Research Article

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Ground state solutions for Kirchhoff-type equations with a general nonlinearity in the critical growth

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Abstract: In this paper, we concern ourselves with the following Kirchhoff-type equations:

$$\begin{cases} -\Big(a + b \displaystyle\int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big)\Delta u + Vu = f(u) & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases}$$

where a, b and V are positive constants and f has critical growth. We use variational methods to prove the existence of ground state solutions. In particular, we do not use the classical Ambrosetti–Rabinowitz condition. Some recent results are extended.

Keywords: Kirchhoff-type equations, critical growth, ground state solutions, variational methods

MSC 2010: 35J20, 35J65, 35J60

1 Introduction and main results

Consider the following Kirchhoff-type problem:

$$\begin{cases} -\Big(a + b \displaystyle\int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big)\Delta u + Vu = f(u) & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where a, b and V are positive constants and $f(u)$ satisfies the following hypotheses:

(f1) $f(u) \in C(\mathbb{R}, \mathbb{R})$ is odd.
(f2) $\lim_{u\to 0^+} f(u)/u = 0$.
(f3) $\lim_{u\to +\infty} f(u)/u^5 = \mu > 0$.
(f4) There exist $M > 0$ and $2 < q < 6$ such that $f(u) \ge \mu u^5 + Mu^{q-1}$ for $u \ge 0$.

Kirchhoff-type problems are related to the stationary analogue of the equation

$$u_{tt} - \Big(a + b \int_{\Omega} |\nabla u|^2\,dx\Big)\Delta u = f(x, u) \quad \text{in } \Omega,$$

where u denotes the displacement, $f(x, u)$ the external force, and b the initial tension while a is related to the intrinsic properties of the string (such as Young's modulus). Equations of this type arise in the study of string or membrane vibration and were first proposed by Kirchhoff in 1883 (see [15]) to describe the transversal

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oscillations of a stretched string, particularly, taking into account the subsequent change in string length caused by oscillations. Kirchhoff-type problems are often referred to as being nonlocal because of the presence of the integral over the entire domain Ω, which provokes some mathematical difficulties. Similar nonlocal problems also model several physical and biological systems, where u describes a process which depends on the average of itself, for example, the population density; see [10, 11], and the references therein.

If we set $a = 1$, $b = 0$, then (1.2) reduces to the following Schrödinger equation:

$$-\Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N.$$

There exist many studies on the existence and multiplicity of solutions for this equation. We refer to [2–5, 20, 21] and the references therein.

There has been a lot of research on the existence of nontrivial solutions to (1.1) with subcritical nonlinearities by variational methods; see, e.g., [17–19, 24] and the references therein. Recently, some researchers have considered the existence of ground state solutions for Kirchhoff-type problems with critical Sobolev exponent. By using the Nehari manifold, Wang and Tian [23] proved the existence and multiplicity of positive ground state solutions for the following semilinear Kirchhoff problem with critical growth:

$$\begin{cases} -\Big(\varepsilon^2 a + b\varepsilon \displaystyle\int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big)\Delta u + M(x)u = f(u) + u^5, & x \in \mathbb{R}^3, \\ u > 0, & u \in H^1(\mathbb{R}^3), \end{cases} \tag{1.2}$$

where ε is a positive parameter and f is a C^1 and subcritical function such that the following hold:
(F1) $f(t) = o(t^3)$ *as* $t \to 0$ and $f(t) \equiv 0$ for all $t \le 0$.
(F2) $f(t)t^{-3}$ is strictly increasing for $t > 0$.
(F3) $f(t) = o(t^5)$ as $|t| \to +\infty$.
We pull the energy level down below the following critical level: $\frac{1}{3}(aS)^{\frac{3}{2}} + \frac{1}{12}b^3S^6$. He and Zou in [13] also considered (1.2), where $f(t)$ satisfies (F1), (F2) and the following:
(F4) $\nu F(t) = \nu\int_0^t f(s)ds \le tf(t)$ holds for some $\nu > 4$.
(F5) $f(t) = o(t^q)$ as $t \to \infty$, $3 < q < 5$.
By the use of variational methods, the authors showed that there exist $\varepsilon^* > 0$, $\lambda^* > 0$ such that for any $\varepsilon > \varepsilon^*$, $\lambda > \lambda^*$, problem (1.2) has at least one positive ground state solution in $H^1(\mathbb{R}^3)$.

Under conditions (F1)–(F4), by variational methods, Li and Ye [16] proved the existence of positive ground state solutions for the following Kirchhoff-type problem:

$$\begin{cases} -\Big(a + b \displaystyle\int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big)\Delta u + u = f(u) + u^5, & x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), & u > 0,\ x \in \mathbb{R}^3. \end{cases}$$

Particularly, Alves and Figueiredo [1] obtained the existence of positive solutions for a periodic Kirchhoff equation with critical or subcritical nonlinearity.

Motivated by the above works described, we borrow an idea from [25] to prove the existence of ground state solutions for problem (1.1) with a general nonlinearity in the critical growth. Our main result is the following.

Theorem 1.1. *Assume that* (f1)–(f4) *hold, then for* $2 < q \le 4$ *with* $M > 0$ *sufficiently large or* $4 < q < 6$, *problem* (1.1) *possesses a radial ground state solution.*

Remark 1.2. Conditions (f1)–(f4) were introduced in [25] to obtain the existence of ground state solutions for a class of Schrödinger–Poisson equations with critical growth. An interesting question now is whether the same existence results occur to the nonlocal problem (1.1) with critical growth. In this paper, we study problem (1.1) and give some positive answers. Theorem 1.1 extends the main result in [25] to the Kirchhoff-type equation.

