

Research Article

Vieri Benci*, Luigi Carlo Berselli and Carlo Romano Grisanti

The Caccioppoli ultrafunctions

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Abstract: Ultrafunctions are a particular class of functions defined on a hyperreal field $\mathbb{R}^* \supset \mathbb{R}$. They have been introduced and studied in some previous works [2, 6, 7]. In this paper we introduce a particular space of ultrafunctions which has special properties, especially in term of localization of functions together with their derivatives. An appropriate notion of integral is then introduced which allows to extend in a consistent way the integration by parts formula, the Gauss theorem and the notion of perimeter. This new space we introduce, seems suitable for applications to Partial Differential Equations and Calculus of Variations. This fact will be illustrated by a simple, but meaningful example.

Keywords: Ultrafunctions, non-Archimedean mathematics, nonstandard analysis, delta function, distributions

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1 Introduction

The Caccioppoli ultrafunctions can be considered as a kind generalized functions. In many circumstances, the notion of real function is not sufficient to the needs of a theory and it is necessary to extend it. Among people working in partial differential equations, the theory of distributions of Schwartz is the most commonly used, but other notions of generalized functions have been introduced by Colombeau [13] and Sato [18, 19]. This paper deals with a new kind of generalized functions, called “ultrafunctions”, which have been introduced recently in [2] and developed in [6–10]. They provide generalized solutions to certain equations which do not have any solution, not even among the distributions.

Actually, the ultrafunctions are pointwise defined on a subset of $(\mathbb{R}^*)^N$, where \mathbb{R}^* is the field of hyperreal numbers, namely the numerical field on which nonstandard analysis (NSA in the sequel) is based. We refer to Keisler [15] for a very clear exposition of NSA and in the following, starred quantities are the natural extensions of the corresponding classical quantities.

The main novelty of this paper is that we introduce the space of Caccioppoli ultrafunctions $V_\Lambda(\Omega)$. They satisfy special properties which are very powerful in applications to Partial Differential Equations and Calculus of Variations. The construction of this space is rather technical, but contains some relevant improvements with respect to the previous notions present in the literature (see e.g. [2, 4–10]).

The main peculiarities of the ultrafunctions in $V_\Lambda(\Omega)$ are the following: there exist a generalized partial derivative D_i and a generalized integral \oint (called pointwise integral) such that the following hold:

- (1) The generalized derivative is a local operator, namely, if $\text{supp}(u) \subset E^*$ (where E is an open set), then $\text{supp}(D_i u) \subset E^*$.
- (2) For all $u, v \in V_\Lambda(\Omega)$,

$$\oint D_i u v \, dx = - \oint u D_i v \, dx. \quad (1.1)$$

***Corresponding author: Vieri Benci**, Dipartimento di Matematica, Università degli Studi di Pisa, Via F. Buonarroti 1/c, 56127 Pisa, Italy, e-mail: vieri.benci@unipi.it

Luigi Carlo Berselli, Carlo Romano Grisanti, Dipartimento di Matematica, Università degli Studi di Pisa, Via F. Buonarroti 1/c, 56127 Pisa, Italy, e-mail: luigi.carlo.berselli@unipi.it, carlo.romano.grisanti@unipi.it.

<https://orcid.org/0000-0001-9479-7492>

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(3) The “generalized” Gauss theorem holds for any measurable set A (see Theorem 4.4)

$$\oint_A D \cdot \varphi \, dx = \oint_{\partial A} \varphi \cdot \mathbf{n}_A \, dS.$$

(4) To any distribution $T \in \mathcal{D}'(\Omega)$ we can associate an equivalence class of ultrafunctions $[u]$ such that, for all $v \in [u]$ and all $\varphi \in \mathcal{D}(\Omega)$,

$$\text{st}\left(\oint v \varphi^\circ \, dx\right) = \langle T, \varphi \rangle,$$

where $\text{st}(\cdot)$ denotes the standard part of an hyperreal number.

The most relevant point, which is not present in the previous approaches to ultrafunctions, is that we are able to extend the notion of partial derivative so that it is a local operator and it satisfies the usual formula valid when integrating by parts, at the price of a suitable extension of the integral as well. In the proof of this fact, the Caccioppoli sets play a fundamental role.

It is interesting to compare the result about the Caccioppoli ultrafunctions with the well-known Schwartz impossibility theorem:

Theorem (Schwartz impossibility theorem). *There does not exist a differential algebra $(\mathfrak{A}, +, \otimes, D)$ in which the distributions can be embedded, where D is a linear operator that extends the distributional derivative and satisfies the Leibniz rule (i.e. $D(u \otimes v) = Du \otimes v + u \otimes Dv$) and \otimes is an extension of the pointwise product on $\mathcal{C}(\mathbb{R})$.*

The ultrafunctions extend the space of distributions; they do not violate the Schwartz theorem since the Leibniz rule, in general, does not hold (see Remark 4.9). Nevertheless, we can prove the integration by parts rule (1.1) and the Gauss’ divergence theorem (with the appropriate extension \oint of the usual integral), which are the main tools used in the applications. These results are a development of the theory previously introduced in [9] and [11].

The theory of ultrafunctions makes deep use of the techniques of NSA presented via the notion of Λ -limit. This presentation has the advantage that a reader, which does not know NSA, is able to follow most of the arguments.

In the last section we present some very simple examples to show that the ultrafunctions can be used to perform a precise mathematical analysis of problems which are not tractable via the distributions.

1.1 Plan of the paper

In Section 2, we present a summary of the theory of Λ -limits and their role in the development of the ultrafunctions using nonstandard methods, especially in the context of transferring as much as possible the language of classical analysis. In Section 3, we define the notion of ultrafunctions, with emphasis on the *pointwise integral*. In Section 4, we define the most relevant notion, namely the generalized derivative, and its connections with the pointwise integral, together with comparison with the classical and distributional derivative. In Section 5, we show how to construct a space satisfying all the properties of the generalized derivative and integrals. This section is the most technical and can be skipped in a first reading. Finally, in Section 6, we present a general result and two very simple variational problems. In particular, the second problem is very elementary but without solutions in the standard H^1 -setting. Nevertheless, it has a natural and explicit candidate as solution. We show how this can be described by means of the language of ultrafunctions.

1.2 Notations

Let X be a set and let Ω be a subset of \mathbb{R}^N .

- $\mathcal{P}(X)$ denotes the power set of X and $\mathcal{P}_{\text{fin}}(X)$ denotes the family of finite subsets of X .
- $\mathfrak{F}(X, Y)$ denotes the set of all functions from X to Y and $\mathfrak{F}(\Omega) = \mathfrak{F}(\Omega, \mathbb{R})$.

- $\mathcal{C}(\Omega)$ denotes the set of continuous functions defined on Ω .
- $\mathcal{C}^k(\Omega)$ denotes the set of functions defined on Ω which have continuous derivatives up to the order k .
- $H^{k,p}(\Omega)$ denotes the usual Sobolev space of functions defined on Ω .
- If $W(\Omega)$ is any function space, then $W_c(\Omega)$ will denote the function space of functions in $W(\Omega)$ having compact support.
- $\mathcal{C}_0(\Omega \cup \Xi)$, $\Xi \subseteq \partial\Omega$, denotes the set of continuous functions in $\mathcal{C}(\Omega \cup \Xi)$ which vanish for $x \in \Xi$.
- $\mathcal{D}(\Omega)$ denotes the set of the infinitely differentiable functions with compact support defined on Ω , and $\mathcal{D}'(\Omega)$ denotes the topological dual of $\mathcal{D}(\Omega)$, namely the set of distributions on Ω .
- For any $\xi \in (\mathbb{R}^N)^*$, $\rho \in \mathbb{R}^*$, we set $\mathfrak{B}_\rho(\xi) = \{x \in (\mathbb{R}^N)^* \mid |x - \xi| < \rho\}$.
- $\text{supp}(f) = (\text{supp}(f))^*$, where supp is the usual notion of support of a function or a distribution.
- $\text{mon}(x) = \{y \in (\mathbb{R}^N)^* \mid x \sim y\}$, where $x \sim y$ means that $x - y$ is infinitesimal.
- $\text{gal}(x) = \{y \in (\mathbb{R}^N)^* \mid x - y \text{ is finite}\}$.
- If W is a generic function space, its topological dual will be denoted by W' and the pairing by $\langle \cdot, \cdot \rangle_W$.
- We denote by χ_X the indicator (or characteristic) function of X , namely,

$$\chi_X(x) = \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{if } x \notin X. \end{cases}$$

- $|X|$ will denote the cardinality of X .

2 Λ -theory

In this section we present the basic notions of Non-Archimedean Mathematics and of Nonstandard Analysis, following a method inspired by [3] (see also [2] and [6]).

2.1 Non-Archimedean fields

Here, we recall the basic definitions and facts regarding non-Archimedean fields. In the following, \mathbb{K} will denote an ordered field. We recall that such a field contains (a copy of) the rational numbers. Its elements will be called numbers.

Definition 2.1. Let \mathbb{K} be an ordered field and $\xi \in \mathbb{K}$. We say that:

- ξ is infinitesimal if, for all positive $n \in \mathbb{N}$, $|\xi| < \frac{1}{n}$.
- ξ is finite if there exists $n \in \mathbb{N}$ such that $|\xi| < n$.
- ξ is infinite if, for all $n \in \mathbb{N}$, $|\xi| > n$ (equivalently, if ξ is not finite).

Definition 2.2. An ordered field \mathbb{K} is called *non-Archimedean* if it contains an infinitesimal $\xi \neq 0$.

It is easily seen that all infinitesimal are finite, that the inverse of an infinite number is a nonzero infinitesimal number, and that the inverse of a nonzero infinitesimal number is infinite.

Definition 2.3. A superreal field is an ordered field \mathbb{K} that properly extends \mathbb{R} .

It is easy to show, due to the completeness of \mathbb{R} , that there are nonzero infinitesimal numbers and infinite numbers in any superreal field. Infinitesimal numbers can be used to formalize a new notion of closeness:

Definition 2.4. We say that two numbers $\xi, \zeta \in \mathbb{K}$ are infinitely close if $\xi - \zeta$ is infinitesimal. In this case, we write $\xi \sim \zeta$.

Clearly, the relation \sim of infinite closeness is an equivalence relation and we have the following theorem.

Theorem 2.5. If \mathbb{K} is a superreal field, every finite number $\xi \in \mathbb{K}$ is infinitely close to a unique real number $r \sim \xi$, called the standard part of ξ .

Given a finite number ξ , we denote its standard part by $\text{st}(\xi)$, and we put $\text{st}(\xi) = \pm\infty$ if $\xi \in \mathbb{K}$ is a positive (negative) infinite number.

Definition 2.6. Let \mathbb{K} be a superreal field, and $\xi \in \mathbb{K}$ a number. The monad of ξ is the set of all numbers that are infinitely close to it, i.e.

$$\text{mon}(\xi) = \{\zeta \in \mathbb{K} \mid \xi \sim \zeta\},$$

and the galaxy of ξ is the set of all numbers that are finitely close to it, i.e.

$$\text{gal}(\xi) = \{\zeta \in \mathbb{K} \mid \xi - \zeta \text{ is finite}\}.$$

By definition, it follows that the set of infinitesimal numbers is $\text{mon}(0)$ and that the set of finite numbers is $\text{gal}(0)$.

2.2 The Λ -limit

In this subsection we introduce a particular non-Archimedean field by means of Λ -theory¹ (for complete proofs and further information the reader is referred to [1, 2, 6]). To recall the basics of Λ -theory we have to recall the notion of superstructure on a set (see also [15]):

Definition 2.7. Let E be an infinite set. The superstructure on E is the set

$$V_\infty(E) = \bigcup_{n \in \mathbb{N}} V_n(E),$$

where the sets $V_n(E)$ are defined by induction setting

$$V_0(E) = E$$

and, for every $n \in \mathbb{N}$,

$$V_{n+1}(E) = V_n(E) \cup \mathcal{P}(V_n(E)).$$

Here $\mathcal{P}(E)$ denotes the power set of E . By identifying the couples with the Kuratowski pairs and the functions and the relations with their graphs, it follows that $V_\infty(E)$ contains almost every usual mathematical object that can be constructed starting with E ; in particular, $V_\infty(\mathbb{R})$, which is the superstructure that we will consider in the following, contains almost every usual mathematical object of analysis.

Throughout this paper we let

$$\mathcal{L} = \mathcal{P}_{\text{fin}}(V_\infty(\mathbb{R}))$$

and we order \mathcal{L} via inclusion. Notice that (\mathcal{L}, \subseteq) is a directed set. We add to \mathcal{L} a *point at infinity* $\Lambda \notin \mathcal{L}$, and we define the following family of neighborhoods of Λ :

$$\{\{\Lambda\} \cup Q \mid Q \in \mathcal{U}\},$$

where \mathcal{U} is a *fine ultrafilter* on \mathcal{L} , namely, a filter such that the following hold:

- For every $A, B \subseteq \mathcal{L}$, if $A \cup B = \mathcal{L}$, then $A \in \mathcal{U}$ or $B \in \mathcal{U}$.
- For every $\lambda \in \mathcal{L}$ the set $Q(\lambda) := \{\mu \in \mathcal{L} \mid \lambda \subseteq \mu\} \in \mathcal{U}$.

In particular, we will refer to the elements of \mathcal{U} as *qualified sets* and we will write $\Lambda = \Lambda(\mathcal{U})$ when we want to highlight the choice of the ultrafilter. A function $\varphi : \mathcal{L} \rightarrow E$ will be called *net* (with values in E). If $\varphi(\lambda)$ is a real net, we have that

$$\lim_{\lambda \rightarrow \Lambda} \varphi(\lambda) = L$$

if and only if for all $\varepsilon > 0$ there exists $Q \in \mathcal{U}$ such that for all $\lambda \in Q$,

$$|\varphi(\lambda) - L| < \varepsilon.$$

¹ Readers expert in nonstandard analysis will recognize that Λ -theory is equivalent to the superstructure constructions of Keisler (see [15] for a presentation of the original constructions of Keisler).

As usual, if a property $P(\lambda)$ is satisfied by any λ in a neighborhood of Λ , we will say that it is *eventually* satisfied.

