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Research Article

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Global $W^{1,p(\cdot)}$ estimate for renormalized solutions of quasilinear equations with measure data on Reifenberg domains

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Abstract: In this paper, we prove the gradient estimate for renormalized solutions to quasilinear elliptic equations with measure data on variable exponent Lebesgue spaces with BMO coefficients in a Reifenberg flat domain.

Keywords: Quasilinear equation, measure data, renormalized solution, generalized Lebesgue space

MSC 2010: 35J60, 35J62

1 Introduction

The gradient estimates for the solution to partial differential equations involving the variable function spaces such as generalized Lebesgue are an interesting topic in both pure and applied mathematics, and have a wide range of applications from non-Newtonian fluids to image restoration. See for example [2, 12, 22, 30, 31]. Motivated from these works, this paper proves the nonlinear Muckenhoupt–Wheeden type estimates of the quasilinear equation with measure data on the generalized Lebesgue spaces. Before coming to details, we recall some preliminaries on the quasilinear equation with measure data.

In this paper, we consider the following Dirichlet with measure data

$$\begin{cases} -\operatorname{div} \mathbf{a}(x, \nabla u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where μ is a signed measure with bounded totally variations in a bounded open domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. The quasilinear elliptic equation with data measure has been studied intensively by many mathematicians. See for example [3–6, 19, 20] and the references therein.

In this paper, we assume that the nonlinearity $\mathbf{a}(x, \xi) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable in x for every ξ and continuous in ξ for a.e. x; moreover, $\mathbf{a}(x, \xi)$ is differentiable in ξ away from the origin. In addition, there exist $2 - 1/n < p \leq n$ and $\alpha, \beta > 0$ so that

$$|\mathbf{a}(x, \xi)| \leq \beta |\xi|^{p-1} \tag{1.2}$$

and

$$\langle \mathbf{a}_\xi(x, \xi)\lambda, \lambda \rangle \geq \alpha |\xi|^{p-2} |\lambda|^2, \quad |\mathbf{a}_\xi(x, \xi)| \leq \beta |\xi|^{p-2} \tag{1.3}$$

for every $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \backslash \{(0, 0)\}$ and a.e. $x \in \mathbb{R}^n$.

It is well known that condition (1.2) implies that $\mathbf{a}(x, 0) = 0$ for a.e. $x \in \mathbb{R}^n$ and condition (1.3) implies that

$$\langle \mathbf{a}(x, \xi) - \mathbf{a}(x, \eta), \xi - \eta \rangle \geq c(\alpha, p)(|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2$$

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for every $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}$ and a.e. $x \in \mathbb{R}^n$. See for example [32].

Note that the standard example of such a nonlinearity $\mathbf{a}(x,\xi)$ satisfying these conditions is the standard p-Laplacian

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}u)$$

with respect to $\mathbf{a}(x,\xi) = |\xi|^{p-2}\xi$.

We now set

$$\Theta(\mathbf{a}, B_r(y)(x) = \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|\mathbf{a}(x,\xi) - \bar{\mathbf{a}}_{B_r(y)})(\xi)|}{|\xi|^{p-1}},$$

where

$$\bar{\mathbf{a}}_{B_r(y)}(\xi) = \fint_{B_r(y)} \mathbf{a}(x,\xi)\, dx.$$

In this paper, we assume that the nonlinearity $\mathbf{a}(x,\xi)$ satisfies the following small BMO semi-norm condition with respect to x.

Definition 1.1. Let $\delta, R > 0$. The nonlinearity $\mathbf{a}$ is said to be (δ, R) small if the following holds true:

$$\sup_{y \in \mathbb{R}^n, 0 < r \leq R} \fint_{B_r(y)} \Theta(\mathbf{a}, B_r(y))(x)^2\, dx \leq \delta^2. \tag{1.4}$$

The small BMO semi-norm condition (1.4) was used in [9, 10, 28]. Note that the nonlinearity $\mathbf{a}(x,\xi)$, which satisfies (1.4), may be merely measurable and discontinuous in x.

In what follows, for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $r > 0$ we denote

$$B_r(x) := \{y \in \mathbb{R}^n : |x - y| < r\}, \quad B_r^+(x) = B_r(x) \cap \{y : y_n > x_n\} \quad \text{and} \quad \Omega_r(x) = \Omega \cap B_r(x).$$

Definition 1.2. Let $\delta, R > 0$. The domain Ω is said to be a (δ, R) Reifenberg flat domain if for every $x \in \partial\Omega$ and $0 < r \leq R$ there exists a coordinate system depending on x and r, whose variables are denoted by $y = (y_1, \dots, y_n)$ such that in this new coordinate system x is the origin and

$$B_r(0) \cap \{y : y_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{y : y_n > -\delta r\}. \tag{1.5}$$

Condition (1.5) is known as the δ-Reifenberg flatness condition which was first introduced in [29]. This condition does not require any smoothness on the boundary of Ω, but is sufficiently flat in the Reifenberg sense. The Reifenberg flat domain includes domains with rough boundaries of fractal nature (see for example [33]), and Lipschitz domains with small Lipschitz constants (see for example [27]). See for example [16, 27, 29]. For our purpose, we always assume that δ is a small constant, and in this paper we always assume $\delta \in (0, 1/2)$.

We define

$$\mathbf{M}_1(\mu)(x) = \sup_{r>0} \frac{|\mu|(B_r(x))}{r^{n-1}}$$

to be the first order fractional function associated to the measure μ. The following gradient estimates for the renormalized solution to problem (1.1) was proved in [28, Theorem 1.4] by adapting the approximation method in [11].

Theorem 1.3. *Let $p \in (2 - 1/n, n]$ and $1 < q < \infty$. Then there exists $\delta = \delta(n, \alpha, \beta, p, q)$ so that if $\mathbf{a}$ is (δ, R_0) small and Ω is a (δ, R_0) Reifenberg flat domain for some $R_0 > 0$, then any renormalized solution u to problem* (1.1) *satisfies the following estimate:*

$$\|\nabla u\|_{L^q(\Omega)} \leq C \|\mathbf{M}_1(\mu)^{\frac{1}{p-1}}\|_{L^q(\Omega)}.$$

Here C depends only on n, p, α, β, q and $\operatorname{diam}\Omega / R_0$.

In Theorem 1.3 we used the notion of renormalized solutions to problem (1.1). For its precise definition, we refer the reader to Section 2.2. Note that the approach of getting estimates for the gradient via fractional operators is not new. This has been introduced in [24, 25]. It is important to stress that the interior version of Theorem 1.3 was obtained in [25].

It is not difficult to see that for a nonnegative locally finite measure ν in $\mathbb{R}^n$ we have

$$\mathbf{M}_1(\nu)(x) \le c_n \mathbf{I}_1(\nu)(x) := c_n \int_{\mathbb{R}^n} \frac{d\nu(y)}{|x-y|^{n-1}}, \quad x \in \mathbb{R}^n.$$

Hence, the estimate in Theorem 1.3 can be viewed as an integral potential bound for gradients of renormalized solutions to problem (1.1). This finding is in line with the result obtained by Wheeden–Muckenhoupt in [26].

The main aim of this paper is to extend the estimate in Theorem 1.3 to generalized Lebesgue spaces $L^{q(\cdot)}(\Omega)$. We first note that the gradient estimates of the solution to certain equations in PDEs on Lebesgue spaces have been widely studied; meanwhile the estimates on the generalized Lebesgue spaces $L^{q(\cdot)}(\Omega)$ have been interesting in their own ways and have been an interesting topic in PDE theory. We now carry out some results in this research direction.

(i) In [18], the authors investigated Calderón–Zygmund theory in the scale of the variable Lebesgue space $L^{p(\cdot)}(\Omega)$ for the first order divergence form problem

$$\begin{cases} \operatorname{div} u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It was also proved in [18] that

$$\||\nabla u|\|_{L^{p(\cdot)}(\Omega)} \le C\|f\|_{L^{p(\cdot)}(\Omega)},$$

provided that $p(\cdot) \in LH$. See Section 2 for the definition of the class LH.

(ii) The $W^{2,p(\cdot)}$ regularity estimates for the Poisson problem in nondivergence form

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

was obtained in [17]. They proved that

$$\||\nabla^2 u|\|_{L^{p(\cdot)}(\Omega)} \le C\|f\|_{L^{p(\cdot)}(\Omega)}$$

holds true for $p(\cdot) \in LH$.

