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Research Article

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Partial regularity up to the boundary for solutions of subquadratic elliptic systems

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Abstract: In this paper, we are concerned with the nonlinear elliptic systems in divergence form under controllable growth condition. We prove that the weak solution u is locally Hölder continuous besides a singular set by using the direct method and classical Morrey-type estimates. Here the Hausdorff dimension of the singular set is less than $n - p$. This result not only holds in the interior, but also holds up to the boundary.

Keywords: Subquadratic elliptic systems, controllable growth condition, higher integrability, partial regularity

MSC 2010: 35J60

1 Introduction

In this paper, we are concerned with partial regularity up to the boundary for weak solutions $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ of a general inhomogeneous system of subquadratic elliptic equations in divergence form:

$$\begin{cases} -\operatorname{div} a(\,\cdot\,, u, Du) = b(\,\cdot\,, u, Du) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $n \geq 2$, $N \geq 2$, and the boundary values $g \in C^1(\bar{\Omega}, \mathbb{R}^N)$. The coefficient a maps $\Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$ into $\mathbb{R}^{nN}$, the inhomogeneity b which maps $\Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$ into $\mathbb{R}^N$ is a Carathéodory map, that means it is continuous with respect to (u, z) and measurable with respect to x. We need to impose certain conditions as boundedness, differentiability, growth, continuity, and uniformly strong ellipticity conditions on a in the following:

$$\begin{cases} |a(x, u, z)| + |D_z a(x, u, z)|(\mu^2 + |z|^2)^{\frac{1}{2}} \leq (\mu^2 + |z|^2)^{\frac{p-1}{2}}, \\ D_z a(x, u, z)\lambda \cdot \lambda \geq \nu(\mu^2 + |z|^2)^{\frac{p-2}{2}} |\lambda^2|, \\ |a(x, u, z) - a(\bar{x}, \bar{u}, z)| \leq L(\mu^2 + |z|^2)^{\frac{p-1}{2}} \omega(|x - \bar{x}| + |u - \bar{u}|), \end{cases} \tag{1.2}$$

where the map $z \mapsto a(\,\cdot\,, \cdot\,, z)$ is a vector field of class

$$C^0(\mathbb{R}^{nN}, \mathbb{R}^{nN}) \cap C^1(\mathbb{R}^{nN} \setminus 0, \mathbb{R}^{nN}),$$

$\lambda \in \mathbb{R}^{nN}$, and $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone nondecreasing concave function bounded by 1 (without loss of generality), continuous at 0 with $\lim_{\rho \to 0} \omega(\rho) = 0$. The parameter $\mu \in [0, 1]$. If $\mu \neq 0$, it means that the system

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is non-degenerate; when $\mu = 0$ the system is degenerate, and the parameter $1 < p < 2$. The third inequality of condition (1.2) means that the coefficients $a(x, u, z)$ are continuous with respect to (x, u). Moreover, we also need to impose the controllable growth condition on the inhomogeneity b:

$$|b(x,u,z)| \le L(\mu^2 + |z|^{p(1-\frac{1}{p^*})} + |u|^{p^*-1}), \quad p^* = \begin{cases} \frac{Np}{N-p}, & N \ne p, \\ \text{any constant}, & N = p, \end{cases} \tag{1.3}$$

where $(x, u, z) \in \bar{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{nN}$.

The regularity for solutions of partial differential equations is a difficult problem. Many authors have studied it such as [1, 2, 18]. There are some previous partial regularity results for elliptic systems. First we can pay attention to some articles that, for dimension n and with the mild continuity assumption to the coefficients such as [13–16] and [17, 24, 27, 31], tell us the regularity results for quasilinear systems, and [12] tells us about the partial regularity for subquadratic growth problems in the interior. For related problems of dimension reduction of the singular set $\mathrm{Sing}_{Du}(\Omega)$ under additional assumptions on $\omega(\cdot)$ see Mingione [28, 29]. For the dimension reduction up to the boundary see Duzaar, Grotowski and Kronz [20]. Furthermore, [19, 22, 26, 30] give some results about the general form of the coefficients for $n \ge 3$. It is well known that we cannot expect full Hölder continuity. In contrast, due to the global higher integrability of the weak solution and the Sobolev embedding theorem, we see that full Hölder regularity up to the boundary holds true provided that p is close to n.

For the setting of low dimension various results have been proved such as [21, 32]. In [8], Campanato obtained local Hölder continuity of the weak solution on the regular set in the interior of Ω under a controllable growth assumption, and that paper also gave the Hausdorff dimension of the singular set up to the boundary. He further achieved similar results for systems of higher order in [9]. Moreover, Campanato [10, 11] presented global Morrey-estimates for the weak solution of systems with coefficients not depending explicitly on u, and Arkhipova [3, 4] proved a partial regularity result up to the boundary for non-degenerate systems in the superquadratic case. An example under a natural growth condition for higher order Morrey-type and Hölder estimates can be found in [33]. In [6], Beck considers weak solutions of second order nonlinear elliptic systems in divergence form under standard subquadratic growth conditions with boundary data of class C^1. Our work here depends on this article.

In this paper, we are concerned with the regularity in the subquadratic case. We prove that the weak solution u to the nonlinear system (1.1) is locally Hölder continuous on $\mathrm{Reg}_u(\bar{\Omega})$ for some Hölder exponent $\theta > 0$ under the assumption that the inhomogeneity obeys a controllable growth condition. Moreover, we show that the Hausdorff dimension of singular sets is strictly less than $n - p$; the same result as in [6].

Notations. We use

$$B_\rho(y) = \{x \in \mathbb{R}^n : |x - y| < \rho\} \quad \text{and} \quad B_\rho^+(y) = \{x \in \mathbb{R}^n : x_n > 0,\ |x - y| < \rho\}$$

to denote a ball and the intersection of a ball with the upper half-space $\mathbb{R}^{n-1} \times \mathbb{R}^n$, respectively. Here the center point $y \in \mathbb{R}^n$ and $y \in \mathbb{R}^{n-1} \times \mathbb{R}_0^+$, respectively, the radius $\rho > 0$. Furthermore, we write

$$\Gamma_\rho(y) = \{x \in \mathbb{R}^n : x_n = 0,\ |x - y| < \rho\} \quad \text{for } y \in \mathbb{R}^{n-1} \times 0.$$

When $y = 0$, we write $B_\rho := B_\rho(0)$, $B := B_1$, $B_\rho^+ := B_\rho^+(0)$, $B^+ = B_1^+$, $\Gamma_\rho := \Gamma_\rho(0)$, and $\Gamma := \Gamma_1$. We use

$$W_\Gamma^{1,p}(B_\rho^+, \mathbb{R}^n) := \left\{u \in W^{1,p}(B_\rho^+, \mathbb{R}^n) : u = 0 \text{ on } \Gamma_{\sqrt{\rho^2 - (y)_n^2}}(x_0^1)\right\}$$

to denote the $W^{1,p}$-functions defined on a half-ball $B_\rho^+(y)$ which vanish on the flat part of the boundary, where $y_n < \rho$ is satisfied and where $y' := (y_1, \dots, y_{n-1}, 0)$ denotes the projection of y onto $\mathbb{R}^{n-1} \times 0$. Sometimes, it will be convenient to treat the tangential derivative $D'(u) = (D_1 u, \dots, D_{n-i} u)$ and the normal derivative $D_n u$ of a function $u \in W_\Gamma^{1,p}(B_\rho^+, \mathbb{R}^n)$ separately. For a given set $X \subset \mathbb{R}^n$ we write $\mathcal{L}^n(X) = |X|$ and $dim_{\mathcal{H}}(X)$ for its n-dimensional Lebesgue-measure and its Hausdorff dimension, respectively. Furthermore, if $f \in L^1(X, \mathbb{R}^n)$ and $0 < |X| < \infty$, we denote the average of f by $(f)_X = \fint_X f\, dx$.