Notice that the Λ -topology satisfies these interesting properties:

Proposition 2.8. *If the net $\varphi(\lambda)$ takes values in a compact set K , then it is a converging net.*

Proof. Suppose that the net $\varphi(\lambda)$ has a subnet converging to $L \in \mathbb{R}$. We fix $\varepsilon > 0$ arbitrarily and we have to prove that $Q_\varepsilon \in \mathcal{U}$ where

$$Q_\varepsilon = \{\lambda \in \mathcal{L} \mid |\varphi(\lambda) - L| < \varepsilon\}.$$

We argue indirectly and we assume that $Q_\varepsilon \notin \mathcal{U}$. Then, by the definition of ultrafilter, $N = \mathcal{L} \setminus Q_\varepsilon \in \mathcal{U}$ and hence

$$|\varphi(\lambda) - L| \geq \varepsilon \quad \text{for all } \lambda \in N.$$

This contradicts the fact that $\varphi(\lambda)$ has a subnet which converges to L . □

Proposition 2.9. *Assume that $\varphi : \mathcal{L} \rightarrow E$, where E is a first countable topological space; then if*

$$\lim_{\lambda \rightarrow \Lambda} \varphi(\lambda) = x_0,$$

there exists a sequence $\{\lambda_n\}$ in \mathcal{L} such that

$$\lim_{n \rightarrow \infty} \varphi(\lambda_n) = x_0.$$

We refer to the sequence $\varphi_n := \varphi(\lambda_n)$ as a subnet of $\varphi(\lambda)$.

Proof. It follows easily from the definitions. □

Example 2.10. Let $\varphi : \mathcal{L} \rightarrow V$ be a net with values in a bounded set of a reflexive Banach space equipped with the weak topology; then

$$v := \lim_{\lambda \rightarrow \Lambda} \varphi(\lambda)$$

is uniquely defined and there exists a sequence $n \mapsto \varphi(\lambda_n)$ which converges to v .

Definition 2.11. The set of the hyperreal numbers $\mathbb{R}^* \supset \mathbb{R}$ is a set equipped with a topology τ such that the following hold:

- Every net $\varphi : \mathcal{L} \rightarrow \mathbb{R}$ has a unique limit in \mathbb{R}^* if \mathcal{L} and \mathbb{R}^* are equipped with the Λ and the τ topology, respectively.
- \mathbb{R}^* is the closure of \mathbb{R} with respect to the topology τ .
- τ is the coarsest topology which satisfies the first property.

The existence of such an \mathbb{R}^* is a well-known fact in NSA. The limit $\xi \in \mathbb{R}^*$ of a net $\varphi : \mathcal{L} \rightarrow \mathbb{R}$ with respect to the τ topology, following [2], is called the Λ -limit of φ and the following notation will be used:

$$\xi = \lim_{\lambda \uparrow \Lambda} \varphi(\lambda), \tag{2.1}$$

namely, we shall use the up-arrow “ \uparrow ” to remind that the target space is equipped with the topology τ .

Given

$$\xi := \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \quad \text{and} \quad \eta := \lim_{\lambda \uparrow \Lambda} \psi(\lambda),$$

we set

$$\xi + \eta := \lim_{\lambda \uparrow \Lambda} (\varphi(\lambda) + \psi(\lambda)), \tag{2.2}$$

and

$$\xi \cdot \eta := \lim_{\lambda \uparrow \Lambda} (\varphi(\lambda) \cdot \psi(\lambda)). \tag{2.3}$$

Then the following well-known theorem holds:

Theorem 2.12. *The definitions in (2.2) and (2.3) are well posed and \mathbb{R}^* , equipped with these operations, is a non-Archimedean field.*

Remark 2.13. We observe that the field of hyperreal numbers is defined as a sort of completion of the real numbers. In fact, \mathbb{R}^* is isomorphic to the ultrapower $\mathbb{R}^{\mathcal{L}}/\mathcal{I}$, where

$$\mathcal{I} = \{\varphi : \mathcal{L} \rightarrow \mathbb{R} \mid \varphi(\lambda) = 0 \text{ eventually}\}.$$

The isomorphism resembles the classical one between the real numbers and the equivalence classes of Cauchy sequences. This method is well known for the construction of real numbers starting from rationals.

2.3 Natural extension of sets and functions

For our purposes it is very important that the notion of Λ -limit can be extended to sets and functions (but also to differential and integral operators) in order to have a much wider set of objects to deal with, to enlarge the notion of variational problem and of variational solution.

So we will define the Λ -limit of any bounded net of mathematical objects in $V_{\infty}(\mathbb{R})$ (a net $\varphi : \mathcal{L} \rightarrow V_{\infty}(\mathbb{R})$ is called *bounded* if there exists $n \in \mathbb{N}$ such that, for all $\lambda \in \mathcal{L}$, $\varphi(\lambda) \in V_n(\mathbb{R})$). To do this, let us consider a net

$$\varphi : \mathcal{L} \rightarrow V_n(\mathbb{R}). \quad (2.4)$$

We will define $\lim_{\lambda \uparrow \Lambda} \varphi(\lambda)$ by induction on n .

Definition 2.14. For $n = 0$, $\lim_{\lambda \uparrow \Lambda} \varphi(\lambda)$ is defined by (2.1). By induction we may assume that the limit is defined for $n - 1$ and we define it for the net (2.4) as follows:

$$\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) = \left\{ \lim_{\lambda \uparrow \Lambda} \psi(\lambda) \mid \psi : \mathcal{L} \rightarrow V_{n-1}(\mathbb{R}), \psi(\lambda) \in \varphi(\lambda) \text{ for all } \lambda \in \mathcal{L} \right\}.$$

A mathematical entity (number, set, function or relation) which is the Λ -limit of a net is called *internal*.

Definition 2.15. If for all $\lambda \in \mathcal{L}$, $E_{\lambda} = E \in V_{\infty}(\mathbb{R})$, we set $\lim_{\lambda \uparrow \Lambda} E_{\lambda} = E^*$, namely,

$$E^* := \left\{ \lim_{\lambda \uparrow \Lambda} \psi(\lambda) \mid \psi(\lambda) \in E \right\}.$$

The set E^* is called the *natural extension* of E .

Notice that, while the Λ -limit of a constant sequence of numbers gives this number itself, a constant sequence of sets gives a larger set, namely E^* . In general, the inclusion $E \subseteq E^*$ is proper.

Given any set E , we can associate to it two sets: its natural extension E^* and the set E^{σ} , where

$$E^{\sigma} = \{X^* \mid X \in E\}.$$

Clearly, E^{σ} is a copy of E , however it might be different as set since, in general, $X^* \neq X$.

Remark 2.16. If $\varphi : \mathcal{L} \rightarrow X$ is a net with values in a topological space, we have the usual limit

$$\lim_{\lambda \rightarrow \Lambda} \varphi(\lambda),$$

which, by Proposition 2.8, always exists in the Alexandrov compactification $X \cup \{\infty\}$. Moreover, we have that the Λ -limit always exists and it is an element of X^* . In addition, the Λ -limit of a net is in X^{σ} if and only if φ is eventually constant. If $X = \mathbb{R}$ and both limits exist, then

$$\lim_{\lambda \rightarrow \Lambda} \varphi(\lambda) = \text{st} \left(\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \right).$$

The above equation suggests the following definition.

Definition 2.17. If X is a topological space equipped with a Hausdorff topology, and $\xi \in X^*$, we set

$$\text{St}_X(\xi) = \lim_{\lambda \rightarrow \Lambda} \varphi(\lambda)$$

if there is a net $\varphi : \mathfrak{L} \rightarrow X$ converging in the topology of X and such that

$$\xi = \lim_{\lambda \uparrow \Lambda} \varphi(\lambda),$$

and

$$\text{St}_X(\xi) = \infty$$

otherwise.

By the above definition we have that

$$\lim_{\lambda \rightarrow \Lambda} \varphi(\lambda) = \text{St}_X\left(\lim_{\lambda \uparrow \Lambda} \varphi(\lambda)\right).$$

Definition 2.18. Let

$$f_\lambda : E_\lambda \rightarrow \mathbb{R}, \quad \lambda \in \mathfrak{L},$$

be a net of functions. We define a function

$$f : \left(\lim_{\lambda \uparrow \Lambda} E_\lambda\right) \rightarrow \mathbb{R}^*$$

as follows: for every $\xi \in (\lim_{\lambda \uparrow \Lambda} E_\lambda)$ we set

$$f(\xi) := \lim_{\lambda \uparrow \Lambda} f_\lambda(\psi(\lambda)),$$

where $\psi(\lambda)$ is a net of numbers such that

$$\psi(\lambda) \in E_\lambda \quad \text{and} \quad \lim_{\lambda \uparrow \Lambda} \psi(\lambda) = \xi.$$

A function which is a Λ -limit is called *internal*. In particular, if, for all $\lambda \in \mathfrak{L}$,

$$f_\lambda = f, \quad f : E \rightarrow \mathbb{R},$$

we set

$$f^* = \lim_{\lambda \uparrow \Lambda} f_\lambda.$$

The function $f^* : E^* \rightarrow \mathbb{R}^*$ is called the *natural extension* of f . If we identify f with its graph, then f^* is the graph of its natural extension.

2.4 Hyperfinite sets and hyperfinite sums

Definition 2.19. An internal set is called *hyperfinite* if it is the Λ -limit of a net $\varphi : \mathfrak{L} \rightarrow \mathfrak{F}$, where \mathfrak{F} is a family of finite sets.

For example, if $E \in V_\infty(\mathbb{R})$, the set

$$\tilde{E} = \lim_{\lambda \uparrow \Lambda} (\lambda \cap E)$$

is hyperfinite. Notice that

$$E^\sigma \subset \tilde{E} \subset E^*,$$

so we can say that every set is contained in a hyperfinite set.

It is possible to add the elements of an hyperfinite set of numbers (or vectors) as follows: let

$$A := \lim_{\lambda \uparrow \Lambda} A_\lambda$$

be an hyperfinite set of numbers (or vectors); then the hyperfinite sum of the elements of A is defined in the following way:

$$\sum_{a \in A} a = \lim_{\lambda \uparrow \Lambda} \sum_{a \in A_\lambda} a.$$

In particular, if $A_\lambda = \{a_1(\lambda), \dots, a_{\beta(\lambda)}(\lambda)\}$ with $\beta(\lambda) \in \mathbb{N}$, then setting

$$\beta = \lim_{\lambda \uparrow \Lambda} \beta(\lambda) \in \mathbb{N}^*,$$

we use the notation

$$\sum_{j=1}^{\beta} a_j = \lim_{\lambda \uparrow \Lambda} \sum_{j=1}^{\beta(\lambda)} a_j(\lambda).$$

3 Ultrafunctions

3.1 Caccioppoli spaces of ultrafunctions

Let Ω be an open bounded set in \mathbb{R}^N , and let $W(\Omega)$ be a (real or complex) vector space such that

$$\mathcal{D}(\overline{\Omega}) \subseteq W(\Omega) \subseteq L^1(\Omega).$$

Definition 3.1. A space of ultrafunctions modeled over the space $W(\Omega)$ is given by

$$W_\Lambda(\Omega) := \lim_{\lambda \uparrow \Lambda} W_\lambda(\Omega) = \left\{ \lim_{\lambda \uparrow \Lambda} f_\lambda \mid f_\lambda \in W_\lambda(\Omega) \right\},$$

where $W_\lambda(\Omega) \subset W(\Omega)$ is an increasing net of finite-dimensional spaces such that

$$W_\lambda(\Omega) \supseteq \text{Span}(W(\Omega) \cap \lambda).$$

So, given any vector space of functions $W(\Omega)$, the space of ultrafunction generated by $\{W_\lambda(\Omega)\}$ is a vector space of hyperfinite dimension that includes $W(\Omega)^\sigma$, as well as other functions in $W(\Omega)^*$. Hence the ultrafunctions are particular internal functions

$$u : \overline{\Omega}^* \rightarrow \mathbb{R}^*.$$

Definition 3.2. Given a space of ultrafunctions $W_\Lambda(\Omega)$, a σ -basis is an internal set of ultrafunctions $\{\sigma_a(x)\}_{a \in \Gamma}$ such that $\Omega \subset \Gamma \subset \Omega^*$ and for all $u \in W_\Lambda(\Omega)$, we can write

$$u(x) = \sum_{a \in \Gamma} u(a) \sigma_a(x).$$

It is possible to prove (see e.g. [2]) that every space of ultrafunctions has a σ -basis. Clearly, if $a, b \in \Gamma$, then $\sigma_a(b) = \delta_{ab}$, where δ_{ab} denotes the Kronecker delta.

Now we will introduce a class of spaces of ultrafunctions suitable for most applications. To do this, we need to recall the notion of Caccioppoli set:

Definition 3.3. A *Caccioppoli set* E is a Borel set such that $\chi_E \in \text{BV}$, namely, such that $\nabla(\chi_E)$ (the distributional gradient of the characteristic function of E) is a finite Radon measure concentrated on ∂E .

The number

$$p(E) := \langle |\nabla(\chi_E)|, 1 \rangle$$

is called *Caccioppoli perimeter* of E . From now on, with some abuse of notation, the above expression will be written as follows:

$$\int |\nabla(\chi_E)| \, dx;$$

this expression makes sense since “ $|\nabla(\chi_E)| \, dx$ ” is a measure.

If $E \subset \overline{\Omega}$ is a measurable set, we define the *density function* of E as follows:

$$\theta_E(x) = \text{st} \left(\frac{m(B_\eta(x) \cap E^*)}{m(B_\eta(x) \cap (\overline{\Omega})^*)} \right), \quad (3.1)$$

where η is a fixed infinitesimal and m is the Lebesgue measure.

Clearly, $\theta_E(x)$ is a function whose value is 1 in $\text{int}(E)$ and 0 in $\mathbb{R}^N \setminus \bar{E}$; moreover, it is easy to prove that $\theta_E(x)$ is a measurable function and we have that

$$\int \theta_E(x) dx = m(E);$$

also, if E is a bounded Caccioppoli set,

$$\int |\nabla \theta_E| dx = p(E).$$

Definition 3.4. A set E is called *special Caccioppoli set* if it is open, bounded and $m(\partial E) = 0$. The family of special Caccioppoli sets will be denoted by $\mathfrak{C}(\Omega)$.

Now we can define a space $V(\Omega)$ suitable for our aims:

Definition 3.5. A function $f \in V(\Omega)$ if and only if

$$f(x) = \sum_{k=1}^n f_k(x) \theta_{E_k}(x),$$

where $f_k \in \mathcal{C}(\mathbb{R}^N)$, $E_k \in \mathfrak{C}(\Omega)$, and n is a number which depends on f . Such a function will be called *Caccioppoli function*.

Notice that $V(\Omega)$ is a module over the ring $\mathcal{C}(\bar{\Omega})$ and that, for all $f \in V(\Omega)$,

$$\left(\int |f(x)| dx = 0 \right) \implies (\forall x \in \mathbb{R}^N, f(x) = 0).$$

Hence, in particular, $(\int |f(x)|^2 dx)^{\frac{1}{2}}$, is a norm (and not a seminorm).

