^2(\Omega)$. A symmetric argument applies to the inequality on the left.

#### Lipschitz continuity with respect to the uniform norm

We show that the map $w_1 \to \Phi(w_1)$ restricted to $L^\infty(\Omega)$ is Lipschitz continuous with Lipschitz constant equal to 1. To this end, we take any $w_1$ and $w_2$ in $N^\perp \cap L^\infty(\Omega)$, and from the definition of $L^\infty(\Omega)$ we deduce that

$$w_1(x) + (\Phi(w_1) + \varepsilon) \leq w_2(x) + (\|w_1 - w_2\|_{L^\infty(\Omega)} + \Phi(w_1) + \varepsilon)$$

for every $\varepsilon > 0$ and almost every $x \in \Omega$. Since the solution with initial datum equal to the left-hand side is positive-slow, the solution with initial datum equal to the right-hand side is positive-slow as well, which proves that

$$\|w_1 - w_2\|_{L^\infty(\Omega)} + \Phi(w_1) + \varepsilon \in \mathcal{K}^+(w_2).$$

In an analogous way we obtain that

$$-\|w_1 - w_2\|_{L^\infty(\Omega)} + \Phi(w_1) - \varepsilon \in \mathcal{K}^-(w_2).$$

Since $\Phi(w_2)$ separates $\mathcal{K}^-(w_2)$ and $\mathcal{K}^+(w_2)$, letting $\varepsilon \to 0^+$ we conclude that

$$\Phi(w_1) - \|w_1 - w_2\|_{L^\infty(\Omega)} \leq \Phi(w_2) \leq \Phi(w_1) + \|w_1 - w_2\|_{L^\infty(\Omega)},$$

which completes the proof.

## 5 Additional results and possible extensions

In this section we describe some additional properties and research directions.

**Strong positivity.** Under the $C^2$ regularity assumption on $\partial\Omega$ which allowed us to set properly the problem, more can be said about the behavior of solutions. Actually, an application of the strong minimum principle gives that for any nonnegative initial value $u_0 \in L^2(\Omega)$ which is positive on a set of positive measure, the solution of (1.1) is uniformly (with respect to $x \in \Omega$) positive for all positive times. This of course means that right after the first time at which a slow solution has a constant sign, it becomes strictly above a positive (time depending) constant or strictly below a time depending negative constant.

**Relaxed regularity.** In principle, in order for (1.1) to be properly set in a reasonable sense (for instance variational or distributional), a $C^1$ or even Lipschitz regularity assumption on $\partial\Omega$ seems to be enough. In such a case, the Sobolev embedding theorem would be applicable, the difficulty might be that the solution does not need to be continuous up to the boundary for $t > 0$. The relevant regularity class for solutions would then be

$$W^{1,\infty}([\delta, +\infty), L^2(\Omega)) \cap L^\infty([\delta, +\infty), H^1(\Omega) \cap L^\infty(\Omega) \cap C(\Omega)),$$

since interior regularity is always true. This is enough to state properly, mutatis mutandis, our various results.

**More general nonlinearities.** For the sake of simplicity, we presented our results for the equation with the model nonlinearity $|u|^p u$. Nevertheless, the theory can be extended with little effort to more general nonlinear terms $f(u(t))$. The essential assumption here is that $f : \mathbb{R} \to \mathbb{R}$ is an increasing function such that $f(x) \sim |x|^p x$

and $f'(x) \sim (p+1)|x|^p$ as $x \to 0$. One can even assume monotonicity just in a neighborhood of the origin, but of course in that case one obtains a description of solutions only for initial data whose norm in $L^\infty(\Omega)$ is small enough.

**The Dirichlet case.** Many results for concrete models have been unified in [7] by developing an abstract theory for evolution inequalities of the form

$$|u'(t) + Au(t)|_H \leq K_0\big(|u(t)|_H^{1+p} + |A^{1/2}u(t)|_H^{1+q}\big) \quad \text{for all } t \geq 0, \tag{5.1}$$

where $A$ is a self-adjoint nonnegative operator with discrete spectrum in a Hilbert space $H$, and $K_0$, $p$, $q$ are positive real numbers. A full description of possible decay rates was provided, showing that all nonzero solutions to (5.1) that decay to 0 are either exponentially fast as solutions to the linearized equation $u'(t) + Au(t) = 0$, or slow as solutions to the ordinary differential inequality $|u'(t)| \leq K_0|u(t)|^{1+p}$. By relying on this kind of general techniques, it will be possible to extend this theory to more general parabolic partial differential equations whose linear part has a nontrivial kernel, for instance the problem "at resonance"

$$u_t - \Delta u + |u|^p u - \lambda_1 u = 0$$

with homogeneous Dirichlet boundary conditions. The proofs will involve more sophisticated tools, since we need to compare solutions not with constants, but constant multiples of the (positive normalized) first eigenfunction of the operator $-\Delta$. In the present paper we decided to limit ourselves to the model example (1.1) in which some of the arguments appear simpler. The other cases will be studied elsewhere.

**Second order equations.** It might be interesting to look for a concrete realization of the slow-fast alternative for second order evolution equations with dissipative terms, in the same way as the results of [7] were extended in [5, 6]. However, this would require completely new ideas, since both the regularizing effect and the comparison principles are specific to parabolic problems and even in the simple case of the ordinary differential equation

$$u'' + u' + u^3 = 0$$

the set of fast solutions has already a rather complicated shape. Moreover, in that case, all nontrivial solutions (including exponentially decaying ones) are asymptotically signed, so that even for the hyperbolic problem

$$u_{tt} + u_t - \Delta u + u^3 = 0$$

with Neumann homogeneous boundary conditions, the slow character is not equivalent to the existence of a constant sign for $t$ large.

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