We now state our main theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $g \in C^1(\bar{\Omega}, \mathbb{R}^N)$. Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, with $p \in (1, 2)$, be a weak solution of* (1.1) *with coefficients $a : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ satisfying assumptions* (1.2)*, and inhomogeneity $b : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^N$ obeying the controllable growth condition* (1.3)*. Then there exists $\delta_1 > 0$, depending only on n, N, p, $\frac{L}{\nu}$, $\|u\|_{L^p}$, $\|Du\|_{L^p}$, $\|Dg\|_{L^\infty}$ for $n \in [2, p + 2 + \delta_1]$, such that there holds*

$$\dim_H(\bar{\Omega} \setminus \operatorname{Reg}_u(\bar{\Omega})) < n - p \quad \text{and} \quad u \in C^{0,\theta}_{\mathrm{loc}}(\operatorname{Reg}_u(\bar{\Omega}), \mathbb{R}^N)$$

for all $\theta \in (0, \min\{1 - \frac{n-2-\delta_1}{p}, 1\})$. Moreover, the singular set $\operatorname{Sing}_u(\bar{\Omega})$ of u is contained in

$$\Sigma := \Big\{x_0 \in \bar{\Omega} : \liminf_{R \searrow 0} R^{p-n} \int\limits_{B_R(x_0 \cap \Omega)} (1 + |Du|^p)\, dx > 0\Big\}.$$

Here the number δ arises from the application of Gehring's lemma on higher integrability (see Lemma 2.4 for an explicit possible choice of the higher integrability exponent). Therefore, the condition $n \in [2, p + 2 + \delta_1)$ mostly means $n \in [2, 3)$ unless p is close to 2 or δ_1 happens to be large.

2 Preliminaries

In this section, we will give the definitions of Morrey and Campanato spaces and some known conclusions which will be used later.

Definition 2.1. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $1 \le p < \infty$. By $\mathcal{L}^{p,\lambda}(\Omega, \mathbb{R}^N)$, $\lambda \ge 0$, we denote the Campanato space of all functions $u \in \mathcal{L}^{p,\lambda}(\Omega, \mathbb{R}^N)$ such that

$$\|u\|^p_{\mathcal{L}^{p,\lambda}(\Omega,\mathbb{R}^N)} := \sup_{y \in \Omega,\, 0<\rho\le \operatorname{diam}\Omega} \rho^{-\lambda} \int\limits_{B_\rho(y)\cap\Omega} |u - (u)_{B_\rho(y)\cap\Omega}|^p\, dx < \infty.$$

The local variant $\mathcal{L}^{p,\lambda}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ can be defined in the usual way, saying that $u \in \mathcal{L}^{p,\lambda}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ if and only if $u \in \mathcal{L}^{p,\lambda}(\Omega', \mathbb{R}^N)$ for every $\Omega' \Subset \Omega$.

Definition 2.2. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $1 \le p < \infty$. By $L^{p,\lambda}(\Omega, \mathbb{R}^N)$, $\lambda \ge 0$, we denote the Morrey space of all functions $u \in L^{p,\lambda}(\Omega, \mathbb{R}^N)$ such that

$$\|u\|^p_{L^{p,\lambda}(\Omega,\mathbb{R}^N)} := \sup_{y \in \Omega,\, 0<\rho\le \operatorname{diam}\Omega} \rho^{-\lambda} \int\limits_{B_\rho(y)\cap\Omega} |u|^p\, dx < \infty.$$

The local variant $L^{p,\lambda}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ can be defined in the usual way, saying that $u \in L^{p,\lambda}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ if and only if $u \in L^{p,\lambda}(\Omega', \mathbb{R}^N)$ for every $\Omega' \Subset \Omega$.

To handle the subquadratic case the V-function is very useful. For $\xi \in \mathbb{R}^k$, $k \in \mathbb{N}$, $\mu \in [0, 1]$, and $p > 1$ it is defined by

$$V_\mu(\xi) = (\mu^2 + |\xi|^2)^{\frac{p-2}{4}} \xi,$$

which is a locally bi-Lipschitz bijection on $\mathbb{R}^k$.

When we deal with the V_μ-function, we will need some technical lemmas.

Lemma 2.3. *Let ξ, η be vectors in $\mathbb{R}^k$, $\mu \in [0, 1]$ and $q > 1$. Then there exist constants $C_1(q), C_2(q) \ge 1$ not depending on μ such that*

$$C_1^{-1}(\mu + |\xi| + |\eta|)^q \le \int_0^1 (\mu + |\xi + t\eta|)^q\, dt \le C_2(\mu + |\xi| + |\eta|)^q.$$

A proof of the latter statement can be found in [3, Lemma 2.1], and for the case $\mu = 1$ in [7]. The next lemma collects some basic inequalities.

Lemma 2.4. *Let ξ, η be vectors in $\mathbb{R}^k$, $\mu \in [0,1]$ and $p \in [1,2]$. Then there exist constants $C_1(k,p)$, $C_2(p)$ such that the following inequalities hold true:*

$$C_1^{-1}|\xi - \eta|(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{4}} \le |V_\mu(\xi) - V_\mu(\eta)| \le C_1|\xi - \eta|(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{4}},$$

$$(\mu^2 + |\xi|^2)^{\frac{p}{2}} \le C_2 + C_2(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{4}}|\xi - \eta|^2,$$

$$(\mu^2 + |\xi|^2)^{\frac{p-2}{2}}|\xi||\eta| \le \varepsilon(\mu^2 + |\eta|^2)^{\frac{p-2}{2}}|\xi|^2 + \varepsilon^{1-p}(\mu^2 + |\eta|^2)^{\frac{p}{2}} \quad \text{for } \varepsilon \in (0,1).$$

The first inequality is proved in [3, Lemma 2.2], while the other inequalities can be easily obtained.

Lemma 2.5 (Sobolev–Poincaré [26]). *Let $p < n$, let $p^* = \frac{np}{n-p}$ and let $B_r(z) \subset \mathbb{R}^n$. Then there exists a constant $C = C(n,N,p)$ such that for every $u \in W^{1,p}(B_r(z), \mathbb{R}^n)$ we have*

$$\Big(\int_{B_r(z)} |u - (u)_{B_r(z)}|^{p^*}\,dx\Big)^{1/p^*} \le C\Big(\int_{B_r(z)^+} |Du|^p\,dx\Big)^{1/p},$$

and such that for every $u \in W_\Gamma^{1,p}(B_r(z)^+, \mathbb{R}^n)$ with $0 \le z_n \le \frac{3}{4}r$ we have

$$\Big(\int_{B_r(z)^+} |u|^{p^*}\,dx\Big)^{1/p^*} \le C\Big(\int_{B_r(z)^+} |Du|^p\,dx\Big)^{1/p}.$$

