

## Research Article

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# Choquard-type equations with Hardy–Littlewood–Sobolev upper-critical growth

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**Abstract:** We are concerned with the existence of ground states and qualitative properties of solutions for a class of nonlocal Schrödinger equations. We consider the case in which the nonlinearity exhibits critical growth in the sense of the Hardy–Littlewood–Sobolev inequality, in the range of the so-called upper-critical exponent. Qualitative behavior and concentration phenomena of solutions are also studied. Our approach turns out to be robust, as we do not require the nonlinearity to enjoy monotonicity nor Ambrosetti–Rabinowitz-type conditions, still using variational methods.

**Keywords:** Ground states, semiclassical states, Choquard equation, Hardy–Littlewood–Sobolev inequality, upper-critical exponent

**MSC 2010:** 35B25, 35B33, 35J61

## 1 Introduction and main results

This paper deals with the following class of nonlinear and nonlocal Schrödinger equations:

$$-\varepsilon^2 \Delta v + V(x)v = \varepsilon^{-\alpha} (I_\alpha * F(v))f(v), \quad v > 0, x \in \mathbb{R}^N, \quad (1.1)$$

where  $\varepsilon > 0$  is the dimensionalized Planck constant,  $N \geq 3$ ,  $\alpha \in (0, N)$ ,  $F$  is the primitive function of  $f$ ,  $I_\alpha$  is the Riesz potential defined for every  $x \in \mathbb{R}^N \setminus \{0\}$  by

$$I_\alpha(x) := \frac{A_\alpha}{|x|^{N-\alpha}}, \quad \text{where } A_\alpha = \frac{\Gamma(\frac{1}{2}(N-\alpha))}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha}, \quad \Gamma \text{ is the Gamma function,}$$

and the external potential  $V$  satisfies:

(V1)  $V \in C(\mathbb{R}^N, \mathbb{R})$  and  $\inf_{x \in \mathbb{R}^N} V(x) > 0$ .

When  $\varepsilon = 1$ ,  $V(x) = a > 0$ , equation (1.1) reduces to the following nonlocal elliptic equation:

$$-\Delta u + au = (I_\alpha * F(u))f(u), \quad u > 0, x \in \mathbb{R}^N, \quad (1.2)$$

which is variational, in the sense that solutions of (1.2) turn out to be critical points of the energy functional

$$L_a(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + au^2 - (I_\alpha * F(u))F(u), \quad u \in H^1(\mathbb{R}^N).$$

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In particular, in the relevant physical case of dimension  $N = 3$ ,  $\alpha = 2$  and  $F(s) = \frac{s^2}{2}$ , (1.2) turns into the so-called Choquard equation

$$-\Delta u + au = (I_2 * u^2)u, \quad x \in \mathbb{R}^3, \quad (1.3)$$

which goes back to the seminal work of Fröhlich [24] and Pekar [50], modeling the quantum Polaron and then used by Choquard [35] to study steady states of the one component plasma approximation in the Hartree–Fock theory [38]. Equation (1.3) appears also in quantum gravity in the form of Schrödinger–Newton systems [51–53] in which a single particle is moving in its own gravitational field (self-gravitating matter), see also [30]. Lieb in [35] proved the existence and uniqueness of positive solutions to (1.3) by using rearrangements techniques. Multiplicity results for (1.3) were then obtained by Lions [39, 40] by means of a variational approach. A class of solutions which turn out to be of great interest in Physics as well as Mathematics are minimal energy solutions, which were predicted by Pekar to have a stochastic characterization in terms of Brownian motion, a conjecture proved just thirty years later by Donsker and Varadhan [21, 22]. We refer to [47] and references therein for an extensive survey on the topic.

Set  $\varepsilon = 1$ ,  $V \equiv 1$  and  $F(u) = \frac{|u|^p}{p}$  in (1.3):

$$-\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N. \quad (1.4)$$

Formally, as  $\alpha \rightarrow 0$ , equation (1.4) yields

$$-\Delta u + u = |u|^{2p-2}u, \quad x \in \mathbb{R}^N, \quad (1.5)$$

which is a prototype in semilinear equations and in particular it is well known since the work of Gidas, Ni and Nirenberg [28] that positive solutions with finite energy are radially symmetric, unique and non-degenerate (in the sense that the kernel of the linearized operator at the solution  $u$  is generated by  $\nabla u$ ), see [28, 49]. In contrast with the local problem (1.5), moving planes methods are somehow difficult to be used and is difficult to be used and the classification of positive solutions to (1.4) (even for  $p = 2$ ) has remained open for a long time. By using a suitable version of the moving planes method developed by Chen, Li and Ou [15], Ma and Zhao [42] gave a breakthrough to this open problem by considering equivalent Bessel–Riesz integral systems. By requiring some involved assumptions on  $\alpha$ ,  $p$  and  $N$ , they proved that positive solutions of (1.4) are, up to translations, radially symmetric and unique. In [44], Moroz and Van Schaftingen established the existence of ground state solutions to (1.4) in the optimal range

$$\frac{N + \alpha}{N} < p < \frac{N + \alpha}{N - 2}. \quad (1.6)$$

The endpoints in the above range of  $p$  are extremal values for the Hardy–Littlewood–Sobolev inequality [36] and sometimes called lower and upper H-L-S critical exponents. From the PDE point of view, a Pohozaev-type identity prevents the existence of finite energy solutions. In the upper critical case, as in the local Sobolev case, the appearance of a group invariance which yields explicit extremal functions to the H-L-S inequality is responsible for the lack of compactness. The lack of compactness can not be recovered by the presence of an external potential. In the lower critical case, equivalent variational characterizations of the ground state level still allow the H-L-S extremal functions to preventing compactness: this casts the problem within the class of Brezis–Nirenberg-type problems [8].

Recently, in [45] the more general Choquard equation (1.2) has been studied by requiring Berestycki–Lions-type conditions, and establishing the existence of ground state solutions in the subcritical case (1.6).

The first purpose of the present work is to investigate the existence of ground state solutions to (1.2) involving the upper H-L-S critical exponent. In presence of lower H-L-S critical exponent, a suitable external potential may lower down the groundstate to the compact region. This turns out to be a Lions-type problem and it is considered in a companion paper [26] as it involves quite different techniques.

**Definition 1.1.** A function  $u$  is said to be a ground state solution of (1.2) if  $u$  is a solution of (1.2) with the least action energy among all nontrivial solutions of (1.2). Namely,

$$L_\alpha(u) = \inf\{L_\alpha(v) : v \in H^1(\mathbb{R}^N) \text{ is a solution to (1.2)}\}.$$

Throughout this paper we assume  $f \in C(\mathbb{R}^+, \mathbb{R})$  which satisfies:

(F1)  $\lim_{t \rightarrow 0^+} \frac{f(t)}{t} = 0,$

(F2)  $\lim_{t \rightarrow +\infty} f(t)t^{-\frac{\alpha+2}{N-2}} = 1,$

(F3) there exist  $\mu > 0$  and  $q \in (2, \frac{N+\alpha}{N-2})$  such that

$$f(t) \geq t^{\frac{2+\alpha}{N-2}} + \mu t^{q-1}, \quad t > 0.$$

Our first main result in this paper is the following:

**Theorem 1.1.** *Assume  $\alpha \in ((N - 4)_+, N)$ ,  $q > \max\{1 + \frac{\alpha}{N-2}, \frac{N+\alpha}{2(N-2)}\}$  and (F1)–(F3). Then, for any  $a > 0$ , (1.2) admits a ground state solution.*

Let us point out that assumption (F3) plays a crucial role. Indeed, under the lonely assumptions (F1) and (F2), equation (1.2) has no solutions for any nontrivial external potential  $V$  by means of a Pohozaev-type identity (Lemma 3.2, Section 2). This fact rules out any perturbative argument and casts the problem into a Brezis–Nirenberg-type.

The second purpose of this paper is to investigate the profile of positive solutions to (1.1) as  $\varepsilon \rightarrow 0$ . Indeed, in quantum physics one expects that as the Planck constant  $\varepsilon \rightarrow 0$ , the dynamic is governed by the external potential  $V$  and an interesting class of solutions show up which develop a spike shape around critical points of  $V$ . From the physical point of view, these solutions are known as semiclassical states, as they describe the transition from quantum mechanics to classical mechanics. For the detailed physical background, we refer to [49] and references therein. By a Lyapunov–Schmidt reduction approach, based on the non-degeneracy condition, in [23, 49] the authors obtained the existence of solutions to the semilinear singularly perturbed Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = f(u), \tag{1.7}$$

which exhibit a single peak or multi peaks concentrating, as  $\varepsilon \rightarrow 0$ , around any given non-degenerate critical points of  $V$ . However, so far, the non-degeneracy condition holds for only a very restricted class of  $f$ . In the last decade, a lot of efforts have been devoted to relax or remove the non-degeneracy condition in this family of singularly perturbed problems. By using a variational approach, Rabinowitz [54] obtained the existence of positive solutions to (1.7) for small  $\varepsilon > 0$  with the following global potential well condition:

$$\liminf_{|x| \rightarrow \infty} V(x) > \inf_{\mathbb{R}^N} V(x).$$

Subsequently, by a penalization approach, del Pino and Felmer [18] weakened the above global potential well condition to the local condition

(V2) there exists a bounded domain  $O \subset \mathbb{R}^N$  such that

$$0 < m \equiv \inf_{x \in O} V(x) < \min_{x \in \partial O} V(x)$$

and proved the existence of a single-peak solution to (1.7). In [18, 54], the non-degeneracy condition is not required. Some related results can be found in [3, 17, 19, 20, 59] and the references therein. In [10] Byeon and Jeanjean introduced a new penalization approach and constructed a spike layered solution of (1.7) under (V2) and the almost optimal Berestycki-Lions conditions [6], see also [9, 11, 12] and [65, 68].

The second main result of this paper is the following:

**Theorem 1.2.** *Assume (V1)–(V2) in addition to the assumptions of Theorem 1.1 and let  $\mathcal{M} \equiv \{x \in O : V(x) = m\}$ . Then, for small  $\varepsilon > 0$ , (1.1) admits a positive solution  $v_\varepsilon$ , which satisfies:*

(i) *There exists a local maximum point  $x_\varepsilon$  of  $v_\varepsilon$  such that*

$$\lim_{\varepsilon \rightarrow 0} \text{dist}(x_\varepsilon, \mathcal{M}) = 0,$$

*and  $w_\varepsilon(x) \equiv v_\varepsilon(\varepsilon x + x_\varepsilon)$  converges (up to a subsequence) uniformly to a ground state solution of the limit equation*

$$-\Delta u + mu = (I_\alpha * F(u))f(u), \quad u > 0, u \in H^1(\mathbb{R}^N).$$

(ii)  $v_\varepsilon(x) \leq C \exp(-\frac{c}{\varepsilon}|x - x_\varepsilon|)$  for some  $c, C > 0$ .

We mention that related results under stronger assumptions have been recently obtained in [5]. For the convenience of the reader let us better contextualize our result within the existing literature on the singularly perturbed problem (1.1).

In [60], Wei and Winter considered the nonlocal equation, equivalent to the Schrödinger–Newton system,

$$-\varepsilon^2 \Delta v + V(x)v = \varepsilon^{-2}(I_2 * v^2)v, \quad x \in \mathbb{R}^3, \tag{1.8}$$

and by using a Lyapunov–Schmidt reduction method under assumption (V1), proved the existence of multi-bump solutions concentrating around local minima, local maxima or non-degenerate critical points of  $V$ . When the potential is allowed to vanish somewhere, thus avoiding (V1), the problem becomes much more difficult. In [56], Secchi considered (1.8) with a positive decaying potential and by means of a perturbative approach, proved the existence and concentration of bound states near local minima (or maxima) points of  $V$  as  $\varepsilon \rightarrow 0$ . Recently, by a nonlocal penalization technique, Moroz and Van Schaftingen [46] obtained a family of single spike solutions for the Choquard equation

$$-\varepsilon^2 \Delta v + V(x)v = \varepsilon^{-\alpha}(I_\alpha * |v|^p)|v|^{p-2}v, \quad x \in \mathbb{R}^N,$$

around the local minimum of  $V$  as  $\varepsilon \rightarrow 0$ . In [46] the assumption on the decay of  $V$  and the range for  $p \geq 2$  are optimal. More recently, using the penalization argument introduced in [10], Yang, Zhang and Zhang [64] investigated the existence and concentration of solutions to (1.1) under the local potential well condition (V2) and mild assumptions on  $f$ . In particular, the *Ambrosetti–Rabinowitz* condition and the *monotonicity* of  $\frac{f(t)}{t}$  are not required. For related results see [4, 7, 16, 43, 48, 56, 58, 63]. All the previous results are subcritical in the sense of the Hardy–Littlewood–Sobolev inequality. In [2], the authors considered the ground state solutions of the Choquard equation (1.1) in  $\mathbb{R}^2$ . By variational methods, the authors proved the existence and concentration of ground states to (1.1) involving critical exponential growth in the sense of the Pohozaev–Trudinger–Moser inequality. A natural open problem which has not been settled before is to establish concentration phenomena for (1.1) in the critical growth regime. Here we give a positive answer to this open problem in Theorem 1.2.

**Overview.** We conclude this section by giving the outline of the paper and pointing out major difficulties. In Section 2 we prove some preliminary results which require some efforts to extend a few well-known results in the local setting, to the nonlocal framework. Section 3 is devoted to proving Theorem 1.1. Here, without the *Ambrosetti–Rabinowitz* condition, to obtain the boundedness of the Palais–Smale sequence becomes a delicate issue. To overcome this difficulty, a possible strategy is to look for a constraint minimization problem. This goes back to Berestycki–Lions [6], in which the authors established the existence of ground state solutions to the scalar mean field equation  $-\Delta u = g(u)$ ,  $u \in H^1(\mathbb{R}^N)$ . By using a similar strategy, Zhang and Zou [67] extended the result in [6] to the critical case. Precisely, in [6, 67], the existence of ground state solutions is reduced to looking at the constraint minimization problem

$$\inf \left\{ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 : \int_{\mathbb{R}^N} G(u) = 1, u \in H^1(\mathbb{R}^N) \right\}$$

and eventually to get rid of the Lagrange multiplier thanks to some appropriate scaling. However, this approach fails for the nonlocal problem (1.2), since  $\int_{\mathbb{R}^N} |\nabla u|^2$ ,  $\int_{\mathbb{R}^N} |u|^2$  and  $\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)$  scale differently in space and hence one has no hope to remove the Lagrange multiplier. The existence of ground state solutions to the nonlocal problem (1.2), in the subcritical case, has been done by Moroz and Van Schaftingen in [45], where they constructed a bounded Palais–Smale sequence satisfying asymptotically the Pohozaev identity and obtained a ground state solution by virtue of a concentration-compactness-type argument and a scaling technique introduced by Jeanjean [31]. Here, to avoid a *Ambrosetti–Rabinowitz*-type condition, we use the Struwe monotonicity trick, in the abstract form due to [32], to get a bounded Palais–Smale sequence. Clearly, due to the presence of a critical H-L-S term, the Palais–Smale condition fails. By a decomposition technique, we recover compactness and obtain the existence of ground state solutions to (1.2). In Section 4, we first prove some qualitative properties of the set of ground states such as compactness, regularity, symmetry and positivity. Then we use a truncation argument as key ingredient to prove Theorem 1.2. In [64], the

authors considered problem (1.1) in the subcritical case and established concentration phenomena. Here, the presence of critical growth prevents to use directly the argument in [64]. We overcome this difficulty by penalizing the problem which is relaxed to a subcritical case. The penalized problem admits a family of spike shaped solutions which develop a concentrating behavior around the local minima of  $V$ . Finally, the analysis carried out in Section 3 enables us to prove the convergence of the penalized solution to a solution of the original problem which preserves the same qualitative properties of the penalized problem.

## 2 Preliminaries

In this section, we are concerned with the existence of ground state solutions to (1.2). Let  $a > 0$  and denote the least energy of (1.2) by

$$E_a = \inf\{L_a(u) : L'_a(u) = 0 \text{ in } H^{-1}(\mathbb{R}^N), u \in H^1(\mathbb{R}^N) \setminus \{0\}\}.$$

In what follows, let  $H^1(\mathbb{R}^N)$  be endowed with the norm

$$\|u\| = \left( \int_{\mathbb{R}^N} |\nabla u|^2 + a|u|^2 \right)^{\frac{1}{2}}, \quad u \in H^1(\mathbb{R}^N).$$

Before proving Theorem 1.1, we prove first some preliminary results. First of all, let us recall the following Hardy–Littlewood–Sobolev inequality which will be frequently used throughout the paper.

**Lemma 2.1** ([37, Theorem 4.3]). *Let  $s, r > 1$  and  $0 < \alpha < N$  with  $\frac{1}{s} + \frac{1}{r} = 1 + \frac{\alpha}{N}$ ,  $f \in L^s(\mathbb{R}^N)$  and  $g \in L^r(\mathbb{R}^N)$ . Then there exists a positive constant  $C(s, N, \alpha)$  (independent of  $f, g$ ) such that*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x)|x - y|^{\alpha - N} g(y) \, dx dy \right| \leq C(s, N, \alpha) \|f\|_s \|g\|_r.$$

*In particular, if  $s = r = \frac{2N}{N + \alpha}$ , the best possible constant is given by*

$$C_\alpha := \pi^{\frac{N - \alpha}{2}} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N + \alpha}{2})} \left[ \frac{\Gamma(\frac{N}{2})}{\Gamma(N)} \right]^{-\frac{\alpha}{N}}.$$

**Remark 2.1.** As a consequence of the Hardy–Littlewood–Sobolev inequality, for any  $v \in L^s(\mathbb{R}^N)$ ,  $s \in (1, \frac{N}{\alpha})$ ,  $I_\alpha * v \in L^{\frac{Ns}{N - \alpha s}}(\mathbb{R}^N)$ . Moreover,  $I_\alpha \in \mathcal{L}(L^s(\mathbb{R}^N), L^{\frac{Ns}{N - \alpha s}}(\mathbb{R}^N))$  and

$$\|I_\alpha * v\|_{\frac{Ns}{N - \alpha s}} \leq C(s, N, \alpha) \|v\|_s.$$

### 2.1 Brezis–Lieb lemma

In this subsection, we prove a nonlocal version of the Brezis–Lieb lemma.

