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Research Article

Sergio Rolando*

Multiple nonradial solutions for a nonlinear elliptic problem with singular and decaying radial potential

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Abstract: Many existence and nonexistence results are known for nonnegative *radial* solutions to the equation

$$-\Delta u + \frac{A}{|x|^{\alpha}} u = f(u) \quad \text{in } \mathbb{R}^N,\ N \geq 3,\ A, \alpha > 0,\ u \in D^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^{-\alpha}\, dx),$$

with the nonlinearities satisfying $|f(u)| \leq (\text{const.}) u^{p-1}$ for some $p > 2$. The existence of *nonradial* solutions, by contrast, is known only for $N \geq 4$, $\alpha = 2$, $f(u) = u^{(N+2)/(N-2)}$ and A large enough. Here we show that the above equation has multiple nonradial solutions as $A \to +\infty$ for $N \geq 4$, $2/(N-1) < \alpha < 2N - 2$ and $\alpha \neq 2$, with the nonlinearities satisfying suitable assumptions. Our argument essentially relies on the compact embeddings between some suitable functional spaces of symmetric functions, which yields the existence of nonnegative solutions of mountain-pass type, and the separation of the corresponding mountain-pass levels from the energy levels associated to radial solutions.

Keywords: Semilinear elliptic PDE, singular vanishing potential, symmetry breaking

MSC 2010: Primary 35J60; secondary 35Q55, 35J20

1 Introduction and main result

This paper is concerned with the following semilinear elliptic problem:

$$\begin{cases} -\Delta u + \dfrac{A}{|x|^{\alpha}} u = f(u) & \text{in } \mathbb{R}^N,\ N \geq 3, \\ u \geq 0 & \text{in } \mathbb{R}^N, \\ u \in H^1_{\alpha},\ u \neq 0. \end{cases} \tag{1.1}$$

where $A, \alpha > 0$ are real constants, $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous nonlinearity satisfying $|f(s)| \leq (\text{const.}) s^{p-1}$ for some $p > 2$ and all $s \geq 0$, and $H^1_{\alpha} := D^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^{-\alpha}\, dx)$ is the natural energy space related to the equation. We will deal with problem (1.1) in the weak sense, that is, when we speak about *solutions* to (1.1) we will always mean *weak solutions*, i.e., functions $u \in H^1_{\alpha} \setminus \{0\}$ such that $u \geq 0$ almost everywhere in $\mathbb{R}^N$ and

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^N} \frac{A}{|x|^{\alpha}} uv \, dx = \int_{\mathbb{R}^N} f(u) v \, dx \quad \text{for all } v \in H^1_{\alpha}. \tag{1.2}$$

As is well known, problems like (1.1) are models for stationary states of reaction diffusion equations in population dynamics (see, e.g., [17]). They also arise in many other branches of mathematical physics, such as nonlinear optics, plasma physics, condensed matter physics and cosmology (see, e.g., [11, 27]), where its nonnegative solutions lead to special solutions (*solitary waves* and *solitons*) for several nonlinear field

***Corresponding author: Sergio Rolando,** Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano bicocca, Via Cozzi 53, 20125 Milano, Italy, e-mail: sergio.rolando@unito.it. https://orcid.org/0000-0002-1679-8198

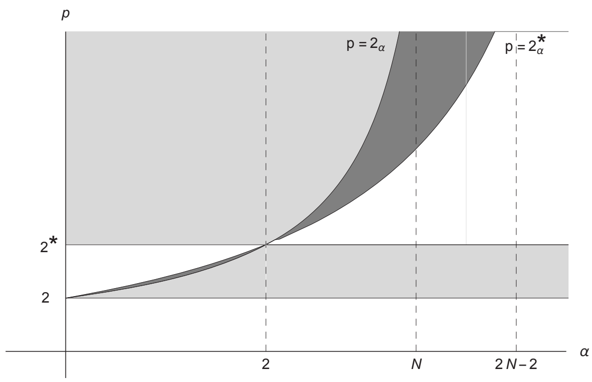

Figure 1: Regions of nonexistence of solutions (light gray), and existence (white with $p > 2$) and nonexistence (dark gray) of radial solutions.

theories like nonlinear Schrödinger (or Gross–Pitaevskii) and Klein–Gordon equations. In this context, (1.1) is a prototype for problems exhibiting radial potentials which are singular at the origin and/or vanishing at infinity (sometimes called the *zero mass case*; see, e.g., [8, 22]).