Definition 3.6. The space $V_\Lambda(\Omega)$ is called *Caccioppoli space of ultrafunctions* if it satisfies the following properties:

- (i) $V_\Lambda(\Omega)$ is modeled on the space $V(\Omega)$.
- (ii) $V_\Lambda(\Omega)$ has a σ -basis $\{\sigma_a(x)\}_{a \in \Gamma}$, $\Gamma \subset (\mathbb{R}^N)^*$, such that, for all $a \in \Gamma$, the support of σ_a is contained in $\text{mon}(a)$.

The existence of a Caccioppoli space of ultrafunctions will be proved in Section 5.

Remark 3.7. Usually in the study of PDEs, the function space where to work depends on the problem or equation which we want to study. The same fact is true in the world of ultrafunctions. However, the Caccioppoli space $V_\Lambda(\Omega)$ has a special position since it satisfies the properties required by a large class of problems. First of all $V_\Lambda(\Omega) \subset (L^1(\Omega))^*$. This fact allows to define the pointwise integral (see the next subsection) for all the ultrafunctions. This integral turns out to be a very good tool. However, the space L^1 is not a good space for modeling ultrafunctions, since they are defined pointwise while the functions in L^1 are defined a.e. Thus, we are lead to the space $L^1(\Omega) \cap \mathcal{C}(\bar{\Omega})$, but this space does not contain functions like $f(x)\theta_E(x)$ which are important in many situations; for example, the Gauss' divergence theorem can be formulated as

$$\int \nabla \cdot F(x) \theta_E(x) dx = \int_{\partial E} \mathbf{n} \cdot F(x) dS$$

whenever the vector field F and E are sufficiently smooth. Thus the space $V_\Lambda(\Omega)$ seems to be the right space for a large class of problems.

3.2 The pointwise integral

From now on we will denote by $V_\Lambda(\Omega)$ a fixed Caccioppoli space of ultrafunctions and by $\{\sigma_a(x)\}_{a \in \Gamma}$ a fixed σ -basis as in Definition 3.6. If $u \in V_\Lambda(\Omega)$, we have that

$$\int^* u(x) dx = \sum_{a \in \Gamma} u(a) \eta_a, \quad (3.2)$$

where

$$\eta_a := \int^* \sigma_a(x) dx.$$

Equality (3.2) suggests the following definition:

Definition 3.8. For any internal function $g : \Omega^* \rightarrow \mathbb{R}^*$, we set

$$\oint g(x) dx := \sum_{q \in \Gamma} g(q) \eta_q.$$

In the sequel we will refer to \oint as to the *pointwise integral*.

From Definition 3.8, we have that

$$\int^* u(x) dx = \oint u(x) dx, \quad u \in V_\Lambda(\Omega),$$

and, in particular,

$$\int f(x) dx = \oint f^*(x) dx, \quad f \in V(\Omega).$$

But in general these equalities are not true for L^1 functions. For example if

$$f(x) = \begin{cases} 1 & \text{if } x = x_0 \in \Omega, \\ 0 & \text{if } x \neq x_0, \end{cases}$$

we have that

$$\int^* f^*(x) dx = \int f(x) dx = 0,$$

while

$$\oint f^*(x) dx = \eta_{x_0} > 0.$$

However, for any set $E \in \mathcal{C}(\Omega)$ and any function $f \in \mathcal{C}(\overline{\Omega})$,

$$\oint f^*(x) \theta_E^*(x) dx = \int_E f(x) dx;$$

in fact,

$$\oint f^*(x) \theta_E(x) dx = \int^* f^*(x) \theta_E^*(x) dx = \int f(x) \theta_E(x) dx = \int_E f(x) dx.$$

Then, if $f(x) \geq 0$ and E is a bounded open set, we have that

$$\oint f^*(x) \chi_E^* dx < \oint f^*(x) \theta_E^*(x) dx < \oint f^*(x) \chi_{\overline{E}}^* dx.$$

since

$$\chi_E < \theta_E < \chi_{\overline{E}}.$$

As we will see in the following part of this paper, in many cases, it is more convenient to work with the pointwise integral \oint rather than with the natural extension of the Lebesgue integral \int^* .

Example 3.9. If ∂E is smooth, we have that, for all $x \in \partial E$, $\theta_E(x) = \frac{1}{2}$ and hence, if E is open,

$$\begin{aligned} \oint f^*(x) \chi_E^*(x) dx &= \oint f^*(x) \theta_E^*(x) dx - \frac{1}{2} \oint f^*(x) \chi_{\partial E}^*(x) dx \\ &= \int_E f(x) dx - \frac{1}{2} \oint f^*(x) \chi_{\partial E}^*(x) dx, \end{aligned}$$

and similarly

$$\oint f^*(x) \chi_{\overline{E}}^*(x) dx = \int_E f(x) dx + \frac{1}{2} \oint f^*(x) \chi_{\partial E}^*(x) dx;$$

of course, the term $\frac{1}{2} \oint f^*(x) \chi_{\partial E}^*(x) dx$ is an infinitesimal number and it is relevant only in some particular problems.

The pointwise integral allows us to define the following scalar product:

$$\oint u(x)v(x) dx = \sum_{q \in \Gamma} u(q)v(q)\eta_q. \quad (3.3)$$

From now on, the norm of an ultrafunction will be given by

$$\|u\| = \left(\oint |u(x)|^2 dx \right)^{\frac{1}{2}}.$$

Notice that

$$\oint u(x)v(x) dx = \int^* u(x)v(x) dx \iff uv \in V_\Lambda(\Omega).$$

Theorem 3.10. *If $\{\sigma_a(x)\}_{a \in \Gamma}$ is a σ -basis, then $\{\frac{\sigma_a(x)}{\sqrt{\eta_a}}\}_{a \in \Gamma}$ is a orthonormal basis with respect to the scalar product (3.3). Hence for every $u \in V_\Lambda(\Omega)$,*

$$u(x) = \sum_{q \in \Gamma} \frac{1}{\eta_q} \left(\oint u(\xi)\sigma_q(\xi) d\xi \right) \sigma_q(x).$$

Moreover, we have that

$$\|\sigma_a\|^2 = \eta_a \quad \text{for all } a \in \Gamma. \quad (3.4)$$

Proof. By (3.3), we have that

$$\oint \sigma_a(x)\sigma_b(x) dx = \sum_{q \in \Gamma} \sigma_a(q)\sigma_b(q)\eta_q = \sum_{q \in \Gamma} \delta_{aq}\delta_{bq}\eta_q = \delta_{ab}\eta_a,$$

and hence the result. By the above equality, taking $b = a$, we get (3.4). \square

3.3 The δ -bases

Next, we will define the *delta ultrafunctions*:

Definition 3.11. Given a point $q \in \Omega^*$, we denote by $\delta_q(x)$ an ultrafunction in $V_\Lambda(\Omega)$ such that

$$\oint v(x)\delta_q(x) dx = v(q) \quad \text{for all } v \in V_\Lambda(\Omega), \quad (3.5)$$

and $\delta_q(x)$ is called *delta (or the Dirac) ultrafunction* concentrated in q .

Let us see the main properties of the delta ultrafunctions:

Theorem 3.12. *The delta ultrafunction satisfies the following properties:*

- (1) *For every $q \in \overline{\Omega}^*$ there exists a unique delta ultrafunction concentrated in q .*
- (2) *For every $a, b \in \overline{\Omega}^*$, $\delta_a(b) = \delta_b(a)$.*
- (3) $\|\delta_q\|^2 = \delta_q(q)$.

Proof. (1) Let $\{e_j\}_{j=1}^\beta$ be an orthonormal real basis of $V_\Lambda(\Omega)$, and set

$$\delta_q(x) = \sum_{j=1}^\beta e_j(q)e_j(x).$$

Let us prove that $\delta_q(x)$ actually satisfies (3.5). Let $v(x) = \sum_{j=1}^\beta v_j e_j(x)$ be any ultrafunction. Then

$$\begin{aligned} \oint v(x)\delta_q(x) dx &= \oint \left(\sum_{j=1}^\beta v_j e_j(x) \right) \left(\sum_{k=1}^\beta e_k(q)e_k(x) \right) dx \\ &= \sum_{j=1}^\beta \sum_{k=1}^\beta v_j e_k(q) \oint e_j(x)e_k(x) dx \\ &= \sum_{j=1}^\beta \sum_{k=1}^\beta v_j e_k(q) \delta_{jk} = \sum_{k=1}^\beta v_k e_k(q) = v(q). \end{aligned}$$

So $\delta_q(x)$ is a delta ultrafunction concentrated in q . It is unique: in fact, if $\gamma_q(x)$ is another delta ultrafunction concentrated in q , then for every $y \in \bar{\Omega}^*$ we have

$$\delta_q(y) - \gamma_q(y) = \oint (\delta_q(x) - \gamma_q(x)) \delta_y(x) dx = \delta_y(q) - \delta_y(q) = 0,$$

and hence $\delta_q(y) = \gamma_q(y)$ for every $y \in \bar{\Omega}^*$.

(2) We have

$$\delta_a(b) = \oint \delta_a(x) \delta_b(x) dx = \delta_b(a).$$

(3) We have

$$\|\delta_q\|^2 = \oint \delta_q(x) \delta_q(x) dx = \delta_q(q).$$

The proof is complete. \square

By the definition of Γ , for all $a, b \in \Gamma$, we have that

$$\oint \delta_a(x) \sigma_b(x) dx = \sigma_a(b) = \delta_{ab}.$$

From this it follows readily the following result.

Proposition 3.13. *The set $\{\delta_a(x)\}_{a \in \Gamma}$ ($\Gamma \subset \Omega^*$) is the dual basis of the sigma-basis; it will be called the δ -basis of $V_\Lambda(\Omega)$.*

Let us examine the main properties of the δ -basis.

Proposition 3.14. *The δ -basis satisfies the following properties:*

(i) One has

$$u(x) = \sum_{q \in \Gamma} \left[\oint \sigma_q(\xi) u(\xi) d\xi \right] \delta_q(x).$$

(ii) For all $a, b \in \Gamma$, $\sigma_a(x) = \eta_a \delta_a(x)$.

(iii) For all $a \in \Gamma$,

$$\|\delta_a\|^2 = \oint \delta_a(x)^2 dx = \delta_a(a) = \eta_a^{-1}.$$

Proof. (i) This is an immediate consequence of the definition of δ -basis.

(ii) By Theorem 3.10, it follows that

$$\delta_a(x) = \sum_{q \in \Gamma} \frac{1}{\eta_q} \left(\oint \delta_a(\xi) \sigma_q(\xi) d\xi \right) \sigma_q(x) = \sum_{q \in \Gamma} \frac{1}{\eta_q} \delta_{aq} \sigma_q(x) = \frac{1}{\eta_a} \sigma_a(x).$$

(iii) This is an immediate consequence of (ii). \square

3.4 The canonical extension of functions

We have seen that every function $f : \Omega \rightarrow \mathbb{R}$ has a natural extension $f^* : \Omega^* \rightarrow \mathbb{R}^*$. However, in general, f^* is not an ultrafunction; in fact, it is not difficult to prove that the natural extension f^* of a function f is an ultrafunction if and only if $f \in V(\Omega)$. So it is useful to define an ultrafunction $f^\circ \in V_\Lambda(\Omega)$ which approximates f^* . More general, for any internal function $u : \Omega^* \rightarrow \mathbb{R}^*$, we will define an ultrafunction u° as follows.

Definition 3.15. If $u : \Omega^* \rightarrow \mathbb{R}^*$ is an internal function, we define $u^\circ \in V_\Lambda(\Omega)$ by the formula

$$u^\circ(x) = \sum_{q \in \Gamma} u(q) \sigma_q(x);$$

if $f : \Omega \rightarrow \mathbb{R}$, with some abuse of notation, we set

$$f^\circ(x) = (f^*)^\circ(x) = \sum_{q \in \Gamma} f^*(q) \sigma_q(x).$$

Since $\Omega \subset \Gamma$, for any internal function u , we have that

$$u(x) = u^\circ(x) \quad \text{for all } x \in \Omega,$$

and

$$u(x) = u^\circ(x) \text{ for all } x \in \Omega^* \iff u \in V_\Lambda(\Omega).$$

Notice that

$$P^\circ : \mathfrak{F}(\Omega)^* \rightarrow V_\Lambda(\Omega)$$

defined by $P^\circ(u) = u^\circ$ is nothing else but the *orthogonal* projection of $u \in \mathfrak{F}(\Omega)^*$ with respect to the semidefinite bilinear form

$$\oint u(x)h(x) dx.$$

Example 3.16. If $f \in \mathcal{C}(\mathbb{R}^N)$, and $E \in \mathcal{C}(\Omega)$, then $f\theta_E \in V(\Omega)$ and hence

$$(f\theta_E)^\circ = f^* \theta_E^*.$$

Definition 3.17. If a function f is not defined on a set $S := \Omega \setminus \Theta$, by convention, we define

$$f^\circ(x) = \sum_{q \in \Gamma \cap \Theta^*} f^*(q) \sigma_q(x).$$

Example 3.18. By the definition above, for all $x \in \Gamma$, we have that

$$\left(\frac{1}{|x|} \right)^\circ = \begin{cases} \frac{1}{|x|} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

If $f \in \mathcal{C}(\Omega)$, then $f^\circ \neq f^*$ unless $f \in V_\Lambda(\Omega)$. Let us examine what f° looks like.

Theorem 3.19. Let $f : \Omega \rightarrow \mathbb{R}$ be continuous in a bounded open set $A \subset \Omega$. Then, for all $x \in A^*$ with $\text{mon}(x) \subset A^*$, we have that

$$f^\circ(x) = f^*(x).$$

Proof. Fix $x_0 \in A$. Since A is bounded, there exists a set $E \in \mathcal{C}(\Omega)$ such that

$$\text{mon}(x_0) \subset E^* \subset A^*.$$

We have that (see Example 3.16)

$$\begin{aligned} f^\circ(x) &= \sum_{a \in \Gamma} f^*(a) \sigma_a(x) \\ &= \sum_{a \in \Gamma} f^*(a) \theta_E^*(a) \sigma_a(x) + \sum_{a \in \Gamma} f^*(a) (1 - \theta_E^*(a)) \sigma_a(x) \\ &= f^*(x) \theta_E^*(x) + \sum_{a \in \Gamma \setminus E^*} f^*(a) (1 - \theta_E^*(a)) \sigma_a(x). \end{aligned}$$

Since $x_0 \in E^*$, it follows that $\theta_E^*(x_0) = 1$; moreover, since $\text{mon}(x_0) \subset E^*$, by Definition 3.6 (ii),

$$\sigma_a(x_0) = \sigma_{x_0}(a) = 0 \quad \text{for all } a \in \Gamma \setminus E^*.$$

Then $f^\circ(x_0) = f^*(x_0)$. □

Corollary 3.20. If $f \in \mathcal{C}(\Omega)$, then, for any $x \in \Omega^*$ such that $|x|$ is finite, we get

$$f^\circ(x) = f^*(x).$$

3.5 Canonical splitting of an ultrafunction

In many applications, it is useful to split an ultrafunction u into a part w° which is the canonical extension of a standard function w and a part ψ which is not directly related to any classical object.