**Lemma 2.2** (Brezis–Lieb Lemma). *Assume  $\alpha \in (0, N)$  and there exists a constant  $C > 0$  such that*

$$|f(t)| \leq C(|t|^{\frac{\alpha}{N}} + |t|^{\frac{\alpha + 2}{N - 2}}), \quad s \in \mathbb{R}.$$

*Let  $\{u_n\} \subset H^1(\mathbb{R}^N)$  be such that  $u_n \rightarrow u$  weakly in  $H^1(\mathbb{R}^N)$  and a.e. in  $\mathbb{R}^N$  as  $n \rightarrow \infty$ . Then*

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_n))F(u_n) = \int_{\mathbb{R}^N} (I_\alpha * F(u_n - u))F(u_n - u) + \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u) + o_n(1),$$

*where  $o_n(1) \rightarrow 0$  as  $n \rightarrow \infty$ .*

In order to prove Lemma 2.2, we recall the following lemma, which states that pointwise convergence of a bounded sequence implies weak convergence.

**Lemma 2.3** ([62, Theorem 4.2.7]). *Let  $\Omega \subseteq \mathbb{R}^N$  be a domain and let  $\{u_n\}$  be bounded in  $L^q(\Omega)$  for some  $q > 1$ . If  $u_n \rightarrow u$  a.e. in  $\Omega$  as  $n \rightarrow \infty$ , then  $u_n \rightarrow u$  weakly in  $L^q(\Omega)$  as  $n \rightarrow \infty$*

*Proof of Lemma 2.2.* Observe that

$$\begin{aligned} & \int_{\mathbb{R}^N} (I_\alpha * F(u_n))F(u_n) - (I_\alpha * F(u_n - u))F(u_n - u) - (I_\alpha * F(u))F(u) \\ &= \int_{\mathbb{R}^N} (I_\alpha * [F(u_n) + F(u_n - u)])[F(u_n) - F(u_n - u)] - (I_\alpha * F(u))F(u) \end{aligned}$$

and there exists  $C > 0$  such that

$$|F(s)| \leq C(|s|^{\frac{N+\alpha}{N}} + |s|^{\frac{N+\alpha}{N-2}}) \quad \text{for all } s \in \mathbb{R},$$

which implies  $F(u) \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ . For any  $\delta > 0$  sufficiently small, by the Hardy–Littlewood–Sobolev inequality there exists  $K_1 > 0$  such that

$$\left| \int_{\Omega_1} (I_\alpha * F(u))F(u) \right| \leq \frac{\delta}{6}, \quad \Omega_1 := \{x \in \mathbb{R}^N : |u(x)| \geq K_1\}.$$

Again by the Hardy–Littlewood–Sobolev inequality we have

$$\begin{aligned} & \left| \int_{\Omega_1} (I_\alpha * [F(u_n) + F(u_n - u)])[F(u_n) - F(u_n - u)] \right| \\ & \leq C \left( \int_{\mathbb{R}^N} |F(u_n) + F(u_n - u)|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \left( \int_{\Omega_1} |F(u_n) - F(u_n - u)|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \\ & \leq C(N, \alpha) \left( \int_{\Omega_1} |F(u_n) - F(u_n - u)|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}}, \end{aligned}$$

where we have used the fact that  $\{u_n\}$  is bounded in  $H^1(\mathbb{R}^N)$ . It is easy to see there exists  $c > 0$  such that

$$|F(u_n) - F(u_n - u)|^{\frac{2N}{N+\alpha}} \leq c \left( |u_n|^{\frac{2\alpha}{N+\alpha}} |u|^{\frac{2N}{N+\alpha}} + |u_n|^{\frac{2+\alpha}{N-2}} |u|^{\frac{2N}{N+\alpha}} + |u|^2 + |u|^{\frac{2N}{N-2}} \right).$$

Then, by Hölder’s inequality,

$$\int_{\Omega_1} |u_n|^{\frac{2\alpha}{N+\alpha}} |u|^{\frac{2N}{N+\alpha}} \leq \left( \int_{\Omega_1} |u_n|^2 \right)^{\frac{\alpha}{N+\alpha}} \left( \int_{\Omega_1} |u|^2 \right)^{\frac{N}{N+\alpha}}$$

and

$$\int_{\Omega_1} |u_n|^{\frac{2+\alpha}{N-2}} |u|^{\frac{2N}{N+\alpha}} \leq \left( \int_{\Omega_1} |u_n|^{\frac{2N}{N-2}} \right)^{\frac{2+\alpha}{N+\alpha}} \left( \int_{\Omega_1} |u|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N+\alpha}}.$$

So for  $\delta$  given above and  $K_1$  fixed but large enough, we get for any  $n$ ,

$$\left| \int_{\Omega_1} (I_\alpha * [F(u_n) + F(u_n - u)])[F(u_n) - F(u_n - u)] \right| \leq \frac{\delta}{6}.$$

Similarly, let  $\Omega_2 := \{x \in \mathbb{R}^N : |x| \geq R\} \setminus \Omega_1$  with  $R > 0$  large enough, we have

$$\left| \int_{\Omega_2} (I_\alpha * F(u))F(u) \right| \leq \frac{\delta}{6}$$

and for any  $n$ ,

$$\left| \int_{\Omega_2} (I_\alpha * [F(u_n) + F(u_n - u)])[F(u_n) - F(u_n - u)] \right| \leq \frac{\delta}{6}.$$

For  $K_2 > K_1$ , let  $\Omega_3(n) := \{x \in \mathbb{R}^N : |u_n(x)| \geq K_2\} \setminus (\Omega_1 \cup \Omega_2)$ . If  $\Omega_3(n) \neq \emptyset$ , then we know that  $|u(x)| < K_1$  and  $|x| < R$  for any  $x \in \Omega_3(n)$ . By noting that  $u_n \rightarrow u$  a.e. in  $\Omega$  as  $n \rightarrow \infty$ , it follows from the Severini-Egoroff theorem that  $u_n$  converges to  $u$  in measure in  $B_R(0)$ , which implies that  $|\Omega_3(n)| \rightarrow 0$  as  $n \rightarrow \infty$ . Similarly we have, for  $n$  large enough,

$$\left| \int_{\Omega_3(n)} (I_\alpha * F(u))F(u) \right| \leq \frac{\delta}{6}$$

and

$$\left| \int_{\Omega_3(n)} (I_\alpha * [F(u_n) + F(u_n - u)])[F(u_n) - F(u_n - u)] \right| \leq \frac{\delta}{6}.$$

Finally, let us estimate

$$\int_{\Omega_4(n)} (I_\alpha * [F(u_n) + F(u_n - u)])[F(u_n) - F(u_n - u)] - (I_\alpha * F(u))F(u),$$

where  $\Omega_4(n) = \mathbb{R}^N \setminus (\Omega_1 \cup \Omega_2 \cup \Omega_3(n))$ . Obviously,  $\Omega_4(n) \subset B_R(0)$ . By Lebesgue's dominated convergence theorem we have

$$\lim_{n \rightarrow \infty} \int_{\Omega_4(n)} |F(u_n - u)|^{\frac{2N}{N+\alpha}} = 0, \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{\Omega_4(n)} |F(u_n) - F(u)|^{\frac{2N}{N+\alpha}} = 0,$$

which implies by the Hardy–Littlewood–Sobolev inequality

$$\left| \int_{\Omega_4(n)} (I_\alpha * [F(u_n) + F(u_n - u)]F(u_n - u) \right| \leq C(N, \alpha) \left( \int_{\Omega_4(n)} |F(u_n - u)|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \rightarrow 0$$

as  $n \rightarrow \infty$ , and

$$\left| \int_{\Omega_4(n)} (I_\alpha * [F(u_n) + F(u_n - u)])[F(u_n) - F(u)] \right| \leq C(N, \alpha) \left( \int_{\Omega_4(n)} |F(u_n) - F(u)|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \rightarrow 0$$

as  $n \rightarrow \infty$ . Now let  $H_n = F(u_n) + F(u_n - u) - F(u)$ ; we have

$$\lim_{n \rightarrow \infty} \int_{\Omega_4(n)} (I_\alpha * [F(u_n) + F(u_n - u)])[F(u_n) - F(u_n - u)] - (I_\alpha * F(u))F(u) = \lim_{n \rightarrow \infty} \int_{\Omega_4(n)} (I_\alpha * H_n)F(u).$$

Noting that  $H_n$  is bounded in  $L^{2N/(N+\alpha)}(\mathbb{R}^N)$  and  $H_n \rightarrow 0$  a.e. in  $\mathbb{R}^N$  as  $n \rightarrow \infty$ , by Lemma 2.3,  $H_n \rightarrow 0$  weakly in  $L^{2N/(N+\alpha)}(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . By Remark 2.1,  $I_\alpha * H_n \rightarrow 0$  weakly in  $L^{2N/(N-\alpha)}(\mathbb{R}^N)$  as  $n \rightarrow \infty$ , which yields

$$\lim_{n \rightarrow \infty} \int_{\Omega_4(n)} (I_\alpha * H_n)F(u) = 0.$$

Thus,

$$\limsup_{n \rightarrow \infty} \left| \int_{\mathbb{R}^N} (I_\alpha * F(u_n))F(u_n) - (I_\alpha * F(u_n - u))F(u_n - u) - (I_\alpha * F(u))F(u) \right| \leq \delta$$

and the arbitrary choice of  $\delta$  concludes the proof. □

## 2.2 Splitting lemma

Next we prove a splitting property for the nonlocal energy.

**Lemma 2.4** (Splitting Lemma). *Assume  $\alpha \in ((N-4)_+, N)$ , (F1)–(F2) and let  $\{u_n\} \subset H^1(\mathbb{R}^N)$  be such that  $u_n \rightarrow u$  weakly in  $H^1(\mathbb{R}^N)$  and a.e. in  $\mathbb{R}^N$  as  $n \rightarrow \infty$ . Then, up to a subsequence if necessary,*

$$\int_{\mathbb{R}^N} ([I_\alpha * F(u_n)]f(u_n) - [I_\alpha * F(u_n - u)]f(u_n - u) - [I_\alpha * F(u)]f(u))\phi = o_n(1)\|\phi\|,$$

where  $o_n(1) \rightarrow 0$  uniformly as  $n \rightarrow \infty$  for any  $\phi \in C_0^\infty(\mathbb{R}^N)$ .

In order to prove Lemma 2.4, we need first to prove Lemma 2.5 and Lemma 2.6 below.

**Lemma 2.5.** *Let  $\Omega \subset \mathbb{R}^N$  be a domain and let  $\{u_n\} \subset H^1(\Omega)$  be such that  $u_n \rightarrow u$  weakly in  $H^1(\Omega)$  and a.e. in  $\Omega$  as  $n \rightarrow \infty$ . Then the following hold:*

(i) *For any  $1 < q \leq r \leq \frac{2N}{N-2}$  and  $r > 2$ ,*

$$\lim_{n \rightarrow \infty} \int_{\Omega} \left| |u_n|^{q-1} u_n - |u_n - u|^{q-1} (u_n - u) - |u|^{q-1} u \right|^{\frac{r}{q}} = 0.$$

(ii) *Assume  $h \in C(\mathbb{R}, \mathbb{R})$  and  $h(t) = o(t)$  as  $t \rightarrow 0$ ,  $|h(t)| \leq c(1 + |t|^q)$  for any  $t \in \mathbb{R}$ , where  $q \in (1, \frac{N+2}{N-2}]$ . The following hold:*

(1) *For any  $r \in [q + 1, \frac{2N}{N-2}]$ ,*

$$\lim_{n \rightarrow \infty} \int_{\Omega} |H(u_n) - H(u_n - u) - H(u)|^{\frac{r}{q+1}} = 0.$$

where  $H(t) = \int_0^t h(s) ds$ ,

(2) *If we further assume that  $\Omega = \mathbb{R}^N$ ,  $\alpha \in ((N - 4)_+, N)$  and  $\lim_{|t| \rightarrow \infty} h(t)|t|^{-\frac{\alpha+2}{N-2}} = 0$ , then*

$$\int_{\mathbb{R}^N} |h(u_n) - h(u_n - u) - h(u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} = o_n(1) \|\phi\|^{\frac{2N}{N+\alpha}},$$

where  $o_n(1) \rightarrow 0$  uniformly for any  $\phi \in C_0^\infty(\mathbb{R}^N)$  as  $n \rightarrow \infty$ .

*Proof.* The proofs of (i) and (1) are similar to [66, Lemma 2.5]. We only give the proof of (2) which is inspired by [1] and [68, Lemma 4.7].

In the following, let  $C$  denote a positive constant (independent of  $\varepsilon, k$ ) which may change from line to line. For any fixed  $\varepsilon \in (0, 1)$ , there exists  $s_0 = s_0(\varepsilon) \in (0, 1)$  such that  $|h(t)| \leq \varepsilon|t|$  for  $|t| \leq 2s_0$ . Choose  $s_1 = s_1(\varepsilon) > 2$  such that

$$|h(t)| \leq \varepsilon|t|^{\frac{2+\alpha}{N-2}}$$

for  $|t| \geq s_1 - 1$ . From the continuity of  $h$ , there exists  $\delta = \delta(\varepsilon) \in (0, s_0)$  such that  $|h(t_1) - h(t_2)| \leq s_0\varepsilon$  for  $|t_1 - t_2| \leq \delta, |t_1|, |t_2| \leq s_1 + 1$ . Moreover, there exists  $c(\varepsilon) > 0$  such that

$$|h(t)| \leq c(\varepsilon)|t| + \varepsilon|t|^{\frac{2+\alpha}{N-2}}$$

for  $t \in \mathbb{R}$ . Noting that  $\alpha \in ((N - 4)_+, N)$ , we have  $2 < \frac{4N}{N+\alpha} < \frac{2N}{N-2}$ . Then there exists  $R = R(\varepsilon) > 0$  such that

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B(0,R)} |h(u)\phi|^{\frac{2N}{N+\alpha}} &\leq C \int_{\mathbb{R}^N \setminus B(0,R)} \left( |u|^{\frac{2N}{N+\alpha}} + \varepsilon|u|^{\frac{2+\alpha}{N-2} \frac{2N}{N+\alpha}} \right) |\phi|^{\frac{2N}{N+\alpha}} \\ &\leq C \left( \int_{\mathbb{R}^N \setminus B(0,R)} |u|^{\frac{4N}{N+\alpha}} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |\phi|^{\frac{4N}{N+\alpha}} \right)^{\frac{1}{2}} \\ &\quad + C\varepsilon \left( \int_{\mathbb{R}^N \setminus B(0,R)} |u|^{\frac{2N}{N-2}} \right)^{\frac{2+\alpha}{N+\alpha}} \left( \int_{\mathbb{R}^N} |\phi|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N+\alpha}} \\ &\leq C\varepsilon \|\phi\|^{\frac{2N}{N+\alpha}}. \end{aligned} \tag{2.1}$$

Setting  $A_n := \{x \in \mathbb{R}^N \setminus B(0, R) : |u_n(x)| \leq s_0\}$ ; then

$$\int_{A_n \cap \{|u| \leq \delta\}} |h(u_n) - h(u_n - u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} \leq C\varepsilon \int_{\mathbb{R}^N} \left( |u_n|^{\frac{2N}{N+\alpha}} + |u_n - u|^{\frac{2N}{N+\alpha}} \right) |\phi|^{\frac{2N}{N+\alpha}} \leq C\varepsilon \|\phi\|^{\frac{2N}{N+\alpha}}.$$

Let  $B_n := \{x \in \mathbb{R}^N \setminus B(0, R) : |u_n(x)| \geq s_1\}$ . Then

$$\int_{B_n \cap \{|u| \leq \delta\}} |h(u_n) - h(u_n - u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} \leq C\varepsilon \int_{\mathbb{R}^N} \left( |u_n|^{\frac{2+\alpha}{N-2} \frac{2N}{N+\alpha}} + |u_n - u|^{\frac{2+\alpha}{N-2} \frac{2N}{N+\alpha}} \right) |\phi|^{\frac{2N}{N+\alpha}} \leq C\varepsilon \|\phi\|^{\frac{2N}{N+\alpha}}.$$

Setting  $C_n := \{x \in \mathbb{R}^N \setminus B(0, R) : s_0 \leq |u_n(x)| \leq s_1\}$ ; then  $|C_n| < \infty$  and

$$\begin{aligned} \int_{C_n \cap \{|u| \leq \delta\}} |h(u_n) - h(u_n - u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} &\leq (s_0 \varepsilon)^{\frac{2N}{N+\alpha}} \int_{C_n \cap \{|u| \leq \delta\}} |\phi|^{\frac{2N}{N+\alpha}} \leq (s_0 \varepsilon)^{\frac{2N}{N+\alpha}} |C_n|^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |\phi|^{\frac{4N}{N+\alpha}} \right)^{\frac{1}{2}} \\ &\leq \varepsilon^{\frac{2N}{N+\alpha}} \left( \int_{C_n} |u_n|^{\frac{4N}{N+\alpha}} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |\phi|^{\frac{4N}{N+\alpha}} \right)^{\frac{1}{2}} \leq C\varepsilon \|\phi\|^{\frac{2N}{N+\alpha}}. \end{aligned}$$