Remark 1.3. Set $V = 1$ in (1.1). Compared to [16, Theorem 1.3], we need not consider the usual Ambrosetti–Rabinowitz (AR) condition (F4), which is restrictive. The lack of the (AR)-condition gives rise to two obstacles to the standard mountain pass arguments both in checking the geometrical assumptions in the functional and in proving the boundedness of its (PS) sequences. In the present paper, we use another method to obtain our results. Moreover, there exist some functions which satisfy our conditions (f1)–(f4), but do not satisfy the conditions of [16]. For example, the function $f(t) = \mu t^5 + Mt^3$, $\mu, M > 0$, satisfies our conditions (f1)–(f4), but does not satisfy conditions (F3) and (F4) in [16]. For this reason, Our main results can be viewed as a partial extension of [16].

The outline of the paper is as follows: In Section 2, we present some preliminary results. In Section 3, we give the proof of Theorem 1.1.

Notations. We use the following notations:
- $H^1(\mathbb{R}^3)$ is the Sobolev space equipped with the norm
$$\|u\|_{H^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)\,dx.$$
- Define
$$\|u\|^2 := \int_{\mathbb{R}^3} (a|\nabla u|^2 + Vu^2)\,dx \quad \text{for } u \in H^1(\mathbb{R}^3).$$
Note that $\|\cdot\|$ is an equivalent norm on $H^1(\mathbb{R}^3)$.
- For any $1 \le s \le \infty$, we denote by
$$\|u\|_{L^s} := \Big(\int_{\mathbb{R}^3} |u|^s\,dx\Big)^{\frac{1}{s}}$$
the usual norm of the Lebesgue space $L^s(\mathbb{R}^3)$.
- For any $x \in \mathbb{R}^3$, we set
$$B_r(x) \triangleq \{y \in \mathbb{R}^3 : |x-y| < r\}.$$
- Let $D^{1,2}(\mathbb{R}^3) := \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$ be the Sobolev space equipped with the norm
$$\|u\|^2_{D^{1,2}(\mathbb{R}^3)} := \int_{\mathbb{R}^3} |\nabla u|^2\,dx.$$
- S denotes the best Sobolev constant
$$S := \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2\,dx}{(\int_{\mathbb{R}^3} u^6\,dx)^{\frac{1}{3}}}.$$
- C denotes various positive constants.

2 Preliminaries

It is clear that problem (1.1) are the Euler–Lagrange equations of the functional $I : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined by
$$I(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big)^2 - \int_{\mathbb{R}^3} F(u)\,dx, \tag{2.1}$$
where $F(u) = \int_0^u f(t)\,dt$. Obviously, by (f1)–(f3), we can obtain that I is a well-defined C^1 functional and satisfies
$$\langle I'(u), v\rangle = \int_{\mathbb{R}^3} (a\nabla u \nabla v + Vuv)\,dx + b\int_{\mathbb{R}^3} |\nabla u|^2\,dx \int_{\mathbb{R}^3} \nabla u \nabla v\,dx - \int_{\mathbb{R}^3} f(u)v\,dx$$
for $v \in H^1(\mathbb{R}^3)$. For simplicity, by (f4), we may assume that $\mu = 1$. Let $g(t) = f(t) - t^5$. Then
$$I(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big)^2 - \int_{\mathbb{R}^3} G(u)\,dx - \frac{1}{6}\int_{\mathbb{R}^3} u^6\,dx,$$

where $G(u) = \int_0^u g(t)\,dt$. It is well known that $u \in H^1(\mathbb{R}^3)$ is a critical point of the functional I if and only if u is a weak solution of (1.1).

Let $E := H_r^1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) : u \text{ is radial}\}$. Then, by the principle of symmetric criticality, a critical point of I on E is a critical point of I on $H^1(\mathbb{R}^3)$. We refer the readers to [6, 12]. Thus, we only need to look for critical points of I on E. To complete the proof of our theorem, the following result will be needed in our argument.

Theorem 2.1 (See [14]). *Let $(X, \|\cdot\|)$ be a Banach space and $h \subset \mathbb{R}_+$ an interval. Consider the following family of C^1 functionals on X:*

$$I_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in h,$$

with B nonnegative and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to \infty$. We assume there are two points v_1, v_2 in X such that

$$c_\lambda = \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\} \quad \text{for all } \lambda \in h,$$

where

$$\Gamma_\lambda = \{\gamma \in C([0,1], X) : \gamma(0) = v_1,\ \gamma(1) = v_2\}.$$

Then for almost every $\lambda \in h$ there is a sequence $\{u_n\} \subset X$ such that
(i) *$\{u_n\}$ is bounded,*
(ii) *$I_\lambda(u_n) \to c_\lambda$,*
(iii) *$I'_\lambda(u_n) \to 0$ in the dual X^{-1} of X.*
Moreover, the map $\lambda \to c_\lambda$ is continuous from the left.

3 Proof of the main result

This section is devoted to the proof of Theorem 1.1. According to Theorem 2.1, we need the following lemmas.

Lemma 3.1. *If u_λ, for $\lambda \in [\frac{1}{2}, 1]$, is a critical point of I_λ, then u_λ satisfies the following Pohožaev-type identity:*

$$\frac{1}{2}\int_{\mathbb{R}^3} a|\nabla u|^2\,dx + \frac{b}{2}\Big(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big)^2 + \frac{3}{2}\int_{\mathbb{R}^3} Vu^2\,dx - 3\lambda\int_{\mathbb{R}^3} F(u)\,dx = 0.$$

Since the proof can be done as in [16], we omit it here.