If $u \in V_\Lambda(\Omega)$, we set

$$\Xi = \{x \in \Omega \mid u(x) \text{ is infinite}\}$$

and

$$\bar{w}(x) = \begin{cases} \text{st}(u(x)) & \text{if } x \in \Omega \setminus \Xi, \\ 0 & \text{if } x \in \Xi. \end{cases}$$

Definition 3.21. For every ultrafunction u consider the splitting

$$u = w^\circ + \psi,$$

where

- $w = \bar{w}|_{\Omega \setminus \Xi}$ and w° , which is defined by Definition 3.17, is called the *functional part* of u ,
- $\psi := u - w^\circ$ is called the *singular part* of u .

We will refer to

$$S := \{x \in \Omega^* \mid \psi(x) \neq 0\}$$

as to the *singular set* of the ultrafunction u .

Notice that w° , the functional part of u , may assume infinite values, but they are determined by the values of w , which is a standard function defined on $\Omega \setminus \Xi$.

Example 3.22. Take $\varepsilon \sim 0$, and

$$u(x) = \frac{1}{x^2 + \varepsilon^2}.$$

In this case

$$w(x) = \frac{1}{x^2}, \quad \psi(x) = \begin{cases} -\frac{\varepsilon^2}{x^2(x^2 + \varepsilon^2)} & \text{if } x \neq 0, \\ \frac{1}{\varepsilon^2} & \text{if } x = 0, \end{cases} \quad S := \{x \in \mathbb{R}^* \mid \psi(x) \neq 0\} \subset \text{mon}(0).$$

We conclude this section with the following trivial propositions which, nevertheless, are very useful in applications:

Proposition 3.23. Let W be a Banach space such that $\mathcal{D}(\Omega) \subset W \subseteq L^1_{\text{loc}}(\Omega)$ and assume that $u_\lambda \in V_\lambda$ is weakly convergent in W . Then if

$$u = w^\circ + \psi$$

is the canonical splitting of $u := \lim_{\lambda \uparrow \Lambda} u_\lambda$, there exists a subnet $u_n := u_{\lambda_n}$ such that

$$\lim_{n \rightarrow \infty} u_n = w \quad \text{weakly in } W$$

and

$$\int \psi v \, dx \sim 0 \quad \text{for all } v \in W.$$

Moreover, if

$$\lim_{n \rightarrow \infty} \|u_n - w\|_W = 0,$$

then $\|\psi\|_W \sim 0$.

Proof. It is an immediate consequence of Proposition 2.9. □

If we use the notation introduced in Definition 2.17, the above proposition can be reformulated as follows:

Proposition 3.24. If $u_\lambda \in V_\lambda$ is weakly convergent to w in W and $u := \lim_{\lambda \uparrow \Lambda} u_\lambda$, then

$$w = \text{St}_{W_{\text{weak}}}(u).$$

If u_λ is strongly convergent to w in W , then

$$w = \text{St}_W(u).$$

An immediate consequence of Proposition 3.23 is the following:

Corollary 3.25. *If $w \in L^1(\Omega)$, then*

$$\oint w^\circ(x) dx \sim \int w(x) dx.$$

Proof. Since $V_\Lambda(\Omega)$ is dense in $L^1(\Omega)$, there is a sequence $u_n \in V_\Lambda(\Omega)$ which converges strongly to w in $L^1(\Omega)$. Now set

$$u := \lim_{\lambda \uparrow \Lambda} u_{|\lambda|}.$$

By Proposition 3.23, we have that

$$u = w^\circ + \psi$$

with $\|\psi\|_{L^{1*}} \sim 0$. Since u and w° are in $V_\Lambda(\Omega)$, it also follows that $\psi \in V_\Lambda(\Omega)$, so that $\oint \psi dx = \int^* \psi dx \sim 0$. Then

$$\oint u(x) dx \sim \oint w^\circ(x) dx.$$

On the other hand,

$$\oint u(x) dx = \int^* u(x) dx = \lim_{\lambda \uparrow \Lambda} \int u_{|\lambda|} dx \sim \lim_{\lambda \rightarrow \Lambda} \int u_{|\lambda|} dx = \lim_{n \rightarrow \infty} \int_\Omega u_n dx = \int_\Omega w(x) dx. \quad \square$$

4 Differential calculus for ultrafunctions

In this section, we will equip the Caccioppoli space of ultrafunctions $V_\Lambda(\Omega)$ with a suitable notion of derivative which generalizes the distributional derivative. Moreover, we will extend the Gauss' divergence theorem to the environment of ultrafunctions and finally we will show the relationship between ultrafunctions and distributions.

4.1 The generalized derivative

If $u \in V_\Lambda(\Omega) \cap [\mathcal{C}^1(\Omega)]^*$, then $\partial_i^* u$ is well defined and hence, using Definition 3.17, we can define an operator

$$D_i : V_\Lambda(\Omega) \cap [\mathcal{C}^1(\Omega)]^* \rightarrow V_\Lambda(\Omega)$$

as follows:

$$D_i u^\circ = (\partial_i^* u)^\circ.$$

However, it would be useful to extend the operator D_i to all the ultrafunctions in $V_\Lambda(\Omega)$ to include in the theory of ultrafunctions also the weak derivative. Moreover, such an extension allows to compare ultrafunctions with distributions. In this section we will define the properties that a generalized derivative must have (Definition 4.1) and in Section 5, we will show that these properties are consistent; we will do that by a construction of the generalized derivative.

Definition 4.1. The generalized derivative

$$D_i : V_\Lambda(\Omega) \rightarrow V_\Lambda(\Omega)$$

is an operator defined on a Caccioppoli ultrafunction space $V_\Lambda(\Omega)$ which satisfies the following properties:

- (I) V_Λ has σ -basis $\{\sigma_a(x)\}_{a \in \Gamma}$ such that, for all $a \in \Gamma$, the support of $D_i \sigma_a$ is contained in $\text{mon}(a)$.
- (II) If $u \in V_\Lambda(\Omega) \cap [\mathcal{C}^1(\Omega)]^*$, then

$$D_i u^\circ = (\partial_i^* u)^\circ.$$

- (III) For all $u, v \in V_\Lambda(\Omega)$,

$$\oint D_i uv dx = - \oint u D_i v dx.$$

(IV) If $E \in \mathfrak{C}(\Omega)$, then for all $v \in V_\Lambda(\Omega)$,

$$\oint D_i \theta_E v \, dx = - \int_{\partial E}^* v (\mathbf{e}_i \cdot \mathbf{n}_E) \, dS,$$

where \mathbf{n}_E is the measure theoretic unit outer normal, integrated on the reduced boundary of E with respect to the $(n-1)$ -Hausdorff measure dS (see e.g. [14, Section 5.7]) and $(\mathbf{e}_1, \dots, \mathbf{e}_N)$ is the canonical basis of \mathbb{R}^N .

We remark that, in the framework of the theory of Caccioppoli sets, the classical formula corresponding to (IV) is the following: for all $v \in \mathcal{C}(\Omega)$,

$$\int \partial_i \theta_E v \, dx = - \int_{\partial E} v (\mathbf{e}_i \cdot \mathbf{n}_E) \, dS.$$

The existence of a generalized derivative will be proved in Section 5.

Now let us define some differential operators:

- $\nabla = (\partial_1, \dots, \partial_N)$ will denote the usual gradient of standard functions,
- $\nabla^* = (\partial_1^*, \dots, \partial_N^*)$ will denote the natural extension of the gradient (in the sense of NSA),
- $D = (D_1, \dots, D_N)$ will denote the canonical extension of the gradient in the sense of the ultrafunctions (Definition 4.1).

Next let us consider the divergence:

- $\nabla \cdot \varphi = \partial_1 \varphi_1 + \dots + \partial_N \varphi_N$ will denote the usual divergence of standard vector fields $\varphi \in [\mathcal{C}^1(\overline{\Omega})]^N$,
- $\nabla^* \cdot \varphi = \partial_1^* \varphi_1 + \dots + \partial_N^* \varphi_N$ will denote the divergence of internal vector fields $\varphi \in [\mathcal{C}^1(\overline{\Omega})^*]^N$,
- $D \cdot \varphi$ will denote the unique ultrafunction $D \cdot \varphi \in V_\Lambda(\Omega)$ such that, for all $v \in V_\Lambda(\Omega)$,

$$\oint D \cdot \varphi v \, dx = - \oint \varphi(x) \cdot Dv \, dx.$$

Finally, we can define the Laplace operator:

- Δ° or D^2 will denote the Laplace operator defined by $D \cdot D$.

4.2 The Gauss' divergence theorem

By Definition 4.1 (IV), for any set $E \in \mathfrak{C}_\Lambda(\Omega)$ and $v \in V_\Lambda(\Omega)$,

$$\oint D_i \theta_E v \, dx = - \sum \int_{\partial E}^* v (\mathbf{e}_i \cdot \mathbf{n}_E) \, dS,$$

and by Definition 4.1 (III),

$$\oint D_i v \theta_E \, dx = \int_{\partial E}^* v (\mathbf{e}_i \cdot \mathbf{n}_E) \, dS.$$

If we take a vector field $\varphi = (v_1, \dots, v_N) \in [V_\Lambda(\Omega)]^N$, by the above identity, we get

$$\oint D \cdot \varphi \theta_E \, dx = \int_{\partial E}^* \varphi \cdot \mathbf{n}_E \, dS. \quad (4.1)$$

Now, if $\varphi \in \mathcal{C}^1(\overline{\Omega})$ and ∂E is smooth, we get the Gauss' divergence theorem:

$$\int_E \nabla \cdot \varphi \, dx = \int_{\partial E} \varphi \cdot \mathbf{n}_E \, dS.$$

Then (4.1) is a generalization of the Gauss' theorem which makes sense for any set $E \in \mathfrak{C}_\Lambda(\Omega)$. Next, we want to generalize Gauss' theorem to any measurable subset $A \subset \Omega$.

First of all we need to generalize the notion of Caccioppoli perimeter $p(E)$ to any arbitrary set. As we have seen in Section 3.1, if $E \in \mathfrak{C}(\Omega)$ is a special Caccioppoli set, we have that

$$p(E) = \int |\nabla \theta_E| \, dx,$$

and it is possible to define an $(n - 1)$ -dimensional measure dS as follows:

$$\int_{\partial E} v(x) dS := \int |\nabla \theta_E| v(x) dx.$$

In particular, if the reduced boundary of E coincides with ∂E , we have that (see [14, Section 5.7])

$$\int_{\partial E} v(x) dS = \int_{\partial E} v(x) d\mathcal{H}^{N-1}.$$

Then the following definition is a natural generalization:

Definition 4.2. If A is a measurable subset of Ω , we set

$$p(A) := \oint |D\theta_A^\circ| dx$$

and for all $v \in V_\Lambda(\Omega)$,

$$\oint_{\partial A} v(x) dS := \oint v(x) |D\theta_A^\circ| dx. \quad (4.2)$$

Remark 4.3. Notice that

$$\oint_{\partial A} v(x) dS \neq \oint v(x) \chi_{\partial A}^\circ(x) dx.$$

In fact, the left-hand term has been defined as

$$\oint_{\partial A} v(x) dS = \sum_{x \in \Gamma} v(x) |D\theta_A^\circ(x)| \eta_x$$

while the right-hand term is

$$\oint v(x) \chi_{\partial A}^\circ(x) dx = \sum_{x \in \Gamma} v(x) \chi_{\partial A}^\circ(x) \eta_x;$$

in particular, if ∂A is smooth and $v(x)$ is bounded, $\sum_{x \in \Gamma} v(x) \chi_{\partial A}^\circ(x) \eta_x$ is an infinitesimal number.

Theorem 4.4. If A is an arbitrary measurable subset of Ω , we have that

$$\oint D \cdot \varphi \theta_A^\circ dx = \oint \varphi \cdot \mathbf{n}_A^\circ dS, \quad (4.3)$$

where

$$\mathbf{n}_A^\circ(x) = \begin{cases} -\frac{D\theta_A^\circ(x)}{|D\theta_A^\circ(x)|} & \text{if } D\theta_A^\circ(x) \neq 0, \\ 0 & \text{if } D\theta_A^\circ(x) = 0. \end{cases}$$

Proof. By Definition 4.1 (III),

$$\oint D \cdot \varphi \theta_A^\circ dx = - \oint \varphi \cdot D\theta_A^\circ dx.$$

Then, using the definition of $\mathbf{n}_A^\circ(x)$ and (4.2), the above formula can be written as follows:

$$\oint D \cdot \varphi \theta_A^\circ dx = \oint \varphi \cdot \mathbf{n}_A^\circ |D\theta_A^\circ| dx = \oint_{\partial A} \varphi \cdot \mathbf{n}_A^\circ dS.$$

The proof is complete. \square

Clearly, if $E \in \mathcal{C}_\Lambda(\Omega)$, then

$$\oint_{\partial E} \varphi \cdot \mathbf{n}_E^\circ dS = \int_{\partial E} \varphi \cdot \mathbf{n}_E dS.$$

Example 4.5. If A is the Koch snowflake, then the usual Gauss' theorem makes no sense since $p(A) = +\infty$; on the other hand equation (4.3) holds true. Moreover, the perimeter in the sense of ultrafunction is an infinite number given by Definition 4.2. In general, if ∂A is a d -dimensional fractal set, it is an interesting open problem to investigate the relation between its Hausdorff measure and the ultrafunction “measure” $dS = |D\theta_A^\circ| dx$.

4.3 Ultrafunctions and distributions

One of the most important properties of the ultrafunctions is that they can be seen (in some sense that we will make precise in this subsection) as generalizations of the distributions.

Definition 4.6. The space of *generalized distributions* on Ω is defined as follows:

$$\mathcal{D}'_G(\Omega) = V_\Lambda(\Omega)/N,$$

where

$$N = \left\{ \tau \in V_\Lambda(\Omega) \mid \oint \tau \varphi \, dx \sim 0 \text{ for all } \varphi \in \mathcal{D}(\Omega) \right\}.$$

The equivalence class of u in $V_\Lambda(\Omega)$ will be denoted by $[u]_{\mathcal{D}}$.

Definition 4.7. Let $[u]_{\mathcal{D}}$ be a generalized distribution. We say that $[u]_{\mathcal{D}}$ is a bounded generalized distribution if, for all $\varphi \in \mathcal{D}(\Omega)$, $\int u \varphi^* \, dx$ is finite. We denote by $\mathcal{D}'_{GB}(\Omega)$ the set of the bounded generalized distributions.