Lemma 2.6 (Gehring's Lemma [20]). *Suppose A to be a closed subset of $\bar{\Omega}$. Consider two nonnegative functions $g, f \in L^1(\Omega)$ and p with $1 < p < \infty$ such that there holds*

$$\fint_{B_{r/2}\cap\Omega} g^p\,dx \le b^p\Big[\Big(\fint_{B_{r/2}\cap\Omega} g\,dx\Big)^p + \fint_{B_{r/2}\cap\Omega} f^p\,dx\Big]$$

for almost all $z \in \Omega \setminus A$ with $B_r^z \cap A = \emptyset$ for some constant b. Then there exist constants

$$C = C(n,p,q,b,k_\Omega) \quad \text{and} \quad \delta = \delta(n,p,b,k_\Omega)$$

such that

$$\Big(\fint_{B_\Omega} \tilde{g}^q\,dx\Big)^{\frac{1}{q}} \le C\Big[\Big(\fint_{B_\Omega} g^p\,dx\Big)^{\frac{1}{p}} + \Big(\fint_{B_\Omega} f^q\,dx\Big)^{\frac{1}{q}}\Big]$$

for all $q \in [p, p+\delta)$, where

$$\tilde{g(x)} = \frac{\mathfrak{L}^n(B_{d(x,A)}(x) \cap \Omega)}{\mathfrak{L}^n(\Omega)},$$

k_Ω is a positive constant such that $|B_\rho(x_0) \cap \Omega| \ge k_\Omega \rho^n$ for all points $x_0 \in \bar{\Omega}$, and every radius $\rho \le \operatorname{diam}(\Omega)$.

Lemma 2.7 ([5, Corollary 4.6]). *Let $v \in W_\Gamma^{(1,p)}(B_R^+(x_0))$ be a weak solution of $\operatorname{div} a_0(Dv) = 0$ under assumptions (1.2). Then there exists a constant $C = C(n,N,p,\frac{L}{\nu})$ independent of v such that for every $B_\rho^+(y) \subset B_R^+(x_0)$ with center $y \in B_R^+(x_0) \cup \Gamma_R(x_0)$ and radius $0 < \rho < R - |x_0 - y|$ there holds*

$$\int_{B_{\tau\rho}^+(y)} (\mu^p + |Dv|^p)\,dx \le \tau\rho \int_{B_\rho^+(y)} (\mu^p + |Dv|^p)\,dx \quad \text{for all } \tau \in (0,1].$$

Further, we have

$$\int_{B_\rho^+(x_0)} (\mu^p + |Dv|^p)\,dx \le C\Big(\frac{\rho}{R}\Big)^{\gamma_0} \int_{B_R^+(x_0)} (\mu^p + |Dv|^p)\,dx.$$

Here $\gamma_0 = \min\{2+\varepsilon, n\}$ for some $\varepsilon = \varepsilon(n,N,p,\frac{L}{\nu}) > 0$.

3 Caccioppoli inequality

In this section, we consider the problem on an upper half-ball, i.e., we consider weak solutions

$$u \in W^{1,p}(B^+, \mathbb{R}^N) \cap L^\infty(B^+, \mathbb{R}^N)$$

of the system

$$\begin{cases} -\operatorname{div} a(\cdot, u, Du) = b(\cdot, u, Du) & \text{in } B^+, \\ u = g & \text{on } \Gamma. \end{cases} \tag{3.1}$$

Next, we will prove the following Caccioppoli inequality.

Theorem 3.1. *Let $u \in g + W^{1,p}_\Gamma(B^+, \mathbb{R}^N)$, $g \in C^1(B^+ \cup \Gamma, \mathbb{R}^N)$, be a weak solution of* (3.1), *where the coefficients $a(\cdot,\cdot,\cdot)$ satisfy the first and second inequalities of* (1.2) *with $\mu \in [0,1]$. Let the inhomogeneity $b(\cdot,\cdot,\cdot)$ obey the controllable growth condition* (1.3). *Then there holds*

$$\fint_{B^+_{r/2}} (1 + |Du|^p)\,dx \le C \fint_{B^+_r} \Big(1 + \Big|\frac{u-g}{r}\Big|^p\Big)\,dx \tag{3.2}$$

for all $z \in B^+ \cup \Gamma$ and $0 < r < 1 - |z|$ with $z_n \le \frac{3}{4}r$, and

$$\fint_{B^+_{r/2}} (1 + |Du|^p)\,dx \le C \fint_{B^+_r} \Big(1 + \Big|\frac{u-(u)_{B_{3r/4}}(z)}{r}\Big|^p\Big)\,dx \tag{3.3}$$

for all $z \in B^+ \cup \Gamma$ and $0 < r < 1 - |z|$ with $z_n > \frac{3}{4}r$. Here C depends only on n, N, p, $\frac{L}{\nu}$, $\|u\|_{L^p}$, $\|Du\|_{L^p}$, $\|Dg\|_{L^\infty}$.

Proof. In order to prove inequality (3.2) close to the boundary, we choose a standard cut-off function $\eta \in C^\infty_0(B_r, [0,1])$ satisfying $\eta \equiv 1$ on $B_{\frac{r}{2}}(z)$ and $|\nabla\eta| \le \frac{4}{r}$. We first note that

$$\varphi = (u-g)\eta^2 \in W^{1,p}_0(B^+, \mathbb{R}^N).$$

Then φ can be taken as a test function in the weak sense of (3.1). Hence, we have

$$\begin{aligned}\fint_{B^+_r(z)} b(\cdot,u,Du)\cdot\varphi\,dx &= \fint_{B^+_r(z)} a(\cdot,u,Du)\cdot D\varphi\,dx \\ &= \fint_{B^+_r(z)} a(\cdot,u,Du)\cdot[(Du-Dg)\eta^2 + 2(u-g)\otimes\nabla\eta\eta]\,dx.\end{aligned}$$

Therefore, one has

$$\begin{aligned}&\fint_{B^+_r(z)} [a(\cdot,u,Du) - a(\cdot,u,0)]\cdot Du\eta^2\,dx \\ &\quad = -\fint_{B^+_r(z)} a(\cdot,u,0)\cdot Du\eta^2\,dx - \fint_{B^+_r(z)} 2a(\cdot,u,Du)(u-g)\otimes\nabla\eta\eta\,dx \\ &\qquad + \fint_{B^+_r(z)} a(\cdot,u,Du)\cdot Dg\eta^2\,dx + \fint_{B^+_r(z)} b(\cdot,u,Du)\cdot(u-g)\eta^2\,dx \\ &\quad = I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{3.4}$$

We use the first inequality of (1.2) and Young's inequality (for a positive ε to be determined later) to obtain

$$I_1 \le L\mu^{p-1} \fint_{B^+_r(z)} |Du|\eta^2\,dx \le \varepsilon \fint_{B^+_r(z)} |Du|^p\eta^2\,dx + \varepsilon^{\frac{1}{1-p}} L^{\frac{p}{p-1}}\mu^p. \tag{3.5}$$

It follows from the first inequality of (1.2), Young's inequality and $\frac{p}{p-1} \geq 2$ that

$$
\begin{aligned}
I_2 &\leq 2L \fint_{B_r^+(z)} (\mu^2 + |Du|^2)^{\frac{p-1}{2}} |u-g||\nabla\eta|\eta \, dx \\
&\leq 8L \fint_{B_r^+(z)} (\mu^2 + |Du|^2)^{\frac{p-1}{2}} \Big|\frac{u-g}{r}\Big|\eta \, dx \\
&\leq \varepsilon \fint_{B_r^+(z)} (\mu^p + |Du|^p)\eta^2 \, dx + 8^p \varepsilon^{1-p} L^p \fint_{B_r^+(z)} \Big|\frac{u-g}{r}\Big|^p \, dx, \qquad (3.6)\\
I_3 &\leq L \fint_{B_r^+(z)} (\mu^2 + |Du|^2)^{\frac{p-1}{2}} Dg\eta^2 \, dx \\
&\leq \varepsilon \fint_{B_r^+(z)} (\mu^p + |Du|^p)\eta^2 \, dx + \varepsilon^{1-p} L^p \fint_{B_r^+(z)} |Dg|^p \, dx. \qquad (3.7)
\end{aligned}
$$