(iii) Recently, the regularity estimates for solutions of elliptic equations in divergence form and nondivergence form were obtained in [8] and [7], respectively.

In this paper, we prove the gradient estimates concerning the measure data problems (1.1) on the variable Lebesgue spaces. More precisely, we prove the following result.

Theorem 1.4. *Let $p \in (2-1/n, n]$ and let $q(\cdot) \in LH$. Then there exists $\delta = \delta(n, \alpha, \beta, p, q(\cdot))$ so that if* **a** *is (δ, R_0) small and Ω is a (δ, R_0) Reifenberg flat domain for some $R_0 > 0$, then for any renormalized solution u to problem* (1.1) *the following estimate holds true:*

$$\|\nabla u\|_{L^{q(\cdot)}(\Omega)} \le C\|\mathbf{M}_1(\mu)^{\frac{1}{p-1}}\|_{L^{q(\cdot)}(\Omega)}.$$

We remark that the maximal function approach in [17, 18] relying on the Calderón–Zygmund theory may not be applicable to our setting for two reasons. Firstly, there is no regularity condition in x which is imposed on the nonlinearity $\mathbf{a}(x,\xi)$. Secondly, without the smoothness condition, the underlying domain Ω may be beyond the Lipschitz category. To overcome this problem, we adapt the technique in [1] to our setting which avoids the use of the maximal functions. Note that this method was also used in [7, 8].

The organization of this paper is as follows: In Section 2, we recall the notion and basic properties of variable Lebesgue spaces. The notion of renormalized solutions to problem (1.1) and approximation results for the solutions to problem (1.1) are also addressed in this section. Section 3 is devoted to give the proof of Theorem 1.4.

Throughout the paper, we always use C and c to denote positive constants that are independent of the main parameters involved but whose values may differ from line to line. We will write $A \lesssim B$ if there is a universal constant C so that $A \le CB$ and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

2 Preliminaries

We will begin with some notations which will be used frequently in the sequel.

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by

$$B_r(x) := \{y \in \mathbb{R}^n : |x - y| < r\}$$

the open ball centered x with radius r in $\mathbb{R}^n$. We also denote $\Omega_r(x) = \Omega \cap B_r(x)$.

For a measurable function f on a measurable subset E (E can be a measurable subset of $\mathbb{R}^n$) we define

$$\bar{f}_E = \fint_E f = \frac{1}{|E|} \int_E f.$$

2.1 Generalized Lebesgue spaces

We now recall some definitions and basic properties concerning the variable Lebesgue spaces in [13]. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $p(\cdot) : \Omega \to [1, \infty)$ be a measurable function. We now define the variable Lebesgue spaces $L^{p(\cdot)}(\Omega)$ to be a generalization of the classical Lebesgue spaces consisting of all measurable functions on Ω satisfying

$$\int_\Omega |f(x)|^{p(x)}\, dx < \infty.$$

Then $L^{p(\cdot)}(\Omega)$ is a Banach function space with the norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\Big\{\lambda > 0 : \int_\Omega \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)} \leq 1\Big\}.$$

It is well known that

$$\|f\|_{L^{p(\cdot)}(\Omega)} \leq 1 \iff \int_\Omega |f(x)|^{p(x)}\, dx \leq 1.$$

The variable Lebesgue spaces have attracted considerable interest since the early 1990s. These spaces have an important role in the theory of function spaces and have a wide range of applications to the calculus of variations and PDEs. For a thorough discussion of these spaces and their history see for example [13, 17].

We say that $p(\cdot) \in LH$ if

$$1 < \inf_{x\in\Omega} p(x) \leq \sup_{x\in\Omega} p(x) < \infty$$

and there exists a constant C_0 such that

$$|p(x) - p(y)| \leq \frac{C_0}{-\log|x - y|} =: \gamma(|x - y|), \quad |x - y| \leq \frac{1}{2}. \tag{2.1}$$

Condition (2.1) is known as the log-Hölder continuity condition. The class of $p(\cdot)$ satisfying the log-Hölder continuity condition plays an important role in harmonic analysis and PDE theory. This class was used to deal with the boundedness of singular integrals in Calderón-Zygmund theory and the generalized Lebesgue–Sobolev space for regularity results for elliptic and parabolic problems. See for example [13, 17, 18] and the references therein.

2.2 The notion of renormalized solutions

In this section, we recall the notion of renormalized solutions to problem (1.1) and its equivalent characterizations. This part is taken from [3, 6, 15, 28].

Let $\mathcal{M}_B(\Omega)$ be the set of all signed measures in Ω with bounded totally variations. Denote by $\mathcal{M}_0(\Omega)$ (resp. $\mathcal{M}_s(\Omega)$) the subset of $\mathcal{M}_b(\Omega)$ consisting of measures which are absolute continuous (resp. singular) with respect to the capacity map $\mathrm{cap}_p(\cdot,\Omega)$ defined by

$$\mathrm{cap}_p(K,\Omega)=\inf\Big\{\int_\Omega |\nabla\phi|^p\,dx : \phi\in C_0^\infty(\Omega),\ \phi\ge 1 \text{ on } K\Big\}$$

for any compact subset $K\subset\Omega$.

It was proved in [21, Lemma 2.1] that for every $\mu\in\mathcal{M}_b(\Omega)$ there exist $\mu_0\in\mathcal{M}_0(\Omega)$ and $\mu_s\in\mathcal{M}_s(\Omega)$ so that $\mu=\mu_0+\mu_s$.

The sequence of measures $\{\mu_k\}$ in $\mathcal{M}_b(\Omega)$ is said to converge in the narrow topology to $\mu\in\mathcal{M}_b(\Omega)$ if

$$\lim_{k\to\infty}\int_\Omega \varphi\,d\mu_k=\int_\Omega\varphi\,d\mu$$

for every bounded and continuous function φ on Ω.

In what follows, for any $\mu\in\mathcal{M}_b(\Omega)$, its positive part and negative part are denoted by μ^+ and μ^-. The notion of renormalized solutions is a generalization of that of entropy solutions for nonlinear elliptic equations and was introduced in [3, 6] where the measures are assumed to be in $L^1(\Omega)$ or in $\mathcal{M}_b(\Omega)$. We now recall two equivalent definitions of renormalized solutions in [15].

Definition 2.1. Suppose $\mu\in\mathcal{M}_b(\Omega)$. Then u is said to be a renormalized solution to (1.1) if the following conditions hold:
(i) The function u is measurable and finite almost everywhere, and $T_k(u)=\max\{-k,\min\{k,u\}\}$ belongs to $W_0^{1,p}(\Omega)$ for every $k>0$.
(ii) The gradient ∇u satisfies $|\nabla u|^{p-1}\in L^q(\Omega)$ for all $q<\frac{n}{n-1}$.
(iii) If $w\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and if there exist $w^{+\infty}, w^{-\infty}\in W^{1,r}(\Omega)\cap L^\infty(\Omega)$ with $r>n$ such that

$$\begin{cases} w=w^{+\infty} & \text{a.e. on the set } \{u>k\},\\ w=w^{-\infty} & \text{a.e. on the set } \{u<-k\}\end{cases}$$

for some $k>0$, then

$$\int_\Omega \mathbf{a}(x,\nabla u)\cdot\nabla w\,dx=\int_\Omega w\,d\mu_0+\int_\Omega w^{+\infty}\,d\mu_s^+-\int_\Omega w^{-\infty}\,d\mu_s^-.$$