We have the following result.

Theorem 4.8. *There is a linear isomorphism*

$$\Phi : \mathcal{D}'_{GB}(\Omega) \rightarrow \mathcal{D}'(\Omega), \quad \langle \Phi([u]_{\mathcal{D}}), \varphi \rangle_{\mathcal{D}(\Omega)} = \text{st} \left(\oint u \varphi^* \, dx \right).$$

Proof. For a proof see e.g. [9]. □

From now on we will identify the spaces $\mathcal{D}'_{GB}(\Omega)$ and $\mathcal{D}'(\Omega)$; so, we will identify $[u]_{\mathcal{D}}$ with $\Phi([u]_{\mathcal{D}})$ and we will write $[u]_{\mathcal{D}} \in \mathcal{D}'(\Omega)$ and

$$\langle [u]_{\mathcal{D}}, \varphi \rangle_{\mathcal{D}(\Omega)} := \langle \Phi([u]_{\mathcal{D}}), \varphi \rangle_{\mathcal{D}(\Omega)} = \text{st} \left(\oint u \varphi^* \, dx \right).$$

Moreover, with some abuse of notation, we will write also that $[u]_{\mathcal{D}} \in L^2(\Omega)$, $[u]_{\mathcal{D}} \in V(\Omega)$, etc., meaning that the distribution $[u]_{\mathcal{D}}$ can be identified with a function f in $L^2(\Omega)$, $V(\Omega)$, etc. By our construction, this is equivalent to saying that $f^* \in [u]_{\mathcal{D}}$. So, in this case, we have that for all $\varphi \in \mathcal{D}(\Omega)$,

$$\langle [u]_{\mathcal{D}}, \varphi \rangle_{\mathcal{D}(\Omega)} = \text{st} \left(\oint u \varphi^* \, dx \right) = \text{st} \left(\oint f^* \varphi^* \, dx \right) = \int f \varphi \, dx.$$

Remark 4.9. Since an ultrafunction $u : \Omega^* \rightarrow \mathbb{R}^*$ is univocally determined by its value in Γ , we may think of ultrafunctions as being defined only on Γ and to denote them by $V_\Lambda(\Gamma)$; the set $V_\Lambda(\Gamma)$ is an algebra which extends the algebra of continuous functions $\mathcal{C}(\Omega)$ if it is equipped with the pointwise product.

Moreover, we recall that, by a well-known theorem of Schwartz, any tempered distribution can be represented as $\partial^\alpha f$, where α is a multi-index and f is a continuous function. If we identify $T = \partial^\alpha f$ with the ultrafunction $D^\alpha f^\circ$, we have that the set of tempered distributions \mathcal{S}' is contained in $V_\Lambda(\Gamma)$. However, the Schwartz impossibility theorem (see introduction) is not violated since $(V_\Lambda(\Gamma), +, \cdot, D)$ is not a differential algebra, because the Leibnitz rule does not hold for some couple of ultrafunctions.

5 Construction of the Caccioppoli space of ultrafunctions

In this section we will prove the existence of Caccioppoli spaces of ultrafunctions (see Definition 3.6) by an explicit construction.

5.1 Construction of the space $V_\Lambda(\Omega)$

In this subsection we will construct a space of ultrafunctions $V_\Lambda(\Omega)$ and in the next subsection we will equip it with a σ -basis in such a way that $V_\Lambda(\Omega)$ becomes a Caccioppoli space of ultrafunctions according to Definition 3.6.

Definition 5.1. Given a family of open sets \mathfrak{R}_0 , we say that a family of open sets $\mathfrak{B} = \{E_k\}_{k \in K}$ is a *basis* for \mathfrak{R}_0 if

- for all $k \neq h$, $E_k \cap E_h = \emptyset$,
- for all $A \in \mathfrak{R}_0$, there is a set of indices $K_E \subset K$ such that

$$A = \text{int} \left(\bigcup_{k \in K_E} \overline{E_k} \right), \quad (5.1)$$

- \mathfrak{B} is the smallest family of sets which satisfies the above properties.

We will refer to the family \mathfrak{R} of *all* the open sets which can be written by the expression (5.1) as to the family *generated* by \mathfrak{R}_0 .

Let us verify that

Lemma 5.2. For any finite family of special Caccioppoli sets \mathfrak{C}_0 , there exists a basis \mathfrak{B} whose elements are special Caccioppoli sets. Moreover, also the set \mathfrak{C} generated by \mathfrak{C}_0 consists of special Caccioppoli sets.

Proof. For any $x \in \Omega$, we set

$$E_x = \bigcap \{A \in \mathfrak{C}_0 \mid x \in A\}.$$

We claim that $\{E_x\}_{x \in \Omega}$ is a basis. Since \mathfrak{C}_0 is a finite family, we also have that $\{E_x\}_{x \in \Omega}$ is a finite family and hence there is a finite set of indices K such that $\mathfrak{B} = \{E_k\}_{k \in K}$. Now it is easy to prove that \mathfrak{B} is a basis and it consists of special Caccioppoli sets. Also the last statement is trivial. \square

We set

$$\mathfrak{C}_{0,\lambda}(\Omega) := \lambda \cap \mathfrak{C}(\Omega),$$

and we denote by $\mathfrak{B}_\lambda(\Omega)$ and $\mathfrak{C}_\lambda(\Omega)$ the relative basis and the generated family which exist by the previous lemma.

Now set

$$\mathfrak{C}_\Lambda(\Omega) = \lim_{\lambda \uparrow \Lambda} \mathfrak{C}_\lambda(\Omega), \quad \mathfrak{B}_\Lambda(\Omega) = \lim_{\lambda \uparrow \Lambda} \mathfrak{B}_\lambda(\Omega). \quad (5.2)$$

Lemma 5.3. The following properties hold true:

- $\mathfrak{C}_\Lambda(\Omega)$ and $\mathfrak{B}_\Lambda(\Omega)$ are hyperfinite.
- $\mathfrak{C}(\Omega)^\sigma \subset \mathfrak{C}_\Lambda(\Omega) \subset \mathfrak{C}(\Omega)^*$.
- If $E \in \mathfrak{C}_\Lambda(\Omega)$, then

$$\theta_E = \sum_{Q \in K(E)} \theta_Q(x),$$

where $K(E) \subset \mathfrak{B}_\Lambda(\Omega)$ is a hyperfinite set and θ_Q is the natural extension to $\mathfrak{C}_\Lambda(\Omega)^*$ of the function $Q \mapsto \theta_Q$ defined on $\mathfrak{C}_\Lambda(\Omega)$ by (3.1).

Proof. It follows trivially by the construction. \square

The next lemma is a basic step for the construction of the space $V_\Lambda(\Omega)$.

Lemma 5.4. For any $Q \in \mathfrak{B}_\Lambda(\Omega)$ there exists a set $\Xi(Q) \subset \overline{Q} \cap \Omega$, and a family of functions $\{\zeta_a\}_{a \in \Xi(Q)}$ such that the following hold:

- (1) $\Xi := \bigcup \{\Xi(Q) \mid Q \in \mathfrak{B}_\Lambda(\Omega)\}$ is a hyperfinite set, and $\Omega \subset \Xi \subset \Omega^*$.
- (2) If $Q, R \in \mathfrak{B}_\Lambda(\Omega)$ and $Q \neq R$, then $\Xi(Q) \cap \Xi(R) = \emptyset$.
- (3) If $a \in \Xi(Q)$, then there exists $f_a \in \mathcal{C}^1(\Omega)^*$ such that $\zeta_a = f_a \cdot \theta_Q$.
- (4) For any $a, b \in \Xi$, $a \neq b$ implies $\text{supp}(\zeta_a) \cap \text{supp}(\zeta_b) = \emptyset$.
- (5) $\zeta_a \geq 0$.
- (6) For any $a \in \Xi$,

$$\zeta_a(a) = 1.$$

Proof. We set

$$r(\lambda) = \frac{1}{3} \min\{d(x, y) \mid x, y \in \lambda \cap \Omega\},$$

and we denote by ρ a smooth bell shaped function having support in $B_1(0)$; then the functions $\rho(\frac{x-a_\lambda}{r(\lambda)})$, $a_\lambda \in \lambda \cap \Omega$, have disjoint support. We set

$$\Xi := \left\{ \lim_{\lambda \uparrow \Lambda} a_\lambda \mid a_\lambda \in \lambda \cap \Omega \right\},$$

so that $\Omega \subset \Xi \subset \Omega^*$ and we divide all points $a \in \Xi$, among sets $\Xi(Q)$, $Q \in \mathfrak{B}_\Lambda$, in such a way that

- if $a \in Q$, then $a \in \Xi(Q)$;
- if $a \in \partial Q_1 \cap \dots \cap \partial Q_l$, there exists a unique Q_j ($j \leq l$) such that $a \in \Xi(Q_j)$.

With this construction, claims (1) and (2) are trivially satisfied.

Now, for any $a \in \Xi(Q)$, set

$$\rho_a(x) := \lim_{\lambda \uparrow \Lambda} \rho\left(\frac{x - a_\lambda}{r(\lambda)}\right),$$

and

$$\zeta_a(x) := \frac{\rho_a(x)\theta_Q(x)}{\rho_a(a)\theta_Q(a)}. \quad (5.3)$$

It is easy to check that the functions ζ_a satisfy (3)–(6). \square

We set

$$V_\Lambda^1(\Omega) = \text{Span}(\{\zeta_a\}_{a \in \Xi}) + \lim_{\lambda \uparrow \Lambda} (\lambda \cap \mathcal{C}^1(\overline{\Omega})),$$

and

$$V_\Lambda^1(Q) = \{u\theta_Q \mid u \in V_\Lambda^1(\Omega)\};$$

so we have that, for any $a \in \Xi(Q)$, $\zeta_a \in V_\Lambda^1(Q)$. Also, we set

$$V_\Lambda^0(\Omega) = \text{Span}\left(\{f, \partial_i f, fg, g\partial_i f \mid f, g \in V_\Lambda^1(\Omega), i = 1, \dots, N\} + \lim_{\lambda \uparrow \Lambda} (\lambda \cap \mathcal{C}(\overline{\Omega}))\right) \quad (5.4)$$

and

$$V_\Lambda^0(Q) = \{u\theta_Q \mid u \in V_\Lambda^0(\Omega)\}.$$

Finally, we can define the $V_\Lambda(\Omega)$ as follows:

$$V_\Lambda(\Omega) = \bigoplus_{Q \in \mathfrak{B}_\Lambda(\Omega)} V_\Lambda^0(Q).$$

Namely, if $u \in V_\Lambda(\Omega)$, then

$$u(x) = \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} u_Q(x)\theta_Q(x) \quad (5.5)$$

with $u_Q \in V_\Lambda^0(Q)$.

5.2 The σ -basis

In this subsection, we will introduce a σ -basis in such a way that $V_\Lambda(\Omega)$ becomes a Caccioppoli space of ultrafunctions, according to Definition 3.6.

Theorem 5.5. *There exists a σ -basis for $V_\Lambda(\Omega)$, $\{\sigma_a(x)\}_{a \in \Gamma}$, such that the following hold:*

- (1) $\Omega \subset \Gamma \subset \Omega^*$.
- (2) $\Gamma = \bigcup_{Q \in \mathfrak{B}_\Lambda(\Omega)} Q_\Gamma$, where $Q \cap \Gamma \subset Q_\Gamma \subset \overline{Q} \cap \Gamma$ and $Q_\Gamma \cap R_\Gamma = \emptyset$ for $Q \neq R$.
- (3) $\{\sigma_a(x)\}_{a \in Q_\Gamma}$ is a σ -basis for $V_\Lambda^0(Q)$.

Proof. First we introduce in $V_\Lambda(\Omega)$ the following scalar product:

$$\langle u, v \rangle = \int^* uv \, dx. \quad (5.6)$$

For any $Q \in \mathfrak{B}_\Lambda(\Omega)$ we set

$$Z(Q) = \left\{ \sum_{a \in \Xi(Q)} \gamma_a \zeta_a(x) \mid \gamma_a \in \mathbb{R}^* \right\},$$

where $\Xi(Q)$ and the functions $\{\zeta_a\}_{a \in \Xi}$ are defined in Lemma 5.4. If we set

$$\mathfrak{d}_a(x) = \frac{\zeta_a(x)}{\int^* |\zeta_a(x)|^2 dx},$$

we have that

$$\{\mathfrak{d}_a(x)\}_{a \in \Xi(Q)}$$

is a δ -basis for $Z(Q) \subset V_\Lambda^0(Q)$ (with respect to the scalar product (5.6)). In fact, if $u \in Z(Q)$, then

$$u(x) = \sum_{b \in \Xi(Q)} u(b) \zeta_b(x),$$

and hence, by Lemma 5.4, it follows that

$$\begin{aligned} \int^* u(x) \mathfrak{d}_a(x) dx &= \int^* \sum_{b \in \Xi(Q)} u(b) \zeta_b(x) \mathfrak{d}_a(x) dx \\ &= \sum_{b \in \Xi(Q)} u(b) \int^* \zeta_b(x) \mathfrak{d}_a(x) dx \\ &= \sum_{b \in \Xi(Q)} u(b) \int^* \zeta_b(x) \left(\frac{\zeta_a(x)}{\int^* |\zeta_a(x)|^2 dx} \right) dx \\ &= \sum_{b \in \Xi(Q)} u(b) \delta_{ab} = u(a). \end{aligned}$$

Next, we want to complete this basis and to get a δ -basis for $V_\Lambda^0(Q)$. To this end, we take an orthonormal basis $\{e_k(x)\}$ of $Z(Q)^\perp$, where $Z(Q)^\perp$ is the orthogonal complement of $Z(Q)$ in $V_\Lambda^0(Q)$ (with respect to the scalar product (5.6)). For every $a \in Q \setminus \Xi$, set

$$\mathfrak{d}_a(x) = \sum_k e_k(a) e_k(x);$$

notice that this definition is not in contradiction with (5.3) since in the latter $a \in \Xi$.