Thus,  $(\mathbb{R}^N \setminus B(0, R)) \cap \{|u| \leq \delta\} = A_n \cup B_n \cup C_n$  and

$$\int_{(\mathbb{R}^N \setminus B(0, R)) \cap \{|u| \leq \delta\}} |h(u_n) - h(u_n - u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} \leq C\varepsilon \|\phi\|^{\frac{2N}{N+\alpha}} \quad \text{for all } n.$$

Clearly, for  $\varepsilon$  given above, there exists  $c(\varepsilon) > 0$  such that

$$|h(u_n) - h(u_n - u)|^{\frac{2N}{N+\alpha}} \leq \varepsilon \left( |u_n|^{\frac{2+\alpha}{N-2} \frac{2N}{N+\alpha}} + |u_n - u|^{\frac{2+\alpha}{N-2} \frac{2N}{N+\alpha}} \right) + c(\varepsilon) \left( |u_n|^{\frac{2N}{N+\alpha}} + |u_n - u|^{\frac{2N}{N+\alpha}} \right)$$

and

$$\begin{aligned} &\int_{(\mathbb{R}^N \setminus B(0, R)) \cap \{|u| \geq \delta\}} |h(u_n) - h(u_n - u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} \\ &\leq \int_{(\mathbb{R}^N \setminus B(0, R)) \cap \{|u| \geq \delta\}} \varepsilon \left( |u_n|^{\frac{2+\alpha}{N-2} \frac{2N}{N+\alpha}} + |u_n - u|^{\frac{2+\alpha}{N-2} \frac{2N}{N+\alpha}} \right) |\phi|^{\frac{2N}{N+\alpha}} + c(\varepsilon) \left( |u_n|^{\frac{2N}{N+\alpha}} + |u_n - u|^{\frac{2N}{N+\alpha}} \right) |\phi|^{\frac{2N}{N+\alpha}} \\ &\leq C\varepsilon \|\phi\|^{\frac{2N}{N+\alpha}} + c(\varepsilon) \int_{(\mathbb{R}^N \setminus B(0, R)) \cap \{|u| \geq \delta\}} \left( |u_n|^{\frac{2N}{N+\alpha}} + |u_n - u|^{\frac{2N}{N+\alpha}} \right) |\phi|^{\frac{2N}{N+\alpha}}. \end{aligned}$$

Noting that  $0 < \alpha + 4 - N < N + \alpha$  and  $|(\mathbb{R}^N \setminus B(0, R)) \cap \{|u| \geq \delta\}| \rightarrow 0$  as  $R \rightarrow \infty$ , there exists  $R = R(\varepsilon)$  large enough, such that

$$\begin{aligned} &\int_{(\mathbb{R}^N \setminus B(0, R)) \cap \{|u| \geq \delta\}} c(\varepsilon) \left( |u_n|^{\frac{2N}{N+\alpha}} + |u_n - u|^{\frac{2N}{N+\alpha}} \right) |\phi|^{\frac{2N}{N+\alpha}} \\ &\leq c(\varepsilon) \left[ \left( \int_{\mathbb{R}^N} |u_n|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N+\alpha}} + \left( \int_{\mathbb{R}^N} |u_n - u|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N+\alpha}} \right] \left( \int_{\mathbb{R}^N} |\phi|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N+\alpha}} |(\mathbb{R}^N \setminus B(0, R)) \cap \{|u| \geq \delta\}|^{\frac{\alpha+4-N}{N+\alpha}} \\ &\leq \varepsilon \|\phi\|^{\frac{2N}{N+\alpha}}. \end{aligned}$$

Then, for any  $n$ ,

$$\int_{(\mathbb{R}^N \setminus B(0, R)) \cap \{|u| \geq \delta\}} |h(u_n) - h(u_n - u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} \leq C\varepsilon \|\phi\|^{\frac{2N}{N+\alpha}}.$$

Thus, by (2.1), for any  $n$ ,

$$\int_{\mathbb{R}^N \setminus B(0, R)} |h(u_n) - h(u) - h(u_n - u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} \leq C\varepsilon \|\phi\|^{\frac{2N}{N+\alpha}}. \tag{2.2}$$

Finally, for  $\varepsilon > 0$  given above, there exists  $C(\varepsilon) > 0$  such that

$$|h(t)|^{\frac{2N}{N+\alpha}} \leq C(\varepsilon) |t|^{\frac{2N}{N+\alpha}} + \varepsilon |t|^{\frac{2N}{N+\alpha} \frac{2+\alpha}{N-2}}, \quad t \in \mathbb{R}. \tag{2.3}$$

Recalling that  $u_n \rightarrow u$  weakly in  $H^1(\mathbb{R}^N)$ , up to a subsequence,  $u_n \rightarrow u$  strongly in  $L^{4N/(N+\alpha)}(B(0, R))$  and there exists  $\omega \in L^{4N/(N+\alpha)}(B(0, R))$  such that  $|u_n(x)|, |u(x)| \leq |\omega(x)|$  a.e.  $x \in B(0, R)$ . Then we easily get for  $n$  large enough,

$$\int_{B(0, R)} |h(u_n - u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} \leq \int_{B(0, R)} \left( C(\varepsilon) |u_n - u|^{\frac{2N}{N+\alpha}} + \varepsilon |u_n - u|^{\frac{2N}{N+\alpha} \frac{2+\alpha}{N-2}} \right) |\phi|^{\frac{2N}{N+\alpha}} \leq C\varepsilon \|\phi\|^{\frac{2N}{N+\alpha}}. \tag{2.4}$$

Moreover, let  $D_n := \{x \in B(0, R) : |u_n(x) - u(x)| \geq 1\}$ , then by (2.3),

$$\begin{aligned} \int_{D_n} |h(u_n) - h(u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} &\leq \int_{D_n} \left[ C(\varepsilon) \left( |u|^{\frac{2N}{N+\alpha}} + |u_n|^{\frac{2N}{N+\alpha}} \right) + \varepsilon \left( |u_n|^{\frac{2N}{N+\alpha} \frac{2+\alpha}{N-2}} + |u|^{\frac{2N}{N+\alpha} \frac{2+\alpha}{N-2}} \right) \right] |\phi|^{\frac{2N}{N+\alpha}} \\ &\leq C\varepsilon \|\phi\|^{\frac{2N}{N+\alpha}} + 2C(\varepsilon) \int_{D_n} |\omega|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} \\ &\leq C\varepsilon \|\phi\|^{\frac{2N}{N+\alpha}} + 2C(\varepsilon) \left( \int_{D_n} |\omega|^{\frac{4N}{N+\alpha}} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |\phi|^{\frac{4N}{N+\alpha}} \right)^{\frac{1}{2}}. \end{aligned}$$

By  $u_n \rightarrow u$  a.e.  $x \in B(0, R)$ , we get  $|D_n| \rightarrow 0$  as  $n \rightarrow \infty$ . Hence,

$$\int_{D_n} |h(u_n) - h(u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} \leq C\varepsilon \|\phi\|^{\frac{2N}{N+\alpha}} \quad \text{for } n \text{ large.} \tag{2.5}$$

On the other hand, for  $\varepsilon$  given above, there exists  $c(\varepsilon) > 0$  such that

$$|h(u_n) - h(u)|^{\frac{2N}{N+\alpha}} \leq \varepsilon \left( |u_n|^{\frac{2+\alpha}{N-2} \frac{2N}{N+\alpha}} + |u_n|^{\frac{2+\alpha}{N-2} \frac{2N}{N+\alpha}} \right) + c(\varepsilon) \left( |u_n|^{\frac{2N}{N+\alpha}} + |u_n|^{\frac{2N}{N+\alpha}} \right).$$

Noting that  $|\{|u| \geq L\}| \rightarrow 0$  as  $L \rightarrow \infty$ , similarly as above, there exists  $L = L(\varepsilon) > 0$  such that for all  $n$ ,

$$\int_{(B(0,R) \setminus D_n) \cap \{|u| \geq L\}} |h(u_n) - h(u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} \leq C\varepsilon \|\phi\|^{\frac{2N}{N+\alpha}}.$$

By the Lebesgue dominated convergence theorem,

$$\int_{(B(0,R) \setminus D_n) \cap \{|u| \leq L\}} |h(u_n) - h(u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} = o_n(1) \|\phi\|^{\frac{2N}{N+\alpha}},$$

where  $o_n(1) \rightarrow 0$  as  $n \rightarrow \infty$  uniformly in  $\phi$ . Then by (2.5),

$$\int_{B(0,R)} |h(u_n) - h(u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} \leq C\varepsilon \|\phi\|^{\frac{2N}{N+\alpha}} \quad \text{for } n \text{ large.}$$

Then, by (2.4) and for  $n$  large,

$$\int_{B(0,R)} |h(u_n) - h(u) - h(u_n - u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} \leq C\varepsilon \|\phi\|^{\frac{2N}{N+\alpha}} \quad \text{for } n \text{ sufficiently large.}$$

Finally, combining the previous estimate with (2.2), we conclude the proof. □

**Lemma 2.6.** *Let  $\alpha \in (0, N)$ ,  $s \in (1, \frac{N}{\alpha})$  and let  $\{g_n\} \in L^1(\mathbb{R}^N) \cap L^s(\mathbb{R}^N)$  be bounded and such that, up to a subsequence, for any bounded domain  $\Omega \subset \mathbb{R}^N$ ,  $g_n \rightarrow 0$  strongly in  $L^s(\Omega)$  as  $n \rightarrow \infty$ . Then, up to a subsequence if necessary,  $(I_\alpha * g_n)(x) \rightarrow 0$  a.e. in  $\mathbb{R}^N$  as  $n \rightarrow \infty$ .*

*Proof.* Let us prove that for any fixed positive  $k \in \mathbb{N}$ , passing to a subsequence if necessary,  $(I_\alpha * g_n)(x) \rightarrow 0$  a.e. in  $B_k(0)$  as  $n \rightarrow \infty$ . Let  $k \in \mathbb{N}^+$  be fixed and for any  $\delta > 0$ , there exists  $K = K(\delta) > k$  such that

$$A_\alpha \int_{\mathbb{R}^N \setminus B_K(x)} \frac{|g_n(y)|}{|x - y|^{N-\alpha}} dy \leq \delta \quad \text{for any } x \in \mathbb{R}^N, n \in \mathbb{N}^+.$$

Obviously,  $B_K(x) \subset B_{2K}(0)$  for any  $x \in B_k(0)$ . Noting that  $g_n \chi_{B_{2K}(0)} \in L^s(\mathbb{R}^N)$ , by Remark 2.1,

$$\|I_\alpha * (|g_n| \chi_{B_{2K}(0)})\|_{L^{\frac{Ns}{N-\alpha s}}(\mathbb{R}^N)} \leq C \|g_n\|_{L^s(B_{2K}(0))},$$

where the constant  $C$  depends only on  $N, \alpha$ . It follows that, up to a subsequence,  $I_\alpha * (|g_n| \chi_{B_{2K}(0)}) \rightarrow 0$  strongly

in  $L^{Ns/(N-\alpha s)}(\mathbb{R}^N)$  and a.e. in  $B_k(0)$  as  $n \rightarrow \infty$ . Then, for almost every  $x \in B_k(0)$ , one has

$$\begin{aligned} \limsup_{n \rightarrow \infty} |(I_\alpha * g_n)(x)| &\leq A_\alpha \limsup_{n \rightarrow \infty} \left( \int_{B_k(x)} \frac{|g_n(y)|}{|x-y|^{N-\alpha}} dy + \int_{\mathbb{R}^N \setminus B_k(x)} \frac{|g_n(y)|}{|x-y|^{N-\alpha}} dy \right) \\ &\leq \delta + A_\alpha \limsup_{n \rightarrow \infty} \int_{B_k(x)} \frac{|g_n(y)|}{|x-y|^{N-\alpha}} dy \\ &\leq \delta + A_\alpha \limsup_{n \rightarrow \infty} \int_{B_{2k}(0)} \frac{|g_n(y)|}{|x-y|^{N-\alpha}} dy \\ &= \delta + \limsup_{n \rightarrow \infty} [I_\alpha * (|g_n| \chi_{B_{2k}(0)})](x) = \delta. \end{aligned}$$

Since  $\delta$  is arbitrary, the proof is completed. □

Now we are set to prove Lemma 2.4.

*Proof of Lemma 2.4.* Set

$$f_1(t) = f(t) - |t|^{\frac{4+\alpha-N}{N-2}} t \quad \text{and} \quad F_1(t) = \int_0^t f_1(s) ds, \quad t \in \mathbb{R}.$$

Observe that for any  $\phi \in C_0^\infty(\mathbb{R}^N)$ ,

$$\int_{\mathbb{R}^N} [I_\alpha * F(u_n)] f(u_n) \phi = \int_{\mathbb{R}^N} [I_\alpha * F(u_n)] f_1(u_n) \phi + \int_{\mathbb{R}^N} [I_\alpha * F(u_n)] |u_n|^{\frac{4+\alpha-N}{N-2}} u_n \phi.$$

**Step 1.** We claim

$$\begin{aligned} \int_{\mathbb{R}^N} [I_\alpha * F(u_n)] |u_n|^{\frac{4+\alpha-N}{N-2}} u_n \phi &= \int_{\mathbb{R}^N} [I_\alpha * F(u_n - u)] |u_n - u|^{\frac{4+\alpha-N}{N-2}} (u_n - u) \phi \\ &\quad + \int_{\mathbb{R}^N} [I_\alpha * F(u)] |u|^{\frac{4+\alpha-N}{N-2}} u \phi + o_n(1) \|\phi\|, \end{aligned}$$

where  $o_n(1) \rightarrow 0$  uniformly for any  $\phi \in C_0^\infty(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . Noting that  $\alpha > N - 4$ , by Lemma 2.5 (ii) (1) with  $h(t) = f(t)$ ,  $q = \frac{2+\alpha}{N-2}$ ,  $r = \frac{2N}{N-2}$ ,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |F(u_n) - F(u_n - u) - F(u)|^{\frac{2N}{N+\alpha}} = 0. \tag{2.6}$$

Then for  $v_n = |u_n|^{\frac{4+\alpha-N}{N-2}} u_n$ , as well as  $v_n = |u_n - u|^{\frac{4+\alpha-N}{N-2}} (u_n - u)$  and also  $v_n = |u|^{\frac{4+\alpha-N}{N-2}} u$ , there exists  $C > 0$  such that

$$\int_{\mathbb{R}^N} |v_n \phi|^{\frac{2N}{N+\alpha}} \leq \left( \int_{\mathbb{R}^N} |v_n|^{\frac{2N}{2+\alpha}} \right)^{\frac{2+\alpha}{N+\alpha}} \left( \int_{\mathbb{R}^N} |\phi|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N+\alpha}} \leq C \left( \int_{\mathbb{R}^N} |\phi|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N+\alpha}},$$

from which it follows

$$\begin{aligned} \left| \int_{\mathbb{R}^N} [I_\alpha * (F(u_n) - F(u_n - u) - F(u))] v_n \phi \right| &\leq C \left( \int_{\mathbb{R}^N} |F(u_n) - F(u_n - u) - F(u)|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \left( \int_{\mathbb{R}^N} |v_n \phi|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \\ &= o_n(1) \left( \int_{\mathbb{R}^N} |v_n \phi|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} = o_n(1) \|\phi\|, \end{aligned}$$

where  $o_n(1) \rightarrow 0$  uniformly for any  $\phi \in C_0^\infty(\mathbb{R}^N)$  as  $n \rightarrow \infty$ .