Lemma 3.2. *Assume that conditions* (f1), (f2)–(f4) *are satisfied. Then the conclusions of Theorem 2.1 hold.*

Proof. By Theorem 2.1, we set

$$X = E, \quad h = \Big[\frac{1}{2}, 1\Big], \quad A(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big)^2, \quad B(u) = \int_{\mathbb{R}^3} F(u)\,dx.$$

By (f3)–(f4), we obtain that $f(u)$ is odd and by the definition of $A(u)$, we can see that $B(u) \geq 0$ for $u \in E$ and $A(u) \to +\infty$ as $\|u\| \to \infty$. By (f1)–(f3), for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$|F(u)| \leq \varepsilon u^2 + C(\varepsilon)u^6. \tag{3.1}$$

Then, by the Sobolev embedding theorem, there holds

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big)^2 - \lambda\int_{\mathbb{R}^3} F(u)\,dx \geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3} \varepsilon u^2\,dx - \int_{\mathbb{R}^3} C(\varepsilon)u^6\,dx \geq C\|u\|^2 - C\|u\|^6,$$

which implies that there exists $\rho > 0$ small enough and $\alpha > 0$ such that

$$I_\lambda(u) \geq \alpha > 0 \quad \text{for all } \|u\| = \rho. \tag{3.2}$$

By (f4) and (2.1), for $\phi \in E$, $\phi \geq 0$ and $\phi \not\equiv 0$, we have

$$I_\lambda(t\phi) \leq \frac{t^2}{2}\|\phi\|^2 + \frac{bt^4}{4}\Big(\int_{\mathbb{R}^3} |\nabla\phi|^2\,dx\Big)^2 - \frac{Mt^q}{2q}\int_{\mathbb{R}^3} |\phi|^q\,dx - \frac{t^6}{12}\int_{\mathbb{R}^3} |\phi|^6\,dx \to -\infty \quad \text{as } t \to +\infty.$$

Taking $v_2 = t_1\phi$ with $t_1 > 0$ large enough, we have $\|v_2\| > \rho$ and

$$I_\lambda(v_2) < 0 \quad \text{for all } \lambda \in \Big[\frac{1}{2}, 1\Big]. \tag{3.3}$$

On the other hand, $I_\lambda(0) = 0$. Set $v_1 = 0$; then inequalities (3.2) and (3.3) imply that the conclusions of Theorem 2.1 hold. □

Lemma 3.3. *Assume that* (f1)–(f4) *hold. Then*

$$0 < c_\lambda < c_\lambda^* := \frac{abS^3}{4\lambda} + \frac{b^3S^6}{24\lambda^2} + \frac{(b^2S^4 + 4a\lambda S)^{\frac{3}{2}}}{24\lambda^2}$$

for $q \in (2, 4]$ with M sufficiently large or $q \in (4, 6)$.

Proof. For $\varepsilon, r > 0$, define

$$u_\varepsilon(x) = \frac{\varphi(x)\varepsilon^{\frac{1}{4}}}{(\varepsilon + |x|^2)^{\frac{1}{2}}},$$

where $\varphi \in C_0^\infty(B_{2r}(0))$, $0 \leq \varphi \leq 1$ and $\varphi|_{B_r(0)} \equiv 1$. Using the method of [9], we obtain

$$\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2\,dx = K_1 + O(\varepsilon^{\frac{1}{2}}), \qquad \int_{\mathbb{R}^3} |u_\varepsilon|^6\,dx = K_2 + O(\varepsilon^{\frac{3}{2}}) \tag{3.4}$$

and

$$\int_{\mathbb{R}^3} |u_\varepsilon|^t\,dx = \begin{cases} K\varepsilon^{\frac{t}{4}}, & t \in [2, 3), \\ K\varepsilon^{\frac{3}{4}}|\ln\varepsilon|, & t = 3, \\ K\varepsilon^{\frac{6-t}{4}}, & t \in (3, 6), \end{cases} \tag{3.5}$$

where K_1, K_2, K are positive constants. Moreover, the best Sobolev constant is $S = K_1K_2^{-1/3}$. By (3.4), we have

$$\frac{\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2\,dx}{(\int_{\mathbb{R}^3} u_\varepsilon^6\,dx)^{\frac{1}{3}}} = S + O(\varepsilon^{\frac{1}{2}}).$$

By Lemma 3.2 and the definition of c_λ, we can deduce that $c_\lambda \leq \sup_{t\geq 0} I_\lambda(tu_\varepsilon)$. Let

$$h(t) = \frac{t^2}{2}\|u_\varepsilon\|^2 + \frac{bt^4}{4}\Big(\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2\,dx\Big)^2 - \frac{\lambda t^6}{6}\int_{\mathbb{R}^3} u_\varepsilon^6\,dx \quad \text{for all } t \geq 0.$$

Note that $h(t)$ attains its maximum at

$$t_0 = \left(\frac{b(\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2\,dx)^2 + \sqrt{b^2(\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2\,dx)^4 + 4\lambda\|u_\varepsilon\|^2\int_{\mathbb{R}^3} u_\varepsilon^6\,dx}}{2\lambda\int_{\mathbb{R}^3} u_\varepsilon^6\,dx}\right)^{\frac{1}{2}}.$$