For every $v \in Z(Q)^\perp$, we have that

$$\int^* v(x) \mathfrak{d}_a(x) dx = v(a);$$

in fact,

$$\begin{aligned} \int^* v(x) \mathfrak{d}_a(x) dx &= \int^* \left(\sum_k v_k e_k(x) \right) \left(\sum_h e_h(a) e_h(x) \right) dx \\ &= \sum_{k,h} v_k e_h(a) \int^* e_k(x) e_h(x) dx \\ &= \sum_{k,h} v_k e_h(a) \delta_{hk} = \sum_k v_k e_k(a) = v(a). \end{aligned}$$

It is not difficult to realize that $\{\mathfrak{d}_a(x)\}_{a \in Q \setminus \Xi}$ generates all $Z(Q)^\perp$ and hence we can select a set $\Xi^*(Q) \subset Q \setminus \Xi$ such that $\{\mathfrak{d}_a(x)\}_{a \in \Xi^*(Q)}$ is a basis for $Z(Q)^\perp$. Taking

$$Q_\Gamma = \Xi^*(Q) \cup \Xi(Q),$$

we have that $\{\mathfrak{d}_a(x)\}_{a \in Q_\Gamma}$ is a basis for $V_\Lambda^0(Q)$.

Now let $\{\sigma_a(x)\}_{a \in Q_\Gamma}$ denote the dual basis of $\{\mathfrak{d}_a(x)\}_{a \in Q_\Gamma}$ namely a basis such that, for all $a, b \in Q_\Gamma$,

$$\int^* \sigma_a(x) \mathfrak{d}_b(x) dx = \delta_{ab}.$$

Clearly, it is a σ -basis for $V_\Lambda^0(Q)$. In fact, if $u \in V_\Lambda^0(Q)$, we have that

$$u(x) = \sum_{a \in Q_\Gamma} \left[\int^* u(t) \mathfrak{d}_a(t) dt \right] \sigma_a(x) = \sum_{a \in Q_\Gamma} u(a) \sigma_a(x).$$

Notice that if $a \in \Xi(Q)$, then $\sigma_a(x) = \zeta_a(x)$. The conclusion follows taking $\Gamma := \bigcup_{Q \in \mathfrak{B}_\Lambda(\Omega)} Q_\Gamma$. \square

By the above theorem, the following corollary follows straightforward.

Corollary 5.6. *The set $V_\Lambda(\Omega)$ is a Caccioppoli space of ultrafunctions in the sense of Definition 3.6.*

If $E \in \mathfrak{C}_\Lambda(\Omega)$ (see (5.2)), we set

$$E_\Gamma = \bigcup_{Q \in \mathfrak{B}_\Lambda(\Omega), Q \subset E} Q_\Gamma.$$

If, for any internal set A , we define

$$\oint_A u(x) dx = \sum_{a \in \Gamma \cap A} u(a) \eta_a,$$

then we have the following result:

Theorem 5.7. *If $u\theta_E \in V_\Lambda(\Omega)$ and $E \in \mathfrak{C}_\Lambda(\Omega)$, then*

$$\oint_{E_\Gamma} u(x) dx = \int_E^* u(x) dx = \oint u(x) \theta_E(x) dx.$$

Proof. We have that

$$\begin{aligned} \oint_{E_\Gamma} u(x) dx &= \sum_{a \in E_\Gamma} u(a) \eta_a \\ &= \int^* \sum_{a \in E_\Gamma} u(a) \sigma_a(x) dx \\ &= \int^* \sum_{Q \subset E} \sum_{a \in Q_\Gamma} u(a) \sigma_a(x) dx. \end{aligned} \quad (5.7)$$

Since $u\theta_E \in V_\Lambda(\Omega)$, it follows from (5.5) that we can write

$$u(x) \theta_E(x) = \sum_{Q \subset E} u_Q(x) \theta_Q(x).$$

By Theorem 5.5 (3),

$$u_Q(x) \theta_Q(x) = \sum_{a \in Q_\Gamma} u(a) \sigma_a(x) \in V_\Lambda^0(Q).$$

Then by (5.7),

$$\oint_{E_\Gamma} u(x) dx = \int^* \sum_{Q \subset E} u_Q(x) \theta_Q(x) dx = \sum_{Q \subset E} \int^* u_Q(x) \theta_Q(x) dx. \quad (5.8)$$

By this equation and the fact that

$$\int^* u_Q(x) \theta_Q(x) dx = \int_Q^* u(x) dx,$$

it follows that

$$\oint_{E_\Gamma} u(x) dx = \sum_{Q \subset E} \int_Q^* u(x) dx = \int_E^* u(x) dx.$$

Moreover, since $u_Q \theta_Q \in V_\Lambda^0(Q) \subset V_\Lambda(\Omega)$,

$$\int^* u_Q(x) \theta_Q(x) dx = \oint u_Q(x) \theta_Q(x) dx,$$

by (5.8) we have

$$\oint_{E_\Gamma} u(x) dx = \sum_{Q \subset E} \oint u_Q(x) \theta_Q(x) dx = \oint \sum_{Q \subset E} u_Q(x) \theta_Q(x) dx = \oint u(x) \theta_E(x) dx. \quad \square$$

5.3 Construction of the generalized derivative

Next we construct a generalized derivative on $V_\Lambda(\Omega)$.

We set

$$U_\Lambda^1 = \bigoplus_{Q \in \mathfrak{B}_\Lambda(\Omega)} V_\Lambda^1(Q),$$

and

$$U_\Lambda^0 = (U_\Lambda^1)^\perp$$

will denote the orthogonal complement of U_Λ^1 in $V_\Lambda(\Omega)$. According to this decomposition, $V_\Lambda(\Omega) = U_\Lambda^1 \oplus U_\Lambda^0$ and we can define the following orthogonal projectors:

$$P_i : V_\Lambda(\Omega) \rightarrow U_\Lambda^i, \quad i = 0, 1,$$

hence, any ultrafunction $u \in V_\Lambda(\Omega)$ has the following orthogonal splitting: $u = u_1 + u_0$, where $u_i = P_i u$.

Now we are able to define the generalized partial derivative for $u \in V_\Lambda^1(\Omega)$.

Definition 5.8. We define the generalized partial derivative

$$D_i : U_\Lambda^1(\Omega) \rightarrow V_\Lambda(\Omega)$$

as follows:

$$\oint D_i u v \, dx = \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \oint \partial_i^* u_Q v_Q \theta_Q \, dx - \frac{1}{2} \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \sum_{R \in \mathfrak{Y}(Q)} \int_{\partial Q \cap \partial R}^* (u_Q - u_R) v_Q (\mathbf{e}_i \cdot \mathbf{n}_Q) \, dS, \quad (5.9)$$

where

$$\mathfrak{Y}(Q) = \{R \in \mathfrak{B}_\Lambda(\Omega) \cup \{Q_\infty\} \mid Q \neq R, \partial Q \cap \partial R \neq \emptyset\}$$

with

$$Q_\infty = \Omega^* \setminus \overline{\bigcup_{Q \in \mathfrak{B}_\Lambda(\Omega)} Q}.$$

Moreover, if $u = u_1 + u_0 \in U_\Lambda^1 \oplus U_\Lambda^0 = V_\Lambda(\Omega)$, we set

$$D_i u = D_i u_1 - (D_i P_1)^\dagger u_0,$$

where, for any linear operator L , L^\dagger denotes the adjoint operator.

Remark 5.9. Notice that if $u, v \in U_\Lambda^1$, by Theorem 5.4 (2), we have

$$\oint D_i u v \, dx = \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \int_Q^* \partial_i^* u_Q v_Q \, dx - \frac{1}{2} \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \sum_{R \in \mathfrak{Y}(Q)} \int_{\partial Q \cap \partial R}^* (u_Q - u_R) v_Q (\mathbf{e}_i \cdot \mathbf{n}_Q) \, dS. \quad (5.10)$$

In fact, if $u, v \in U_\Lambda^1$, then $u_Q, v_Q \in V_\Lambda^1(\Omega)$, and hence, by (5.4), $\partial_i^* u_Q v_Q \in V_\Lambda^0(\Omega)$ and so

$$\oint \partial_i^* u_Q v_Q \theta_Q \, dx = \int_Q^* \partial_i^* u_Q v_Q \, dx.$$

Theorem 5.10. The operator $D_i : V_\Lambda(\Omega) \rightarrow V_\Lambda(\Omega)$, given by Definition 5.8, satisfies the requests (I)–(III) of Definition 4.1.

Proof. Let us prove property (I). If $u \theta_Q, v \theta_R \in V_\Lambda(\Omega)$ and $\overline{Q} \cap \overline{R} = \emptyset$, by Definition 5.8,

$$\oint D_i (u \theta_Q) v \theta_R \, dx = 0.$$

Set

$$\delta := \max\{\text{diam}(Q) \mid Q \in \mathfrak{B}_\Lambda(\Omega)\}.$$

If $q \in Q$ and $r \in R$, then

$$|q - r| > 2\delta \implies \overline{Q} \cap \overline{R} = \emptyset,$$

so, if $\sigma_q \in V_\Lambda^0(Q)$, and $r \in R$, then

$$|q - r| > 2\delta \implies \overline{Q} \cap \overline{R} = \emptyset,$$

and hence, if we set $\varepsilon_0 > 3\delta$,

$$\bigcup \{R \in \mathfrak{B}_\Lambda(\Omega) \mid \overline{Q} \cap \overline{R} \neq \emptyset\} \subset B_{\varepsilon_0}(q).$$

Since $\sigma_q \in V_\Lambda^0(Q)$,

$$\text{supp}(D_i \sigma_q) \subset \overline{\bigcup \{R \in \mathfrak{B}_\Lambda(\Omega) \mid \overline{Q} \cap \overline{R} \neq \emptyset\}} \subset B_{\varepsilon_0}(q).$$

We prove property (II). If $u \in [\mathcal{C}^1(\Omega)]^* \cap V_\Lambda(\Omega)$, then

$$u = \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} u \theta_Q,$$

and hence for all $x \in \partial Q \cap \partial R$, $u_Q(x) - u_R(x) = u(x) - u(x) = 0$. Then, by (5.9), we have, for all $v \in V_\Lambda(\Omega)$,

$$\begin{aligned} \oint D_i u v \, dx &= \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \oint \partial_i^* u v \theta_Q \, dx \\ &= \oint \partial_i^* u \left(\sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} v \theta_Q \right) dx \\ &= \oint \partial_i^* u v \, dx. \end{aligned}$$

The conclusion follows from the arbitrariness of v .

Next let us prove property (III). First we prove this property if $u, v \in U_\Lambda^1$. By (5.10), we have that

$$\begin{aligned} \oint D_i u v \, dx &= \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \int_Q^* \partial_i^* u v \, dx - \frac{1}{2} \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \sum_{R \in \mathfrak{I}(Q)} \int_{\partial Q \cap \partial R}^* u_Q v_Q (\mathbf{e}_i \cdot \mathbf{n}_Q) \, dS \\ &\quad + \frac{1}{2} \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \sum_{R \in \mathfrak{I}(Q)} \int_{\partial Q \cap \partial R}^* u_R v_Q (\mathbf{e}_i \cdot \mathbf{n}_Q) \, dS \\ &= \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \int_Q^* \partial_i^* u v \, dx - \frac{1}{2} \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \int_{\partial Q}^* u_Q v_Q (\mathbf{e}_i \cdot \mathbf{n}_Q) \, dS \\ &\quad + \frac{1}{2} \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \sum_{R \in \mathfrak{I}(Q)} \int_{\partial Q \cap \partial R}^* u_R v_Q (\mathbf{e}_i \cdot \mathbf{n}_Q) \, dS. \end{aligned} \quad (5.11)$$

Next we will compute $\oint u D_i v \, dx$ and we will show that it is equal to $-\oint D_i u v \, dx$. So we replace u with v , in the above equality and we get

$$\begin{aligned} \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \int_Q^* u D_i v \, dx &= \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \int_Q^* \partial_i^* v u_Q \, dx - \frac{1}{2} \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \int_{\partial Q}^* u_Q v_Q (\mathbf{e}_i \cdot \mathbf{n}_Q) \, dS \\ &\quad + \frac{1}{2} \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \sum_{R \in \mathfrak{I}(Q)} \int_{\partial Q \cap \partial R}^* v_R u_Q (\mathbf{e}_i \cdot \mathbf{n}_Q) \, dS. \end{aligned} \quad (5.12)$$

Now, we compute $\sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \int_Q^* \partial_i^* v u_Q \, dx$ and the last term of the above expression separately. We have that

$$\sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \int_Q^* \partial_i^* v u_Q \, dx = - \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \int_Q^* \partial_i^* u_Q v_Q \, dx + \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \int_{\partial Q}^* u_Q v_Q (\mathbf{e}_i \cdot \mathbf{n}_Q) \, dS. \quad (5.13)$$

Moreover, the last term in (5.12), changing the order on which the terms are added, becomes

$$\sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \sum_{R \in \mathfrak{I}(Q)} \int_{\partial Q \cap \partial R}^* v_R u_Q (\mathbf{e}_i \cdot \mathbf{n}_Q) \, dS = \sum_{R \in \mathfrak{B}_\Lambda(\Omega)} \sum_{Q \in \mathfrak{I}(R)} \int_{\partial Q \cap \partial R}^* v_R u_Q (\mathbf{e}_i \cdot \mathbf{n}_Q) \, dS.$$

On the right-hand side we can change the name of the variables Q and R :

$$\begin{aligned} \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \sum_{R \in \mathfrak{I}(Q)} \int_{\partial Q \cap \partial R}^* v_R u_Q (\mathbf{e}_i \cdot \mathbf{n}_Q) \, dS &= \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \sum_{R \in \mathfrak{I}(Q)} \int_{\partial Q \cap \partial R}^* v_Q u_R (\mathbf{e}_i \cdot \mathbf{n}_R) \, dS \\ &= - \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \sum_{R \in \mathfrak{I}(Q)} \int_{\partial Q \cap \partial R}^* v_Q u_R (\mathbf{e}_i \cdot \mathbf{n}_Q) \, dS. \end{aligned} \quad (5.14)$$

In the last step we have used the fact that $x \in \partial Q \cap \partial R$ implies $\mathbf{n}_R(x) = -\mathbf{n}_Q(x)$. Replacing (5.13) and (5.14) in (5.12), we get

$$\begin{aligned} \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \int_Q^* u D_i v \, dx &= - \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \int_Q^* \partial_i^* u_Q v_Q \, dx + \frac{1}{2} \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \int_{\partial Q}^* u_Q v_Q (\mathbf{e}_i \cdot \mathbf{n}_Q) \, dS \\ &\quad - \frac{1}{2} \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \sum_{R \in \mathfrak{N}(Q)} \int_{\partial Q \cap \partial R}^* v_Q u_R (\mathbf{e}_i \cdot \mathbf{n}_Q) \, dS. \end{aligned}$$

Comparing (5.11) and the above equation, we get that

$$\oint D_i u v \, dx = - \oint u D_i v \, dx \quad \text{for all } u, v \in U_\Lambda^1. \quad (5.15)$$