By (1.3), Young's inequality and Sobolev's inequality, we have

$$
\begin{aligned}
\int_{B_r^+(z)} b(\cdot, u, Du)\cdot(u-g)\eta^2 \, dx &\leq L \int_{B_r^+(z)} L(\mu^2 + |Du|^{p(1-\frac{1}{p^*})} + |u|^{p^*-1}) \\
&\leq \varepsilon \int_{B_r^+(z)} (\mu^p + |Du|^p)\eta^2 \, dx + \varepsilon^{1-p^*} L^{p^*} \int_{B_r^+(z)} |u-g|^{p^*} \, dx \\
&\quad + \varepsilon \int |u-g|^{p^*}\eta^2 \, dx + \varepsilon^{\frac{1}{1-p^*}} L^{\frac{p^*}{p^*-1}} \int_{B_r^+(z)} |u|^{p^*} \, dx \\
&\leq \varepsilon \int_{B_r^+(z)} (\mu^p + |Du|^p)\eta^2 \, dx + \varepsilon^{1-p^*} L^{p^*} \Big(\int_{B_r^+(z)} |Du - Dg|^p \, dx\Big)^{\frac{p^*}{p}} \\
&\quad + \varepsilon \Big(\int_{B_r^+(z)} |Du - Dg|^p \, dx\Big)^{\frac{p^*}{p}} + \varepsilon^{\frac{1}{1-p^*}} L^{\frac{p^*}{p^*-1}} \Big(\int_{B_r^+(z)} |u|^p \, dx + \int_{B_r^+(z)} |Du|^p \, dx\Big)^{\frac{p^*}{p}} \\
&\leq \varepsilon \int_{B_r^+(z)} (\mu^p + |Du|^p)\eta^2 \, dx + \varepsilon^{1-p^*} L^{p^*} |Du|_{L^p}^{p^*} + \varepsilon^{1-p^*} L^{p^*} \Big(\int_{B_r^+(z)} |Dg|^p \, dx\Big)^{\frac{p^*}{p}} \\
&\quad + \varepsilon \|Du\|_{L^p}^{p^*} + \varepsilon\Big(\int_{B_r^+(z)} |Dg|^p \, dx\Big)^{\frac{p^*}{p}} + \varepsilon^{\frac{1}{1-p^*}} L^{\frac{p^*}{p^*-1}} (\|u\|_{L^p}^{p^*} + \|Du\|_{L^p}^{p^*}).
\end{aligned}
$$

Then we can infer that

$$
\begin{aligned}
I_4 &= \fint_{B_r^+(z)} b(\cdot, u, Du)\cdot(u-g)\eta^2 \, dx \\
&\leq \varepsilon \fint_{B_r^+(z)} (\mu^p + |Du|^p)\eta^2 \, dx \\
&\quad + \varepsilon^{1-p^*} L^{p^*} \|Du\|_{L^p}^{p^*} \fint_{B_r^+(z)} 1 \, dx + \varepsilon^{1-p^*} L^{p^*} \Big(\fint_{B_r^+(z)} |Dg|^p \, dx\Big)^{\frac{p^*}{p}} \\
&\quad + \varepsilon \|Du\|_{L^p}^{p^*} \fint_{B_r^+(z)} 1 \, dx + \varepsilon\Big(\fint_{B_r^+(z)} |Dg|^p \, dx\Big)^{\frac{p^*}{p}} \\
&\quad + \varepsilon^{\frac{1}{1-p^*}} L^{\frac{p^*}{p^*-1}} \Big(\|u\|_{L^p}^{p^*} \fint_{B_r^+(z)} 1 \, dx + \|Du\|_{L^p}^{p^*} \fint_{B_r^+(z)} 1 \, dx\Big). \qquad (3.8)
\end{aligned}
$$

By setting $\varepsilon \le \frac{\nu}{8}$, estimates (3.4)–(3.8) imply that

$$\begin{aligned}\nu^{-1} \fint_{B_r^+(z)} (a(\cdot, u, Du) - a(\cdot, u, 0)) \cdot Du\eta^2\,dx &\le \frac{1}{2} \fint_{B_r^+(z)} (\mu^p + |Du|^p)\eta^2\,dx \\ &\quad + C\Big(N, p, \frac{L}{\nu}\Big) \fint_{B_r^+(z)} \Big(\mu^p + \|u\|_{L^p}^{p^*} + \|Du\|_{L^p}^{p^*} + \Big|\frac{u-g}{r}\Big|^p\Big)\,dx \\ &\quad + C\Big(N, p, \frac{L}{\nu}\Big)\Big(\fint_{B_r^+(z)} |Dg|^p\,dx\Big)^{\frac{p^*}{p}}. \end{aligned} \tag{3.9}$$

On the other hand, we use the second inequality of (1.2) to obtain

$$\begin{aligned}\fint_{B_r^+(z)} (a(\cdot, u, Du) - a(\cdot, u, 0)) \cdot Du\eta^2\,dx &= \fint_{B_r^+(z)} \int_0^1 D_z a(\cdot, u, Du)Du \cdot Du\eta^2\,dt\,dx \\ &\ge \fint_{B_r^+(z)} \int_0^1 \nu(\mu^2 + t^2|Du|^2)^{\frac{p-2}{2}} |Du|^2\eta^2\,dt\,dx \\ &\ge \nu \fint_{B_r^+(z)} |V_\mu(Du)|^2\eta^2\,dx. \end{aligned} \tag{3.10}$$

By virtue of $\mu^p + |Du|^p \le 2(\mu^p + |V_\mu(Du)|^2)$ and inequalities (3.9) and (3.10), we have

$$\begin{aligned}\fint_{B_r^+(z)} (\mu^p + |Du|^p)\eta^2\,dx &\le 2 \fint_{B_r^+(z)} (\mu^p + |V_\mu(Du)|^2)\eta^2 \\ &\le 2 \fint_{B_r^+(z)} (\mu^p + \nu^{-1}) \fint_{B_r^+(z)} (a(\cdot, u, Du) - a(\cdot, u, 0)) \cdot Du\eta^2\,dx \\ &\le \frac{1}{2} \fint_{B_r^+(z)} \Big(\mu^p + \|u\|_{L^p}^{p^*} + \|Du\|_{L^p}^{p^*} + \Big|\frac{u-g}{r}\Big|^p\Big)\,dx. \end{aligned}$$

Then

$$\fint_{B_r^+(z)} (\mu^p + |Du|^p)\eta^2\,dx \le C\Big(N, p, \frac{L}{\nu}\Big) \fint_{B_r^+(z)} \Big(\mu^p + \|u\|_{L^p}^{p^*} + \|Du\|_{L^p}^{p^*} + |\frac{u-g}{r}|^p + \|Dg\|_{L^\infty}^{p^*}\Big)\,dx. \tag{3.11}$$

Setting $\mu = 1$ in (3.11), we have

$$\fint_{B_r^+(z)} (1 + |Du|^p)\eta^2\,dx \le C\Big(n, N, p, \frac{L}{\nu}, \|u\|_{L^p}, \|Du\|_{L^p}, \|Dg\|_{L^\infty}\Big) \fint_{B_r^+(z)} \Big(1 + \Big|\frac{u-g}{r}\Big|^p\Big)\,dx.$$

Then we obtain the result (3.2) by using the fact $\eta \equiv 1$ on $B_{\frac{r}{2}(z)}$.