On the other hand, by virtue of (i) of Lemma 2.5 with  $q = \frac{2+\alpha}{N-2}$  and  $r = \frac{2N}{N-2}$ ,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left| |u_n|^{\frac{4+\alpha-N}{N-2}} u_n - |u_n - u|^{\frac{4+\alpha-N}{N-2}} (u_n - u) - |u|^{\frac{4+\alpha-N}{N-2}} u \right|^{\frac{2N}{2+\alpha}} = 0.$$

For  $w_n = F(u_n)$ , as well as  $w_n = F(u_n - u)$  and also  $w_n = F(u)$ , one easily gets  $\{w_n\}$  bounded in  $L^{2N/(N+\alpha)}(\mathbb{R}^N)$ . By the Hardy–Littlewood–Sobolev inequality and Hölder’s inequality, there exists  $C > 0$  such that

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} [I_\alpha * w_n] [|u_n|^{\frac{4+\alpha-N}{N-2}} u_n - |u_n - u|^{\frac{4+\alpha-N}{N-2}} (u_n - u) - |u|^{\frac{4+\alpha-N}{N-2}} u] \phi \right| \\ & \leq C \left( \int_{\mathbb{R}^N} \left| |u_n|^{\frac{4+\alpha-N}{N-2}} u_n - |u_n - u|^{\frac{4+\alpha-N}{N-2}} (u_n - u) - |u|^{\frac{4+\alpha-N}{N-2}} u \right|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \\ & \leq C \left( \int_{\mathbb{R}^N} \left| |u_n|^{\frac{4+\alpha-N}{N-2}} u_n - |u_n - u|^{\frac{4+\alpha-N}{N-2}} (u_n - u) - |u|^{\frac{4+\alpha-N}{N-2}} u \right|^{\frac{2+\alpha}{2N}} \right)^{\frac{2+\alpha}{2N}} \left( \int_{\mathbb{R}^N} |\phi|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{2N}} \\ & = o_n(1) \|\phi\|, \end{aligned}$$

where  $o_n(1) \rightarrow 0$  uniformly for any  $\phi \in C_0^\infty(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . Then we get

$$\begin{aligned} \int_{\mathbb{R}^N} [I_\alpha * F(u_n)] |u_n|^{\frac{4+\alpha-N}{N-2}} u_n \phi &= \int_{\mathbb{R}^N} [I_\alpha * F(u_n - u)] |u_n - u|^{\frac{4+\alpha-N}{N-2}} (u_n - u) \phi + \int_{\mathbb{R}^N} [I_\alpha * F(u)] |u|^{\frac{4+\alpha-N}{N-2}} u \phi \\ &+ \int_{\mathbb{R}^N} [I_\alpha * F(u_n - u)] |u|^{\frac{4+\alpha-N}{N-2}} u \phi \\ &+ \int_{\mathbb{R}^N} [I_\alpha * F(u)] |u_n - u|^{\frac{4+\alpha-N}{N-2}} (u_n - u) \phi + o_n(1) \|\phi\|, \end{aligned}$$

where  $o_n(1) \rightarrow 0$  uniformly for any  $\phi \in C_0^\infty(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . Noting that  $F(u) \in L^{2N/(N+\alpha)}(\mathbb{R}^N)$ , by Remark 2.1,  $|I_\alpha * F(u)|^{\frac{2N}{N+2}} \in L^{(N+2)/(N-\alpha)}(\mathbb{R}^N)$ . By virtue of Lemma 2.3,  $|u_n - u|^{\frac{2N(2+\alpha)}{(N-2)(N+2)}} \rightarrow 0$  weakly in  $L^{(N+2)/(2+\alpha)}(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . This yields

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |I_\alpha * F(u)|^{\frac{2N}{N+2}} |u_n - u|^{\frac{2N(2+\alpha)}{(N-2)(N+2)}} = 0,$$

which implies that

$$\begin{aligned} \left| \int_{\mathbb{R}^N} [I_\alpha * F(u)] |u_n - u|^{\frac{4+\alpha-N}{N-2}} (u_n - u) \phi \right| &\leq \left( \int_{\mathbb{R}^N} |I_\alpha * F(u)|^{\frac{2N}{N+2}} |u_n - u|^{\frac{2N(2+\alpha)}{(N-2)(N+2)}} \right)^{\frac{N+2}{2N}} \left( \int_{\mathbb{R}^N} |\phi|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{2N}} \\ &= o_n(1) \|\phi\|, \end{aligned}$$

where  $o_n(1) \rightarrow 0$  uniformly for any  $\phi \in C_0^\infty(\mathbb{R}^N)$  as  $n \rightarrow \infty$ .

At the same time, since  $\alpha \in ((N - 4)_+, N)$ , for  $s \in (1, \frac{2N}{N+\alpha}) \subset (1, \frac{N}{\alpha})$ , by Rellich’s theorem, up to a subsequence, for any bounded domain  $\Omega \subset \mathbb{R}^N$ ,  $F(u_n - u) \rightarrow 0$  strongly in  $L^s(\Omega)$  as  $n \rightarrow \infty$ . By Lemma 2.6, up to a subsequence,  $I_\alpha * F(u_n - u) \rightarrow 0$  a.e. in  $\mathbb{R}^N$  as  $n \rightarrow \infty$ . By Remark 2.1 we have

$$\sup_n \left\| |I_\alpha * F(u_n - u)|^{\frac{2N}{N+2}} \right\|_{L^{(N+2)/(N-\alpha)}(\mathbb{R}^N)} \leq C \sup_n \|F(u_n - u)\|_{L^{2N/(N+\alpha)}(\mathbb{R}^N)} < \infty,$$

which yields, by Lemma 2.3,  $|I_\alpha * F(u_n - u)|^{\frac{2N}{N+2}} \rightarrow 0$  weakly in  $L^{(N+2)/(N-\alpha)}(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . Noting that  $|u|^{\frac{2+\alpha}{N-2}} \frac{2N}{N+2} \in L^{(N+2)/(2+\alpha)}(\mathbb{R}^N)$ ,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |I_\alpha * F(u_n - u)|^{\frac{2N}{N+2}} |u|^{\frac{2+\alpha}{N-2}} \frac{2N}{N+2} = 0 \tag{2.7}$$

and by Hölder’s inequality,

$$\left| \int_{\mathbb{R}^N} [I_\alpha * F(u_n - u)] |u|^{\frac{4+\alpha-N}{N-2}} u \phi \right| \leq \left( \int_{\mathbb{R}^N} |I_\alpha * F(u_n - u)|^{\frac{2N}{N+2}} |u|^{\frac{2+\alpha}{N-2}} \frac{2N}{N+2} \right)^{\frac{N+2}{2N}} \left( \int_{\mathbb{R}^N} |\phi|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{2N}} = o_n(1) \|\phi\|,$$

where  $o_n(1) \rightarrow 0$  uniformly for any  $\phi \in C_0^\infty(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . The claim is thus proved.

Step 2. We claim

$$\int_{\mathbb{R}^N} [I_\alpha * F(u_n)]f_1(u_n)\phi = \int_{\mathbb{R}^N} [I_\alpha * F(u_n - u)]f_1(u_n - u)\phi + \int_{\mathbb{R}^N} [I_\alpha * F(u)]f_1(u)\phi + o_n(1)\|\phi\|, \tag{2.8}$$

where  $o_n(1) \rightarrow 0$  uniformly for any  $\phi \in C_0^\infty(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . The following hold:

$$\left\{ \begin{array}{l} \int_{\mathbb{R}^N} [I_\alpha * (F(u_n) - F(u_n - u) - F(u))]f_1(u_n)\phi = o_n(1)\|\phi\|, \\ \int_{\mathbb{R}^N} [I_\alpha * (F(u_n) - F(u_n - u) - F(u))]f_1(u_n - u)\phi = o_n(1)\|\phi\|, \\ \int_{\mathbb{R}^N} [I_\alpha * (F(u_n) - F(u_n - u) - F(u))]f_1(u)\phi = o_n(1)\|\phi\|, \end{array} \right. \tag{2.9}$$

where  $o_n(1) \rightarrow 0$  uniformly for any  $\phi \in C_0^\infty(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . Let us only prove the first identity in (2.9), the remaining ones being similar. Observe that there exists  $\delta \in (0, 1)$  and  $C(\delta) > 0$  such that  $|f_1(t)| \leq |t|$  for  $|t| \leq \delta$  and  $|f_1(t)| \leq C(\delta)|t|^{\frac{2+\alpha}{N-2}}$  for  $|t| \geq \delta$ . Noting that  $\alpha \in ((N - 4)_+, N)$ , we have  $2 < \frac{4N}{N+\alpha} < \frac{2N}{N-2}$ . Then, for any  $\phi \in C_0^\infty(\mathbb{R}^N)$ , there exists  $C > 0$  (independent of  $\phi, n$ ) such that

$$\begin{aligned} \int_{\mathbb{R}^N} |f_1(u_n)\phi|^{\frac{2N}{N+\alpha}} &= \int_{\{x \in \mathbb{R}^N : |u_n(x)| \leq \delta\}} |f_1(u_n)\phi|^{\frac{2N}{N+\alpha}} + \int_{\{x \in \mathbb{R}^N : |u_n(x)| \geq \delta\}} |f_1(u_n)\phi|^{\frac{2N}{N+\alpha}} \\ &\leq \int_{\{x \in \mathbb{R}^N : |u_n(x)| \leq \delta\}} |u_n\phi|^{\frac{2N}{N+\alpha}} + [C(\delta)]^{\frac{2N}{N+\alpha}} \int_{\{x \in \mathbb{R}^N : |u_n(x)| \geq \delta\}} |u_n|^{\frac{2N(2+\alpha)}{(N-2)(N+\alpha)}} |\phi|^{\frac{2N}{N+\alpha}} \\ &\leq \left( \int_{\mathbb{R}^N} |u_n|^{\frac{4N}{N+\alpha}} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |\phi|^{\frac{4N}{N+\alpha}} \right)^{\frac{1}{2}} + [C(\delta)]^{\frac{2N}{N+\alpha}} \left( \int_{\mathbb{R}^N} |u_n|^{\frac{2N}{N-2}} \right)^{\frac{2+\alpha}{N+\alpha}} \left( \int_{\mathbb{R}^N} |\phi|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N+\alpha}} \\ &\leq C\|\phi\|^{\frac{2N}{N+\alpha}} \quad \text{for all } n \geq 1. \end{aligned}$$

Thus

$$\left( \int_{\mathbb{R}^N} |f_1(u_n)\phi|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \leq C\|\phi\| \quad \text{uniformly for all } \phi \in C_0^\infty(\mathbb{R}^N), n = 1, 2, \dots$$

Then by the Hardy–Littlewood–Sobolev inequality and (2.6),

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} [I_\alpha * (F(u_n) - F(u_n - u) - F(u))]f_1(u_n)\phi \right| \\ &\leq C \left( \int_{\mathbb{R}^N} |F(u_n) - F(u_n - u) - F(u)|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \left( \int_{\mathbb{R}^N} |f_1(u_n)\phi|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \\ &= o_n(1)\|\phi\|, \end{aligned}$$

where  $o_n(1) \rightarrow 0$  uniformly for any  $\phi \in C_0^\infty(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . So (2.9) holds.

Similarly we prove

$$\left\{ \begin{array}{l} \int_{\mathbb{R}^N} (I_\alpha * F(u_n))[f_1(u_n) - f_1(u_n - u) - f_1(u)]\phi = o_n(1)\|\phi\|, \\ \int_{\mathbb{R}^N} (I_\alpha * F(u_n - u))[f_1(u_n) - f_1(u_n - u) - f_1(u)]\phi = o_n(1)\|\phi\|, \\ \int_{\mathbb{R}^N} (I_\alpha * F(u))[f_1(u_n) - f_1(u_n - u) - f_1(u)]\phi = o_n(1)\|\phi\|, \end{array} \right. \tag{2.10}$$

where  $o_n(1) \rightarrow 0$  uniformly for any  $\phi \in C_0^\infty(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . By the Hardy–Littlewood–Sobolev inequality and (ii)<sub>2</sub> of Lemma 2.5, there exists  $C > 0$  such that

$$\begin{aligned} \left| \int_{\mathbb{R}^N} (I_\alpha * F(u_n))[f_1(u_n) - f_1(u_n - u) - f_1(u)]\phi \right| &\leq C \left( \int_{\mathbb{R}^N} |f_1(u_n) - f_1(u_n - u) - f_1(u)|^{\frac{2N}{N+\alpha}} |\phi|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \\ &= o_n(1)\|\phi\|, \end{aligned}$$

where  $o_n(1) \rightarrow 0$  uniformly for any  $\phi \in C_0^\infty(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . So the first identity of (2.10) holds and the remaining can be proved in a similar fashion.

Combine (2.9) and (2.10) to have

$$\begin{aligned} \int_{\mathbb{R}^N} [I_\alpha * F(u_n)]f_1(u_n)\phi &= \int_{\mathbb{R}^N} [I_\alpha * F(u_n - u)]f_1(u_n - u)\phi + \int_{\mathbb{R}^N} [I_\alpha * F(u)]f_1(u)\phi \\ &\quad + \int_{\mathbb{R}^N} [I_\alpha * F(u_n - u)]f_1(u)\phi + \int_{\mathbb{R}^N} [I_\alpha * F(u)]f_1(u_n - u)\phi + o_n(1)\|\phi\|, \end{aligned}$$

where  $o_n(1) \rightarrow 0$  uniformly for any  $\phi \in C_0^\infty(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . To conclude the proof of (2.8), it remains to prove

$$\int_{\mathbb{R}^N} [I_\alpha * F(u_n - u)]f_1(u)\phi = o_n(1)\|\phi\|$$

and

$$\int_{\mathbb{R}^N} [I_\alpha * F(u)]f_1(u_n - u)\phi = o_n(1)\|\phi\|, \tag{2.11}$$

where  $o_n(1) \rightarrow 0$  uniformly for any  $\phi \in C_0^\infty(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . Notice that for any  $\varepsilon \in (0, 1)$ , there exist  $\delta \in (0, 1)$  and  $C_\varepsilon > 0$  such that  $|f_1(t)| \leq \varepsilon|t|$  for  $|t| \leq \delta$  and  $|f_1(t)| \leq C_\varepsilon|t|^{\frac{2+\alpha}{N-2}}$  for  $|t| \geq \delta$ . Then, for any  $\phi \in C_0^\infty(\mathbb{R}^N)$ , by the Hardy–Littlewood–Sobolev inequality and Hölder’s inequality,

$$\begin{aligned} \left| \int_{\mathbb{R}^N} [I_\alpha * F(u_n - u)]f_1(u)\phi \right| &\leq \varepsilon \int_{\{x \in \mathbb{R}^N : |u(x)| \leq \delta\}} |I_\alpha * F(u_n - u)||u\phi| + C_\varepsilon \int_{\{x \in \mathbb{R}^N : |u(x)| \geq \delta\}} |I_\alpha * F(u_n - u)||u|^{\frac{2+\alpha}{N-2}}|\phi| \\ &\leq \varepsilon \|F(u_n - u)\|_{L^{2N/(N+\alpha)}(\mathbb{R}^N)} \left( \int_{\{x \in \mathbb{R}^N : |u(x)| \leq \delta\}} |u\phi|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \\ &\quad + C_\varepsilon \left( \int_{\mathbb{R}^N} |I_\alpha * F(u_n - u)|^{\frac{2N}{N+2}} |u|^{\frac{2+\alpha}{N-2} \frac{2N}{N+2}} \right)^{\frac{N+2}{2N}} \left( \int_{\mathbb{R}^N} |\phi|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{2N}}. \end{aligned}$$

There exists  $c > 0$  (independent of  $\phi, \delta, \varepsilon$ ) such that

$$\int_{\{x \in \mathbb{R}^N : |u(x)| \leq \delta\}} |u\phi|^{\frac{2N}{N+\alpha}} \leq c\|\phi\|^{\frac{2N}{N+\alpha}}.$$

Then by (2.7), there exists  $\tilde{C} > 0$  (independent of  $\phi, \varepsilon$ ) such that

$$\limsup_{n \rightarrow \infty} \left| \int_{\mathbb{R}^N} [I_\alpha * F(u_n - u)]f_1(u)\phi \right| \leq \tilde{C}\varepsilon\|\phi\|.$$

It follows that

$$\int_{\mathbb{R}^N} [I_\alpha * F(u_n - u)]f_1(u)\phi = o_n(1)\|\phi\|,$$

where  $o_n(1) \rightarrow 0$  uniformly for any  $\phi \in C_0^\infty(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . Similarly, (2.11) can be proved and the proof of Lemma 2.4 is complete. □

### 3 Ground state solutions: Proof of Theorem 1.1

Since we are looking for positive ground state solutions to (1.2), we may assume that  $f$  is odd in  $\mathbb{R}^N$ . In this section, a key tool is a monotonicity trick, originally due to Struwe [57] and which here we borrow in the abstract form due to Jeanjean and Toland [32, 34].

For  $\lambda \in [\frac{1}{2}, 1]$ , we consider the following family of functionals:

$$I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + au^2 - \frac{\lambda}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u), \quad u \in H^1(\mathbb{R}^N).$$

Obviously, if  $f$  satisfies the assumptions of Theorem 1.1, for  $\lambda \in [\frac{1}{2}, 1]$ ,  $I_\lambda \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$  and every critical point of  $I_\lambda$  is a weak solution of

$$-\Delta u + au = \lambda(I_\alpha * F(u))f(u), \quad u \in H^1(\mathbb{R}^N). \tag{3.1}$$

The existence of critical points to  $I_\lambda$  is a consequence of the following abstract result

**Theorem A** (see [32]). *Let  $X$  be a Banach space equipped with a norm  $\|\cdot\|_X$ , let  $J \subset \mathbb{R}^+$  be an interval and let a family of  $C^1$ -functionals  $\{I_\lambda\}_{\lambda \in J}$  be given on  $X$  of the form*

$$I_\lambda(u) = A(u) - \lambda B(u), \quad u \in X.$$

*Assume that  $B(u) \geq 0$  for any  $u \in X$ , at least one between  $A$  and  $B$  is coercive on  $X$  and there exist two points  $v_1, v_2 \in X$  such that for any  $\lambda \in J$ ,*

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\},$$

*where  $\Gamma := \{\gamma \in C([0, 1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}$ . Then, for almost every  $\lambda \in J$ , the  $C^1$ -functional  $I_\lambda$  admits a bounded Palais–Smale sequence at level  $c_\lambda$ . Moreover,  $c_\lambda$  is left-continuous with respect to  $\lambda \in [\frac{1}{2}, 1]$ .*

In the following, set  $X = H^1(\mathbb{R}^N)$  and

$$A(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + au^2, \quad B(u) = \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u).$$

Obviously,  $A(u) \rightarrow +\infty$  as  $\|u\| \rightarrow \infty$ . Thanks to (F3),  $B(u) \geq 0$  for any  $u \in H^1(\mathbb{R}^N)$ . Moreover, by (F1)–(F2), for any  $\varepsilon > 0$ , there exists  $C_\varepsilon > 0$  such that  $F(t) \leq \varepsilon|t|^{(N+\alpha)/N} + C_\varepsilon|t|^{(N+\alpha)/(N-2)}$  for any  $t \in \mathbb{R}$ . Then, as in [45], there exists  $\delta > 0$  such that

$$\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u) \leq \frac{1}{2}\|u\|^2 \quad \text{if } \|u\| \leq \delta,$$

and therefore for any  $u \in H^1(\mathbb{R}^N)$  and  $\lambda \in J$ ,

$$I_\lambda(u) \geq \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u|^2 + au^2 > 0 \quad \text{if } 0 < \|u\| \leq \delta. \tag{3.2}$$

On the other hand, for fixed  $0 \neq u_0 \in H^1(\mathbb{R}^N)$  and for any  $\lambda \in J, t > 0$ , by (F3),

$$I_\lambda(\lambda u_0) \leq \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u_0|^2 + a|u_0|^2 - \frac{t^{\frac{2N+2\alpha}{N-2}}}{4} \left(\frac{N-2}{N+\alpha}\right)^2 \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{\frac{N+\alpha}{N-2}})|u_0|^{\frac{N+\alpha}{N-2}}$$

and  $I_\lambda(tu_0) \rightarrow -\infty$  as  $t \rightarrow \infty$ . Then there exists  $t_0 > 0$  (independent of  $\lambda$ ) such that  $I_\lambda(t_0u_0) < 0, \lambda \in J$  and  $\|t_0u_0\| > \delta$ . Let

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)),$$

where  $\Gamma := \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = t_0u_0\}$ .