Definition 2.2. Suppose $\mu\in\mathcal{M}_b(\Omega)$. Then u is said to be a renormalized solution to (1.1) if the following conditions hold:
(i) For every $k>0$ there exist two nonnegative measures $\lambda_k^+,\lambda_k^-\in\mathcal{M}_0(\Omega)$ concentrated on the sets $\{u=k\}$ and $\{u=-k\}$, respectively, so that $\lambda_k^+\to\mu_s^+$ and $\lambda_k^-\to\mu_s^-$ in the narrow topology of measures.
(ii) For every $k>0$ we have

$$\int_{\{|u|<k\}}\mathbf{a}(x,\nabla u)\cdot\nabla\varphi\,dx=\int_{\{|u|<k\}}\varphi\,d\mu_0+\int_\Omega\varphi\,d\lambda_k^+-\int_\Omega\varphi\,d\lambda_k^- \tag{2.2}$$

for every $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Remark 2.3 ([15]). (i) Note that if u is a renormalized solution to (1.1), then u is finite everywhere with respect to $\mathrm{cap}_p(\cdot,\Omega)$. Therefore, u is finite μ_0-almost everywhere.
(ii) By (ii) of Definition 2.2, if u is a renormalized solution to (1.1), then (2.2) is equivalent to

$$-\mathrm{div}\,\mathbf{a}(x,\nabla T_k(u))=\mu_k \quad \text{in } \mathcal{D}'(\Omega)$$

with

$$\mu_k=\chi_{\{|u|<k\}}\mu_0+\lambda_k^+-\lambda_k^-.$$

It was proved that $T_k(u) \in W_0^{1,p}(\Omega)$ and $|\mu_k|$ converges to $|\mu|$ in the narrow topology of measures as well. As a consequence, if u is a renormalized solution to (1.1) then u is a solution to (1.1). A solution u to problem (1.1) is understood in the following sense in [28]: For each $k > 0$ the truncation

$$T_k(u) = \max\{-k, \min\{k, u\}\} \in W_0^{1,p}(\Omega)$$

and satisfies

$$-\operatorname{div} \mathbf{a}(x, \nabla T_k(u)) = \mu_k$$

in the sense of distributions in Ω for a finite measure μ_k in Ω. Moreover, if we extend both μ and μ_k by zero to $\mathbb{R}^n \backslash \Omega$ then μ_k^+ and μ_k^- converges to μ^+ and μ_k^- weakly as measures in $\mathbb{R}^n$, respectively. Note that for the existence and the uniqueness for the solution and the renormalized solution to the measure data problem (1.1), we refer to [15]. Alternatively, the notion of solutions obtained by limit of approximations (SOLA) can be employed in this situation. See for example [4, 5, 14, 19, 20].

2.3 Approximation results for the solutions to measure data problems

We first recall the approximation results related to problem (1.1). The first result (Lemma 2.4) gives an interior approximation result for weak solutions to problem (1.1), whereas the latter (Lemma 2.5) gives a boundary approximation result for weak solutions to problem (1.1). We refer the reader to [20] (see also [19, 28]) for the proof of Lemma 2.4, and [28, Corollary 2.13] for the proof of Lemma 2.5. Set

$$F(\mu, u, B_r(x)) = \Big[\frac{|\mu|(B_r(x))}{r^{n-1}}\Big]^{\frac{1}{p-1}} + \Big[\frac{|\mu|(B_r(x))}{r^{n-1}}\Big]\Big(\fint_{B_r(x)} |\nabla u|\Big)^{2-p} \chi_{\{p<2\}}.$$

Lemma 2.4. *For any $\varepsilon > 0$ there exists $\delta > 0$ so that the following holds: If $\mathbf{a}$ is (δ, R_0) small and Ω is a (δ, R_0) Reifenberg flat domain for some $R_0 > 0$, and $u \in W_0^{1,p}(\Omega)$ is a weak solution to (1.1) in $B_{2R} \equiv B_{2R}(x_0) \subset \Omega$ with $2R \le R_0$, then there exist a function $v \in W^{1,p}(B_R) \cap W^{1,\infty}(B_{R/2})$ and $C = C(n, p, \alpha, \beta)$ so that*

$$\|\nabla v\|_{L^\infty(B_{R/2})} \le C \fint_{B_{2R}} |\nabla u|\, dx + CF(\mu, u, B_{2R}(x_0)),$$

and

$$\fint_{B_R} |\nabla(u - v)| \le CF(\mu, u, B_{2R}(x_0)) + \varepsilon \fint_{B_{2R}} |\nabla u|\, dx.$$

Lemma 2.5. *For any $\varepsilon > 0$ there exists $\delta > 0$ so that the following holds: If $\mathbf{a}$ is (δ, R_0) small and Ω is a (δ, R_0) Reifenberg flat domain for some $R_0 > 0$, and $u \in W_0^{1,p}(\Omega)$ is a weak solution to (1.1) with $x_0 \in \partial\Omega$ and $0 < 10R < R_0$, then there exist a function $v \in W^{1,\infty}(B_{R/10}(x_0))$ and $C = C(n, p, \alpha, \beta)$ so that*

$$\|\nabla v\|_{L^\infty(B_{R/10}(x_0))} \le C \fint_{\Omega_{10R}(x_0)} |\nabla u|\, dx + CF(\mu, u, B_{10R}(x_0)),$$

and

$$\fint_{\Omega_R(x_0)} |\nabla(u - v)| \le CF(\mu, u, B_{10R}(x_0)) + \varepsilon \fint_{\Omega_{10R}(x_0)} |\nabla u|\, dx.$$

3 Proof of Theorem 1.4

We first note that for $x \in \Omega$ and $0 < r \le R_0$, from (1.5), we have

$$\frac{|B_r(x)|}{|\Omega_r(x)|} \le \Big(\frac{2}{1-\delta}\Big)^n \le 4^n. \tag{3.1}$$

This simple observation will be used frequently.

To prove Theorem 1.4, it suffices to prove that for any renormalized solution u to problem (1.1), we have

$$\int_\Omega |\nabla u|^{q(x)}\,dx \le c(n,\alpha,\beta,q(\cdot),\Omega,\delta) \tag{3.2}$$

uniformly under the assumption

$$\int_\Omega \mathbf{M}_1(\mu)(x)^{\frac{q(x)}{p-1}}\,dx \le 1. \tag{3.3}$$

Indeed, once (3.2) is proved, by using the rescaled maps

$$\tilde{\mathbf{a}} = \frac{\mathbf{a}(x, r_0\xi)}{r_0^{p-1}}, \quad \tilde{u} = \frac{u(x)}{r_0} \quad \text{and} \quad \tilde{\mu} = \frac{\mu}{r_0^{p-1}},$$

where $r_0 = \|\mathbf{M}_1(\mu)^{\frac{1}{p-1}}\|_{L^{q(\cdot)}(\Omega)}$, we get that $\tilde{u}$ is a renormalized solution to the problem

$$\begin{cases} -\operatorname{div}\tilde{\mathbf{a}}(x,\nabla\tilde{u}) = \tilde{\mu} & \text{in } \Omega,\\ \tilde{u} = 0 & \text{on } \partial\Omega.\end{cases}$$

Note that under these rescaled maps $\tilde{\mathbf{a}}$ is (δ, R_0) small and Ω is a (δ, R_0) Reifenberg flat domain. Therefore, from (3.2) we get that

$$\int_\Omega |\nabla\tilde{u}|^{q(x)}\,dx \le c(n,\alpha,\beta,q(\cdot),\Omega,\delta),$$

which implies that

$$\|\nabla u\|_{L^{q(\cdot)}(\Omega)} \le C\|\mathbf{M}_1(\mu)^{\frac{1}{p-1}}\|_{L^{q(\cdot)}(\Omega)}.$$

We now prove (3.2). Let τ be a positive number so that $1 < \tau < q_1 < q(x) < q_2 < \infty$ for all $x \in \Omega$, where

$$q_1 := \inf_{x\in\Omega} q(x) \quad \text{and} \quad q_2 := \sup_{x\in\Omega} q(x).$$

Then from (3.3) we have

$$\int_\Omega \mathbf{M}_1(\mu)(x)^{\frac{\tau}{p-1}}\,dx + \int_\Omega \mathbf{M}_1(\mu)(x)^{\frac{q_1}{p-1}}\,dx \le 2\int_\Omega \big[\mathbf{M}_1(\mu)(x)^{\frac{q(x)}{p-1}} + 1\big]\,dx \le 2(1+|\Omega|). \tag{3.4}$$

On the other hand, Theorem 1.3 tells us that there exists δ_0 so that if $\mathbf{a}$ is (δ, R_0) small and Ω is a (δ, R_0) Reifenberg flat domain for some $R_0 > 0$, then for any renormalized solution u to problem (1.1) the following estimates hold:

$$\|\nabla u\|_{L^{q_1}(\Omega)} \le C\|\mathbf{M}_1(\mu)^{\frac{1}{p-1}}\|_{L^{q_1}(\Omega)} \tag{3.5}$$

and

$$\|\nabla u\|_{L^{\tau}(\Omega)} \le C\|\mathbf{M}_1(\mu)^{\frac{1}{p-1}}\|_{L^{\tau}(\Omega)}. \tag{3.6}$$