Then

$$\begin{aligned}\max_{t\geq 0} h(t) &= \frac{b\|u_\varepsilon\|^2(\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2\,dx)^2}{4\lambda\int_{\mathbb{R}^3} u_\varepsilon^6\,dx} + \frac{b^3(\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2\,dx)^6}{24\lambda^2(\int_{\mathbb{R}^3} u_\varepsilon^6\,dx)^2} + \frac{[b^2(\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2\,dx)^4 + 4\lambda\int_{\mathbb{R}^3} u_\varepsilon^6\,dx\|u_\varepsilon\|^2]^{\frac{3}{2}}}{24\lambda^2(\int_{\mathbb{R}^3} u_\varepsilon^6\,dx)^2} \\ &= \frac{abS^3}{4\lambda} + \frac{b^3S^6}{24\lambda^2} + \frac{(b^2S^4 + 4a\lambda S)^{\frac{3}{2}}}{24\lambda^2} + O(\varepsilon^{\frac{1}{2}}) := c_\lambda^* + O(\varepsilon^{\frac{1}{2}})\end{aligned}$$

for $\varepsilon > 0$ small enough.

Obviously, we see that there exists $0 < t_1 < 1$ such that for $\varepsilon < 1$, we have

$$\sup_{0\le t\le t_1} I_\lambda(tu_\varepsilon) \le \sup_{0\le t\le t_1}\Big[\frac{1}{2}t^2\|u_\varepsilon\|^2 + \frac{b}{4}t^4\Big(\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2\,dx\Big)^2\Big] \le \sup_{0\le t\le t_1}\Big(\frac{1}{2}t^2\|u_\varepsilon\|^2 + Ct^4\|u_\varepsilon\|^4\Big) \le c_\lambda^*. \tag{3.6}$$

By (f4) , we have

$$I_\lambda(tu_\varepsilon) = h(t) - \lambda\int_{\mathbb{R}^3} G(tu_\varepsilon)\,dx \le h(t) - \lambda\frac{M}{q}t^q\int_{\mathbb{R}^3}|u_\varepsilon|^q\,dx \le h(t) - CMt^q\int_{\mathbb{R}^3}|u_\varepsilon|^q\,dx. \tag{3.7}$$

It follows from (3.4), (3.5) and (3.7) that there exists $0 < \varepsilon_0 < 1$ such that for $\varepsilon < \varepsilon_0$, we have

$$\lim_{n\to\infty} I_\lambda(tu_\varepsilon) = -\infty.$$

Thus, there exists $t_2 > 0$ such that

$$\sup_{t\ge t_2} I_\lambda(tu_\varepsilon) \le c_\lambda^*. \tag{3.8}$$

From (3.7) and the definition of $h(t)$, we have

$$\sup_{t_1\le t\le t_2} I_\lambda(tu_\varepsilon) \le \sup_{t\ge 0} h(t) - CM\int_{\mathbb{R}^3}|u_\varepsilon|^q\,dx \le \frac{abS^3}{4\lambda} + \frac{b^3S^6}{24\lambda^2} + \frac{(b^2S^4 + 4a\lambda S)^{\frac{3}{2}}}{24\lambda^2} + O(\varepsilon^{\frac{1}{2}}) - CM\int_{\mathbb{R}^3}|u_\varepsilon|^q\,dx. \tag{3.9}$$

For $q \in (2, 4]$, fix $\varepsilon \in (0, \varepsilon_0)$, it follows from (3.9) that

$$\sup_{t_1\le t\le t_2} I_\lambda(tu_\varepsilon) < c_\lambda^* \quad \text{for } M \text{ sufficiently large.} \tag{3.10}$$

For $q \in (4, 6)$, by (3.5) and (3.9), we obtain

$$\sup_{t_1\le t\le t_2} I_\lambda(tu_\varepsilon) \le c_\lambda^* + O(\varepsilon^{\frac{1}{2}}) - CO(\varepsilon^{\frac{6-q}{4}}). \tag{3.11}$$

Since $\frac{6-q}{4} < \frac{1}{2}$, then there exists $\varepsilon_1 \in (0, \varepsilon_0)$ small enough such that for $\varepsilon \in (0, \varepsilon_1)$, we have

$$\sup_{t_1\le t\le t_2} I_\lambda(tu_\varepsilon) \le c_\lambda^*. \tag{3.12}$$

By (3.6), (3.8) and (3.10)–(3.12), the proof of Lemma 3.3 is complete. □

Lemma 3.4. *Set*

$$\Lambda = \lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^2\,dx \quad \text{and} \quad J_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{b\Lambda^2}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx - \lambda\int_{\mathbb{R}^3}F(u)\,dx.$$

If $J'_\lambda(u) = 0$, where $\lambda \in [\frac{1}{2}, 1]$, then $J_\lambda(u) \ge 0$.

Proof. Since $\langle J'_\lambda(u), u\rangle = 0$, by Lemma 3.1, we get the following the Pohožaev-type identity:

$$\frac{1}{2}\int_{\mathbb{R}^3} a|\nabla u|^2\,dx + \frac{b\Lambda^2}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx + \frac{3}{2}\int_{\mathbb{R}^3}Vu^2\,dx - 3\lambda\int_{\mathbb{R}^3}F(u)\,dx = 0.$$

Then we get that

$$\frac{1}{6}\int_{\mathbb{R}^3} a|\nabla u|^2\,dx + \frac{b\Lambda^2}{6}\int_{\mathbb{R}^3}|\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3}Vu^2\,dx - \lambda\int_{\mathbb{R}^3}F(u)\,dx = 0. \tag{3.13}$$

Combining (3.13) with the definition of $J_\lambda(u)$, we obtain that

$$J_\lambda(u) = \frac{1}{3}\int_{\mathbb{R}^3} a|\nabla u|^2\,dx + \frac{b\Lambda^2}{3}\int_{\mathbb{R}^3}|\nabla u|^2\,dx \ge 0.$$ □

Using a notion similar to [16, Lemma 3.6], we can obtain the following result.