**Remark 3.1.** Here we remark that  $c_\lambda$  is independent of  $u_0$ . In fact, let

$$d_\lambda := \inf_{\gamma \in \Gamma_1} \max_{t \in [0,1]} I_\lambda(\gamma(t)),$$

where  $\Gamma_1 := \{\gamma \in C([0, 1], X) : \gamma(0) = 0, I_\lambda(\gamma(1)) < 0\}$ . Clearly,  $d_\lambda \leq c_\lambda$ . On the other hand, for any  $\gamma \in \Gamma_1$ , it follows from (3.2) that  $\|\gamma(1)\| > \delta$ . Due to the path connectedness of  $H^1(\mathbb{R}^N)$ , there exists  $\tilde{\gamma} \in C([0, 1], H^1(\mathbb{R}^N))$  such that  $\tilde{\gamma}(t) = \gamma(2t)$  if  $t \in [0, \frac{1}{2}]$ ,  $\|\tilde{\gamma}(t)\| > \delta$  if  $t \in [\frac{1}{2}, 1]$  and  $\tilde{\gamma}(1) = t_0 u_0$ . Then  $\tilde{\gamma} \in \Gamma$  and

$$\max_{t \in [0,1]} I_\lambda(\tilde{\gamma}(t)) = \max_{t \in [0,1]} I_\lambda(\gamma(t)),$$

which implies that  $c_\lambda \leq d_\lambda$  and so  $d_\lambda = c_\lambda$  for any  $\lambda \in J$ .

By (3.2),  $c_\lambda > \frac{\delta^2}{4}$  for any  $\lambda \in J$ . Then, as a consequence of Theorem A, we have:

**Lemma 3.1.** Assume (F1)–(F3). Then, for almost every  $\lambda \in J = [\frac{1}{2}, 1]$ , problem (3.1) possesses a bounded Palais–Smale sequence at the level  $c_\lambda$ . Namely, there exists  $\{u_n\} \subset H^1(\mathbb{R}^N)$  such that

- (i)  $\{u_n\}$  is bounded in  $H^1(\mathbb{R}^N)$ ,
- (ii)  $I_\lambda(u_n) \rightarrow c_\lambda$  and  $I'_\lambda(u_n) \rightarrow 0$  in  $H^{-1}(\mathbb{R}^N)$  as  $n \rightarrow \infty$ .

Next, in the spirit of [33, 41], we establish a decomposition of such a Palais–Smale sequence  $\{u_n\}$ , which will play a crucial role in proving the existence of ground states to (1.2). However, some extra difficulties with respect to the local case are carried over by the presence the nonlocal as well as critical H-L-S nonlinearity.

**Proposition 3.1.** With the same assumptions in Theorem 1.2, let  $\lambda \in [\frac{1}{2}, 1]$  and  $\{u_n\}$  given by Lemma 3.1. Assume  $u_n \rightharpoonup u_\lambda$  weakly in  $H^1(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . Then, up to a subsequence, there exist  $k \in \mathbb{N}^+$ ,  $\{x_n^j\}_{j=1}^k \subset \mathbb{R}^N$  and  $\{v_\lambda^j\}_{j=1}^k \subset H^1(\mathbb{R}^N)$  such that:

- (i)  $I'_\lambda(u_\lambda) = 0$  in  $H^{-1}(\mathbb{R}^N)$ ,
- (ii)  $v_\lambda^j \neq 0$  and  $I'_\lambda(v_\lambda^j) = 0$  in  $H^{-1}(\mathbb{R}^N)$ ,  $j = 1, 2, \dots, k$ ,
- (iii)  $c_\lambda = I_\lambda(u_\lambda) + \sum_{j=1}^k I_\lambda(v_\lambda^j)$ ,
- (iv)  $\|u_n - u_\lambda - \sum_{j=1}^k v_\lambda^j(\cdot - x_n^j)\| \rightarrow 0$  as  $n \rightarrow \infty$ ,
- (v)  $|x_n^j| \rightarrow \infty$  and  $|x_n^i - x_n^j| \rightarrow \infty$  as  $n \rightarrow \infty$  for any  $i \neq j$ .

Before proving Proposition 3.1, we need a few preliminary lemmas.

**Lemma 3.2.** Let  $\lambda \in [\frac{1}{2}, 1]$  and let  $u_\lambda$  be any nontrivial weak solution of (3.1). Then  $u_\lambda$  satisfies the following Pohožäev identity:

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + \frac{N}{2} a \int_{\mathbb{R}^N} |u_\lambda|^2 = \frac{N+\alpha}{2} \lambda \int_{\mathbb{R}^N} (I_\alpha * F(u_\lambda)) F(u_\lambda). \tag{3.3}$$

Moreover, there exist  $\beta, \gamma > 0$  (independent of  $\lambda \in [\frac{1}{2}, 1]$ ) such that  $\|u_\lambda\| \geq \beta$  and  $I_\lambda(u_\lambda) \geq \gamma$  for any nontrivial solution  $u_\lambda, \lambda \in [\frac{1}{2}, 1]$ .

*Proof.* For the proof of the Pohožäev-type identity (3.3) we refer to [45, Theorem 3]. Let  $\lambda \in [\frac{1}{2}, 1]$  and let  $u_\lambda$  be any nontrivial weak solution to (3.1). Then

$$\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + a|u_\lambda|^2 \leq \int_{\mathbb{R}^N} (I_\alpha * F(u_\lambda)) f(u_\lambda) u_\lambda. \tag{3.4}$$

Thanks to (F1)–(F2), for any  $\varepsilon > 0$ , there exists  $C_\varepsilon > 0$  such that  $F(t), tf(t) \leq \varepsilon|t|^{\frac{N+\alpha}{N}} + C_\varepsilon|t|^{\frac{N+\alpha}{N-2}}$  for any  $t \in \mathbb{R}$ . Moreover, as in [45], there exists  $\beta > 0$  such that

$$\int_{\mathbb{R}^N} (I_\alpha * F(u)) f(u) u \leq \frac{\|u\|^2}{2} \quad \text{if } \|u\| \leq \beta,$$

which yields by (3.4),  $\|u_\lambda\| \geq \beta$ . By Pohožäev’s identity (3.3),

$$I_\lambda(u_\lambda) = \frac{2+\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + \frac{\alpha a}{2(N+\alpha)} \int_{\mathbb{R}^N} |u_\lambda|^2$$

and this concludes the proof. □

Let  $\alpha \in (0, N)$ . For any  $u \in D^{1,2}(\mathbb{R}^N)$ , combining the Hardy–Littlewood–Sobolev inequality with Sobolev’s inequality, we have

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N-2}}) |u|^{\frac{N+\alpha}{N-2}} \leq A_\alpha C_\alpha \left( \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} \right)^{\frac{N+\alpha}{N}} \leq A_\alpha C_\alpha S^{-\frac{N+\alpha}{N-2}} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \right)^{\frac{N+\alpha}{N-2}},$$

where

$$S := \inf_{0 \neq u \in D^{1,2}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left( \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}}}.$$

Then

$$S_\alpha := \inf_{0 \neq u \in D^{1,2}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left( \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N-2}}) |u|^{\frac{N+\alpha}{N-2}} \right)^{\frac{N-2}{N+\alpha}}} \geq \frac{S}{(A_\alpha C_\alpha)^{\frac{N-2}{N+\alpha}}}.$$

Minimizers for  $S_\alpha$  are explicitly known from [37, Theorem 4.3] (see also [27, Lemma 1.2]). Actually,

$$S_\alpha = \frac{S}{(A_\alpha C_\alpha)^{\frac{N-2}{N+\alpha}}}$$

and it is achieved by the instanton

$$U(x) = \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}.$$

Now, we use this information to prove an upper estimate for  $c_\lambda$ .

**Lemma 3.3.** *Let  $\lambda \in [\frac{1}{2}, 1]$ ,  $\alpha \in (0, N)$  and assume*

$$q > \max \left\{ 1 + \frac{\alpha}{N-2}, \frac{N+\alpha}{2(N-2)} \right\}.$$

*Then the following upper bound holds:*

$$c_\lambda < \frac{2+\alpha}{2(N+\alpha)} \left( \frac{N+\alpha}{N-2} \right)^{\frac{N-2}{2+\alpha}} \lambda^{\frac{2-N}{2+\alpha}} S_\alpha^{\frac{N+\alpha}{2+\alpha}}.$$

*Proof.* Let  $\varphi \in C_0^\infty(\mathbb{R}^N)$  be a cut-off function with support  $B_2$  such that  $\varphi \equiv 1$  on  $B_1$  and  $0 \leq \varphi \leq 1$  on  $B_2$ , where  $B_r$  denotes the ball in  $\mathbb{R}^N$  of center at origin and radius  $r$ . Given  $\varepsilon > 0$ , we set  $\psi_\varepsilon(x) = \varphi(x)U_\varepsilon(x)$ , where

$$U_\varepsilon(x) = \frac{(N(N-2)\varepsilon^2)^{\frac{N-2}{4}}}{(\varepsilon^2 + |x|^2)^{\frac{N-2}{2}}}.$$

By [8] (see also [61, Lemma 1.46]), we have the following estimates:

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^2 &= S^{\frac{N}{2}} + \begin{cases} O(\varepsilon^{N-2}) & \text{if } N \geq 4, \\ K_1 \varepsilon + O(\varepsilon^3) & \text{if } N = 3, \end{cases} \\ \int_{\mathbb{R}^N} |\psi_\varepsilon|^{\frac{2N}{N-2}} &= S^{\frac{N}{2}} + O(\varepsilon^N) \quad \text{if } N \geq 3, \\ \int_{\mathbb{R}^N} |\psi_\varepsilon|^2 &= \begin{cases} K_2 \varepsilon^2 + O(\varepsilon^{N-2}) & \text{if } N \geq 5, \\ K_2 \varepsilon^2 |\ln \varepsilon| + O(\varepsilon^2) & \text{if } N = 4, \\ K_2 \varepsilon + O(\varepsilon^2) & \text{if } N = 3, \end{cases} \end{aligned}$$

where  $K_1, K_2 > 0$ . Then we get

$$\int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^2 + a |\psi_\varepsilon|^2 = S^{\frac{N}{2}} + \begin{cases} aK_2 \varepsilon^2 + O(\varepsilon^{N-2}) & \text{if } N \geq 5, \\ aK_2 \varepsilon^2 |\ln \varepsilon| + O(\varepsilon^2) & \text{if } N = 4, \\ (K_1 + aK_2) \varepsilon + O(\varepsilon^2) & \text{if } N = 3. \end{cases} \tag{3.5}$$

By direct computation, we know

$$\left( \int_{\mathbb{R}^N} |\psi_\varepsilon|^{\frac{2Nq}{N+\alpha}} \right)^{\frac{N+\alpha}{N}} = K_3 \varepsilon^{N+\alpha-(N-2)q} + o(\varepsilon^{N+\alpha-(N-2)q}),$$

and then by the Hardy–Littlewood–Sobolev inequality,

$$\int_{\mathbb{R}^N} (I_\alpha * |\psi_\varepsilon|^{\frac{N+\alpha}{N-2}}) |\psi_\varepsilon|^q \leq C_\alpha \left( \int_{\mathbb{R}^N} |\psi_\varepsilon|^{\frac{2N}{N-2}} \right)^{\frac{N+\alpha}{2N}} \left( \int_{\mathbb{R}^N} |\psi_\varepsilon|^{\frac{2Nq}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \leq K_4 \varepsilon^{\frac{N+\alpha-(N-2)q}{2}} + o(\varepsilon^{\frac{N+\alpha-(N-2)q}{2}}), \tag{3.6}$$

where  $K_3, K_4 > 0$ . Moreover, similar as in [25, 27], by direct computation, for some  $K_5 > 0$ ,

$$\int_{\mathbb{R}^N} (I_\alpha * |\psi_\varepsilon|^{\frac{N+\alpha}{N-2}}) |\psi_\varepsilon|^{\frac{N+\alpha}{N-2}} \geq (A_\alpha C_\alpha)^{\frac{N}{2}} S_\alpha^{\frac{N+\alpha}{2}} - K_5 \varepsilon^{\frac{N+\alpha}{2}} + o(\varepsilon^{\frac{N+\alpha}{2}}). \tag{3.7}$$

We also have

$$\begin{aligned} \int_{\mathbb{R}^N} (I_\alpha * |\psi_\varepsilon|^{\frac{N+\alpha}{N-2}}) |\psi_\varepsilon|^q &\geq A_\alpha \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{U_\varepsilon^{\frac{N+\alpha}{N-2}}(x) U_\varepsilon^q(y)}{|x-y|^{N-\alpha}} dx dy - \int_{\mathbb{R}^N \setminus B_1} \int_{B_1} \frac{U_\varepsilon^{\frac{N+\alpha}{N-2}}(x) U_\varepsilon^q(y)}{|x-y|^{N-\alpha}} dx dy \right. \\ &\quad \left. - \int_{B_1} \int_{\mathbb{R}^N \setminus B_1} \frac{U_\varepsilon^{\frac{N+\alpha}{N-2}}(x) U_\varepsilon^q(y)}{|x-y|^{N-\alpha}} dx dy - \int_{\mathbb{R}^N \setminus B_1} \int_{\mathbb{R}^N \setminus B_1} \frac{U_\varepsilon^{\frac{N+\alpha}{N-2}}(x) U_\varepsilon^q(y)}{|x-y|^{N-\alpha}} dx dy \right), \end{aligned}$$

where for some  $\tilde{K}_i > 0, i = 1, 2, 3, 4$ ,

$$\left\{ \begin{aligned} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{U_\varepsilon^{\frac{N+\alpha}{N-2}}(x) U_\varepsilon^q(y)}{|x-y|^{N-\alpha}} dx dy &= \tilde{K}_1 \varepsilon^{\frac{N+\alpha-(N-2)q}{2}}, \\ \int_{\mathbb{R}^N \setminus B_1} \int_{B_1} \frac{U_\varepsilon^{\frac{N+\alpha}{N-2}}(x) U_\varepsilon^q(y)}{|x-y|^{N-\alpha}} dx dy &\leq \tilde{K}_2 \varepsilon^{N+\alpha-\frac{N-2}{2}q} + o(\varepsilon^{N+\alpha-\frac{N-2}{2}q}), \\ \int_{B_1} \int_{\mathbb{R}^N \setminus B_1} \frac{U_\varepsilon^{\frac{N+\alpha}{N-2}}(x) U_\varepsilon^q(y)}{|x-y|^{N-\alpha}} dx dy &\leq \tilde{K}_3 \varepsilon^{\frac{N-2}{2}q} + o(\varepsilon^{\frac{N-2}{2}q}), \\ \int_{\mathbb{R}^N \setminus B_1} \int_{\mathbb{R}^N \setminus B_1} \frac{U_\varepsilon^{\frac{N+\alpha}{N-2}}(x) U_\varepsilon^q(y)}{|x-y|^{N-\alpha}} dx dy &\leq \tilde{K}_4 \varepsilon^{\frac{N+\alpha+(N-2)q}{2}} + o(\varepsilon^{\frac{N+\alpha+(N-2)q}{2}}). \end{aligned} \right.$$

Thus for some  $K_6 > 0$ , we have

$$\int_{\mathbb{R}^N} (I_\alpha * |\psi_\varepsilon|^{\frac{N+\alpha}{N-2}}) |\psi_\varepsilon|^q \geq K_6 \varepsilon^{\frac{N+\alpha-(N-2)q}{2}} + o(\varepsilon^{\frac{N+\alpha-(N-2)q}{2}}). \tag{3.8}$$

Here, we used the fact that  $q > \frac{N+\alpha}{2(N-2)}$ . Then for any  $t > 0$ ,

$$\begin{aligned} I_\lambda(t\psi_\varepsilon) &\leq \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^2 + a |\psi_\varepsilon|^2 - \frac{\mu\lambda}{q} \frac{N-2}{N+\alpha} t^{q+\frac{N+\alpha}{N-2}} \int_{\mathbb{R}^N} (I_\alpha * \psi_\varepsilon^{\frac{N+\alpha}{N-2}}) \psi_\varepsilon^q \\ &\quad - \frac{t^{\frac{2(N+\alpha)}{N-2}}}{2} \left( \frac{N-2}{N+\alpha} \right)^2 \lambda \int_{\mathbb{R}^N} (I_\alpha * \psi_\varepsilon^{\frac{N+\alpha}{N-2}}) \psi_\varepsilon^{\frac{N+\alpha}{N-2}} =: g_\varepsilon(t). \end{aligned}$$

One has  $g_\varepsilon(t) \rightarrow -\infty$  as  $t \rightarrow +\infty$  and  $g_\varepsilon(t) > 0$  for  $t > 0$  small. Following [55, Lemma 3.3],  $g_\varepsilon$  has a unique critical point  $t_\varepsilon$  in  $(0, +\infty)$ , which is the maximum point of  $g_\varepsilon$ . From  $g'_\varepsilon(t_\varepsilon) = 0$ ,

$$\begin{aligned} t_\varepsilon \int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^2 + a |\psi_\varepsilon|^2 - \left( q + \frac{N+\alpha}{N-2} \right) \frac{\mu\lambda}{q} \frac{N-2}{N+\alpha} t_\varepsilon^{q+\frac{N+\alpha}{N-2}-1} \int_{\mathbb{R}^N} (I_\alpha * \psi_\varepsilon^{\frac{N+\alpha}{N-2}}) \psi_\varepsilon^q \\ = t_\varepsilon^{\frac{2(N+\alpha)}{N-2}-1} \frac{N-2}{N+\alpha} \lambda \int_{\mathbb{R}^N} (I_\alpha * \psi_\varepsilon^{\frac{N+\alpha}{N-2}}) \psi_\varepsilon^{\frac{N+\alpha}{N-2}}. \end{aligned} \tag{3.9}$$

**Claim.** *There exist  $t_0, t_1 > 0$  (both independent of  $\varepsilon$ ) such that  $t_\varepsilon \in [t_0, t_1]$  for  $\varepsilon > 0$  small.*

Consider first the case,  $t_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Then by (3.5), (3.6) and (3.7), there exist  $c_1, c_2 > 0$  (independent of  $\varepsilon$ ) such that for  $\varepsilon$  small,

$$c_1 t_\varepsilon \leq c_2 \varepsilon^{\frac{N+\alpha-(N-2)q}{2}} t_\varepsilon^{q+\frac{N+\alpha}{N-2}-1} + t_\varepsilon^{q+\frac{N+\alpha}{N-2}-1} \leq 2t_\varepsilon^{q+\frac{N+\alpha}{N-2}-1},$$

where we used the fact that  $q < \frac{N+\alpha}{N-2}$ : hence a contradiction and  $t_\varepsilon \geq t_0$ . By (3.9), one has

$$\int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^2 + a|\psi_\varepsilon|^2 \geq t_\varepsilon^{\frac{2(N+\alpha)}{N-2}-2} \frac{N-2}{N+\alpha} \lambda \int_{\mathbb{R}^N} \left( I_\alpha * \psi_\varepsilon^{\frac{N+\alpha}{N-2}} \right) \psi_\varepsilon^{\frac{N+\alpha}{N-2}},$$

which implies, combining (3.5) and (3.7), that  $t_\varepsilon \leq t_1$  for some  $t_1 > 0$  and  $\varepsilon$  small.