Hence, from (3.4), (3.5) and (3.6) we have

$$\int_\Omega |\nabla u|^\tau\,dx + \int_\Omega |\nabla u|^{q_1}\,dx \le C_1\bigg[\int_\Omega \mathbf{M}_1(\mu)(x)^{\frac{\tau}{p-1}}\,dx + \int_\Omega \mathbf{M}_1(\mu)(x)^{\frac{q_1}{p-1}}\,dx\bigg] \le 2C_1(1+|\Omega|).$$

Let $x_0 \in \overline{\Omega}$ and $R > 0$. We set

$$1 < \tau < q_1 \le q^- = \inf_{x\in\Omega_{2R}(x_0)} q(x) \le q^+ = \sup_{x\in\Omega_{2R}(x_0)} q(x) \le q_2 < \infty.$$

We will choose the constant R as follows: Since $\lim_{r\to 0}\gamma(4r) = 0$ and $\gamma(\cdot)$ is increasing on (0,1/2) with γ defined in (2.1), there exists R_γ so that

$$\frac{\tau(q^- + \gamma(4R_\gamma))}{q^-} < q_1.$$

In the rest of this paper, we assume that $0 < R < \min\{R_0, R_\gamma, 1/10\}$. Hence,

$$\frac{\tau(q^- + \gamma(4R))}{q^-} < q_1.$$

Moreover, from (2.1), for $x \in \Omega_{2R}(x_0)$ we have

$$q(x) - q^- \le \gamma(4R).$$

This implies that for $x \in \Omega_{2R}(x_0)$ we have

$$\frac{\tau q(x)}{q^-} \le \frac{\tau(q^- + \gamma(4R))}{q^-} \le q_1. \tag{3.7}$$

We now define

$$\Lambda_\delta = \fint_{\Omega_{2R}(x_0)} \Big\{|\nabla u(x)|^{\frac{\tau q(x)}{q^-}} + \frac{1}{\delta}\big[\mathbf{M}_1(\mu)(x)^{\frac{1}{p-1}}\big]^{\frac{\tau q(x)}{q^-}}\Big\}\,dx + \frac{1}{\delta}.$$

Note that Λ_δ is well-defined since $\frac{\tau q(x)}{q^-} \le q_1$ for $x \in \Omega_{2R}(x_0)$. Fix $1 \le s_1 \le s_2 \le 2$. For $\lambda > 0$ we define the level set

$$E(\lambda) = \{x \in \Omega_{s_1 R}(x_0) : |\nabla u(x)|^{\frac{\tau q(x)}{q^-}} > \lambda\}.$$

For $y \in E(\lambda)$ and $r > 0$ we define

$$\Phi_y(r) = \fint_{\Omega_r(y)} \Big\{|\nabla u(x)|^{\frac{\tau q(x)}{q^-}} + \frac{1}{\delta}\big[\mathbf{M}_1(\mu)(x)^{\frac{1}{p-1}}\big]^{\frac{\tau q(x)}{q^-}}\Big\}\,dx.$$

Hence, for $0 < r < (s_2 - s_1)R$, by (3.1) we have

$$\begin{aligned}\Phi_y(r) &= \frac{|\Omega_{2R}(x_0)|}{|\Omega_r(y)|} \fint_{\Omega_{2R}(x_0)} \Big\{|\nabla u(x)|^{\frac{\tau q(x)}{q^-}} + \frac{1}{\delta}\big[\mathbf{M}_1(\mu)(x)^{\frac{1}{p-1}}\big]^{\frac{\tau q(x)}{q^-}}\Big\}\,dx \\ &\le \frac{|B_{2R}(x_0)|}{|B_r(y)|}\frac{|B_r(x_0)|}{|\Omega_r(y)|}\Lambda_\delta \le 4^n\Big(\frac{2R}{r}\Big)^n \Lambda_\delta.\end{aligned} \tag{3.8}$$

This implies that

$$\Phi_y(r) \le \Big(\frac{8 \times 10^6}{s_2 - s_1}\Big)^n \Lambda_\delta,$$

as long as

$$\frac{(s_2 - s_1)R}{10^6} < r < (s_2 - s_1)R.$$

We now fix

$$\lambda > \Big(\frac{8 \times 10^6}{s_2 - s_1}\Big)^n \Lambda_\delta.$$

Then from (3.8) we can conclude that for all $y \in E(\lambda)$ and

$$\frac{(s_2 - s_1)R}{10^6} < r < (s_2 - s_1)R$$

we have

$$\Phi_y(r) < \lambda. \tag{3.9}$$

On the other hand, by the Lebesgue differentiation theorem, we also get, for $y \in E(\lambda)$, that

$$\lim_{r \to 0^+} \Phi_y(r) > \lambda.$$

This along with (3.9) implies that for $y \in E(\lambda)$ there exists $0 < r_y \le \frac{(s_2 - s_1)R}{10^6}$ so that

$$\Phi_y(r_y) = \lambda \quad \text{and} \quad \Phi_y(r) < \lambda \quad \text{for all } r \in (r_y, (s_2 - s_1)R).$$

We now apply the Besicovitch covering theorem to pick up the finite overlap family $\{\Omega_{r_i}(y_i)\}_{i=1}^{\infty}$ with $y_i \in E(\lambda)$ and $0 < r_i < \frac{(s_2-s_1)R}{10^6}$ so that

$$E(\lambda) \subset \bigcup_i \Omega_{5r_i}(y_i), \qquad \sum_i \chi_{\Omega_{5r_i}(y_i)} \le \kappa := \kappa(n),$$

and

$$\Phi_{y_i}(r_i) = \lambda \quad \text{and} \quad \Phi_{y_i}(r) < \lambda \quad \text{for all } r \in (r_i, (s_2 - s_1)R). \tag{3.10}$$

For $t > 0$ we define

$$F_{\mathbf{M}_1(\mu)}(t) = \Big\{x : [\mathbf{M}_1(\mu)(x)^{\frac{1}{p-1}}]^{\frac{\tau q(x)}{q^-}} > t\Big\}$$

and

$$F_u(t) = \Big\{x : |\nabla u|^{\frac{\tau q(x)}{q^-}} > t\Big\}.$$

Then, from (3.10), we obtain

$$\begin{aligned}|\Omega_{r_i}(y_i)| &= \frac{1}{\lambda} \int\limits_{\Omega_{r_i}(y_i)} \Big\{|\nabla u(x)|^{\frac{\tau q(x)}{q^-}} + \frac{1}{\delta}[\mathbf{M}_1(\mu)(x)^{\frac{1}{p-1}}]^{\frac{\tau q(x)}{q^-}}\Big\}\, dx \\ &\le \frac{1}{\lambda}\Big\{ \int\limits_{\Omega_{r_i}(y_i)\cap F_u(\lambda/4)} |\nabla u(x)|^{\frac{\tau q(x)}{q^-}}\, dx + \int\limits_{\Omega_{r_i}(y_i)\cap F_{\mathbf{M}_1(\mu)}(\delta\lambda/4)} \frac{1}{\delta}[\mathbf{M}_1(\mu)(x)^{\frac{1}{p-1}}]^{\frac{\tau q(x)}{q^-}}\, dx\Big\} + \frac{|\Omega_{r_i}(y_i)|}{2},\end{aligned}$$

which implies

$$|\Omega_{r_i}(y_i)| \le \frac{2}{\lambda}\Big\{ \int\limits_{\Omega_{r_i}(y_i)\cap F_u(\lambda/4)} |\nabla u(x)|^{\frac{\tau q(x)}{q^-}}\, dx + \int\limits_{\Omega_{r_i}(y_i)\cap F_{\mathbf{M}_1(\mu)}(\delta\lambda/4)} \frac{1}{\delta}[\mathbf{M}_1(\mu)(x)^{\frac{1}{p-1}}]^{\frac{\tau q(x)}{q^-}}\, dx\Big\}. \tag{3.11}$$

We now estimate the level set $E(\lambda)$ by making use of Lemma 2.4 and Lemma 2.5. However, we can not apply these two approximation results directly, since a renormalized solution is not a weak solution. To do this, from the definition of the renormalized solution we observe that for each $k > 0$, we have that $u_k = T_k(u) \in W_0^{1,p}(\Omega)$ is a weak solution to the problem

$$-\mathrm{div}\, \mathbf{a}(x, \nabla u_k) = \mu_k$$

for a finite measure μ_k in Ω. Moreover, if we extend both μ and μ_k by zero to $\mathbb{R}^n\backslash\Omega$, then μ_k^+ and μ_k^- converge to μ^+ and μ^- weakly as measures in $\mathbb{R}^n$, respectively. This implies that

$$\lim_{k\to\infty} |\mu_k|(B) \le |\mu|(\overline{B})$$

for all balls $B \subset \mathbb{R}^n$. Hence,

$$\limsup_{k\to\infty} \mathbf{M}_1(\mu_k)(x) \le c(n)\mathbf{M}_1(\mu)(x), \quad x \in \mathbb{R}^n.$$