Estimate (3.3) in the interior is achieved in the same way by using a standard cut-off function η with support in the ball $B_{\frac{3r}{4}(z)} \subset B^+$ and choosing $\varphi = (u - (u)_{B_{\frac{3r}{4}}})\eta^2$ as a test function instead of $\varphi = (u - g)\eta^2$. □

4 Higher integrability

Theorem 4.1. *Let $u \in g + W_\Gamma^{(1,p)}(B^+, \mathbb{R}^N)$, $g \in C^1(B^+ \cup \Gamma, \mathbb{R}^N)$, be a weak solution of* (3.1), *where the coefficients $a(\cdot, \cdot, \cdot)$ satisfy assumptions* (1.2) *with $\mu \in [0, 1]$. Let the inhomogeneity $b(\cdot, \cdot, \cdot)$ obey the controllable growth condition* (1.3). *Then there exists an exponent $q > p$ depending only on $p, n, N, \frac{L}{\nu}, \|Dg\|_{L^\infty}, \|u\|_{L^p}, \|Du\|_{L^p}$*

such that $u \in W^{(1,q)}(B_\rho^+, \mathbb{R}^N)$ *for all* $\rho < 1$*. Furthermore, for* $y \in B^+ \cap \Gamma$ *and* $0 < \rho < 1 - |y|$ *there holds*

$$\Big(\fint_{B_{\rho/2}^+(z)} (1 + |Du|)^{\frac{p}{q}}\,dx \Big)^q \le C\Big(\fint_{B_{\rho/2}^+(z)} (1 + |Du|^p)\,dx \Big),$$

with constants $C = C(p, n, N, \frac{L}{\nu}, \|Dg\|L^\infty, \|u\|_{L^p}, \|Du\|_{L^p})$.

Proof. Applying Lemma 2.5 in the zero-boundary-data version to inequalities (3.2), for $z \in B^+ \cup \Gamma$ and $0 < r < 1 - |z|$ with $z_n \le \frac{3}{4}r$, we get

$$\fint_{B_{r\backslash 2}^+} (1 + |Du|^p)\,dx \le C \fint_{B_r^+} \Big(1 + \Big|\frac{u-g}{r}\Big|^p\Big)\,dx \le C\Big[1 + \Big(\fint_{B_r^+} |Du - Dg|^{\frac{np}{n+p}}\,dx\Big)\Big]^{\frac{n+p}{n}},$$

where C depends only on p, n, N, $\frac{L}{\nu}$, $\|Dg\|_{L^\infty}$, $\|u\|_{L^p}$, $\|Du\|_{L^p}$ in the interior (for $z_n > \frac{3}{4}r$). We first apply Lemma 2.3 in the mean value version to (3.3), and then increase the domain of integration to B_r^+. Next, from Lemma 2.6, we know that there exist a constant C and an exponent $q > p$ depending on p, n, N, $\frac{L}{\nu}$, $\|Dg\|_{L^\infty}$, $\|u\|_{L^p}$, $\|Du\|_{L^p}$ such that

$$(1 + |Du|^p)^{\frac{n}{n+p}} \in L^{\frac{n+p}{n}\frac{2}{p}}(B_{\rho/2}(y))$$

with the estimate

$$\begin{aligned}
\Big(\fint_{B_{\rho/2}^+(z)} (1 + |Du|)^q\,dx \Big)^{\frac{p}{q}} &\le 2^p \Big(\fint_{B_{\rho/2}^+(z)} (1 + |Du|)^{\frac{q}{p}}\,dx \Big)^{\frac{p}{q}} \\
&\le 2^{n(1+\frac{p}{q})+p} \Big(\fint_{B_{\rho/2}^+(z)} \frac{\mathfrak{L}^n(B_{d(x,A)}(x) \cap B_\rho^+(y))}{\mathfrak{L}^n(B_\rho^+(y))} (1 + |Du|)^{\frac{q}{p}}\,dx \Big)^{\frac{p}{q}} \\
&\le C \fint_{B_{\rho/2}^+(z)} (1 + |Du|^p)\,dx.
\end{aligned}$$

Hence, we have finished the proof of the desired higher integrability estimate. □

5 Decay estimate for the solution

This section will give an appropriate decay estimate for the solution u of the original system (3.1) by comparing u with the solution

$$v \in W^{(1,p)}(B_R^+(x_0), \mathbb{R}^N) \cap L^\infty(B^+, \mathbb{R}^N)$$

of the system

$$\begin{cases} -\operatorname{div} a_0(Dv) = 0 & \text{in } B_R^+(x_0), \\ u = g & \text{on } \partial B_R^+(x_0), \end{cases} \tag{5.1}$$

where $u = a_0(z) = (x_0, (u)_{B_R^+(x_0)}, z)$, $x_0 \in \Gamma$ and $2R < 1 - |x_0|$. Testing the latter system with $u - g - v$, we obtain

$$\begin{aligned}
0 &= \int_{B_R^+(x_0)} a_0(Dv) \cdot (Du - Dg - Dv)\,dx \\
&= \int_{B_R^+(x_0)} \int_0^1 D_z a_0(tDv)(Dv) \cdot (Du - Dg - Dv)\,dt\,dx \\
&= \int_{B_R^+(x_0)} a_0(Dv) \cdot (Du - Dg - Dv)\,dx.
\end{aligned}$$

Using the first and second inequalities of (1.2), Young's inequality, Lemma 2.3, and the second inequality of Lemma 2.4, we have

$$\begin{aligned}
\nu \int_{B_R^+(x_0)} (\mu^2 + |Dv|^2)^{\frac{p-2}{2}} |Dv|^2\,dx &\le C(p)\nu \int_{B_R^+(x_0)} \int_0^1 (\mu^2 + |tDv|^2)^{\frac{p-2}{2}} |Dv|^2\,dt\,dx \\
&\le C(p) \int_{B_R^+(x_0)} \int_0^1 D_z a_0(tDv)Dv \cdot Dv\,dt\,dx \\
&= \int_{B_R^+(x_0)} \int_0^1 D_z a_0(tDv)D(u-g) \cdot Dv\,dt\,dx \\
&\le \varepsilon \int_{B_R^+(x_0)} (\mu^2 + |Dv|^2)^{\frac{p-2}{2}} |Dv|^2\,dx + C(p)\varepsilon^{1-p}L^p \int_{B_R^+(x_0)} (\mu^p + |Du - Dg|^p)\,dx.
\end{aligned}$$

Choosing $\varepsilon \le \frac{\nu}{2}$, we have

$$\int_{B_R^+(x_0)} |Dv|^p\,dx \le C \int_{B_R^+(x_0)} (\mu^p + |Du - Dg|^p)\,dx \le C \int_{B_R^+(x_0)} (1 + |Du|^p)\,dx. \tag{5.2}$$

Note that

$$\operatorname{div}\big(a_0(Dv + Dg) - a_0(Du)\big) = \operatorname{div}\big(a_0(Dv + Dg) - a_0(Dv)\big) + \operatorname{div}\big(a(\cdot\,, u, Du) - a_0(Du)\big) + b(\cdot\,, u, Du).$$