By the Claim just proved and (3.8), we have for some  $K_7 > 0$ ,

$$\frac{\mu \lambda}{q} \frac{N-2}{N+\alpha} t_\varepsilon^{q+\frac{N+\alpha}{N-2}} \int_{\mathbb{R}^N} \left( I_\alpha * \psi_\varepsilon^{\frac{N+\alpha}{N-2}} \right) \psi_\varepsilon^q \geq K_7 \varepsilon^{\frac{N+\alpha-(N-2)q}{2}} + o(\varepsilon^{\frac{N+\alpha-(N-2)q}{2}})$$

and hence on the one hand the following:

$$\begin{aligned} \max_{t \geq 0} I_\lambda(t\psi_\varepsilon) &= g_\varepsilon(t_\varepsilon) \leq \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^2 + a|\psi_\varepsilon|^2 - K_7 \varepsilon^{\frac{N+\alpha-(N-2)q}{2}} \\ &\quad - \frac{t_\varepsilon^{\frac{2(N+\alpha)}{N-2}}}{2} \left( \frac{N-2}{N+\alpha} \right)^2 \lambda \int_{\mathbb{R}^N} \left( I_\alpha * \psi_\varepsilon^{\frac{N+\alpha}{N-2}} \right) \psi_\varepsilon^{\frac{N+\alpha}{N-2}} + o(\varepsilon^{\frac{N+\alpha-(N-2)q}{2}}) \\ &\leq \max_{t \geq 0} \left[ \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^2 + a|\psi_\varepsilon|^2 - \frac{t^{\frac{2(N+\alpha)}{N-2}}}{2} \left( \frac{N-2}{N+\alpha} \right)^2 \lambda \int_{\mathbb{R}^N} \left( I_\alpha * \psi_\varepsilon^{\frac{N+\alpha}{N-2}} \right) \psi_\varepsilon^{\frac{N+\alpha}{N-2}} \right] \\ &\quad - K_7 \varepsilon^{\frac{N+\alpha-(N-2)q}{2}} + o(\varepsilon^{\frac{N+\alpha-(N-2)q}{2}}) \\ &= \frac{2+\alpha}{2(N+\alpha)} \left( \frac{N+\alpha}{N-2} \right)^{\frac{N-2}{2+\alpha}} \lambda^{\frac{2-N}{2+\alpha}} \frac{\left( \int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^2 + a|\psi_\varepsilon|^2 \right)^{\frac{N+\alpha}{2+\alpha}}}{\left( \int_{\mathbb{R}^N} \left( I_\alpha * \psi_\varepsilon^{\frac{N+\alpha}{N-2}} \right) \psi_\varepsilon^{\frac{N+\alpha}{N-2}} \right)^{\frac{N-2}{2+\alpha}}} \\ &\quad - K_7 \varepsilon^{\frac{N+\alpha-(N-2)q}{2}} + o(\varepsilon^{\frac{N+\alpha-(N-2)q}{2}}). \end{aligned}$$

On the other hand, by (3.5) and (3.7), for some  $K_8 > 0$ ,

$$\frac{\left( \int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^2 + a|\psi_\varepsilon|^2 \right)^{\frac{N+\alpha}{2+\alpha}}}{\left( \int_{\mathbb{R}^N} \left( I_\alpha * \psi_\varepsilon^{\frac{N+\alpha}{N-2}} \right) \psi_\varepsilon^{\frac{N+\alpha}{N-2}} \right)^{\frac{N-2}{2+\alpha}}} \leq S_\alpha^{\frac{N+\alpha}{2+\alpha}} + \begin{cases} K_8 \varepsilon^{\min\{2, \frac{N+\alpha}{2}\}} + o(\varepsilon^{\min\{2, \frac{N+\alpha}{2}\}}) & \text{if } N \geq 5, \\ K_8 \varepsilon^2 |\ln \varepsilon| + o(\varepsilon^2 |\ln \varepsilon|) & \text{if } N = 4, \\ K_8 \varepsilon + o(\varepsilon) & \text{if } N = 3. \end{cases}$$

Then, for some  $K_9, K_{10} > 0$ ,

$$\begin{aligned} \max_{t \geq 0} I_\lambda(t\psi_\varepsilon) &\leq \frac{2+\alpha}{2(N+\alpha)} \left( \frac{N+\alpha}{N-2} \right)^{\frac{N-2}{2+\alpha}} \lambda^{\frac{2-N}{2+\alpha}} S_\alpha^{\frac{N+\alpha}{2+\alpha}} + \begin{cases} K_9 \varepsilon^{\min\{2, \frac{N+\alpha}{2}\}} - K_{10} \varepsilon^{\frac{N+\alpha-(N-2)q}{2}} + o(\varepsilon^{\frac{N+\alpha-(N-2)q}{2}}) & \text{if } N \geq 5, \\ K_9 \varepsilon^2 |\ln \varepsilon| - K_{10} \varepsilon^{\frac{N+\alpha-(N-2)q}{2}} + o(\varepsilon^{\frac{N+\alpha-(N-2)q}{2}}) & \text{if } N = 4, \\ K_9 \varepsilon - K_{10} \varepsilon^{\frac{N+\alpha-(N-2)q}{2}} + o(\varepsilon^{\frac{N+\alpha-(N-2)q}{2}}) & \text{if } N = 3, \end{cases} \\ &< \frac{2+\alpha}{2(N+\alpha)} \left( \frac{N+\alpha}{N-2} \right)^{\frac{N-2}{2+\alpha}} \lambda^{\frac{2-N}{2+\alpha}} S_\alpha^{\frac{N+\alpha}{2+\alpha}} \quad \text{if } \varepsilon > 0 \text{ is sufficiently small,} \end{aligned}$$

where we used the fact  $N + \alpha - (N - 2)q < \min\{2, \frac{N+\alpha}{2}\}$ . Therefore, for any  $\lambda \in [\frac{1}{2}, 1]$  and  $\varepsilon > 0$  small enough, we get

$$c_\lambda \leq \max_{t \geq 0} I_\lambda(t\psi_\varepsilon) < \frac{2+\alpha}{2(N+\alpha)} \left( \frac{N+\alpha}{N-2} \right)^{\frac{N-2}{2+\alpha}} \lambda^{\frac{2-N}{2+\alpha}} S_\alpha^{\frac{N+\alpha}{2+\alpha}}. \quad \square$$

*Proof of Proposition 3.1.* Let  $\lambda \in [\frac{1}{2}, 1]$  and assume  $u_n \rightharpoonup u_\lambda$  weakly in  $H^1(\mathbb{R}^N)$  and satisfy  $I_\lambda(u_n) \rightarrow c_\lambda$  and  $I'_\lambda(u_n) \rightarrow 0$  in  $H^{-1}(\mathbb{R}^N)$  as  $n \rightarrow \infty$ .

Step 1. We claim  $I'_\lambda(u_\lambda) = 0$  in  $H^{-1}(\mathbb{R}^N)$ . As a consequence of Lemma 2.4, it is enough to show, up to a subsequence, that for any fixed  $\phi \in C_0^\infty(\mathbb{R}^N)$ ,

$$\int_{\mathbb{R}^N} [I_\alpha * F(u_n - u)]f(u_n - u)\phi \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

In fact, by (F1)–(F2), there exists  $C > 0$  such that

$$|f(t)|^{\frac{2N}{N+\alpha}} \leq C(|t|^{\frac{2N}{N+\alpha}} + |t|^{\frac{2+\alpha}{N-2} \frac{2N}{N+\alpha}}), \quad t \in \mathbb{R}.$$

By virtue of the Hardy–Littlewood–Sobolev inequality and Rellich’s theorem, up to a subsequence, for some  $C$  (independent of  $n$ ) we have

$$\left| \int_{\mathbb{R}^N} [I_\alpha * F(u_n - u)]f(u_n - u)\phi \right| \leq C \left( \int_{\mathbb{R}^N} |f(u_n - u)\phi|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Step 2. Set  $v_n^1 := u_n - u_\lambda$ ; we claim

$$\limsup_{n \rightarrow \infty} \sup_{z \in \mathbb{R}^N} \int_{B_1(z)} |v_n^1|^2 > 0. \tag{3.10}$$

Indeed, arguing by contradiction, if not, by Lions’ lemma [41, Lemma I.1],  $v_n^1 \rightarrow 0$  strongly in  $L^t(\mathbb{R}^N)$  as  $n \rightarrow \infty$  for any  $t \in (2, \frac{2N}{N-2})$ . Noting that  $\langle I'_\lambda(u_n), v_n^1 \rangle \rightarrow 0$  as  $n \rightarrow \infty$  and  $\langle I'_\lambda(u_\lambda), v_n^1 \rangle = 0$  for any  $n$ , by virtue of Lemma 2.2 and Lemma 2.4, we get

$$c_\lambda = I_\lambda(u_\lambda) + I_\lambda(v_n^1) + o_n(1), \quad \|v_n^1\|^2 = \lambda \int_{\mathbb{R}^N} [I_\alpha * F(v_n^1)]f(v_n^1)v_n^1 + o_n(1), \tag{3.11}$$

where  $o_n(1) \rightarrow 0$  as  $n \rightarrow \infty$ . Next, we show that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [I_\alpha * F_1(v_n^1)]F_1(v_n^1) = 0,$$

where

$$f_1(t) = f(t) - |t|^{\frac{4+\alpha-N}{N-2}} t, \quad F_1(t) = \int_0^t f_1(s) ds, \quad t \in \mathbb{R}.$$

Notice that  $\frac{4N}{N+\alpha} \in (2, \frac{2N}{N-2})$  and  $f_1(t) = o(t)$  as  $|t| \rightarrow 0$ ,  $\lim_{|t| \rightarrow \infty} |f_1(t)||t|^{-\frac{2+\alpha}{N-2}} = 0$ . It is easy to see that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |F_1(v_n^1)|^{\frac{2N}{N+\alpha}} = 0,$$

which yields by the Hardy–Littlewood–Sobolev inequality that there exists some  $C > 0$  (independent of  $n$ ) such that

$$\left| \int_{\mathbb{R}^N} [I_\alpha * F_1(v_n^1)]F_1(v_n^1) \right| \leq C \left( \int_{\mathbb{R}^N} |F_1(v_n^1)|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Similarly,

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [I_\alpha * F_1(v_n^1)]|v_n^1|^{\frac{N+\alpha}{N-2}} &= 0, \\ \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [I_\alpha * F_1(v_n^1)]f_1(v_n^1)v_n^1 &= 0. \end{aligned}$$

Then by (3.11), we get

$$\begin{cases} c_\lambda = I_\lambda(u_\lambda) + \frac{1}{2} \|v_n^1\|^2 - \frac{\lambda}{2} \left(\frac{N-2}{N+\alpha}\right)^2 \int_{\mathbb{R}^N} [I_\alpha * |v_n^1|^{\frac{N+\alpha}{N-2}}] |v_n^1|^{\frac{N+\alpha}{N-2}} + o_n(1), \\ \|v_n^1\|^2 = \lambda \frac{N-2}{N+\alpha} \int_{\mathbb{R}^N} [I_\alpha * |v_n^1|^{\frac{N+\alpha}{N-2}}] |v_n^1|^{\frac{N+\alpha}{N-2}} + o_n(1), \end{cases} \quad (3.12)$$

where  $o_n(1) \rightarrow 0$  as  $n \rightarrow \infty$ . Recalling that  $v_n^1 \rightharpoonup 0$  strongly in  $H^1(\mathbb{R}^N)$  as  $n \rightarrow \infty$ , let

$$\lim_{n \rightarrow \infty} \|v_n^1\|^2 = \lambda \frac{N-2}{N+\alpha} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [I_\alpha * |v_n^1|^{\frac{N+\alpha}{N-2}}] |v_n^1|^{\frac{N+\alpha}{N-2}} = b;$$

then  $b > 0$ . From

$$\int_{\mathbb{R}^N} |\nabla v_n^1|^2 \geq S_\alpha \left( \int_{\mathbb{R}^N} [I_\alpha * |v_n^1|^{\frac{N+\alpha}{N-2}}] |v_n^1|^{\frac{N+\alpha}{N-2}} \right)^{\frac{N-2}{N+\alpha}},$$

we have

$$b \geq \left(\frac{N+\alpha}{N-2}\right)^{\frac{N-2}{2+\alpha}} \lambda^{\frac{2-N}{2+\alpha}} S_\alpha^{\frac{N+\alpha}{2+\alpha}}.$$

By Lemma 3.2 and (3.12),

$$c_\lambda \geq \frac{2+\alpha}{2(N+\alpha)} \left(\frac{N+\alpha}{N-2}\right)^{\frac{N-2}{2+\alpha}} \lambda^{\frac{2-N}{2+\alpha}} S_\alpha^{\frac{N+\alpha}{2+\alpha}},$$

which is a contradiction. Thus (3.10) holds true.