Without loss of generality we may assume that for every $k > 0$ we have

$$\mathbf{M}_1(\mu_k)(x) \le c(n)\mathbf{M}_1(\mu)(x), \quad x \in \mathbb{R}^n.$$

Since $\mathbf{a}$ is (δ, R_0) small and Ω is a (δ, R_0) Reifenberg flat domain for some $R_0 > 0$ and some $\delta \le \delta_0$, from Theorem 1.3, for any $k > 0$ we have

$$\|\nabla u_k\|_{L^{q_1}(\Omega)} \le C\|\mathbf{M}_1(\mu_k)^{\frac{1}{p-1}}\|_{L^{q_1}(\Omega)} \le C\|\mathbf{M}_1(\mu)^{\frac{1}{p-1}}\|_{L^{q_1}(\Omega)}$$

and

$$\|\nabla u_k\|_{L^{\tau}(\Omega)} \le C\|\mathbf{M}_1(\mu_k)^{\frac{1}{p-1}}\|_{L^{\tau}(\Omega)} \le C\|\mathbf{M}_1(\mu)^{\frac{1}{p-1}}\|_{L^{\tau}(\Omega)}.$$

Hence,

$$\int\limits_{\Omega} |\nabla u_k|^{\tau}\, dx + \int\limits_{\Omega} |\nabla u_k|^{q_1}\, dx \le C_2\Big[\int\limits_{\Omega} \mathbf{M}_1(\mu)(x)^{\frac{\tau}{p-1}}\, dx + \int\limits_{\Omega} \mathbf{M}_1(\mu)(x)^{\frac{q_1}{p-1}}\, dx\Big] \le 2C_2(1 + |\Omega|). \tag{3.12}$$

For $y \in E(\lambda)$ and $r > 0$ we define

$$\Phi_y^k(r) = \fint_{\Omega_r(y)} \Big\{ |\nabla u_k(x)|^{\frac{\tau q(x)}{q^-}} + \frac{1}{\delta}\big[\mathbf{M}_1(\mu_k)(x)^{\frac{1}{p-1}}\big]^{\frac{\tau q(x)}{q^-}} \Big\}\, dx.$$

Since $\Phi_y^k(r) < \Phi_y(r)$, from (3.10) we have

$$\Phi_{y_i}^k(r) < \lambda \quad \text{for all } r \in (r_i, (s_2 - s_1)R). \tag{3.13}$$

We now prove the following approximation results.

Lemma 3.1. *If $B_{20r_i}(y_i) \subset \Omega$, then for any $\varepsilon > 0$ there exists $\delta > 0$ so that for $k > 0$ there is $v_{k,i}$ satisfying the following:*

$$\|v_{k,i}\|_{L^\infty(B_{5r_i}(y_i))} \le c_1 \lambda^{\frac{q^-}{\tau q_i^+}}$$

and

$$\fint_{B_{10r_i}(y_i)} |\nabla(u_k - v_{k,i})| \le \varepsilon \lambda^{\frac{q^-}{\tau q_i^+}},$$

where $q_i^+ = \sup_{x\in B_{20r_i}(y_i)} q(x)$.

Proof. Set $q_i^- = \inf_{x\in B_{20r_i}(y_i)} q(x)$ and $q_i^+ = \sup_{x\in B_{20r_i}(y_i)} q(x)$. We then have from (2.1) that

$$q_i^+ - q_i^- \le \gamma(40r_i).$$

Observe that

$$\begin{aligned}
\fint_{B_{20r_i}(y_i)} |\nabla u_k|^\tau\, dx &= \Big(\fint_{B_{20r_i}(y_i)} |\nabla u_k|^\tau\, dx\Big)^{\frac{q_i^-}{q_i^+}} \Big(\fint_{B_{20r_i}(y_i)} |\nabla u_k|^\tau\, dx\Big)^{1-\frac{q_i^-}{q_i^+}} \\
&\le \Big(\fint_{B_{20r_i}(y_i)} |\nabla u_k|^{\frac{\tau q_i^-}{q^-}}\, dx\Big)^{\frac{q^-}{q_i^+}} \Big(\fint_{B_{20r_i}(y_i)} |\nabla u_k|^\tau\, dx\Big)^{1-\frac{q_i^-}{q_i^+}} \\
&\le \Big[\Big(\fint_{B_{20r_i}(y_i)} |\nabla u_k|^{\frac{\tau q(x)}{q^-}}\, dx\Big)^{\frac{q^-}{q_i^+}} + 1\Big]\Big(\fint_{B_{20r_i}(y_i)} |\nabla u_k|^\tau\, dx\Big)^{1-\frac{q_i^-}{q_i^+}},
\end{aligned}$$

which along with (3.13) yields that

$$\fint_{B_{20r_i}(y_i)} |\nabla u_k|^\tau\, dx \le \big(\lambda^{\frac{q^-}{q_i^+}} + 1\big)\Big(\fint_{B_{20r_i}(y_i)} |\nabla u_k|^\tau\, dx\Big)^{1-\frac{q_i^-}{q_i^+}} \le 2\lambda^{\frac{q^-}{q_i^+}}\Big(\fint_{B_{20r_i}(y_i)} |\nabla u_k|^\tau\, dx\Big)^{1-\frac{q_i^-}{q_i^+}}.$$

On the other hand, we have

$$\Big(\fint_{B_{20r_i}(y_i)} |\nabla u_k|^\tau\, dx\Big)^{q_i^+ - q_i^-} \lesssim \Big(\frac{1}{r_i^n}\Big)^{\gamma(40r_i)} \Big(\int_{B_{20r_i}(y_i)} |\nabla u_k|^\tau\, dx\Big)^{q_i^+ - q_i^-}.$$

Note that

$$\Big(\frac{1}{r_i^n}\Big)^{\gamma(40r_i)} = 40^{n\gamma(40r_i)}\Big(\frac{1}{40r_i}\Big)^{nC_0 \log_{\frac{1}{40r_i}} e} \le 40^{n\gamma(R)} e^{nC_0} \le 40^{n\gamma(1/2)} e^{nC_0}.$$

Hence,

$$\Big(\fint_{B_{20r_i}(y_i)} |\nabla u_k|^\tau\, dx\Big)^{q_i^+ - q_i^-} \lesssim \Big(\int_{B_{20r_i}(y_i)} |\nabla u_k|^\tau\, dx\Big)^{q_i^+ - q_i^-} \lesssim (1 + |\Omega|)^{\gamma(40r_i)},$$

where we used (3.12).

Since $40r_i \le \frac{2}{5}R \le R_0$, we further obtain

$$\Big(\fint_{B_{20r_i}(y_i)} |\nabla u_k|^\tau\,dx\Big)^{q_i^+-q_i^-} \lesssim (1+|\Omega|)^{\gamma(R)} \lesssim (1+|\Omega|)^{\gamma(1/2)}.$$

Hence,

$$\fint_{B_{20r_i}(y_i)} |\nabla u_k|^\tau\,dx \le c\lambda^{\frac{q^-}{q_i^+}}. \tag{3.14}$$

By a similar argument, we can prove that

$$\begin{aligned}\fint_{B_{20r_i}(y_i)} \mathbf{M}_1(\mu_k)(x)^{\frac{\tau}{p-1}}\,dx &\le \fint_{B_{20r_i}(y_i)} \mathbf{M}_1(\mu)(x)^{\frac{\tau}{p-1}}\,dx\\ &\le c\Big[\Big(\fint_{B_{20r_i}(y_i)} \mathbf{M}_1(\mu)(x)^{\frac{\tau q(x)}{q^-}}\,dx\Big)^{\frac{q^-}{q_i^+}}+1\Big]\\ &\le c\Big[(\delta\lambda)^{\frac{q^-}{q_i^+}}+1\Big]\\ &\le c(\delta\lambda)^{\frac{q^-}{q_i^+}}, \quad \text{since } \delta\lambda > \delta\Lambda_\delta \ge 1.\end{aligned} \tag{3.15}$$