Lemma 3.5. *For $s, t > 0$, the system*

$$\begin{cases} \Gamma(t,s) = t - aS\left(\frac{s+t}{\lambda}\right)^{\frac{1}{3}} = 0, \\ \Upsilon(t,s) = s - bS^2\left(\frac{s+t}{\lambda}\right)^{\frac{2}{3}} = 0 \end{cases}$$

has a unique solution (t_0, s_0). Moreover, if

$$\begin{cases} \Gamma(t,s) \geq 0, \\ \Upsilon(t,s) \geq 0, \end{cases}$$

then $t \geq t_0$ and $s \geq s_0$, where

$$t_0 = \frac{abS^3 + a\sqrt{b^2S^6 + 4\lambda aS^3}}{2\lambda}, \quad s_0 = \frac{bS^6 + 2\lambda abS^3 + b^2S^3\sqrt{b^3S^6 + 4\lambda aS^3}}{2\lambda^2}.$$

Lemma 3.6. *Assume that* (f1)–(f3) *hold. If $\{u_n\} \subset E$, for $\lambda \in [\frac{1}{2}, 1]$, is a sequence such that $\|u_n\| < C$, $I_\lambda(u_n) \to c_\lambda$, $I'_\lambda(u_n) \to 0$, and, moreover, $c_\lambda < c^*_\lambda$, then $\{u_n\}$ has a strong convergent subsequence in E.*

Proof. Since $\|u_n\| < C$ in E, there exists a $u \in E$ such that

$$u_n \rightharpoonup u \text{ weakly in } H, \quad u_n \to u \text{ strongly in } L^s(\mathbb{R}^3) \text{ for all } s \in (2,6).$$

Set

$$\Lambda = \lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx.$$

By Lemma 3.4 and the fact that $c_\lambda < c^*_\lambda$, we have

$$c_\lambda - J_\lambda(u) \leq c^*_\lambda. \tag{3.14}$$

Using an argument similar to [7, Radial Lemma A.II.], by the boundedness of $\{u_n\}$, we have $\lim_{|x|\to\infty} u_n(x) = 0$. Since

$$\lim_{|t|\to\infty} \frac{G(t)}{t^2 + t^6} = 0 \quad \text{and} \quad \lim_{t\to 0} \frac{G(t)}{t^2 + t^6} = 0,$$

we also get

$$\int_{\mathbb{R}^3} (u_n^2 + u_n^6) \, dx < \infty.$$

By the compactness lemma of Strass [22], one has

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} (G(u_n) - G(u)) \, dx = 0. \tag{3.15}$$

Similarly,

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} (g(u_n)u_n - g(u)u) \, dx = 0. \tag{3.16}$$

Setting $\omega_n = u_n - u$, due to Brezis–Lieb (see [8]), we have

$$\int_{\mathbb{R}^3} |\nabla \omega_n|^2 \, dx = \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx - \int_{\mathbb{R}^3} |\nabla u_\lambda|^2 \, dx + o(1),$$
$$\int_{\mathbb{R}^3} |\omega_n|^6 \, dx = \int_{\mathbb{R}^3} |u_n|^6 \, dx - \int_{\mathbb{R}^3} |u_\lambda|^6 \, dx + o(1),$$

and

$$\Lambda^2 + o(1) = \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx = \int_{\mathbb{R}^3} |\nabla \omega_n|^2 \, dx + \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + o(1).$$

Then, from (3.15) and (3.16), we have

$$
\begin{aligned}
I_\lambda(u_n) - \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx\Big)^2 + \frac{b\Lambda^2}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx - J_\lambda(u)
&= J_\lambda(u_n) - J_\lambda(u)\\
&= \frac{1}{2}\|\omega_n\|^2 + \frac{b\Lambda^2}{4}\int_{\mathbb{R}^3} |\nabla \omega_n|^2\,dx\\
&\quad + \frac{b}{4}\Big[\Big(\int_{\mathbb{R}^3} |\nabla \omega_n|^2\,dx\Big)^2 + \int_{\mathbb{R}^3} |\nabla \omega_n|^2\,dx \int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big] - \frac{\lambda}{6}\int_{\mathbb{R}^3} |\omega_n|^6\,dx + o(1),
\end{aligned}
\tag{3.17}
$$

and

$$J_\lambda'(u) = 0.$$

Then, by (3.16) and the fact that $I_\lambda'(u_n) \to 0$, we have

$$
\begin{aligned}
o(1) &= \langle J_\lambda'(u_n), u_n\rangle - \langle J_\lambda'(u), u\rangle\\
&= \|\omega_n\|^2 + b\Big(\int_{\mathbb{R}^3} |\nabla \omega_n|^2\,dx\Big)^2 + b\int_{\mathbb{R}^3} |\nabla \omega_n|^2\,dx \int_{\mathbb{R}^3} |\nabla u|^2\,dx - \lambda\int_{\mathbb{R}^3} \omega_n^6\,dx + o(1).
\end{aligned}
\tag{3.18}
$$

We can assume that there exists $l_i \geq 0$ ($i = 1, 2, 3$) such that

$$\|\omega_n\|^2 \to l_1, \quad b\Big(\int_{\mathbb{R}^3} |\nabla \omega_n|^2\,dx\Big)^2 + b\int_{\mathbb{R}^3} |\nabla \omega_n|^2\,dx \int_{\mathbb{R}^3} |\nabla u|^2\,dx \to l_2, \quad \lambda\int_{\mathbb{R}^3} \omega_n^6\,dx \to l_3.$$