**Step 3.** By (3.10) and  $v_n^1 \rightharpoonup 0$  weakly in  $H^1(\mathbb{R}^N)$  as  $n \rightarrow \infty$ , there exists  $\{z_n^1\} \subset \mathbb{R}^N$  such that  $|z_n^1| \rightarrow \infty$  as  $n \rightarrow \infty$  and

$$\lim_{n \rightarrow \infty} \int_{B_1(z_n^1)} |v_n^1|^2 > 0.$$

Let  $u_n^1 = v_n^1(\cdot + z_n^1)$ . Then, up to a subsequence,  $u_n^1 \rightharpoonup v_\lambda^1$  weakly in  $H^1(\mathbb{R}^N)$  as  $n \rightarrow \infty$  for some  $v_\lambda^1 \neq 0$ . By Lemma 2.2 and Lemma 2.4, we have

$$I_\lambda(u_n^1) \rightarrow c_\lambda - I_\lambda(u_\lambda), \quad I'_\lambda(u_n^1) \rightarrow 0 \quad \text{in } H^{-1}(\mathbb{R}^N) \text{ as } n \rightarrow \infty.$$

Similarly as above,  $I'_\lambda(v_\lambda^1) = 0$ . Let  $v_n^2 = u_n^1 - v_\lambda^1$ . Then

$$u_n = u_\lambda + v_\lambda^1(\cdot - z_n^1) + v_n^2(\cdot - z_n^1).$$

If  $v_n^2 \rightarrow 0$ , i.e.  $u_n^1 \rightarrow v_\lambda^1$  strongly in  $H^1(\mathbb{R}^N)$  as  $n \rightarrow \infty$ , then

$$c_\lambda = I_\lambda(u_\lambda) + I_\lambda(v_\lambda^1), \quad \|u_n - u_\lambda - v_\lambda^1(\cdot - z_n^1)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and we are done. Otherwise, if  $v_n^2 \not\rightarrow 0$  strongly in  $H^1(\mathbb{R}^N)$  as  $n \rightarrow \infty$ , similarly as above

$$\lim_{n \rightarrow \infty} \sup_{z \in \mathbb{R}^N} \int_{B_1(z)} |v_n^2|^2 > 0.$$

Then there exists  $\{z_n^2\} \subset \mathbb{R}^N$  such that  $|z_n^2| \rightarrow \infty$  as  $n \rightarrow \infty$  and

$$\lim_{n \rightarrow \infty} \int_{B_1(z_n^2)} |v_n^2|^2 > 0.$$

Let  $u_n^2 = v_n^2(\cdot + z_n^2)$ . Then, up to a subsequence,  $u_n^2 \rightharpoonup v_\lambda^2$  weakly in  $H^1(\mathbb{R}^N)$  as  $n \rightarrow \infty$  for some  $v_\lambda^2 \neq 0$ . We have  $I'_\lambda(v_\lambda^2) = 0$  and

$$I_\lambda(u_n^2) \rightarrow c_\lambda - I_\lambda(u_\lambda) - I_\lambda(v_\lambda^1), \quad I'_\lambda(u_n^2) \rightarrow 0 \quad \text{in } H^{-1}(\mathbb{R}^N) \text{ as } n \rightarrow \infty.$$

Let  $v_n^3 = u_n^2 - v_\lambda^2$ . Then

$$u_n = u_\lambda + v_\lambda^1(\cdot - z_n^1) + v_\lambda^2(\cdot - z_n^1 - z_n^2) + v_n^3(\cdot - z_n^1 - z_n^2).$$

If  $v_n^3 \rightarrow 0$ , i.e.,  $u_n^2 \rightarrow v_\lambda^2$  strongly in  $H^1(\mathbb{R}^N)$  as  $n \rightarrow \infty$ , then

$$c_\lambda = I_\lambda(u_\lambda) + I_\lambda(v_\lambda^1) + I_\lambda(v_\lambda^2), \quad \|u_n - u_\lambda - v_\lambda^1(\cdot - z_n^1) - v_\lambda^2(\cdot - z_n^1 - z_n^2)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and we are done. Otherwise, we can iterate the above procedure and by Lemma 3.2, we will end up in a finite number  $k$  of steps. Namely, let  $x_n^j = \sum_{i=1}^j z_n^i$  to have

$$c_\lambda = I_\lambda(u_\lambda) + \sum_{j=1}^k I_\lambda(v_\lambda^j), \quad \left\| u_n - u_\lambda - \sum_{j=1}^k v_\lambda^j(\cdot - x_n^j) \right\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

**Step 4.** Clearly,  $|x_n^j - x_n^{j-1}| = |z_n^j| \rightarrow \infty$  as  $n \rightarrow \infty$  for  $j = 2, 3, \dots, k$ . However, it is not clear that if  $\{x_n^j\}_{j=1}^k$  repels each other as  $n \rightarrow \infty$ , i.e.,  $|x_n^j - x_n^i| \rightarrow \infty$  as  $n \rightarrow \infty$  for any  $i, j = 1, 2, \dots, k$  and  $i \neq j$ . Let us show that after extracting a subsequence from  $\{x_n^j\}$  and redefining  $\{v_\lambda^j\}$  if necessary, properties (iii), (iv), (v) hold. Let  $\Lambda_1, \Lambda_2 \subset \{1, 2, \dots, k\}$  and satisfy  $\Lambda_1 \cup \Lambda_2 = \{1, 2, \dots, k\}$  and let  $\{x_n^j\}_n$  be bounded if  $j \in \Lambda_1$ , whereas  $|x_n^j| \rightarrow \infty$  as  $n \rightarrow \infty$  if  $j \in \Lambda_2$ . Then, for any  $j \in \Lambda_1$  if  $\Lambda_1 \neq \emptyset$ , there exists  $0 \neq v^j \in H^1(\mathbb{R}^N)$  such that, up to a subsequence,  $v_\lambda^j(\cdot - x_n^j) \rightharpoonup v^j$  weakly in  $H^1(\mathbb{R}^N)$  as  $n \rightarrow \infty$  and  $I'_\lambda(v^j) = 0$  in  $H^{-1}(\mathbb{R}^N)$ . By Rellich's theorem, for any  $t \in [2, \frac{2N}{N-2})$ , we have  $v_\lambda^j(\cdot - x_n^j) \rightarrow v^j$  strongly in  $L^t(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . Noting that  $I'_\lambda(v_\lambda^j(\cdot - x_n^j)) = 0$  in  $H^{-1}(\mathbb{R}^N)$  and  $I_\lambda(v_\lambda^j(\cdot - x_n^j)) \leq c_\lambda$ , similar to Step 2, we know that  $v_\lambda^j(\cdot - x_n^j) \rightarrow v^j$  strongly in  $H^1(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . Then, up to a subsequence, there exists  $\tilde{v}^j \in H^1(\mathbb{R}^N)$  such that  $\sum_{j \in \Lambda_1} v_\lambda^j(\cdot - x_n^j) \rightarrow \tilde{v}^j$  strongly in  $H^1(\mathbb{R}^N)$  as  $n \rightarrow \infty$ , which eventually implies

$$\left\| u_n - u_\lambda - \sum_{j \in \Lambda_2} v_\lambda^j(\cdot - x_n^j) \right\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Recalling that  $\|u_n - u_\lambda\| \not\rightarrow 0$  as  $n \rightarrow \infty$ , we have  $\Lambda_2 \neq \emptyset$ . Let  $x_n^i \in \Lambda_2$  and

$$\Lambda_2^i := \{j \in \Lambda_2 : |x_n^i - x_n^j| \text{ stays bounded}\}.$$

Then similarly as above, up to a subsequence, for some  $\tilde{v}_\lambda^i \in H^1(\mathbb{R}^N)$ , we have  $\sum_{j \in \Lambda_2^i} v_\lambda^j(\cdot + x_n^i - x_n^j) \rightarrow \tilde{v}_\lambda^i$  strongly in  $H^1(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . Then, as  $n \rightarrow \infty$ ,

$$\left\| u_n - u_\lambda - \tilde{v}_\lambda^i(\cdot - x_n^i) - \sum_{j \in (\Lambda_2 \setminus \Lambda_2^i)} v_\lambda^j(\cdot - x_n^j) \right\| \rightarrow 0.$$

Without loss of generality, we may assume that  $\tilde{v}_\lambda^i \neq 0$ . Noting that  $u_n(\cdot + x_n^i) \rightarrow \tilde{v}_\lambda^i$  a.e. in  $\mathbb{R}^N$  as  $n \rightarrow \infty$ , we get  $I'_\lambda(\tilde{v}_\lambda^i) = 0$  in  $H^{-1}(\mathbb{R}^N)$ . Then we redefine  $v_\lambda^i := \tilde{v}_\lambda^i$  and as  $n \rightarrow \infty$ ,

$$\left\| u_n - u_\lambda - \sum_{j \in (\Lambda_2 \setminus \Lambda_2^i) \cup \{i\}} v_\lambda^j(\cdot - x_n^j) \right\| \rightarrow 0.$$

By repeating the argument above at most  $(k - 1)$  times and redefining  $\{v_\lambda^j\}$  if necessary, we end up with  $\Lambda \subset \Lambda_2$  such that

$$\begin{cases} |x_n^j| \rightarrow \infty \quad \text{and} \quad |x_n^i - x_n^j| \rightarrow \infty \quad \text{as } n \rightarrow \infty \text{ for any } i, j \in \Lambda \text{ and } i \neq j, \\ \left\| u_n - u_\lambda - \sum_{j \in \Lambda} v_\lambda^j(\cdot - x_n^j) \right\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{cases}$$

Finally, by Lemma 2.2 one has  $c_\lambda = I_\lambda(u_\lambda) + \sum_{j \in \Lambda} I_\lambda(v_\lambda^j)$ . The proof is now complete. □

*Proof of Theorem 1.1.* As a consequence of Lemma 3.1, Proposition 3.1 and Lemma 3.2, one has that for almost every  $\lambda \in J = [\frac{1}{2}, 1]$ , problem (3.1) admits a nontrivial solution  $u_\lambda$  satisfying  $\|u_\lambda\| \geq \beta$ ,  $\gamma \leq I_\lambda(u_\lambda) \leq c_\lambda$ , where  $\beta, \gamma > 0$  (independent of  $\lambda$ ). Then there exist  $\{\lambda_n\} \subset [\frac{1}{2}, 1]$  and  $\{u_n\} \subset H^1(\mathbb{R}^N)$  such that, as  $n \rightarrow \infty$ ,

$$\lambda_n \rightarrow 1, \quad \gamma \leq I_{\lambda_n}(u_n) \leq c_{\lambda_n}, \quad I'_{\lambda_n}(u_n) = 0 \quad \text{in } H^{-1}(\mathbb{R}^N). \tag{3.13}$$

By Pohožev's identity (3.3) we have

$$I_{\lambda_n}(u_n) = \frac{2 + \alpha}{2(N + \alpha)} \int_{\mathbb{R}^N} |\nabla u_n|^2 + \frac{\alpha a}{2(N + \alpha)} \int_{\mathbb{R}^N} |u_n|^2$$

and  $\{u_n\}$  is bounded in  $H^1(\mathbb{R}^N)$ . Notice that

$$L_a(u) = I_\lambda(u) + \frac{1}{2}(\lambda - 1) \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u), \quad u \in H^1(\mathbb{R}^N).$$

Then by (3.13), up to a sequence, there exists  $c_0 \in [\gamma, c_1]$  such that

$$c_0 := \lim_{n \rightarrow \infty} L_a(u_n) = \lim_{n \rightarrow \infty} I_{\lambda_n}(u_n) \leq \lim_{n \rightarrow \infty} c_{\lambda_n} = c_1,$$

where we used the fact that  $c_\lambda$  is continuous from the left at  $\lambda$ . Moreover, by (3.13), for any  $\phi \in C_0^\infty(\mathbb{R}^N)$ ,

$$\langle L'_a(u_n), \phi \rangle = (\lambda_n - 1) \int_{\mathbb{R}^N} [I_\alpha * F(u_n)]f(u_n)\phi.$$

Similarly as above, there exists some  $C > 0$  such that

$$\left( \int_{\mathbb{R}^N} |f(u_n)\phi|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \leq C\|\phi\| \quad \text{uniformly for all } \phi \in C_0^\infty(\mathbb{R}^N), n = 1, 2, \dots,$$

and by the Hardy–Littlewood–Sobolev inequality

$$\begin{aligned} |\langle L'_a(u_n), \phi \rangle| &= (1 - \lambda_n) \left| \int_{\mathbb{R}^N} [I_\alpha * F(u_n)]f(u_n)\phi \right| \\ &\leq C(1 - \lambda_n) \left( \int_{\mathbb{R}^N} |F(u_n)|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \left( \int_{\mathbb{R}^N} |f(u_n)\phi|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \\ &= o_n(1)\|\phi\|, \end{aligned}$$

where  $o_n(1) \rightarrow 0$  uniformly for any  $\phi \in C_0^\infty(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . Namely,  $L'_a(u_n) \rightarrow 0$  in  $H^{-1}(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . Finally, we obtain

$$\|u_n\| \geq \beta, \quad L_a(u_n) \rightarrow c_0 \leq c_1, \quad L'_a(u_n) \rightarrow 0 \quad \text{in } H^{-1}(\mathbb{R}^N) \text{ as } n \rightarrow \infty.$$

If  $u_n \rightarrow u_0$  strongly in  $H^1(\mathbb{R}^N)$ , then  $\|u_0\| \geq \beta$ ,  $L_a(u_0) = c_0 \leq c_1$  and  $L'_a(u_0) = 0$  in  $H^{-1}(\mathbb{R}^N)$ . Otherwise, as a consequence of Proposition 3.1 with  $\lambda = 1$ ,  $c_\lambda = c_0$ ,  $u_\lambda = u_0$ , there exist  $k \in \mathbb{N}^+$  and  $\{v^j\}_{j=1}^k \subset H^1(\mathbb{R}^N)$  such that  $v^j \neq 0$ ,  $L'_a(v^j) = 0$  in  $H^{-1}(\mathbb{R}^N)$  for all  $j$  and  $c_0 = L_a(u_0) + \sum_{j=1}^k L_a(v^j)$ . So let

$$\mathcal{N} := \{u \in H^1(\mathbb{R}^N \setminus \{0\}) : L'_a(u) = 0 \quad \text{in } H^{-1}(\mathbb{R}^N)\}.$$

Then  $\mathcal{N} \neq \emptyset$  and  $\inf_{u \in \mathcal{N}} L_a(u) = E_a \in [\gamma, c_1]$ .

We conclude the proof of Theorem 1.1 by showing that  $E_a$  is achieved. Clearly, there exists  $\{v_n\} \subset \mathcal{N}$  such that as  $n \rightarrow \infty$ ,  $L_a(v_n) \rightarrow E_a$  and  $L'_a(v_n) = 0$  in  $H^{-1}(\mathbb{R}^N)$ . Thus  $\{v_n\}$  is bounded in  $H^1(\mathbb{R}^N)$ . Assume that  $v_n \rightharpoonup v_0$  weakly in  $H^1(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . Then  $L'_a(v_0) = 0$  in  $H^{-1}(\mathbb{R}^N)$ . If  $v_n \rightarrow v_0$  strongly in  $H^1(\mathbb{R}^N)$ , then  $L_a(v_0) = E_a$ . Namely,  $v_0$  is a ground state solution of (1.2). Otherwise, there exist  $k \in \mathbb{N}^+$  and  $\{v^j\}_{j=1}^k \subset H^1(\mathbb{R}^N)$  such that  $v^j \neq 0$ ,  $L'_a(v^j) = 0$  in  $H^{-1}(\mathbb{R}^N)$  for all  $j$  and  $E_a = L_a(v_0) + \sum_{j=1}^k L_a(v^j)$ . By the definition of  $E_a$ ,  $v_0 = 0$ ,  $k = 1$  and  $L_a(v^1) = E_a$ , which yields  $v^1$  as a ground state solution to (1.2). The proof is now complete.  $\square$

## 4 Towards semiclassical states

### 4.1 Compactness of the set of ground state solutions

Denote the set of ground state solutions to (1.2) by

$$\mathcal{N}_a := \{u \in H^1(\mathbb{R}^N) : L_a(u) = E_a, L'_a(u) = 0 \text{ in } H^{-1}(\mathbb{R}^N)\}.$$

Then by Theorem 1.1,  $\mathcal{N}_a \neq \emptyset$  for any  $a > 0$ . Since  $L_a$  is invariant by translations,  $\mathcal{N}_a$  cannot be compact in  $H^1(\mathbb{R}^N)$ . However, this turns out to be the only way to loose compactness as we have the following result.

**Proposition 4.1.** For any  $a > 0$ , up to translations,  $\mathcal{N}_a$  is compact in  $H^1(\mathbb{R}^N)$ .

*Proof.* Let  $\{u_n\} \subset \mathcal{N}_a$ . Then  $L_a(u_n) = E_a$  and  $L'_a(u_n) = 0$  in  $H^{-1}(\mathbb{R}^N)$ . Similarly as above  $\{u_n\}$  is bounded in  $H^1(\mathbb{R}^N)$ . Assume that  $u_n \rightharpoonup u_0$  weakly in  $H^1(\mathbb{R}^N)$  as  $n \rightarrow \infty$ ; then  $L'_a(u_0) = 0$  in  $H^{-1}(\mathbb{R}^N)$ . If  $u_n \rightarrow u_0$  strongly in  $H^1(\mathbb{R}^N)$ , we are done. Otherwise, by virtue of Proposition 3.1, up to a subsequence, there exists  $k \in \mathbb{N}^+$ ,  $\{x_n^j\}_{j=1}^k \subset \mathbb{R}^N$  and  $\{v^j\}_{j=1}^k \subset H^1(\mathbb{R}^N)$  such that  $v^j \neq 0$ ,  $L'_a(v^j) = 0$  in  $H^{-1}(\mathbb{R}^N)$  for all  $j$  and

$$E_a = L_a(u_0) + \sum_{j=1}^k L_a(v^j), \quad \left\| u_n - u_0 - \sum_{j=1}^k v_\lambda^j(\cdot - x_n^j) \right\| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which implies that  $u_0 = 0$ ,  $k = 1$ ,  $v^1 \in \mathcal{N}_a$  and  $\|u_n(\cdot + x_n^1) - v_\lambda^1\| \rightarrow 0$  as  $n \rightarrow \infty$ . □

### 4.2 Regularity, positivity and symmetry

Here we borrow some ideas from [4, 45] to establish boundedness, decay, positivity and symmetry of ground state solutions to (1.2).

**Proposition 4.2.** Let  $a > 0$ . The following hold:

- (i)  $0 < \inf\{\|u\|_\infty : u \in \mathcal{N}_a\} \leq \sup\{\|u\|_\infty : u \in \mathcal{N}_a\} < \infty$ .
- (ii) For any  $u \in \mathcal{N}_a$ ,  $u \in C^{1,\gamma}_{\text{loc}}(\mathbb{R}^N)$  for  $\gamma \in (0, 1)$ .
- (iii) For any  $u \in \mathcal{N}_a$ ,  $u$  has constant sign and is radially symmetric about a point.
- (iv)  $E_a$  coincides with the mountain pass value.
- (v) There exist  $C, c > 0$ , independent of  $u \in \mathcal{N}_a$ , such that  $|D^{\alpha_1} u(x)| \leq C \exp(-c|x - x_0|)$ ,  $x \in \mathbb{R}^N$ , for  $|\alpha_1| = 0, 1$ , where  $|u(x_0)| = \max_{x \in \mathbb{R}^N} |u(x)|$ .

*Proof.* First, by Pohozaev’s inequality it follows that  $\mathcal{N}_a$  is bounded in  $H^1(\mathbb{R}^N)$ .