Hence, for every $k > 0$, applying Lemma 2.4, we obtain that for any $\varepsilon' > 0$ there exists $0 < \delta \le \delta_0$ and a function

$$v_{k,i} \in W^{1,p}(B_{20r_i}(y_i)) \cap W^{1,\infty}(B_{20r_i}(y_i))$$

and $C = C(n,p,\alpha,\beta)$ so that

$$\|\nabla v_{k,i}\|_{L^\infty(B_{5r_i}(y_i))} \le C\fint_{B_{10r_i}(y_i)} |\nabla u_k|\,dx + CF(\mu_k, u_k, B_{10r_i}(y_i)) \tag{3.16}$$

and

$$\fint_{B_{10r_i}(y_i)} |\nabla(u_k - v_{k,i})| \le CF(\mu_k, u_k, B_{20r_i}(y_i)) + \varepsilon'\fint_{B_{20r_i}(y_i)} |\nabla u_k|\,dx. \tag{3.17}$$

Note that for $x \in B_{20r_i}(y_i)$ we have

$$\frac{|\mu_k|(B_{20r_i}(y_i))}{r_i^{n-1}} \le c(n)\frac{|\mu_k|(B_{40r_i}(x))}{r_i^{n-1}} \le c_1(n)\mathbf{M}_1(\mu_k)(x) \le c_2(n)\mathbf{M}_1(\mu)(x),$$

which implies

$$\frac{|\mu_k|(B_{20r_i}(y_i))}{r_i^{n-1}} \lesssim \Big[\fint_{B_{20r_i}(y_i)} \mathbf{M}_1(\mu)(x)^{\frac{\tau}{p-1}}\,dx\Big]^{\frac{p-1}{\tau}}.$$

This along with (3.14)–(3.17) yields that

$$\|\nabla v_{k,i}\|_{L^\infty(B_{5r_i}(y_i))} \le C\Big[\lambda^{\frac{q^-}{\tau q_i^+}} + (\delta\lambda)^{\frac{q^-}{\tau q_i^+}} + (\delta\lambda)^{\frac{q^-}{q_i^+}\frac{p-1}{\tau}}\lambda^{\frac{q^-}{q_i^+}\frac{2-p}{\tau}}\chi_{\{p<2\}}\Big]$$

and

$$\fint_{B_{10r_i}(y_i)} |\nabla(u_k - v_{k,i})| \le C\Big[\varepsilon'\lambda^{\frac{q^-}{\tau q_i^+}} + (\delta\lambda)^{\frac{q^-}{\tau q_i^+}} + (\delta\lambda)^{\frac{q^-}{q_i^+}\frac{p-1}{\tau}}\lambda^{\frac{q^-}{q_i^+}\frac{2-p}{\tau}}\chi_{\{p<2\}}\Big]. \qquad \square$$

Lemma 3.2. *If $B_{20r_i}(y_i) \cap \Omega^c \ne \emptyset$, then for any $\varepsilon > 0$ there exists $\delta > 0$ so that for every $k > 0$ there is $v_{k,i}$ satisfying the following:*

$$\|v_{k,i}\|_{L^\infty(B_{5r_i}(y_i))} \le c_2\lambda^{\frac{q^-}{\tau q_i^+}}$$

and

$$\fint_{\Omega_{50r_i}(y_i)} |\nabla(u_k - v_{k,i})| \le \varepsilon\lambda^{\frac{q^-}{\tau q_i^+}},$$

where $q_i^+ = \sup_{x\in B_{3000r_i}(y_i)} q(x)$.

Proof. The proof of this lemma is similar to that of Lemma 3.1 and hence we just sketch it here.

Set $q_i^- = \inf_{x\in\Omega_{3000r_i}(y_i)} q(x)$ and $q_i^+ = \sup_{x\in\Omega_{3000r_i}(y_i)} q(x)$. We then have

$$q_i^+ - q_i^- \le \gamma(6000r_i).$$

Let $x_0^i \in B_{20r_i}(y_i)\cap\partial\Omega$. Then we have

$$B_{5r_i}(y_i) \subset B_{25r_i}(x_0^i) \subset B_{50r_i}(y_i) \subset B_{250r_i}(x_i^0) \subset B_{2500r_i}(x_i^0) \subset B_{3000r_i}(y_i). \tag{3.18}$$

Similarly to Lemma 3.2 , we also obtain that

$$\fint_{\Omega_{3000r_i}(x_0^i)} |\nabla u_k|^\tau \, dx \le c\lambda^{\frac{q^-}{q_i^+}}$$

and

$$\fint_{\Omega_{3000r_i}(x_0^i)} \mathbf{M}_1(\mu_k)(x)^{\frac{\tau}{p-1}} \, dx \le c(\delta\lambda)^{\frac{q^-}{q_i^+}}.$$

At this stage, we use the argument in the proof of Lemma 3.2, which makes use of Lemma 2.5 instead of Lemma 2.4, to conclude that for any $\varepsilon' > 0$ there exists $\delta > 0$ and a function

$$v_{k,i} \in W^{1,p}(B_{25r_i}(x_0^i)) \cap W^{1,\infty}(B_{250r_i}(x_0^i))$$

and $C = C(n, p, \alpha, \beta)$ so that

$$\|\nabla v_{k,i}\|_{L^\infty(B_{25r_i}(x_0^i))} \le C\Big[\lambda^{\frac{q^-}{\tau q_i^+}} + (\delta\lambda)^{\frac{q^-}{\tau q_i^+}} + (\delta\lambda)^{\frac{q^-}{q_i^+}\frac{p-1}{\tau}}\lambda^{\frac{q^-}{q_i^+}\frac{2-p}{\tau}}\chi_{\{p<2\}}\Big]$$

and

$$\fint_{\Omega_{250r_i}(x_0^i)} |\nabla(u_k - v_{k,i})| \le C\Big[\varepsilon'\lambda^{\frac{q^-}{\tau q_i^+}} + (\delta\lambda)^{\frac{q^-}{\tau q_i^+}} + (\delta\lambda)^{\frac{q^-}{q_i^+}\frac{p-1}{\tau}}\lambda^{\frac{q^-}{q_i^+}\frac{2-p}{\tau}}\chi_{\{p<2\}}\Big].$$

This along with (3.18) gives the desired estimates. □

Proposition 3.3. *For any $\varepsilon > 0$ there exists $0 < \delta \le \delta_0$ such that if* **a** *is (δ, R_0) small and Ω is a (δ, R_0) Reifenberg flat domain, and u is a renormalized solution to* (1.1), *then for any $1 \le s_1 \le s_2 \le 2$ we have*

$$|E(A\lambda)| \le C\frac{\varepsilon}{\lambda}\Big\{\int_{\Omega_{s_2R}(x_0)\cap F_u(\lambda/4)} |\nabla u|^{\frac{\tau q(x)}{q^-}} \, dx + \frac{1}{\delta}\int_{\Omega_{s_2R}(x_0)\cap F_{\mathbf{M}_1(\mu)}(\delta\lambda/4)} \big[\mathbf{M}_1(\mu)(x)^{\frac{1}{p-1}}\big]^{\frac{\tau q(x)}{q^-}} \, dx\Big\}$$

for all

$$\lambda > \Big(\frac{8\times 10^6}{s_2 - s_1}\Big)^n \Lambda_\delta,$$

where $A \ge 3(1 + c_1 + c_2)$.