Although it can be considered as a quite recent investigation, the study of problem (1.1) has already some history, which probably started in [26] and continued in [5, 9, 15, 16, 24, 25] (see [4] for a similar cylindrical problem). Currently, the problem of existence and nonexistence of *radial* solutions is essentially solved in the pure-power case $f(u) = u^{p-1}$, where the results obtained rest upon compatibility conditions between α and p. These can be summarized as follows (for a chronological overview of these results see [5]): the problem has a radial solution for $(\alpha, p) = (2, 2^*)$ (see [26]) and for all the pairs (α, p) satisfying

$$\begin{cases} 0 < \alpha < 2, \\ 2^*_\alpha < p < 2^*, \end{cases} \quad \text{or} \quad \begin{cases} 2 < \alpha < 2N-2, \\ 2^* < p < 2^*_\alpha, \end{cases} \quad \text{or} \quad \begin{cases} \alpha \geq 2N-2, \\ p > 2^*, \end{cases} \quad \text{with } 2^*_\alpha := 2\frac{2N-2+\alpha}{2N-2-\alpha} \tag{1.3}$$

(see [25]), while it has no solution if

$$\begin{cases} 0 < \alpha < 2, \\ p \notin (2_\alpha, 2^*), \end{cases} \quad \text{or} \quad \begin{cases} \alpha = 2, \\ p \neq 2^*, \end{cases} \quad \text{or} \quad \begin{cases} 2 < \alpha < N, \\ p \notin (2^*, 2_\alpha), \end{cases} \quad \text{or} \quad \begin{cases} \alpha \geq N, \\ p \leq 2^*, \end{cases} \quad \text{with } 2_\alpha := \frac{2N}{N-\alpha}$$

(see [9]) and no radial solution for both

$$\begin{cases} 0 < \alpha < 2, \\ 2_\alpha < p \leq 2^*_\alpha, \end{cases} \quad \text{and} \quad \begin{cases} 2 < \alpha < 2N-2, \\ 2^*_\alpha \leq p < 2_\alpha \end{cases}$$

(see [5] and [15], respectively). As usual, $2^* := 2N/(N-2)$ denotes the critical exponent for the Sobolev embedding in dimension $N \geq 3$. All these results are portrayed in the picture of the αp-plane given in Figure 1, where nonexistence regions are shaded in gray (nonexistence of radial solutions) and light gray (nonexistence of solutions at all, which includes both the lines $p = 2^*$ and $p = 2_\alpha$ except for the pair $(\alpha, p) = (2, 2^*)$), whereas white color (of course above the line $p = 2$) means existence of radial solutions. As to nonradial solutions, the only result available is the one contained in [26, Theorem 0.5], where Terracini proves that problem (1.1), with $N \geq 4$, $\alpha = 2$ and $f(u) = u^{2^*-1}$, has at least a nonradial solution for every A large enough. This brought Catrina to say, in the introduction of his paper [15]: "*Two questions still remain: whether one can find non-radial solutions in the case when radial solutions do not exist, or in the case when radial solutions exist*".

Su, Wang and Willem [25] covered also the case where problem (1.1) has general nonlinearities satisfying the power growth condition $|f(u)| \leq (\text{const.})u^{p-1}$ for some $p > 2$, and ensured that, under some rather standard additional assumptions on f (precisely (f1) and (f2) below), problem (1.1) has a *radial* solution for all the pairs (α, p) satisfying (1.3). To be precise, they only concerned themselves with radial weak solutions in the sense of the dual space of the radial subspace of H^1_α (where the energy functional of the problem is

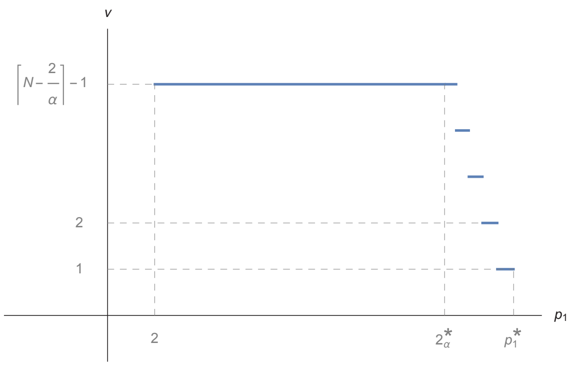

Figure 2: v as a function of $p_1 \in (2, p_1^*)$ for $N \geq 3$ and $\alpha \in (0, 2)$ fixed.