**Claim 1.** For any  $p \in [2, \frac{N}{\alpha} \frac{2N}{N-2})$ , there exists  $C_p > 0$  such that

$$\|u\|_p \leq C_p \|u\|_2 \quad \text{for all } u \in \mathcal{N}_a. \tag{4.1}$$

In fact, for any fixed  $u \in \mathcal{N}_a$ , let  $H(u) = \frac{F(u)}{u}$  and  $K(u) = f(u)$  in  $\{x \in \mathbb{R}^N : u(x) \neq 0\}$ . Let  $R > 0$  and  $\phi_R \in C^\infty_0(\mathbb{R})$  be such that  $\phi_R(t) \in [0, 1]$  for  $t \in \mathbb{R}$ ,  $\phi_R(t) = 1$  for  $|t| \leq R$  and  $\phi_R(t) = 0$  for  $|t| \geq 2R$ . Set

$$\begin{aligned} H^*(u) &= \phi_R(u)H(u), & H_*(u) &= H(u) - H^*(u), \\ K^*(u) &= \phi_R(u)K(u), & K_*(u) &= K(u) - K^*(u). \end{aligned}$$

By (F1)–(F2), there exists  $C > 0$  (depending only on  $R$ ) such that for any  $x \in \mathbb{R}^N$ ,

$$\begin{aligned} |H^*(u)| &\leq C|u|^{\frac{\alpha}{N}}, & |K^*(u)| &\leq C|u|^{\frac{\alpha}{N}}, \\ |H_*(u)| &\leq C|u|^{\frac{\alpha+2}{N-2}}, & |K_*(u)| &\leq C|u|^{\frac{\alpha+2}{N-2}}. \end{aligned}$$

Note that  $H^*(u), K^*(u)$  are uniformly bounded in  $L^{2N/\alpha}(\mathbb{R}^N)$  and so are  $H_*(u), K_*(u)$  in  $L^{2N/(\alpha+2)}(\mathbb{R}^N)$  for any  $u \in \mathcal{N}_a$ . Thanks to the compactness of  $\mathcal{N}_a$ , for any  $\varepsilon > 0$  we can choose  $R$  depending only on  $\varepsilon$  such that

$$\left( \int_{\mathbb{R}^N} |H_*(u)|^{\frac{2N}{\alpha+2}} \int_{\mathbb{R}^N} |K_*(u)|^{\frac{2N}{\alpha+2}} \right)^{\frac{\alpha+2}{2N}} \leq \varepsilon^2 \quad \text{for all } u \in \mathcal{N}_a.$$

Then repeating line by line the argument as in [45, Proposition 3.1], (4.1) follows.

**Claim 2.** The map  $I_\alpha * F(u)$  is uniformly bounded in  $L^\infty(\mathbb{R}^N)$  for all  $u \in \mathcal{N}_a$ .

By (F1)–(F2) and the very definition of  $I_\alpha * F(u)$ , there exists  $C(\alpha)$  (depending only  $N, \alpha$ ) such that for any  $x \in \mathbb{R}^N$  and  $u \in \mathcal{N}_a$ ,

$$(I_\alpha * |F(u)|)(x) \leq C(\alpha) \int_{\mathbb{R}^2} (|u|^2 + |u|^{\frac{N+\alpha}{N-2}}) dy + C(\alpha) \int_{|x-y| \leq 1} \frac{|u|^2 + |u|^{\frac{N+\alpha}{N-2}}}{|x-y|^{N-\alpha}} dy.$$

Thanks to (4.1), for some  $c$  (independent of  $u$ ) such that for any  $x \in \mathbb{R}^N$ ,

$$(I_\alpha * |F(u)|)(x) \leq c + C(\alpha) \int_{|x-y| \leq 1} \frac{|u|^2 + |u|^{\frac{N+\alpha}{N-2}}}{|x-y|^{N-\alpha}} dy.$$

As in [64, Proposition 2.2], we can choose  $t \in (\frac{N}{\alpha}, \frac{N}{\alpha} \frac{N}{N-2})$  with  $2t \in (2, \frac{N}{\alpha} \frac{2N}{N-2})$  and  $s \in (\frac{N}{\alpha}, \frac{N}{\alpha} \frac{2N}{N+\alpha})$  with  $s \frac{N+\alpha}{N-2} \in (2, \frac{N}{\alpha} \frac{2N}{N-2})$ , and there exist  $C_1, C_2 > 0$  (independent of  $u$ ) such that

$$\int_{|x-y| \leq 1} \frac{|u|^2 + |u|^{\frac{N+\alpha}{N-2}}}{|x-y|^{N-\alpha}} dy \leq C_1 \|u\|_{2t}^2 + C_2 \|u\|_{s \frac{N+\alpha}{N-2}}^{\frac{N+\alpha}{N-2}},$$

which combining with (4.1) implies the claim.

Now let  $\tilde{f}(x, u) := (I_\alpha * F(u))(x)f(u)$ . Then by (F1)–(F2), for any  $u \in \mathcal{N}_a$ ,  $u$  satisfies that for any  $\delta > 0$ , there exists  $C_\delta > 0$  (independent of  $u$ ) such that

$$|\tilde{f}(x, u)u| \leq (\delta|u|^2 + C_\delta|u|^{\frac{N+\alpha}{N-2}}), \quad x \in \mathbb{R}^N,$$

and

$$-\Delta u + au = \tilde{f}(x, u), \quad u \in H^1(\mathbb{R}^N).$$

Noting that  $\frac{N+\alpha}{N-2} < \frac{2N}{N-2}$ , by means of a standard Moser iteration [29] (see also [14]),  $\mathcal{N}_a$  is uniformly bounded in  $L^\infty(\mathbb{R}^N)$ . Since  $|\tilde{f}(x, u)| = o(1)|u|$  if  $\|u\|_\infty \rightarrow 0$  and  $E_a > 0$ , one also has  $\inf\{\|u\|_\infty : u \in \mathcal{N}_a\} > 0$ .

Since  $u \in L^\infty(\mathbb{R}^N)$  for any  $u \in \mathcal{N}_a$ , it follows from the elliptic regularity estimates (see [29]) that  $u \in C_{loc}^{1,\gamma}(\mathbb{R}^N)$  for some  $\gamma \in (0, 1)$ . From the proof of Theorem 1.1, we know that  $E_a \leq c_1$ , where

$$c_1 := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} L_a(\gamma(t)),$$

where  $\Gamma := \{\gamma \in C([0, 1], X) : \gamma(0) = 0, L_a(\gamma(1)) < 0\}$ . Following [45], for any  $u \in \mathcal{N}_a$ , there exists a path  $\gamma \in \Gamma$  such that  $\gamma(\frac{1}{2}) = u$  and  $L_a(\gamma)$  achieves its maximum at  $\frac{1}{2}$ . Thereby,  $c_1 = E_a$ . Namely,  $E_a$  is also a mountain pass value. Moreover, for any  $u \in \mathcal{N}_a$ ,  $u$  has a constant sign and is radially symmetric about some point. If  $u$  is positive, then  $u$  is decreasing at  $r = |x - x_0|$ , where  $x_0$  is the maximum point of  $u$ . Finally, by the radial lemma,  $u(x) \rightarrow 0$  uniformly as  $|x - x_0| \rightarrow \infty$  for  $u \in \mathcal{N}_a$ . By the comparison principle, there exist  $C, c > 0$ , independent of  $u \in \mathcal{N}_a$  such that  $|D^{\alpha_1} u(x)| \leq C \exp(-c|x - x_0|)$ ,  $x \in \mathbb{R}^N$  for  $|\alpha_1| = 0, 1$ .  $\square$

### 4.3 Proof of Theorem 1.2

Let  $u(x) = v(\varepsilon x)$ ,  $V_\varepsilon(x) = V(\varepsilon x)$  and consider the following problem:

$$-\Delta u + V_\varepsilon(x)u = (I_\alpha * F(u))f(u), \quad x \in \mathbb{R}^N. \tag{4.2}$$

Let  $H_\varepsilon$  be the completion of  $C_0^\infty(\mathbb{R}^N)$  with respect to the norm

$$\|u\|_\varepsilon = \left( \int_{\mathbb{R}^N} (|\nabla u|^2 + V_\varepsilon u^2) \right)^{\frac{1}{2}}.$$

For any set  $B \subset \mathbb{R}^N$  and  $\varepsilon > 0$ , we define  $B_\varepsilon \equiv \{x \in \mathbb{R}^N : \varepsilon x \in B\}$  and  $B^\delta \equiv \{x \in \mathbb{R}^N : \text{dist}(x, B) \leq \delta\}$ . Since we are looking for positive solutions of (1.1), from now on, we may assume that  $f(t) = 0$  for  $t \leq 0$ . For  $u \in H_\varepsilon$ , let

$$P_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V_\varepsilon u^2 - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u).$$

Fix an arbitrary  $\nu > 0$  and define

$$\chi_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in O_\varepsilon, \\ \varepsilon^{-\nu} & \text{if } x \in \mathbb{R}^N \setminus O_\varepsilon, \end{cases}$$

as well as

$$Q_\varepsilon(u) = \left( \int_{\mathbb{R}^N} \chi_\varepsilon u^2 \, dx - 1 \right)_+^2.$$

Let  $\Gamma_\varepsilon : H_\varepsilon \rightarrow \mathbb{R}$  be given by

$$\Gamma_\varepsilon(u) = P_\varepsilon(u) + Q_\varepsilon(u).$$

To find solutions of (4.2) which concentrate inside  $O$  as  $\varepsilon \rightarrow 0$ , we look for critical points  $u_\varepsilon$  of  $\Gamma_\varepsilon$  satisfying  $Q_\varepsilon(u_\varepsilon) = 0$ . The functional  $Q_\varepsilon$  that was first introduced in [13] will act as a penalization to forcing the concentration phenomena inside  $O$ . In what follows, we seek the critical points of  $\Gamma_\varepsilon$  in some neighborhood of ground state solutions to (1.2) with  $a = m$ .

### 4.4 The truncated problem

Denote  $S_m$  by the set of positive ground state solutions of (1.2) with  $a = m$  satisfying  $u(0) = \max_{x \in \mathbb{R}^N} u(x)$ , where  $m$  is given in Section 1.

**Lemma 4.1.** *The set  $S_m$  is compact in  $H^1(\mathbb{R}^N)$ .*

*Proof.* By Proposition 4.2,  $S_m \neq \emptyset$ . For any  $\{u_n\} \subset S_m$ , without loss of generality, we assume that  $u_n \rightharpoonup u_0$  weakly in  $H^1(\mathbb{R}^N)$  and a.e. in  $\mathbb{R}^N$  as  $n \rightarrow \infty$ . Let us first prove that  $u_0 \neq 0$ . Indeed, by (v) of Proposition 4.2, there exist  $c, C > 0$  (independent of  $n$ ) such that  $|u_n(x)| \leq C \exp(-c|x|)$  for any  $x \in \mathbb{R}^N$ . By the Lebesgue dominated convergence theorem,  $u_n \rightarrow u_0$  strongly in  $L^p(\mathbb{R}^N)$  as  $n \rightarrow \infty$  for any  $p \in [2, \frac{2N}{N-2}]$ . So if  $u_0 = 0$ , one has  $u_n \rightarrow 0$  strongly in  $H^1(\mathbb{R}^N)$  as  $n \rightarrow \infty$ , which contradicts the fact  $E_m > 0$ . We claim  $u_n \rightarrow u_0$  strongly in  $H^1(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . Indeed, if not, by Proposition 3.1, there exist  $k \in \mathbb{N}^+$  and  $\{v^j\}_{j=1}^k \subset H^1(\mathbb{R}^N)$  such that  $v^j \neq 0$ ,  $L'_m(v^j) = 0$  in  $H^{-1}(\mathbb{R}^N)$  for all  $j$  and  $E_m = L_m(u_0) + \sum_{j=1}^k L_m(v^j)$ . Noting that  $L_m(u_0) \geq E_m$  and  $L_m(v^j) \geq E_m$ , we get a contradiction. Finally,  $u_0 \in S_m$ . Clearly,  $u_0 \in \mathcal{N}_m$  is positive and radially symmetric. Recalling that 0 is the same maximum point  $u_n$  for any  $n$ , by the local elliptic estimate, 0 is also a maximum point of  $u_0$ . The proof is complete.  $\square$

By Proposition 4.2, let  $\kappa > 0$  be fixed and satisfy

$$\sup_{U \in S_m} \|U\|_\infty < \kappa. \tag{4.3}$$

For  $k > \max_{t \in [0, \kappa]} f(t)$  fixed, let  $f_k(t) := \min\{f(t), k\}$  and consider the truncated problem

$$-\varepsilon^2 \Delta v + V(x)v = \varepsilon^{-\alpha} (I_\alpha * F_k(v)) f_k(v), \quad v \in H^1(\mathbb{R}^N), \tag{4.4}$$

whose associated limit problem is

$$-\Delta u + mu = (I_\alpha * F_k(u)) f_k(u), \quad u \in H^1(\mathbb{R}^N), \tag{4.5}$$

where  $F_k(t) = \int_0^t f_k(s) \, ds$ . Denote by  $S_m^k$  be the set of positive ground state solutions  $U$  of (4.5) satisfying  $U(0) = \max_{x \in \mathbb{R}^N} U(x)$ . Then by [45, Theorem 2],  $S_m^k \neq \emptyset$ . As in Lemma 4.1,  $S_m^k$  is compact in  $H^1(\mathbb{R}^N)$ .

**Lemma 4.2.** *We have  $S_m \subset S_m^k$ .*

*Proof.* Denote by  $E_m^k$  the least energy of (4.5). Notice that any  $u \in S_m$  is also a solution to (4.5). Then  $E_m^k \leq E_m$ . By [45],  $E_m^k$  is a mountain pass value. Combining (iv) of Proposition 4.2 with the fact  $f_k(t) \leq f(t)$  for  $t > 0$  and  $f_k(t) = f(t) = 0$  for  $t \leq 0$ , we have  $E_m^k \geq E_m$  and so  $E_m^k = E_m$ , which yields  $S_m \subset S_m^k$ .  $\square$

### 4.5 Proof of Theorem 1.2

In the following, we use the truncation approach to prove Theorem 1.2. First, we consider the truncated problem (4.4). By Lemma 4.2,  $S_m$  is a compact subset of  $S_m^k$ . Inspired from [10] we show that (4.4) admits

a nontrivial positive solution  $v_\varepsilon$  in some neighborhood of  $S_m$  for small  $\varepsilon$ . Then we show that there exists  $\varepsilon_0 > 0$  such that

$$\|v_\varepsilon\|_\infty < \kappa \quad \text{for } \varepsilon \in (0, \varepsilon_0).$$

As a consequence,  $v_\varepsilon$  turns out to be a solution to the original problem (1.1).

For this purpose, set

$$\delta = \frac{1}{10} \min\{\text{dist}(\mathcal{M}, O^c)\}.$$

Let  $\beta \in (0, \delta)$  and consider a cut-off  $\varphi \in C_0^\infty(\mathbb{R}^N)$  such that  $0 \leq \varphi \leq 1$ ,  $\varphi(x) = 1$  for  $|x| \leq \beta$  and  $\varphi(x) = 0$  for  $|x| \geq 2\beta$ . Set  $\varphi_\varepsilon(y) = \varphi(\varepsilon y)$ ,  $y \in \mathbb{R}^N$ , and for some  $x \in (\mathcal{M})^\beta$  and  $U \in S_m$ , we define

$$U_\varepsilon^x(y) = \varphi_\varepsilon\left(y - \frac{x}{\varepsilon}\right)U\left(y - \frac{x}{\varepsilon}\right)$$

and

$$X_\varepsilon = \{U_\varepsilon^x : x \in (\mathcal{M})^\beta, U_i \in S_m\}.$$

In the following, we show that (4.4) admits a solution in  $X_\varepsilon^d \subset X_\varepsilon$  for  $\varepsilon, d > 0$  small enough, where

$$X_\varepsilon^d = \left\{u \in H_\varepsilon : \inf_{v \in X_\varepsilon} \|u - v\|_\varepsilon \leq d\right\}.$$

In fact, since  $f_k$  satisfies all the hypotheses of [64, Theorem 2.1], for  $\varepsilon, d > 0$  small, (4.4) admits a positive solution  $v_\varepsilon \in X_\varepsilon^d$  for which there exist  $U \in S_m$  and a maximum point  $x_\varepsilon$  of  $v_\varepsilon$  such that  $\lim_{\varepsilon \rightarrow 0} \text{dist}(x_\varepsilon, \mathcal{M}) = 0$  and  $v_\varepsilon(\varepsilon \cdot + x_\varepsilon) \rightarrow U(\cdot + z_0)$  in  $H^1(\mathbb{R}^N)$  as  $\varepsilon \rightarrow 0$  for some  $z_0 \in \mathbb{R}^N$ . We have

$$-\Delta w_\varepsilon + V_\varepsilon\left(x + \frac{x_\varepsilon}{\varepsilon}\right)w_\varepsilon = (I_\alpha * F_k(w_\varepsilon))f_k(w_\varepsilon), \quad x \in \mathbb{R}^N,$$

where  $w_\varepsilon(\cdot) = v_\varepsilon(\varepsilon \cdot + x_\varepsilon)$ . As in Proposition 4.2,  $I_\alpha * F_k(w_\varepsilon)$  is uniformly bounded in  $L^\infty(\mathbb{R}^N)$  for all  $\varepsilon$ . Then, noting that  $0 \leq f_k(w_\varepsilon(x)) \leq k$  for all  $x \in \mathbb{R}^N$ , local elliptic estimates (see [29]) yield  $w_\varepsilon(0) \rightarrow U(z_0)$  as  $\varepsilon \rightarrow 0$ . It follows from (4.3) that  $\|v_\varepsilon\|_\infty = w_\varepsilon(0) < \kappa$  uniformly for small  $\varepsilon > 0$ . Therefore, for small  $\varepsilon > 0$ ,  $f_k(v_\varepsilon(x)) \equiv f(v_\varepsilon(x))$ ,  $x \in \mathbb{R}^N$ , and then  $v_\varepsilon$  is a positive solution to (1.1).  $\square$

**Conflict of interest.** The authors declare they have no conflict of interest.

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