Proof. From the definition of $E(A\lambda)$, for every $k > 0$ we have

$$\begin{aligned}
|E(A\lambda)| &= \Big|\Big\{x \in \Omega_{s_1R}(x_0) : |\nabla u|^{\frac{\tau q(x)}{q^-}} > A\lambda\Big\}\Big| \\
&\le \sum_i \Big|\Big\{x \in \Omega_{s_1R}(x_0) \cap B_{5r_i}(y_i) : |\nabla u|^{\frac{\tau q(x)}{q^-}} > A\lambda\Big\}\Big| \\
&\le \sum_i \Big|\Big\{x \in \Omega_{s_1R}(x_0) \cap B_{5r_i}(y_i) : |\nabla v_{k,i}|^{\frac{\tau q(x)}{q^-}} > A\lambda/3\Big\}\Big| \\
&\quad + \sum_i \Big|\Big\{x \in \Omega_{s_1R}(x_0) \cap B_{5r_i}(y_i) : |\nabla(u_k - v_{k,i})|^{\frac{\tau q(x)}{q^-}} > A\lambda/3\Big\}\Big| \\
&\quad + \sum_i \Big|\Big\{x \in \Omega_{s_1R}(x_0) \cap B_{5r_i}(y_i) : |\nabla(u - u_k)|^{\frac{\tau q(x)}{q^-}} > A\lambda/3\Big\}\Big|.
\end{aligned}$$

By Lemma 3.1 and Lemma 3.2, we have

$$\|\nabla v_{k,i}\|^{\frac{\tau q(x)}{q^-}}_{L^\infty(\Omega_{5r_i}(y_i))} \le (c_1 + c_2)\lambda \le A\lambda/3.$$

Hence, we have

$$|E(A\lambda)| \leq \sum_i \Big|\{x \in \Omega_{s_1R}(x_0) \cap B_{5r_i}(y_i) : |\nabla(u_k - v_{k,i})|^{\frac{\tau q(x)}{q^-}} > A\lambda/3\}\Big| + \sum_i \Big|\{x \in \Omega_{s_1R}(x_0) \cap B_{5r_i}(y_i) : |\nabla(u - u_k)|^{\frac{\tau q(x)}{q^-}} > A\lambda/3\}\Big| =: E_{1,k}(A\lambda) + E_{2,k}(A\lambda).$$

For the first term, by Lemma 3.1 and Lemma 3.2 we have

$$\begin{aligned} E_{1,k}(A\lambda) &\leq \sum_i \Big|\{x \in \Omega_{s_1R}(x_0) \cap B_{5r_i}(y_i) : |\nabla(u_k - v_{k,i})| > (A\lambda/2)^{\frac{q^-}{\tau q_i^+}}\}\Big| \\ &\leq \sum_i \frac{1}{(A\lambda/2)^{q^-/\tau q_i^+}} \int\limits_{\Omega_{s_1R}(x_0)\cap B_{5r_i}(y_i)} |\nabla(u_k - v_{k,i})|\,dx \\ &\leq \sum_i \frac{\varepsilon}{(A\lambda/2)^{q^-/\tau q_i^+}} \lambda^{\frac{q^-}{\tau q_i^+}} |\Omega_{5r_i}(y_i)| \\ &\leq c\varepsilon \sum_i |\Omega_{5r_i}(y_i)| \leq c\varepsilon \sum_i |\Omega_{r_i}(y_i)|. \end{aligned}$$

On the other hand, since $\{B_{5r_i}(y_i)\}_i$ is a finite overlap, we have

$$\begin{aligned} E_{2,k}(A\lambda) &\leq C\Big|\{x \in \Omega : |\nabla(u - u_k)|^{\frac{\tau q(x)}{q^-}} > A\lambda/3\}\Big| \\ &\leq C\Big|\{x \in \Omega : |\nabla(u - u_k)|^{\frac{\tau q_1}{q^-}} > A\lambda/6\}\Big| + C\Big|\{x \in \Omega : |\nabla(u - u_k)|^{\frac{\tau q_2}{q^-}} > A\lambda/6\}\Big| \\ &\leq C\Big|\{x \in \Omega : |\nabla(u - u_k)| > (A\lambda/6)^{\frac{q^-}{\tau q_1}}\}\Big| + C\Big|\{x \in \Omega : |\nabla(u - u_k)| > (A\lambda/6)^{\frac{q^-}{\tau q_2}}\}\Big|. \end{aligned}$$

This implies

$$E_{2,k}(A\lambda) \to 0 \quad \text{as } k \to \infty.$$

Hence, by letting $k \to \infty$, we obtain that

$$|E(A\lambda)| \leq c\varepsilon \sum_i |\Omega_{5r_i}(y_i)| \leq c\varepsilon \sum_i |\Omega_{r_i}(y_i)|.$$

This together with (3.11) implies that

$$|E(A\lambda)| \leq \frac{c\varepsilon}{\lambda} \sum_i \Big\{ \int\limits_{\Omega_{r_i}(y_i)\cap F_u(\lambda/4)} |\nabla u(x)|^{\frac{\tau q(x)}{q^-}}\,dx + \int\limits_{\Omega_{r_i}(y_i)\cap F_{\mathbf{M}_1(\mu)}(\delta\lambda/4)} \frac{1}{\delta}\big[\mathbf{M}_1(\mu)(x)^{\frac{1}{p-1}}\big]^{\frac{\tau q(x)}{q^-}}\,dx\Big\}.$$

Since $\{\Omega_{5r_i}(y_i)\}_i$ is a finite overlap and $\Omega_{r_i}(y_i) \subset \Omega_{s_2R}(x_0)$, we further obtain

$$|E(A\lambda)| \leq \frac{c\varepsilon}{\lambda} \Big\{ \int\limits_{\Omega_{s_2R}(x_0)\cap F_u(\lambda/4)} |\nabla u(x)|^{\frac{\tau q(x)}{q^-}}\,dx + \int\limits_{\Omega_{s_2R}(x_0)\cap F_{\mathbf{M}_1(\mu)}(\delta\lambda/4)} \frac{1}{\delta}\big[\mathbf{M}_1(\mu)(x)^{\frac{1}{p-1}}\big]^{\frac{\tau q(x)}{q^-}}\,dx\Big\}. \qquad \square$$

We are now ready to give the proof of Theorem 1.4.

Proof of Theorem 1.4. By a standard approximation, we may assume that

$$\int\limits_\Omega |\nabla u|^{q(x)}\,dx < \infty.$$

We first observe that

$$\begin{aligned} \int\limits_{\Omega_{s_1R}(x_0)} |\nabla u|^{q(x)}\,dx &= \int\limits_{\Omega_{s_1R}(x_0)} \Big[|\nabla u|^{\frac{\tau q(x)}{q^-}}\Big]^{\frac{q^-}{\tau}}\,dx \\ &\leq c_p A^{\frac{q^-}{\tau}} \int_0^\infty \lambda^{\frac{q^-}{\tau}-1} |E(A\lambda)|\,d\lambda \\ &\leq c_p \int_0^{A_0\Lambda_\delta} \cdots + c_p \int_{A_0\Lambda_\delta}^\infty \cdots =: E(s_1, R) + F(s_1, R), \end{aligned} \tag{3.19}$$

where $A = 3(1 + c_1 + c_2)$ and

$$A_0 = \Big(\frac{8\times 10^6}{s_2 - s_1}\Big)^n.$$

For the first term $E(s_1, R)$, we have

$$\begin{aligned} E(s_1,R) &\le c_p A^{\frac{q^-}{\tau}}(A_0\Lambda_\delta)^{\frac{q^-}{\tau}}|\Omega_{s_1R}(x_0)| \\ &\le c_p(A_0A)^{\frac{q^-}{\tau}}|\Omega_{s_1R}(x_0)|\Big[\fint_{\Omega_{2R}(x_0)}\Big\{|\nabla u(x)|^{\frac{\tau q(x)}{q^-}} + \frac{1}{\delta}[\mathbf{M}_1(\mu)(x)^{\frac{1}{p-1}}]^{\frac{\tau q(x)}{q^-}}\Big\}\,dx + \frac{1}{\delta}\Big]^{\frac{q^-}{\tau}}. \end{aligned} \tag{3.20}$$

From (3.7), we have

$$\fint_{\Omega_{2R}(x_0)} |\nabla u(x)|^{\frac{\tau q(x)}{q^-}} \le \fint_{\Omega_{2R}(x_0)} [|\nabla u(x)|^{q_1} + 1]\,dx \lesssim \frac{1+|\Omega|}{|\Omega_{2R}(x_0)|}. \tag{3.21}$$

On the other hand, we have

$$\fint_{\Omega_{2R}(x_0)} [\mathbf{M}_1(\mu)(x)^{\frac{1}{p-1}}]^{\frac{\tau q(x)}{q^-}}\,dx \le \fint_{\Omega_{2R}(x_0)} [\mathbf{M}_1(\mu)(x)^{\frac{q(x)}{p-1}} + 1]\,dx \le \frac{1+|\Omega|}{|\Omega_{2R}(x_0)|}. \tag{3.22}$$