Let us prove property (III) in the general case. We have that

$$D_i u = D_i u_1 - (D_i P_1)^\dagger u_0,$$

hence

$$\begin{aligned} \oint D_i u v \, dx &= \oint D_i u_1 v \, dx - \oint (D_i P_1)^\dagger u_0 v \, dx \\ &= \oint D_i u_1 v_1 \, dx + \oint D_i u_1 v_0 \, dx - \oint u_0 D_i P_1 v \, dx \\ &= \oint D_i u_1 v_1 \, dx + \oint D_i u_1 v_0 \, dx - \oint u_0 D_i v_1 \, dx. \end{aligned} \quad (5.16)$$

Now, replacing u with v and applying property (5.15) for $u_1, v_1 \in U_\Lambda^1$, we get

$$\begin{aligned} \oint u D_i v \, dx &= \oint u_1 D_i v_1 \, dx - \oint D_i u_1 v_0 \, dx + \oint u_0 D_i v_1 \, dx \\ &= - \oint D_i u_1 v_1 \, dx - \oint D_i u_1 v_0 \, dx + \oint u_0 D_i v_1 \, dx. \end{aligned}$$

Comparing the above equation with (5.16), we get that

$$\oint D_i u v \, dx = - \oint u D_i v \, dx.$$

The proof is complete. \square

Before proving property (IV), we need the following lemma:

Lemma 5.11. *The following identity holds true: for all $E \in \mathfrak{C}_\Lambda(\Omega)$, all $u \in V_\Lambda(\Omega) \cap [\mathcal{C}^1(\Omega)]^*$ and all $v \in V_\Lambda(\Omega)$,*

$$\oint D_i(u \theta_E) v \, dx = \oint \partial_i u v \theta_E \, dx - \int_{\partial E}^* u v (\mathbf{e}_i \cdot \mathbf{n}_E) \, dS. \quad (5.17)$$

Proof. We can write

$$u \theta_E = \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} h_Q u \theta_Q,$$

where

$$h_Q = \begin{cases} 1 & \text{if } Q \subset E, \\ 0 & \text{if } Q \not\subset E. \end{cases}$$

Then we have that

$$h_Q u - h_R u = u$$

if and only if

$$R \in \mathfrak{N}_{Q,E}^+ := \{R \in \mathfrak{B}_\Lambda(\Omega) \cup \{Q_\infty\} \mid \partial R \cap \partial E \neq \emptyset, Q \subset E, R \subset \Omega \setminus E\}.$$

Moreover, we have that

$$h_Q u - h_R u = -u$$

if and only if

$$R \in \mathfrak{Y}_{Q,E}^- := \{R \in \mathfrak{B}_\Lambda(\Omega) \cup \{Q_\infty\} \mid \partial R \cap \partial E \neq \emptyset, Q \subset \Omega \setminus E, R \subset E\}.$$

Otherwise, we have that

$$h_Q u - h_R u = 0 \quad \text{or} \quad \partial R \cap \partial E = \emptyset.$$

Then, by Theorem 5.7,

$$\begin{aligned} \oint D_i(u_E \theta_E) v \, dx &= \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \oint \partial_i^* (h_Q u) v \theta_Q \, dx - \frac{1}{2} \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \sum_{R \in \mathfrak{Y}(Q)} \int_{\partial Q \cap \partial R}^* (h_Q u - h_R u) v (\mathbf{e}_i \cdot \mathbf{n}_Q) \, dS \\ &= \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} h_Q \oint_{Q_\Gamma} \partial_i^* u v \, dx - \frac{1}{2} \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \sum_{R \in \mathfrak{Y}_{Q,E}^+} \int_{\partial Q \cap \partial R}^* u v (\mathbf{e}_i \cdot \mathbf{n}_Q) \, dS \\ &\quad + \frac{1}{2} \sum_{Q \in \mathfrak{B}_\Lambda(\Omega)} \sum_{R \in \mathfrak{Y}_{Q,E}^-} \int_{\partial Q \cap \partial R}^* u v (\mathbf{e}_i \cdot \mathbf{n}_Q) \, dS \\ &= \oint_{E_\Gamma} \partial_i^* u v \, dx - \frac{1}{2} \sum_{Q \in \mathfrak{B}_\Lambda(\Omega), Q \subset E} \int_{\partial Q \cap \partial E}^* u v (\mathbf{e}_i \cdot \mathbf{n}_E) \, dS \\ &\quad + \frac{1}{2} \sum_{Q \in \mathfrak{B}_\Lambda(\Omega), Q \subset \Omega \setminus E} \int_{\partial Q \cap \partial E}^* u v (\mathbf{e}_i \cdot (-\mathbf{n}_E)) \, dS \\ &= \oint \partial_i^* u v \theta_E \, dx - \frac{1}{2} \int_{\partial E}^* u v (\mathbf{e}_i \cdot \mathbf{n}_E) \, dS + \frac{1}{2} \int_{\partial E}^* u v (\mathbf{e}_i \cdot (-\mathbf{n}_E)) \, dS \\ &= \oint \partial_i^* u v \theta_E \, dx - \int_{\partial E}^* u v (\mathbf{e}_i \cdot \mathbf{n}_E) \, dS. \end{aligned}$$

Then (5.17) holds true. \square

Corollary 5.12. *The operator $D_i : V_\Lambda(\Omega) \rightarrow V_\Lambda(\Omega)$ given by Definition 5.8 satisfies the request (IV) of Definition 4.1.*

Proof. The result follows straightforward from (5.17) just taking $u = 1$. \square

6 Some examples

We present a general minimization result and two very basic examples which can be analyzed in the framework of ultrafunctions. We have chosen these examples for their simplicity and also because we can give explicit solutions.

6.1 A minimization result

In this subsection we will consider a minimization problem. Let Ω be an open bounded set in \mathbb{R}^N and let $\Xi \subset \partial\Omega$ be any nonempty portion of the boundary. We consider the following problem: minimize

$$J(u) = \int_{\Omega} \left[\frac{1}{2} a(u) |\nabla u(x)|^p + f(x, u) \right] dx, \quad p > 1,$$

in the set $\mathcal{C}^1(\Omega) \cap \mathcal{C}_0(\Omega \cup \Xi)$. We make the following assumptions:

- (1) $a(u) \geq 0$ and $a(u) \geq k > 0$ for u sufficiently large.
- (2) $a(u)$ is lower semicontinuous.
- (3) $f(x, u)$ is a lower semicontinuous function in u , measurable in x , such that $|f(x, u)| \leq M|u|^q$ with $0 < q < p$ and $M \in \mathbb{R}^+$.

Clearly, the above assumptions are not sufficient to guarantee the existence of a solution, not even in a Sobolev space. We refer to [20] for a survey of this problem in the framework of Sobolev spaces. On the other hand, we have selected this problem since it can be solved in the framework of the ultrafunctions.

More exactly, this problem becomes:

Problem. Find an ultrafunction $u \in V_\Lambda(\Omega)$ which vanishes on Ξ^* and minimizes

$$J^\circ(u) := \oint_\Omega \left[\frac{1}{2} a^*(u) |Du(x)|^p - f^*(x, u) \right] dx, \quad p > 1.$$

We have the following result:

Theorem 6.1. *If assumptions (1)–(3) are satisfied, then the functional $J^\circ(u)$ has a minimizer in the space*

$$\{v \in V_\Lambda(\Omega) \mid v(x) = 0 \text{ for all } x \in \Xi^*\}.$$

Moreover, if $J(u)$ has a minimizer w in $V(\Omega)$, then $u = w^\circ$.

Proof. The proof is based on a standard approximation by finite-dimensional spaces. Let us observe that, for each finite-dimensional space V_λ , we can consider the approximate problem:

Problem. Find $u_\lambda \in V_\lambda$ such that

$$J^\circ(u_\lambda) = \min_{v_\lambda \in V_\lambda} J^\circ(v_\lambda).$$

The above minimization problem has a solution, being the functional coercive, due to the hypotheses on $a(\cdot)$ and the fact that $p > q$. If we take a minimizing sequence $u_\lambda^n \in V_\lambda$, then we can extract a subsequence weakly converging to some $u_\lambda \in V_\lambda$. By observing that in finite-dimensional spaces all norms are equivalent, it follows also that $u_\lambda^n \rightarrow u_\lambda$ pointwise. Then, by the lower-semicontinuity of a and f , it follows that the pointwise limit satisfies

$$J^\circ(u_\lambda) \leq \liminf J^\circ(u_\lambda^n).$$

Next, we use the very general properties of Λ -limits, as introduced in Section 2.2. We set

$$u := \lim_{\lambda \uparrow \Lambda} u_\lambda.$$

Then, taking a generic $v := \lim_{\lambda \uparrow \Lambda} v_\lambda$, from the inequality $J^\circ(u_\lambda) \leq J^\circ(v_\lambda)$, we get

$$J^\circ(u) \leq J^\circ(v) \quad \text{for all } v \in V_\Lambda(\Omega).$$

The last statement is trivial. □

Clearly, under this generality, the solution u could be very wild; however, we can state a regularization result which allows the comparison with variational and classical solutions.

Theorem 6.2. *Let the assumptions of the above theorem hold. If $\mathcal{H}^{N-1}(\Xi) > 0$ and there exists $v \in \mathbb{R}$ such that $a(u) \geq v > 0$, then the minimizer has the following form:*

$$u = w^\circ + \psi,$$

where $w \in H^{1,p}(\Omega)$ and ψ is null in the sense of distributions, namely,

$$\oint \psi \varphi^* dx \sim 0 \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

In this case

$$J^\circ(u) \sim \inf_{v \in V(\Omega)} J^\circ(v)$$

with $V(\Omega)$ as in Definition 3.5. Moreover, if in addition $a(u) < M$, with $M \in \mathbb{R}$, we have that

$$\|\psi\|_{H^{1,p}(\Omega)} \sim 0$$

and $J^\circ(u) \sim J(w)$. Finally, if $J(u)$ has a minimizer in $w \in H^{1,p}(\Omega) \cap \mathcal{C}(\Omega)$, then $u = w^\circ$ and $J^\circ(u) = J(w)$.

Proof. Under the above hypotheses the minimization problem has an additional a priori estimate in $H^{1,p}(\Omega)$, due to the fact that $a(\cdot)$ is bounded away from zero. Moreover, the fact that the function vanishes on a non-trascurable $(N - 1)$ -dimensional part of the boundary shows that the generalized Poincaré inequality holds true. Hence, by Proposition 3.23, the approximating net $\{u_\lambda\}$ has a subnet $\{u_n\}$ such that

$$u_n \rightarrow u \quad \text{weakly in } H^{1,p}(\Omega).$$

This proves the first statement, since obviously $\psi := u - w^\circ$ vanishes in the sense of distributions. In this case, in general the minimum is not achieved in $V(\Omega)$ and hence $J^\circ(w^\circ + \psi) < J(w)$.

Next, if $a(\cdot)$ is bounded also from above, by classical results of semicontinuity of De Giorgi (see Boccardo [12, Section 9, Theorem 9.3]) J is weakly l.s.c. Thus u is a minimizer and, by well-known results, $u_n \rightarrow u$ strongly in $H^{1,p}(\Omega)$. This implies, by Proposition 3.23, that $u \sim w^\circ$, and hence ψ is infinitesimal in $H^{1,p}(\Omega)$, proving the second part.

Finally, if the minimizer is a function $w \in H^{1,p}(\Omega) \cap \mathcal{C}(\Omega) \subset V_\Lambda(\Omega)$, we have that $u_\lambda = w^\circ$ eventually; then $\|\psi\|_{H^{1,p}(\Omega)} = 0$. \square

6.2 The Poisson problem in \mathbb{R}^2

Now we consider the very classical problem

$$-\Delta u = \varphi(x), \quad \varphi \in \mathcal{D}(\mathbb{R}^N). \quad (6.1)$$

If $N \geq 3$, the solution is given by

$$\varphi(x) * \frac{|x|^{-N+2}}{(N-2)\omega_N}$$

and it can be characterized in several ways.

First of all, it is the only solution in the Schwartz space \mathcal{S}' of tempered distributions obtained via the equation

$$\hat{u}(\xi) = \frac{\hat{\varphi}(\xi)}{|\xi|^2}, \quad (6.2)$$

where \hat{T} denotes the Fourier transform of T . Moreover, it is the minimizer of the Dirichlet integral

$$J(u) = \int \left[\frac{1}{2} |\nabla u(x)|^2 - \varphi(x)u(x) \right] dx$$

in the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ which is defined as the completion of $\mathcal{C}^1(\mathbb{R}^N)$ with respect to the Dirichlet norm

$$\|u\| = \sqrt{\int |\nabla u(x)|^2 dx}.$$

Each of these characterizations provides a different method to prove its existence.

The situation is completely different when $N = 2$. In this case, it is well known that the fundamental class of solutions is given by

$$2\pi \cdot \varphi(x) * \log |x|;$$

however, none of the previous characterization makes sense. In fact, we cannot use equation (6.2), since $\frac{1}{|\xi|^2} \notin L^1_{\text{loc}}(\mathbb{R}^2)$ and hence $\frac{1}{|\xi|^2}$ does not define a tempered distribution. Also, the space $\mathcal{D}^{1,2}(\mathbb{R}^2)$ is not an Hilbert space and the functional $J(u)$ is not bounded from below in $\mathcal{D}^{1,2}(\mathbb{R}^2)$.

On the contrary, using the theory of ultrafunctions, we can treat equation (6.1) independently of the dimension.

First of all, we recall that in equation (6.1) with $N \geq 3$, the boundary conditions are replaced by the condition $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$. This is a sort of Dirichlet boundary condition. In the theory of ultrafunctions it is not necessary to replace the Dirichlet boundary condition with such a trick. In fact, we can reformulate the problem in the following way.

Problem. Find $u \in V_\Lambda(B_R)$ such that

$$\begin{aligned} -\Delta^\circ u &= \varphi^\circ(x) && \text{in } B_R, \\ u &= 0 && \text{on } \partial B_R, \end{aligned}$$

where Δ° is the “generalized” Laplacian defined in Section 4.1 and R is an infinite number such that $\chi_{B_R} \in V_\Lambda(\mathbb{R}^N)$.²

Clearly, the solutions of the above problem are the minimizers of the Dirichlet integral

$$J^\circ(u) = \oint \left[\frac{1}{2} |Du(x)|^2 - \varphi^\circ(x)u(x) \right] dx$$

in the space $u \in V_\Lambda(B_R)$, with the Dirichlet boundary condition. Notice that, in the case of ultrafunctions, the problem has the same structure independently of N . In order to prove the existence, we can use Theorem 6.1.³ The fact that $J^\circ(u)$ may assume infinite values does not change the structure of the problem and shows the utility of the use of infinite quantities. The relation between the classical solution w and the ultrafunction u is given by

$$u = w^\circ + \psi$$

with

$$\text{St}_{\mathcal{G}'} \psi = 0.$$

Some people might be disappointed that u depends on R and it is not a standard function; if this is the case, it is sufficient to take

$$w = \text{St}_{\mathcal{G}'} u$$

and call w the *standard solution of the Poisson problem with Dirichlet boundary condition at ∞* . In this way we get the usual fundamental class of solutions and they can be characterized in the usual way also in the case $N = 2$. Concluding, in the framework of ultrafunctions, the Poisson problem with Dirichlet boundary condition is the *same* problem independently of the space dimension and it is very similar to the same problem when R is finite.