Then by (3.18) and (3.17), we have

$$
\begin{cases}
l_1 + l_2 - l_3 = 0,\\
\frac{1}{2}l_1 + \frac{1}{4}l_2 - \frac{1}{6}l_3 + \frac{b\Lambda^2}{4}\lim\limits_{n\to\infty}\int_{\mathbb{R}^3} |\nabla \omega_n|^2\,dx = c_\lambda + \frac{b\Lambda^2}{4} - J_\lambda(u).
\end{cases}
$$

By the definition of S, we see that

$$\int_{\mathbb{R}^3} |\nabla \omega_n|^2\,dx \geq \frac{S}{\lambda^{1/3}}\Big(\lambda\int_{\mathbb{R}^3} |\omega_n|^6\,dx\Big)^{\frac{1}{3}}, \quad b\Big(\int_{\mathbb{R}^3} |\nabla \omega_n|\,dx\Big)^2 \geq b\frac{S^2}{\lambda^{2/3}}\Big(\lambda\int_{\mathbb{R}^3} |\omega_n|^6\,dx\Big)^{\frac{2}{3}}.$$

Then

$$l_1 \geq aS\Big(\frac{l_1 + l_2}{\lambda}\Big)^{\frac{1}{3}} \quad \text{and} \quad l_2 \geq bS^2\Big(\frac{l_1 + l_2}{\lambda}\Big)^{\frac{2}{3}}.$$

Obviously, if $l_1 > 0$, then $l_2, l_3 > 0$. By Lemma 3.5, we have that

$$
\begin{aligned}
c_\lambda + \frac{b\Lambda^2}{4} - J_\lambda(u) &= \frac{1}{3}l_1 + \frac{1}{12}l_2 + \frac{b\Lambda^2}{4}\lim_{n\to\infty}\int_{\mathbb{R}^3} |\nabla \omega_n|^2\,dx\\
&\geq \frac{1}{3}\,\frac{abS^3 + a\sqrt{b^2S^6 + 4\lambda aS^3}}{2\lambda} + \frac{1}{12}\,\frac{bS^6 + 2\lambda abS^3 + b^2S^3\sqrt{b^3S^6 + 4\lambda aS^3}}{2\lambda^2} + \frac{b\Lambda^2}{4}\\
&= \frac{abS^3}{4\lambda} + \frac{b^3S^6}{24\lambda^2} + \frac{(b^2S^4 + 4a\lambda S)^{\frac{3}{2}}}{24\lambda^2} + \frac{b\Lambda^2}{4}\\
&= c_\lambda^* + \frac{b\Lambda^2}{4},
\end{aligned}
$$

which is contrary to (3.14). Therefore, $\|\omega_n\| \to 0$ and Lemma 3.6 is complete. □

Lemma 3.7. *Under the assumptions of Theorem 1.1, for almost every $\lambda \in [\frac{1}{2}, 1]$ there exists $u_\lambda \in E$, $u_\lambda \neq 0$ such that $I_\lambda(u_\lambda) = c_\lambda$ and $I_\lambda'(u_\lambda) = 0$.*

Proof. By Lemma 3.2, there is a sequence$\{u_n\} \subset E$ satisfying $\|u_n\| < C$, $I_\lambda(u_n) \to c_\lambda$ and $I'_\lambda(u_n) \to 0$. Moreover, $0 < c_\lambda < c^*_\lambda$. Then, by Lemma 3.6, the sequence $\{u_n\}$ has a strong convergent subsequence, still denoted by$\{u_n\}$. In other words, there exists $u_\lambda \in E$ such that $I_\lambda(u_\lambda) = c_\lambda$ and $I'_\lambda(u_\lambda) = 0$. □

Lemma 3.8. *Under the assumptions of Theorem 1.1, the functional I admits nontrivial critical points.*

Proof. From Lemma 3.7, there exist $c_{\lambda_n} \in (0, c^*_{\lambda_n})$, $\lambda_n \in [\frac{1}{2}, 1]$ and $u_{\lambda_n} \subset E$ with $u_{\lambda_n} \neq 0$ such that $\lambda_n \to 1$, $I'_{\lambda_n}(u_{\lambda_n}) = 0$ and $I_{\lambda_n}(u_{\lambda_n}) = c_{\lambda_n}$. Then, by Lemma 3.1, we have

$$3c_{\lambda_n} = \int_{\mathbb{R}^3} a|\nabla u_{\lambda_n}|^2\,dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u_{\lambda_n}|^2\,dx\Big)^2.$$

The Sobolev embedding theorem implies the boundedness of $\int_{\mathbb{R}^3} |u_{\lambda_n}|^6\,dx$. From (3.1) and Lemma 3.1 we get

$$\frac{1}{2}\int_{\mathbb{R}^3} a|\nabla u_{\lambda_n}|^2\,dx + \frac{3}{2}\int_{\mathbb{R}^3} Vu^2_{\lambda_n}\,dx + \frac{b}{2}\Big(\int_{\mathbb{R}^3} |\nabla u_{\lambda_n}|^2\,dx\Big)^2 = 3\lambda_{\lambda_n}\int_{\mathbb{R}^3} F(u_{\lambda_n})\,dx \le \varepsilon\int_{\mathbb{R}^3} |u_{\lambda_n}|^2\,dx + C(\varepsilon)\int_{\mathbb{R}^3} |u_{\lambda_n}|^6\,dx$$