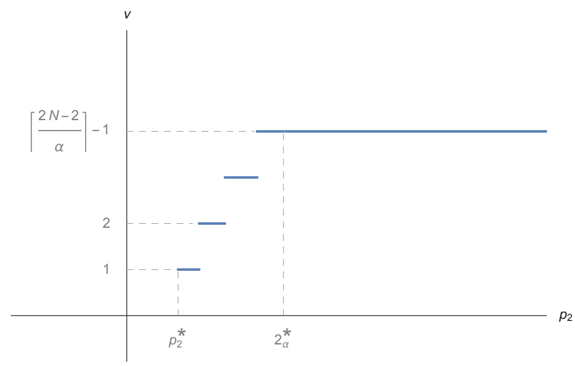

Figure 3: v as a function of $p_2 > p_2^*$ for $N \geq 3$ and $\alpha \in (2, 2N - 2)$ fixed.

well defined by the embeddings they proved), but the symmetric criticality type results of [6] actually apply, yielding solutions in the sense of our definition (1.2). No results are known in the literature about nonradial solutions.

This general lack of symmetry breaking results is the motivation of this paper, where we prove that problem (1.1) has multiple nonradial solutions as $A \to +\infty$, provided that $N \geq 4$, $\alpha \in (2/(N-1), 2N-2) \setminus \{2\}$ and f belongs to a suitable class of nonlinearities satisfying a power growth condition. We observe straight away that such a class of nonlinearities does not unfortunately contain pure powers (which does not satisfy our assumption (f0), where $p_1 \neq p_2$).

The main assumptions characterizing our class of nonlinearities are the following, where we define $F(s) := \int_0^s f(t)\,dt$:

(f0) $\sup_{s>0} \frac{|f(s)|}{\min\{s^{p_1-1}, s^{p_2-1}\}} < +\infty$ for some $2 < p_1 < 2^* < p_2$.
(f1) There exists $\theta > 2$ such that $\theta F(s) \leq f(s)s$ for all $s > 0$.
(f2) $F(s) > 0$ for all $s > 0$.
(f3) The function $\frac{f(s)}{s}$ is strictly increasing on $(0, +\infty)$.
(f4) There exists $\mu > 2$ such that the function $\frac{F(s)}{s^\mu}$ is decreasing on $(0, +\infty)$.

For $N \geq 3$, $\alpha \in (0, 2N-2)$, $\alpha \neq 2$ and $2 < p_1 < 2^* < p_2$, we define

$$v := v_{N,\alpha,p_1,p_2} := \begin{cases} \left\lceil 2\min\left\{\frac{N-1}{\alpha}, \frac{N-2}{2-\alpha}\frac{2^*-p_1}{p_1-2}\right\} - 2N\left(\frac{1}{\alpha} - \frac{1}{2}\right)\right\rceil - 1 & \text{if } 0 < \alpha < 2, \\ \left\lceil 2\min\left\{\frac{N-1}{\alpha}, \frac{N-2}{\alpha-2}\frac{p_2-2^*}{p_2-2}\right\}\right\rceil - 1 & \text{if } 2 < \alpha < 2N-2, \end{cases} \tag{1.4}$$

where $\lceil \cdot \rceil$ denotes the ceiling function (i.e., $\lceil x \rceil := \min\{n \in \mathbb{Z} : n \geq x\}$).

Our main result is the following theorem.

Theorem 1.1. *Let $N \geq 4$ and $\alpha \in (2/(N-1), 2N-2)$, $\alpha \neq 2$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying* (f1)–(f4). *Assume that* (f0) *holds with*

$$p_1 < p_1^* := 2\frac{\alpha^2(N-1) - 2\alpha(N-1) + 4N}{\alpha^2(N-1) - 2\alpha(N+1) + 4N} \quad \text{or} \quad p_2 > p_2^* := 2\frac{2N+2-\alpha}{2N-2-\alpha} \tag{1.5}$$

if $\alpha \in (2/(N-1), 2)$ or $\alpha \in (2, 2N-2)$, respectively. Then there exists $A_* > 0$ such that for every $A > A_*$, problem (1.1) *has both a radial solution and v different nonradial solutions.*

Some comments on Theorem 1.1 are in order. First of all, under the assumptions of the theorem, v is positive (see Lemma 5.2 below), and so at least one nonradial solution actually exists. On the other hand, it is easy to check that for every fixed N and α, the behavior of v as a function of p_1 and p_2 is the one portrayed in Figures 2 and 3, respectively, whence one sees that the number v of nonradial solutions may assume every natural value (as $N \to \infty$).