By Young's inequality, we deduce from the condition of the second inequality of (1.2) that

$$\begin{aligned}
&2^{\frac{p-2}{2}}\nu \int_{B_R^+(x_0)} (\mu^2 + |Du|^2 + |Dv + Dg|^2)^{\frac{p-2}{2}} |Du - Dv - Dg|^2\,dx \\
&\quad\le \nu \int_{B_R^+(x_0)} \int_0^1 (\mu^2 + |Du + t(Dv + Dg - Du)|^2)^{\frac{p-2}{2}} |Du - Dv - Dg|^2\,dt\,dx \\
&\quad\le \int_{B_R^+(x_0)} \int_0^1 D_z a_0(Du + t(Du - Dv - Dg))(Du - Dv - Dg) \cdot (Du - Dv - Dg)\,dt\,dx \\
&\quad= \int_{B_R^+(x_0)} \big(a_0(Dv + Dg) - a_0(Du)\big) \cdot (Du - Dv - Dg)\,dx \\
&\quad\le \int_{B_R^+(x_0)} \big(a_0(Dv + Dg) - a_0(Dv)\big) \cdot (Du - Dv - Dg)\,dx \\
&\qquad\quad + \int_{B_R^+(x_0)} \big(a(\cdot\,, u, Du) - a_0(Du)\big) \cdot (Du - Dv - Dg)\,dx \\
&\qquad\quad - \int_{B_R^+(x_0)} \big(b(\cdot\,, u, Du) - a_0(Du)\big) \cdot (Du - Dv - Dg)\,dx \\
&\quad=: I_1 + I_2 + I_3.
\end{aligned} \tag{5.3}$$

First, we will estimate the first term I_1. It follows from the first inequality of (1.2), Lemma 2.3, Young's inequality, $p < 2$, and (5.2) that

$$\begin{aligned}
I_1 &\le cL \int_{B_R^+(x_0)} (\mu^2 + |Dv|^2 + |Dg|^2)^{\frac{p-2}{2}} |Du - Dv - Dg||Dg|\,dx, \\
&\le cL\Big(\delta \int_{B_R^+(x_0)} (1 + |Du|^p)\,dx + R^n\delta^{1-p}\Big)
\end{aligned} \tag{5.4}$$

for every $\delta \in (0, 1)$.

For the second term, using the third inequality of (1.2), Hölder's inequality, Young's inequality, the higher integrability estimate for $1 + |Du|^p$ from the energy estimate (5.2), Jensen's inequality and Poincaré's inequality, we have

$$\begin{aligned}
I_1 &\le cL \int_{B_R^+(x_0)} \omega(|x-x_0| - |u-(u)_{B_R^+(x_0)}|)(\mu^2 + |Dv|^2 + |Dg|^2)^{\frac{p-2}{2}} |Du - Dv - Dg|\, dx \\
&\le |B_R^+(x_0)| L \Big(\fint_{B_R^+(x_0)} \omega(R + |u - (u)_{B_R^+}|)\, dx \Big)^{\frac{p-1}{p}\frac{q-p}{q}} \\
&\quad \times \Big(\fint_{B_R^+(x_0)} (\mu^p + |Du|^p)^{\frac{q}{p}}\, dx \Big)^{\frac{p-1}{p}\frac{p}{q}} \Big(3^{p-1} \fint_{B_R^+(x_0)} (|Du|^p + |Dg|^p_{L^\infty})\, dx \Big)^{\frac{1}{p}} \\
&\le |B_R^+(x_0)| Lc\omega^\beta \Big(\fint_{B_R^+(x_0)} (R + |u - (u)_{B_R^+}|)\, dx \Big) \\
&\quad \times \Big(\fint_{B_{2R}^+(x_0)} (1 + |Du|^p)\, dx \Big)^{\frac{p-1}{p}} \Big(\fint_{B_R^+(x_0)} (1 + |Du|^p)\, dx \Big)^{\frac{1}{p}} \\
&\le Lc\omega^\beta \Big(\int_{B_R^+(x_0)} (R^p + |u - (u)_{B_R^+}|^p)\, dx \Big)^{\frac{1}{p}} \int_{B_{2R}^+(x_0)} (1 + |Du|^p)\, dx \\
&\le Lc\omega^\beta \Big(\Big(R^{p-n} \int_{B_R^+(x_0)} (1 + |Du|^p)\, dx \Big)^{\frac{1}{p}} \Big) \int_{B_{2R}^+(x_0)} (1 + |Du|^p)\, dx.
\end{aligned} \tag{5.5}$$

Here

$$\beta := \frac{p-1}{p}\frac{q-p}{q} \tag{5.6}$$

and $q > p$ denotes the (up-to-the-boundary) higher integrability exponent of the gradient Du from Theorem 4.1 depending only on $p, n, N, \frac{L}{\nu}, \|Dg\|_{L^\infty}, \|u\|_{L^p}, \|Du\|_{L^p}$.

Finally, by the growth condition on $b(x, u, Du)$, Hölder's inequality, Poincaré's inequality, Sobolev's inequality, and (5.2), we have

$$\begin{aligned}
I_3 &\le L \int_{B_R^+(z)} (\mu^2 + |Du|^{p(1-\frac{1}{p^*})} + |u|^{p^*-1})|v + g - u|\, dx \\
&\le C(p, N)L \int_{B_R^+(z)} (\mu^2 + (u)_{z,R}^{p^*-1} + |Du|^{p(1-\frac{1}{p^*})} + |u - (u)_{z,R}|^{p^*-1})|v + g - u|\, dx \\
&\le cL\Big[\delta \int_{B_R^+(z)} (\mu^2 + (u)_{z,R}^{p^*} + |Du|^p + |u - (u)_{z,R}|^{p^*})\, dx + \delta^{1-p^*} \int_{B_R^+(z)} |v + g - u|^{p^*}\, dx \Big] \\
&\le cL\Big[\delta \int_{B_R^+(x_0)} (1 + |Du|^p)\, dx + \delta \Big(\int_{B_R^+(x_0)} (1 + |Du|^p)\, dx \Big)^{\frac{p*}{p}} + \delta^{1-p^*} \Big(\int_{B_R^+(x_0)} (1 + |Du|^p)\, dx \Big)^{\frac{p*}{p}} \Big].
\end{aligned}$$

It follows from (5.3)–(5.7) that

$$\begin{aligned}
&\int_{B_R^+(x_0)} (\mu^2 + |Du|^2 + |Dv + Dg|^2)^{\frac{p-2}{2}} |Du - Dv - Dg|^2\, dx \\
&\quad \le C\Big[\omega^\beta \Big(\Big(R^{p-n} \int_{B_R^+(x_0)} (1 + |Du|^p)\, dx \Big)^{\frac{1}{p}} \Big) + \delta \Big(\int_{B_R^+(x_0)} (1 + |Du|^p)\, dx \Big)^{\frac{p*-p}{p}} \\
&\qquad + \delta^{1-p^*} \Big(\int_{B_R^+(x_0)} (1 + |Du|^p)\, dx \Big)^{\frac{p*-p}{p}} + \delta \Big] \int_{B_{2R^+(z)}} (1 + |Du|^p)\, dx + R^n \delta^{1-p}.
\end{aligned}$$

From Lemma 2.7, one has

$$\int_{B_\rho^+(x_0)} |Dv|^p\,dx \le C\Big(\frac{\rho}{R}\Big)^{\gamma_0} \int_{B_R^+(x_0)} |Dv|^p\,dx. \tag{5.7}$$

Note that

$$\int_{B_\rho^+(x_0)} (1+|Dg|^p)\,dx \le C\Big(\frac{\rho}{R}\Big)^{\gamma_0} \int_{B_R^+(x_0)} 1\,dx. \tag{5.8}$$