for any $\varepsilon > 0$ and some $C(\varepsilon) > 0$. Hence, $\|u_{\lambda_n}\|$ is bounded. Since

$$I(u_{\lambda_n}) = I_{\lambda_n}(u_{\lambda_n}) + (\lambda_n - 1)\int_{\mathbb{R}^3} F(u_{\lambda_n})\,dx,$$

we have

$$\langle I'(u_{\lambda_n}), u_{\lambda_n}\rangle = \langle I'_{\lambda_n}(u_{\lambda_n}), u_{\lambda_n}\rangle + (\lambda_n - 1)\int_{\mathbb{R}^3} f(u_{\lambda_n})u_{\lambda_n}\,dx$$

and

$$\lim_{n\to\infty} c_{\lambda_n} = c_1 \in (0, c^*), \quad \text{where } c^* = \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(b^2S^4 + 4aS)^{\frac{3}{2}}}{24}.$$

By a standard argument, we obtain that $\lim_{n\to\infty} I(u_{\lambda_n}) = c_1$ and $\lim_{n\to\infty} I'(u_{\lambda_n}) = 0$. By the boundedness of $\|u_{\lambda_n}\|$, similarly to the proof of Lemma 3.6, we can prove that there exists $u_0 \in E$ such that $I'(u_0) = 0$. We claim that $u_0 \neq 0$. Otherwise, if $u_0 = 0$, then $u_{\lambda_n} \to 0$ weakly in E and

$$u_{\lambda_n} \to 0 \text{ strongly in } L^s(\mathbb{R}^3), \quad s \in (2, 6). \tag{3.19}$$

From (3.19) and (f1)–(f3), we have

$$\int_{\mathbb{R}^3} G(u_{\lambda_n}) = o(1) \quad \text{and} \quad \int_{\mathbb{R}^3} g(u_{\lambda_n})u_{\lambda_n} = o(1).$$

From $\lim_{n\to\infty} I(u_{\lambda_n}) = c_1$ and $\lim_{n\to\infty} I'(u_{\lambda_n}) = 0$, we have that

$$c_1 + o(1) = \frac{1}{2}\|u_{\lambda_n}\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u_{\lambda_n}|^2\,dx\Big)^2 - \frac{1}{6}\int_{\mathbb{R}^3} |u_{\lambda_n}|^6\,dx \tag{3.20}$$

and

$$o(1) = \|u_{\lambda_n}\|^2 + b\Big(\int_{\mathbb{R}^3} |\nabla u_{\lambda_n}|^2\,dx\Big)^2 - \int_{\mathbb{R}^3} |u_{\lambda_n}|^6\,dx. \tag{3.21}$$

Assume that

$$\|u_{\lambda_n}\|^2 \to h_1 \ge 0, \quad b\Big(\int_{\mathbb{R}^3} |\nabla u_{\lambda_n}|^2\,dx\Big)^2 \to h_2 \ge 0, \quad \int_{\mathbb{R}^3} u^6_{\lambda_n}\,dx \to h_3 \ge 0.$$

Then by (3.21) and (3.20), we have

$$\begin{cases} h_1 + h_2 - h_3 = 0, \\ \qquad c_1 = \frac{1}{2}h_1 + \frac{1}{4}h_2 - \frac{1}{6}h_3, \end{cases}$$

where $c_1 > 0$ implies that $h_1, h_2, h_3 > 0$. By the definition of S, we see that

$$h_1 \geq aS(h_1 + h_2)^{\frac{1}{3}} \text{ and } h_2 \geq bS^2(h_1 + h_2)^{\frac{2}{3}}.$$

By Lemma 3.5, we have that

$$\begin{aligned} c_1 &= \frac{1}{3}h_1 + \frac{1}{12}h_2 \\ &\geq \frac{1}{3}\frac{abS^3 + a\sqrt{b^2S^6 + 4aS^3}}{2} + \frac{1}{12}\frac{bS^6 + 2abS^3 + b^2S^3\sqrt{b^3S^6 + 4aS^3}}{2} \\ &= \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(b^2S^4 + 4aS)^{\frac{3}{2}}}{24} = c^*, \end{aligned}$$

which is contrary to $c_1 < c^*$. □

In the following, we prove $c_1 \geq I(u_0)$. From $I'_{\lambda_n}(u_{\lambda_n}) = 0$, $I'(u_0) = 0$ and by Lemma 3.1, we obtain that

$$c_{\lambda_n} = I_{\lambda_n}(u_{\lambda_n}) = \frac{a}{3}\int_{R^3} |\nabla u_{\lambda_n}|^2\,dx + \frac{b}{12}\Big(\int_{R^3} |\nabla u_{\lambda_n}|^2\,dx\Big)^2$$

and

$$I(u_0) = \frac{a}{3}\int_{\mathbb{R}^3} |\nabla u_0|^2\,dx + \frac{b}{12}\Big(\int_{\mathbb{R}^3} |\nabla u_0|^2\,dx\Big)^2.$$

Thus, by Fatou's Lemma,

$$c_1 = \lim_{n\to\infty} c_{\lambda_n} \geq \frac{a}{3}\int_{\mathbb{R}^3} |\nabla u_0|^2\,dx + \frac{b}{12}\Big(\int_{\mathbb{R}^3} |\nabla u_0|^2\,dx\Big)^2 = I(u_0).$$

Proof of Theorem 1.1. Let

$$m = \inf\{I(u) : u \in E,\ u \neq 0,\ I'(u) = 0 \text{ in } E^{-1}\}.$$

By Lemma 3.8 and Lemma 3.4, we have $0 \leq m \leq I(u_0) \leq c_1 < c^*$. We choose a minimizing sequence $\{u_n\}$ for m, i.e. $u_n \neq 0$, $I(u_n) \to m$ and $I'(u_n) = 0$. Now, we prove that $\{u_n\}$ is bounded. The proof is divided into two steps.