We now insert (3.21) and (3.22) into (3.20) to obtain that

$$\begin{aligned} E(s_1,R) &\le c_p(A_0A)^{\frac{q^-}{\tau}}|\Omega_{s_1R}(x_0)|\Big[\frac{1+|\Omega|}{|\Omega_{2R}(x_0)|} + \frac{1+|\Omega|}{\delta|\Omega_{2R}(x_0)|}\Big]^{\frac{q^-}{\tau}} \\ &\le \frac{C}{(s_2-s_1)^{nq_2/\tau}}|\Omega_{s_1R}(x_0)|\Big[\frac{1+|\Omega|}{|\Omega_{2R}(x_0)|} + \frac{1+|\Omega|}{\delta|\Omega_{2R}(x_0)|}\Big]^{\frac{q_2}{\tau}} \\ &\le \frac{C}{(s_2-s_1)^{nq_2/\tau}}|\Omega_{2R}(x_0)|\Big[\frac{1+|\Omega|}{|\Omega_{R}(x_0)|} + \frac{1+|\Omega|}{\delta|\Omega_{R}(x_0)|}\Big]^{\frac{q_2}{\tau}}. \end{aligned}$$

For the second term, using Proposition 3.3, we have

$$\begin{aligned} F(s_1,R) &\le c_p\varepsilon A^{\frac{q^-}{\tau}}\int_0^\infty \lambda^{\frac{q^-}{\tau}-2}\int_{\Omega_{s_2R}(x_0)\cap F_u(\lambda/4)} |\nabla u|^{\frac{\tau q(x)}{q^-}}\,dx\,d\lambda \\ &\quad + \frac{c_p\varepsilon A^{\frac{q^-}{\tau}}}{\delta}\int_0^\infty \lambda^{\frac{q^-}{\tau}-2}\int_{\Omega_{s_2R}(x_0)\cap F_{\mathbf{M}_1(\mu)}(\delta\lambda/4)} [\mathbf{M}_1(\mu)(x)^{\frac{1}{p-1}}]^{\frac{\tau q(x)}{q^-}}\,dx\,d\lambda, \end{aligned}$$

which together with the Fubini theorem implies that

$$\begin{aligned} F(s_1,R) &\le c_p\varepsilon A^{\frac{q^-}{\tau}}\int_{\Omega_{s_2R}(x_0)}\int_0^{4|\nabla u|^{\frac{\tau q(x)}{q^-}}} \lambda^{\frac{q^-}{\tau}-2}|\nabla u|^{\frac{\tau q(x)}{q^-}}\,d\lambda\,dx \\ &\quad + \frac{c_p\varepsilon A^{\frac{q^-}{\tau}}}{\delta}\int_{\Omega_{s_2R}(x_0)}\int_0^{\frac{4}{\delta}[\mathbf{M}_1(\mu)(x)^{\frac{1}{p-1}}]^{\frac{\tau q(x)}{q^-}}} \lambda^{\frac{q^-}{\tau}-2}[\mathbf{M}_1(\mu)(x)^{\frac{1}{p-1}}]^{\frac{\tau q(x)}{q^-}}\,d\lambda\,dx \\ &\le c_p\varepsilon A^{\frac{q^-}{\tau}}\int_{\Omega_{s_2R}(x_0)}|\nabla u|^{q(x)}\,dx + \frac{c_p\varepsilon A^{\frac{q^-}{\tau}}}{\delta^{\frac{q^-}{\tau}}}\int_{\Omega_{s_2R}(x_0)}\mathbf{M}_1(\mu)(x)^{\frac{q(x)}{p-1}}\,dx \\ &\le c_p\varepsilon A^{\frac{q^-}{\tau}}\int_{\Omega_{s_2R}(x_0)}|\nabla u|^{q(x)}\,dx + \frac{c_p\varepsilon A^{\frac{q^-}{\tau}}}{\delta^{\frac{q_2}{\tau}}} \\ &\le c_3\varepsilon\int_{\Omega_{s_2R}(x_0)}|\nabla u|^{q(x)}\,dx + \frac{c_3\varepsilon}{\delta^{\frac{q_2}{\tau}}}. \end{aligned}$$

We now insert the estimates of $E(s_1, R)$ and $F(s_1, R)$ into (3.19) to get that

$$\int\limits_{\Omega_{s_1R}(x_0)} |\nabla u|^{q(x)}\,dx \le \frac{c}{(s_2-s_1)^{nq_2/\tau}}|\Omega_{2R}(x_0)|\Big[\frac{1+|\Omega|}{|\Omega_R(x_0)|}+\frac{1+|\Omega|}{\delta|\Omega_R(x_0)|}\Big]^{\frac{q_2}{\tau}} + c_3\varepsilon\int\limits_{\Omega_{s_2R}(x_0)} |\nabla u|^{q(x)}\,dx + \frac{c_3\varepsilon}{\delta^{\frac{q_2}{\tau}}}.$$

We recall the following auxiliary lemma in [23, Lemma 4.3].

Lemma 3.4. *Let f be a bounded nonnegative function on $[a_1, a_2]$ with $0 < a_1 < a_2$. Assume that for any $a_1 \le x_1 \le x_2 \le a_2$ we have*

$$f(x_1) \le \theta_1 f(x_2) + \frac{A_1}{(x_2-x_1)^{\theta_2}} + A_2,$$

where $A_1, A_2 > 0$, $0 < \theta_1 < 1$ and $\theta_2 > 0$. Then there exists $c = c(\theta_1, \theta_2)$ so that

$$f(x_1) \le c\Big[\frac{A_1}{(x_2-x_1)^{\theta_2}} + A_2\Big].$$

We now apply this lemma for $a_1 = 1$, $a_2 = 2$ and

$$f(s) = \int\limits_{\Omega_{sR}(x_0)} |\nabla u|^{q(x)}\,dx$$

to conclude that

$$\begin{aligned}\int\limits_{\Omega_{s_1R}(x_0)} |\nabla u|^{q(x)}\,dx &\lesssim \frac{c}{(s_2-s_1)^{nq_2/\tau}}|\Omega_{2R}(x_0)|\Big[\frac{1+|\Omega|}{|\Omega_R(x_0)|}+\frac{1+|\Omega|}{\delta|\Omega_R(x_0)|}\Big]^{\frac{q_2}{\tau}} + \frac{c_3\varepsilon}{\delta^{\frac{q_2}{\tau}}}\\ &\lesssim \frac{c}{(s_2-s_1)^{nq_2/\tau}}|\Omega_{2R}(x_0)|\Big[\frac{1+|\Omega|}{|\Omega_R(x_0)|}+\frac{1+|\Omega|}{\delta|\Omega_R(x_0)|}\Big]^{\frac{q_2}{\tau}} + \frac{c_3\varepsilon}{\delta^{\frac{q_2}{\tau}}},\end{aligned}$$

provided $c_3\varepsilon < 1$.

We now insert $s_1 = 1$ and $s_2 = 2$ into the above expression to obtain that

$$\int\limits_{\Omega_R(x_0)} |\nabla u|^{q(x)}\,dx \lesssim |\Omega_R(x_0)|\Big[\frac{1+|\Omega|}{|\Omega_R(x_0)|}\Big]^{\frac{q_2}{\tau}} + 1. \tag{3.23}$$

We now fix $0 < R < \min\{R_0, R_\gamma, 1/10\}$. Then there exist $x_1, \dots, x_N \in \Omega$ so that

$$\Omega = \bigcup_{k=1}^N \Omega_R(x_k).$$

Hence, from (3.23) we obtain

$$\begin{aligned}\int\limits_{\Omega} |\nabla u|^{q(x)}\,dx &\le \sum_{k=1}^N \int\limits_{\Omega_R(x_k)} |\nabla u|^{q(x)}\,dx\\ &\lesssim \sum_{k=1}^N \Big[|\Omega_R(x_k)|\Big(\frac{1+|\Omega|}{|\Omega_R(x_k)|}\Big)^{\frac{q_2}{\tau}} + 1\Big]\\ &\lesssim \sum_{k=1}^N \Big[R^{-n(\frac{q_2}{\tau}-1)}(1+|\Omega|)^{\frac{q_2}{\tau}}\Big(\frac{B_R(x_k)}{\Omega_R(x_k)}\Big)^{\frac{q_2}{\tau}-1} + 1\Big],\end{aligned}$$

which along with (3.1) implies that

$$\int\limits_{\Omega} |\nabla u|^{q(x)}\,dx \lesssim \sum_{k=1}^N \Big[R^{-n(\frac{q_2}{\tau}-1)}(1+|\Omega|)^{\frac{q_2}{\tau}}4^{n(\frac{q_2}{\tau}-1)} + 1\Big] \lesssim 1. \qquad \square$$

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