It follows from Lemma 2.3 that

$$1+|Du|^p \le C[1+|Dv+Dg|^p + (\mu^2+|Du|^2+|Dv+Dg|^2)^{\frac{p-2}{2}}|Du-Dv-Dg|^2]. \tag{5.9}$$

Then we deduce from (5.7)–(5.10) that

$$\begin{aligned}
\int_{B_\rho^+(x_0)} (1+|Du|^p)\,dx &\le C\Big(\frac{\rho}{R}\Big)^{\gamma_0} \int_{B_\rho^+(x_0)} (1+|Du|^p)\,dx \\
&\quad + C\bigg[\omega^\beta\bigg(\bigg(R^{p-n}\int_{B_R^+(x_0)} (1+|Du|^p)\,dx\bigg)^{\frac{1}{p}}\bigg) + \delta\bigg(\int_{B_R^+(x_0)} (1+|Du|^p)\,dx\bigg)^{\frac{p*-p}{p}} \\
&\quad + \delta^{1-p^*}\bigg(\int_{B_R^+(x_0)} (1+|Du|^p)\,dx\bigg)^{\frac{p*-p}{p}} + \delta\bigg]\int_{B_{2R^+(z)}} (1+|Du|^p)\,dx + R^n\delta^{1-p} \\
&\le C\bigg[\Big(\frac{\rho}{R}\Big)^{\gamma_0}\omega^\beta\bigg(\bigg((2R)^{p-n}\int_{B_R^+(x_0)} (1+|Du|^p)\,dx\bigg)^{\frac{1}{p}}\bigg) + \delta\bigg(\int_{B_R^+(x_0)} (1+|Du|^p)\,dx\bigg)^{\frac{p*-p}{p}} \\
&\quad + \delta^{1-p^*}\bigg(\int_{B_R^+(x_0)} (1+|Du|^p)\,dx\bigg)^{\frac{p*-p}{p}} + \delta\bigg] \times \int_{B_{2R^+(x_0)}} (1+|Du|^p)\,dx + R^n\delta^{1-p}.
\end{aligned} \tag{5.10}$$

Define the Excess function as

$$\Phi(x_0,r) := \int_{B_r^+(x_0)} (1+|Du|^p)\,dx.$$

Then (5.10) can be rewritten as follows:

$$\Phi(x_0,\rho) \le C\Big[\Big(\frac{\rho}{R}\Big)^{\gamma_0}\omega^\beta\Big(((2R)^{p-n}\Phi(x_0,2R))^{\frac{1}{p}}\Big) + \delta\Phi(x_0,\rho)^{\frac{p*-p}{p}} + \delta^{1-p^*}\Phi(x_0,\rho)^{\frac{p*-p}{p}} + \delta\Big]\Phi(x_0,2R) + R^n\delta^{1-p}.$$

To conclude, we have the following lemma of decay estimate.

Lemma 5.1. *Let β, γ_0 be chosen as above in Lemma 2.5 and let $\delta \in (0,1)$. Furthermore, let $u \in g + W_\Gamma^{1,p}$, $1 < p < 2$, be a weak solution of system (5.1) under assumptions (1.2) and (1.3) with $g \in C^1(B^+ \cup \Gamma, \mathbb{R}^N)$. If $x_0 \in \Gamma$ and $R < 1 - |x_0|$, or if $x_0 \in B^+$ and $R < \min\{1-|x_0|, (x_0)_n\}$, there holds*

$$\Phi(x_0,\rho) \le C\Big[\Big(\frac{\rho}{R}\Big)^{\gamma_0}\omega^\beta(((R)^{p-n}\Phi(x_0,R))^{\frac{1}{p}}) + \delta\Phi(x_0,\rho)^{\frac{p*-p}{p}} + \delta^{1-p^*}\Phi(x_0,\rho)^{\frac{p*-p}{p}} + \delta\Big]\Phi(x_0,R) + R^n\delta^{1-p}$$

for every $\rho \in (0,R)$ and the constant C depends only on p, n, N, $\frac{L}{\nu}$, $\|Dg\|_{L^\infty}$, $\|u\|_{L^p}$, $\|Du\|_{L^p}$.

6 Proof of Theorem 1.1

Now, we prove the partial regularity result of the system on the unit half-ball. Afterwards, we can use a transformation which flattens the boundary locally, and a covering argument in a standard way to yield the statement of Theorem 1.1.

Theorem 6.1. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, $p \in (1,2)$, be a weak solution of*

$$\begin{cases} -\operatorname{div} a(\cdot\,, u, Du) = b(\cdot\,, u, Du) & \text{in } B^+, \\ u = g & \text{on } \Gamma, \end{cases}$$

where we assume that the coefficients $a : B^+ \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ satisfy assumptions (1.2)*, the inhomogeneity $b : B^+ \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^N$ obeys the controllable growth condition* (1.3) *and $g \in C^1(B^+ \cup \Gamma, \mathbb{R}^N)$. Then there exists $\delta_2 > 0$, depending only on n, N, p, $\frac{L}{\nu}$, $\|u\|_{L^p}$, $\|Du\|_{L^p}$, and $\|Dg\|_{L^\infty}$ for $n \in [2, p+2+\delta_2]$, such that there holds*

$$\dim_{\mathcal{H}}(B^+ \cup \Gamma) \setminus \operatorname{Reg}_u(B^+ \cup \Gamma) < n - p \quad \text{and} \quad u \in C^{0,\theta}_{\mathrm{loc}}(\operatorname{Reg}_u(B^+ \cup \Gamma), \mathbb{R}^N)$$

for all $\theta \in (0, \min\{1 - \frac{n-2-\delta_2}{p}, 1\})$. Moreover, the singular set $\operatorname{Sing}_u(B^+ \cup \Gamma)$ of u is contained in

$$\Sigma := \Big\{x_0 \in B^+ \cup \Gamma : \liminf_{R \searrow 0} R^{p-n} \int\limits_{B_R(x_0) \cup B^+} (1 + |Du|^p)\,dx > 0\Big\}.$$

Proof. Firstly, we give some denotions for some coefficients and constants that will be used later. Let q be the higher integrability exponent depending only on p, n, $\frac{L}{\nu}$, $\|Dg\|_{L^\infty}$, $\|Du\|_{L^p}$, $\|Du\|_{L^p}$ from Theorem 4.1, and $\beta = \frac{p-1}{p}\frac{q-p}{q}$. Let ε be the positive number mentioned in the proof of Theorem 3.1. If $n = 2$, we set $\varepsilon = 2p(1-\gamma)$, $\gamma \in (0,1)$. If $n \geq 3$, it stems from the application of Gehring's Lemma (see Lemma 2.6) and depends on n, N, p, and L, ν. We set $\gamma_0 = \min\{2 + \varepsilon, n\}$. Choosing the constant $\varepsilon_0 < 1$ in [24, Chapter III, Lemma 2.1] depends on γ_0, $\gamma_0 - \frac{\varepsilon}{2}$. We use γ_0, $\gamma_0 - \frac{\varepsilon}{2}$, C instead of α, β, A which comes from Lemma 5.1. Furthermore, we set $\delta = \frac{\varepsilon_0}{4}$. Since $\omega(\cdot)$ is a modulus of continuity, we can find a positive number ς such that

$$\omega^\beta(\tau^{\frac{1}{p}}) < \frac{\varepsilon_0}{4} \quad \text{and} \quad (\delta + \delta^{1-p^*})\tau^{\frac{p^*-p}{p}} < \frac{\varepsilon_0}{4}.$$