Step 1: $\{\|u_n\|_{L^2}\}$ is bounded. By contradiction, we assume that $\|u_n\|_{L^2} \to \infty$ as $n \to \infty$. Set

$$v_n = \frac{u_n}{\|u_n\|_{L^2}}, \quad X_n = \int_{\mathbb{R}^3} |\nabla u_n|^2\,dx\|u_n\|_{L^2}^{-2}, \quad Y_n = \Big(\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx\Big)^2\|u_n\|_{L^2}^{-2}, \quad Z_n = \int_{\mathbb{R}^3} F(u_n)\,dx\|u_n\|_{L^2}^{-2}.$$

Since $I(u_n) \to m$ and $I'(u_n) = 0$, using (2.1) and Lemma 3.1, we have

$$\begin{cases} \frac{1}{2}\int_{\mathbb{R}^3} a|\nabla u_n|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} V|u_n|^2\,dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx\Big)^2 - \int_{\mathbb{R}^3} F(u_n)\,dx = m + o(1), \\ \frac{1}{2}\int_{\mathbb{R}^3} a|\nabla u_n|^2\,dx + \frac{3}{2}\int_{\mathbb{R}^3} V|u_n|^2\,dx + \frac{b}{2}\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx\Big)^2 - 3\int_{\mathbb{R}^3} F(u_n)\,dx = 0, \end{cases} \tag{3.22}$$

and m is bounded. Multiplying (3.22) by $\frac{1}{\|u_n\|_{L^2}}$, we get

$$\begin{cases} \frac{1}{2}aX_n + \frac{1}{2}V + \frac{b}{4}Y_n - Z_n = o(1), \\ \frac{1}{2}aX_n + \frac{3}{2}V + \frac{b}{2}Y_n - 3Z_n = 0, \end{cases} \tag{3.23}$$

where $o(1)$ denotes that the quantity tends to zero as $n \to \infty$. Solving (3.23), we have

$$X_n = -\frac{b}{4a}Y_n + o(1). \tag{3.24}$$

Since $Y_n \geq 0$, $X_n \geq 0$ and $a, b > 0$ for all $n \in \mathrm{N}$, equation (3.24) is a contradiction for n large enough. Thus, $\{\|u_n\|_{L^2}\}$ is bounded.

Step 2: $\|\nabla u_n\|_{L^2}$ is bounded. Similarly, by contradiction, we can assume that $\|\nabla u_n\|_{L^2} \to \infty$ as $n \to \infty$. Set

$$v_n = \frac{u_n}{\|\nabla u_n\|_{L^2}}, \quad M_n = \int_{\mathbb{R}^3} u_n^2\, dx \|\nabla u_n\|_{L^2}^{-2}, \quad N_n = \Big(\int_{\mathbb{R}^3} |\nabla u_n|^2\, dx\Big)^2 \|\nabla u_n\|_{L^2}^{-2}, \quad S_n = \int_{\mathbb{R}^3} F(u_n)\, dx \|\nabla u_n\|_{L^2}^{-2}.$$

Then, multiplying (3.22) by $\frac{1}{\|\nabla u_n\|_{L^2}}$, we get

$$\begin{cases} \frac{1}{2}a + \frac{1}{2}VM_n + \frac{b}{4}N_n - S_n = o(1), \\ \frac{1}{2}a + \frac{3}{2}VM_n + \frac{b}{2}N_n - 3S_n = 0. \end{cases} \tag{3.25}$$

Solving (3.25), we have

$$N_n = -\frac{4a}{b} + o(1). \tag{3.26}$$

Since $N_n \geq 0$ and $a, b > 0$, equation (3.26) is a contradiction for n large enough. Thus, $\{\|\nabla u_n\|_{L^2}\}$ is bounded. Thus, we prove the boundedness of $\{u_n\}$. Similarly to the proofs of Lemma 3.6 and Lemma 3.8, we can prove that there exists $u \neq 0 \in E$ such that $I'(u) = 0$.

Next, we will give the proof of $m \geq I(u)$. In fact, by $I'(u) = 0$, $I'(u_n) = 0$ and Lemma 3.1, we have that

$$I(u) = \frac{a}{3}\int_{\mathbb{R}^3} |\nabla u|^2\, dx + \frac{b}{12}\Big(\int_{\mathbb{R}^3} |\nabla u|^2\, dx\Big)^2$$

and

$$I(u_n) = \frac{a}{3}\int_{\mathbb{R}^3} |\nabla u_n|^2\, dx + \frac{b}{12}\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2\, dx\Big)^2.$$

Then

$$m + o(1) = I(u_n) = \frac{a}{3}\int_{\mathbb{R}^3} |\nabla u_n|^2\, dx + \frac{b}{12}\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2\, dx\Big)^2.$$

Thus, by Fatou's Lemma,

$$m \geq \frac{a}{3}\int_{\mathbb{R}^3} |\nabla u|^2\, dx + \frac{b}{12}\Big(\int_{\mathbb{R}^3} |\nabla u|^2\, dx\Big)^2 = I(u).$$

By combination with the definition of m, there exists $u \neq 0$ satisfying $m = I(u)$ and $I'(u) = 0$, which completes the proof of Theorem 1.1. □

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