This fact proves that the use of infinite numbers is an advantage which people should not ignore.

6.3 An explicit example

If the assumptions of Theorem 6.2 do not hold true, the solution could not be related to any standard object. For example, if $\mathcal{H}^{N-1}(\Xi) = 0$ and $f(x, u) > k|u|^s$ ($p < N$, $k > 0$, $0 < s < q$), the generalized solution $u(x)$ takes infinite values for every $x \in \Omega$. However, there are cases in which $u(x)$ can be identified with a standard and meaningful function, but the minimization problem makes no sense in the usual mathematics. In the example which we will present here, we deal with a functional which might very well represent a physical model, even if the explicit solution cannot be interpreted in a standard world, since it involves the square of a measure (namely δ^2).

Let us consider, for $\gamma > 0$, the one-dimensional variational problem of finding the minimum of the functional

$$J(u) = \int_0^1 \frac{1}{2} a(u) |u'(x)|^2 - \gamma u(x) dx \quad (6.3)$$

among the functions such that $u(0) = 0$. In particular, we are interested in the case in which a is the following degenerate function:

$$a(s) = \begin{cases} 1 & \text{if } s \in (-\infty, 1) \cup (2, +\infty), \\ 0 & \text{if } s \in [1, 2]. \end{cases}$$

² Such an R exists by overspilling (see e.g. [15–17]); in fact, for any $r \in \mathbb{R}$, $\chi_{B_r} \in V_\Lambda(\mathbb{R}^N)$.

³ The fact that Ω is a standard set while B_R is an internal set does not change the proof.

Formally, the Euler equation, if $u \notin [1, 2]$, is

$$u''(x) = -\gamma.$$

We recall that, by standard arguments,

$$u(1) \neq 1 \implies u'(1) = 0.$$

Hence, if $\gamma < 2$, the solution is explicitly computed

$$u(x) = \frac{\gamma}{2}(2x - x^2),$$

since it turns out that $0 \leq u(x) < 1$ for all $x \in (0, 1)$ and then the degeneracy does not take place.

If $\gamma > 2$, we see that the solution does not live in $H^1(0, 1)$, hence the problem has not a “classical” weak solution. More exactly, we have the following result:

Theorem 6.3. *If $\gamma > 2$, then the functional (6.3) has a unique minimizer given by*

$$u(x) = \begin{cases} \frac{1}{2}(2\gamma x - \gamma x^2), & 0 < x < \xi, \\ \frac{1}{2}(-\gamma x^2 + 2\gamma x + 2), & \xi < x < 1, \end{cases}$$

where $\xi \in (0, 1)$ is a suitable real number which depends on γ (see Figure 1).

Proof. First, we show that the generalized solution has at most one discontinuity. In fact, for $\gamma > 2$ the solution satisfies $u(\xi) = 1$, for some $0 < \xi < 1$, and at that point the classical Euler equations are not anymore valid. On the other hand, where $u > 2$, the solution satisfies a regular problem, hence we are in the situation of having at least the following possible candidate as solution with a jump at

$$\xi = \frac{\gamma - \sqrt{\gamma^2 - 2\gamma}}{\gamma} = 1 - \sqrt{1 - \frac{2}{\gamma}}$$

and a discontinuity of derivatives at some $\xi < \eta < 1$.

In the specific case, we have (see Figure 1)

$$u(x) = \begin{cases} \frac{1}{2}(2\gamma x - \gamma x^2), & 0 < x < \xi, \\ 2, & \xi < x < \eta, \\ \frac{\gamma\eta^2}{2} - \gamma\eta - \frac{\gamma x^2}{2} + \gamma x + 2, & \eta < x < 1. \end{cases}$$

We now show that this is not possible because the functional takes a lower value on the solution with only

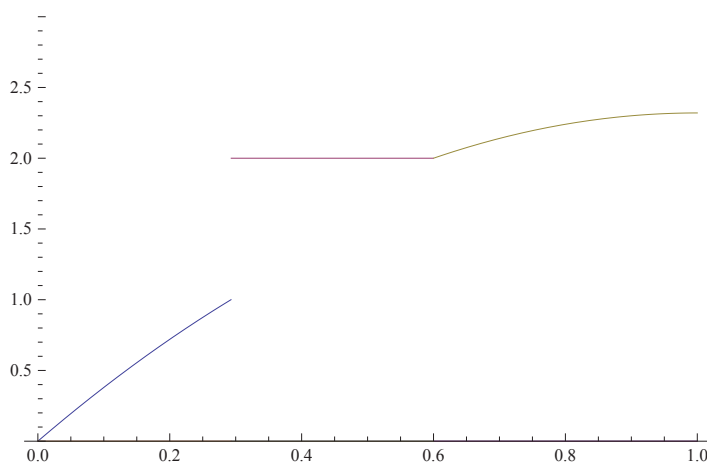


Figure 1: The function $u(x)$ for $\gamma = 4$.

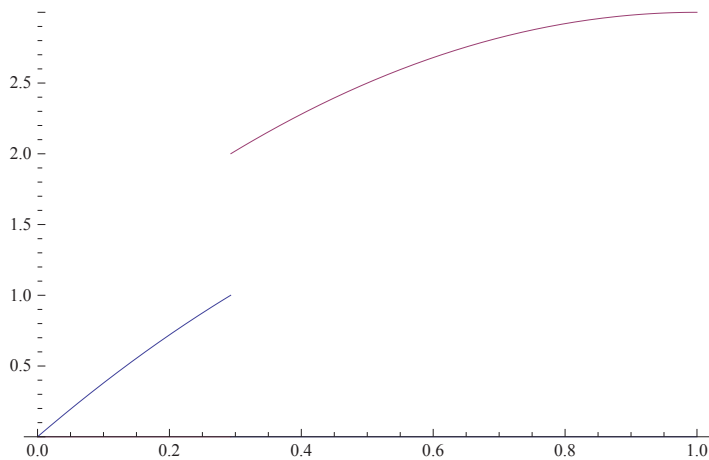


Figure 2: The function $\tilde{u}(x)$ for $\gamma = 4$.

a jump at $x = \xi$. In fact, if we consider the function $\tilde{u}(x)$ defined as

$$\tilde{u}(x) = \begin{cases} \frac{1}{2}(2\gamma x - \gamma x^2), & 0 < x < \xi, \\ \frac{1}{2}(-\gamma x^2 + 2\gamma x + 2), & \xi < x < 1, \end{cases}$$

we observe that $u = \tilde{u}$ in $[0, \xi]$, while $u' = \tilde{u}' = \gamma(1 - x)$ for all $x \notin [\xi, \eta]$ and, by explicit computations, we have

$$J(\tilde{u}) - J(u) = \gamma^2 \left[-\frac{\eta}{2} + \frac{\eta^2}{2} - \frac{\eta^3}{6} + \frac{\xi}{2} - \frac{\xi^2}{2} + \frac{\xi^3}{6} \right] = \gamma^2 (\Phi(\eta) - \Phi(\xi)) < 0, \quad \xi < \eta,$$

where $\Phi(s) = -\frac{s}{2} + \frac{s^2}{2} - \frac{s^3}{6}$ is strictly decreasing, since $\Phi'(s) = -\frac{1}{2}(s - 1)^2 \leq 0$.

Actually, the solution is the one shown in Figure 2. Next we show that there exists a unique point ξ such that the minimum is attained. We write the functional $J(u)$, on a generic solution with a single jump from the value $u = 1$ to the value $u = 2$ at the point $0 < \xi < 1$ and such that the Euler equation is satisfied before and after ξ . We obtain the following value for the functional (in terms of the point ξ):

$$J(u) = F(\xi) = \frac{\gamma^2 \xi^3}{8} - \frac{\gamma^2 \xi^2}{2} + \frac{\gamma^2 \xi}{2} - \frac{\gamma^2}{6} + \frac{3\gamma \xi}{2} - 2\gamma + \frac{1}{2\xi}.$$

We observe that, for all $\gamma > 2$,

$$F(0^+) = +\infty \quad \text{and} \quad F(1) = -\frac{\gamma^2}{24} - \frac{\gamma}{2} + \frac{1}{2} < 0.$$

To study the behavior of $F(\xi)$, one has to solve some fourth-order equations (this could be possible in an explicit but cumbersome way), so we prefer to make a qualitative study. We evaluate

$$F'(\xi) = \frac{3\gamma^2 \xi^2}{8} - \gamma^2 \xi + \frac{\gamma^2}{2} + \frac{3\gamma}{2} - \frac{1}{2\xi^2},$$

$$F''(\xi) = \frac{3\gamma^2 \xi}{4} - \gamma^2 + \frac{1}{\xi^3},$$

$$F'''(\xi) = \frac{3\gamma^2}{4} - \frac{3}{\xi^4},$$

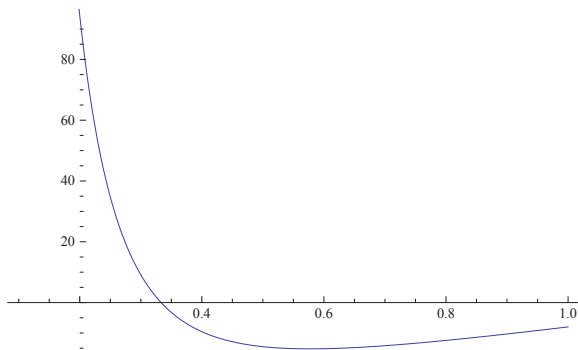
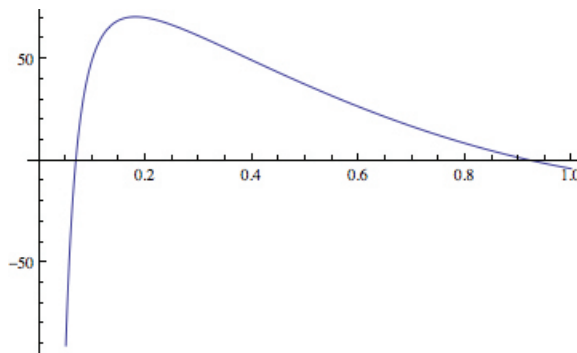
hence we have that

$$F'''(\xi) < 0 \quad \text{if and only if} \quad 0 < \xi < \sqrt{\frac{2}{\gamma}} < 1.$$

Consequently, the function $F''(\xi)$, which nevertheless satisfies, for all $\gamma > 2$,

$$F''(0^+) = +\infty, \quad F''(1) = 1 - \frac{\gamma^2}{4} < 0,$$

has a unique negative minimum at the point $\sqrt{2/\gamma}$.

Figure 3: The function $F''(\xi)$ for $\gamma = 4$.Figure 4: The function $F'(\xi)$ for $\gamma = 14$.

From this, we deduce that there exists one and only one point $0 < \xi_0 < \sqrt{2/\gamma}$ such that

$$\begin{aligned} F''(\xi) &> 0, & 0 < \xi < \xi_0, \\ F''(\xi) &< 0, & \xi_0 < \xi \leq 1. \end{aligned}$$

From the sign of F'' we get that F' is strictly increasing in $(0, \xi_0)$ and decreasing in $(\xi_0, 1)$ (see Figure 3). Next for all $\gamma > 2$,

$$F'(0^+) = -\infty, \quad F'(1) = -\frac{\gamma^2}{8} + \frac{3\gamma}{2} - \frac{1}{2},$$

hence in the case that

$$-\frac{\gamma^2}{8} + \frac{3\gamma}{2} - \frac{1}{2} > 0, \quad \text{that is, } \gamma < 2(3 + 2\sqrt{2}) \cong 11.656\dots,$$

then F' has a single zero $\xi_1 \in (0, \xi_0)$ and, being a change of sign, ξ_1 is a point of absolute minimum for $F(\xi)$.

If $\gamma \geq 2(3 + 2\sqrt{2})$, the above argument fails. In this case we can observe that

$$F'\left(\frac{1}{\gamma}\right) = \frac{\gamma}{2} + \frac{3}{8} > 0,$$

hence $F'(1)$, which is negative at $\xi = 1$ and near $\xi = 0$ vanishes exactly two times, at the point ξ_1 , which is a point of local minimum and at another point $\xi_2 > \xi_1$, which is a point of local maximum (see Figure 4). Hence, to find the absolute minimum, we have to compare the value of $F(\xi_1)$ with that of $F(1)$.

In particular, we have that $\xi_1 < \sqrt{2/\gamma}$ and hence we can show that the minimum is not at $\xi = 1$ simply by observing that we can find at least a point where $F(\xi) < F(1)$ and this point is $\sqrt{2/\gamma}$. In fact,

$$M(\gamma) := F\left(\sqrt{\frac{2}{\gamma}}\right) - F(1) = \frac{\gamma^{\frac{3}{2}}}{\sqrt{2}} - \frac{\gamma^2}{8} - \frac{5\gamma}{2} + 2\sqrt{2}\sqrt{\gamma} - \frac{1}{2} \leq 0.$$

In particular, $M(2) = 0$ and

$$M'(\gamma) = \frac{1}{4}\left(-\gamma + 3\sqrt{2}\sqrt{\gamma} + \frac{4\sqrt{2}}{\sqrt{\gamma}} - 10\right) < 0.$$

This follows since replacing $\sqrt{\gamma}$ with χ , we have to control the sign of the cubic

$$\tilde{M}(\chi) = -\chi^3 + 3\sqrt{2}\chi^2 - 10\chi + 4\sqrt{2},$$

which is negative for all $\chi \geq 1$, since $\tilde{M}(1) = -11 + 7\sqrt{2}$, while

$$\tilde{M}'(\chi) = -3\chi^2 + 6\sqrt{2}\chi - 10$$

is a parabola with negative minimum. □

Remark 6.4. It could be interesting to study this problem in dimension greater than one, namely, to minimize

$$J^o(u) = \int_{\Omega} \left(\frac{1}{2} a(u) |Du(x)|^2 - \gamma u(x) \right) dx$$

in the set

$$\{v \in V_{\Lambda}(\Omega) : u(x) = 0 \text{ for all } x \in \Xi^*\}$$

and in particular to investigate the structure of the singular set of u , both in the general case and in some particular situations in which it is possible to find explicit solutions (e.g., $\Omega = B_R(0)$).

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