We now consider a point $x_0 \in B^+_{1-\delta} \setminus \Sigma$. This means that the excess quantity $R_0^{p-n}\Phi(x_0, R)$ becomes very small when $R \searrow 0$. Hence there exists a radius $R_0 > 0$ such that

$$B^+_{R_0}(x_0) \Subset B_{1-\sigma}, \quad R_0^{p-n} > \delta + \delta^{1-p^*}, \quad R_0^{p-n} \int\limits_{B^+_{R_0}(x_0)} (1 + |Du|^p)\,dx = R_0^{p-n}\Phi(x_0, R_0) < \tau.$$

Because the function $z \mapsto R_0^{p-n}\Phi(z, R_0)$ is continuous, for all $z \in B_r(x_0) \cap (B^+ \cap \Gamma)$ there exists a ball $B_r(x_0)$ such that

$$B_{R_0}(z) \Subset B_{1-\delta} \quad \text{and} \quad R_0^{p-n} \int\limits_{B^+_{R_0}(z)} (1 + |Du|^p)\,dx = R_0^{p-n}\Phi(z, R_0) < \tau.$$

Our next goal is to show the following Morrey-type estimates:

$$\Phi(z, \rho) \leq C\Big[\Big(\frac{\rho}{R_0}\Big)^{\gamma_0 - \varepsilon\backslash 2}\Phi(z, R_0) + \rho^{\gamma_0 - \varepsilon\backslash 2}\Big] \tag{6.1}$$

for all balls $B^+_\rho(z)$ with center $z \in B_r(x_0) \cap (B^+ \cap \Gamma)$ and a constant C which depends only on p, n, $\frac{L}{\nu}$, $\|Dg\|_{L^\infty}$, $\|u\|_{L^p}$, $\|Du\|_{L^p}$. This leads us to achieving our goal that the gradient Du belongs to a Morrey space on $B_r(x_0) \cap (B^+ \cup \Gamma)$. In order to prove (6.1), we need to distinguish several cases by combining the estimates at the boundary and in the interior.

Case 1: $z \in \Gamma$, $0 < \rho \leq R_0$. From the choices of δ, ε_0, τ, R_0 made above, considering Lemma 5.1 on the boundary version, we have

$$\begin{aligned} \Phi(z, \rho) &\leq C\Big[\Big(\frac{\rho}{R_0}\Big)^{\gamma_0} + \frac{3\varepsilon_0}{4}\Big]\Phi(z, R_0) + 4^{p-1}cR_0^n\varepsilon_0^{1-p} \\ &\leq C\Big[\Big(\frac{\rho}{R_0}\Big)^{\gamma_0} + \frac{3\varepsilon_0}{4}\Big]\Phi(z, R_0) + cR_0^{\gamma_0 - \varepsilon/2} \end{aligned}$$

for all $\rho \leq R_0$, and the constant C has the dependencies stated above. Then we apply [23, Chapter III, Lemma 2.1] to deduce inequality (6.1) for every such center z.

Case 2: $z \in B^+$, $0 < \rho \le R_0 \le z_n$. There holds $B_{R_0}(z) \subset B^+$, hence we consider Lemma 5.1 on the interior version. Then inequality (6.1) follows as in Case 1.

Case 3: $z \in B^+$, $0 < z_n < \rho \le R_0$. Without loss of generality, we may assume $\rho \le \frac{R_0}{4}$. Otherwise, (6.1) is trivially satisfied. Then we have the following relationship:

$$B^+_\rho(z) \subset B^+_{2\rho}(\tilde{z}) \subset B^+_{R_0/2}(\tilde{z}) \subset B^+_{R_0}(z),$$

where $\tilde{z}$ denotes the projection of z on $R^{n-1} \times 0$. We use the boundary estimates in Case 1 and the monotonicity to obtain

$$\begin{aligned}\Phi(z,\rho) \le \Phi(\tilde{z}, 2\rho) &\le C\Big[\Big(\frac{4\rho}{R_0}\Big)^{\gamma_0-\varepsilon/2}\Phi\Big(\tilde{z}, \frac{1}{2}R_0\Big) + (2\rho)^{\gamma_0-\varepsilon/2}\Big] \\ &\le C\Big[\Big(\frac{\rho}{R_0}\Big)^{\gamma_0-\varepsilon/2}\Phi(z, R_0) + \rho^{\gamma_0-\varepsilon/2}\Big].\end{aligned}$$

Case 4: $z \in B^+$, $0 < \rho \le z_n < R_0$. Without loss of generality, we assume $z_n < R_0/4$. Then we have

$$B_\rho(z) \subset B_{z_n}(z) \subset B^+_{2z_n}(z'') \subset B^+_{R_0/2}(z'') \subset B^+_{R_0}(z).$$

We use the interior estimates in Case 2 and the boundary estimates in Case 1 to find

$$\begin{aligned}\Phi(z,\rho) &\le C\Big[\Big(\frac{\rho}{z_n}\Big)^{\gamma_0-\varepsilon/2}\Phi(z, z_n) + \rho^{\gamma_0-\varepsilon/2}\Big] \\ &\le C\Big[\Big(\frac{\rho}{z_n}\Big)^{\gamma_0-\varepsilon/2}\Phi(z'', 2z_n) + \rho^{\gamma_0-\varepsilon/2}\Big] \\ &\le C\Big[\Big(\frac{\rho}{z_n}\Big)^{\gamma_0-\varepsilon/2} C\Big[\Big(\frac{4z_n}{R_0}\Big)^{\gamma_0-\varepsilon/2}\Phi\Big(z'', \frac{1}{2}R_0\Big) + (2z_n)^{\gamma_0-\varepsilon/2}\Big] + \rho^{\gamma_0-\varepsilon/2}\Big] \\ &\le C\Big[\Big(\frac{\rho}{R_0}\Big)^{\gamma_0-\varepsilon/2}\Phi(z, R_0) + \rho^{\gamma_0-\varepsilon/2}\Big].\end{aligned}$$

Then we proved inequality (6.1) for all cases required. This tells us that

$$Du \in L^{p,\gamma_0-\varepsilon/2}(B_r(x_0) \cap (B^+ \cap \Gamma), \mathbb{R}^{nN}).$$

We set $\delta_2 = \frac{\varepsilon}{2}$ and observe that $n < p + 2 + \delta_2 = p + 2 + \frac{\varepsilon}{2}$. We recall that if $n = 2$, we have $\gamma_0 = 2$; if $n > 2$, we have $\gamma_0 = 2 + \varepsilon$. This leads us to having $\gamma_0 - \frac{\varepsilon}{2} \in (n - p, n]$. According to the Campanato–Meyer embedding theorem (see [25, Theorem 2.2]), we get the conclusion that

$$u \in C^{0,\theta}(B_r(x_0) \cap (B^+ \cap \Gamma), \mathbb{R}^N) \quad \text{with } \theta = 1 - \frac{n - \gamma_0 + \varepsilon/2}{p}.$$

Since the higher integrability of Du has been shown in Theorem 4.1, we can improve the condition of x being a regular point via

$$R^{p-n} \int_{B^+_R(x)} (1 + |Du|^p)\,dx \le C\Big(R^{q-n} \int_{B^+_R(x)} (1 + |Du|^q)\,dx\Big)^{\frac{p}{q}}$$

for R sufficiently small. As a consequence we get

$$B^+ \setminus \Sigma \supseteq \Big\{x \in B^+ \cap \Gamma : \liminf_{R\to 0} R^{q-n} \int_{B_R(x)\cap B^+} (1 + |Du|^q)\,dx = 0\Big\}.$$

By [25, Proposition 2.7], we know that the Hausdorff dimension of the singular set is less than $n - p$. □

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