Regarding assumptions (1.5) and (f0), it is worth observing that $\alpha \in (0, 2)$ implies $2 < p_1^* < 2^*$, while $2 < \alpha < 2N - 2$ implies $p_2^* > 2^*$, and so (1.5) and (f0) are consistent with each other. Assumption (f0) is the so-called *double-power growth condition* and seems to be typical in nonlinear problems with potentials vanishing at infinity (see, e.g., [2, 3, 6–8, 12–14, 18, 19, 22]). Such an assumption obviously implies the single-power growth condition $|f(s)| \le (\text{const.})s^{p-1}$ for all $p \in [p_1, p_2]$ and $s \ge 0$, but it is actually more stringent than that, since it requires $p_1 \ne p_2$. We finally observe that (f0) still remains true if one raises p_1 and lowers p_2, but this decreases ν (see Figures 2 and 3), and therefore it is convenient to apply Theorem 1.1 with p_1 as small as possible and p_2 as large as possible (which is also consistent with assumption (1.5)).

The plan of the paper is the following. In Section 2 we define the variational setting and introduce the argument we will use in the proof of Theorem 1.1, which will be given in Section 5. Observe that we cannot use the technique used in [26], where the homogeneity of the nonlinearity is exploited and a nonradial solution is obtained as a global minimizer of the Sobolev type quotient associated to the problem. Our argument, instead, essentially relies on the following two main elements:

(i) The compact embeddings between some suitable functional spaces of symmetric functions, which yield the existence of ν different solutions of mountain-pass type.
(ii) The separation of the corresponding mountain-pass levels from the energy levels associated to radial solutions.

Sections 3 and 4 are devoted to the estimation of these levels in order to separate them. As a conclusion, we get ν nonradial solutions on which the energy functional of the equation has a lower value than the energy levels of radial solutions.

We end this introductory section by giving some examples of nonlinearities to which Theorem 1.1 applies, and by presenting some notations we use throughout the paper.

Example 1.2. Let $N \ge 4$ and $\alpha \in (2/(N-1), 2N-2)$, $\alpha \ne 2$, and let $2 < p_1 < 2^* < p_2$ be such that (1.5) holds. The most obvious nonlinearity to which Theorem 1.1 applies is $f(s) = \min\{|s|^{p_1-1}, |s|^{p_2-1}\}$, which satisfies (f1) and (f4) for $\theta = p_1$ and any $\mu > p_2$. Other simple examples are

$$f(s) = \frac{|s|^{p_2-1}}{1 + |s|^{p_2-p_1}}, \quad f(s) = \frac{d}{ds}\left(\frac{|s|^{p_2}}{1 + |s|^{p_2-p_1}}\right),$$

both of which satisfy (f1) with $\theta = p_1$. In the latter case, (f4) clearly holds for any $\mu > p_2$. We leave it to the reader to check that (f4) also holds in the former case for μ large enough.

Notations.

- σ_d denotes the $(d-1)$-dimensional measure of the unit sphere of $\mathbb{R}^d$.
- $C_c^\infty(\Omega)$ is the space of infinitely differentiable real functions with compact support in the open set $\Omega \subseteq \mathbb{R}^d$.
- $D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$ is the usual Sobolev space, which identifies with the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm of the gradient.

2 Preliminaries

Let $N \ge 3$ and $A, \alpha > 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (f0)–(f2). In this section we define the functional setting and introduce the argument we will use in proving Theorem 1.1.

As already mentioned in the introduction, we define the Hilbert space

$$H_\alpha^1 := \left\{u \in D^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \frac{u^2}{|x|^\alpha}\, dx < \infty\right\},$$

which we endow with the following scalar product and related norm:

$$(u, v)_A := \int_{\mathbb{R}^N} \left(\nabla u \cdot \nabla v + \frac{A}{|x|^\alpha} uv\right) dx, \quad \|u\|_A^2 := \int_{\mathbb{R}^N} \left(|\nabla u|^2 + \frac{A}{|x|^\alpha} u^2\right) dx.$$

Of course, the embedding $H_\alpha^1 \hookrightarrow D^{1,2}(\mathbb{R}^N)$ is continuous.

Given any integer K such that $1 \le K \le N-1$, we write every $x \in \mathbb{R}^N$ as $x = (y,z) \in \mathbb{R}^K \times \mathbb{R}^{N-K}$, and in the space H^1_α, we consider the following closed subspaces of symmetric functions:

$$H_{\mathrm{r}} := \{u \in H : u(x) = u(|x|)\} \quad \text{and} \quad H_K := \{u \in H : u(x) = u(y,z) = u(|y|,|z|)\}.$$

Of course $u(y,z) = u(|y|,|z|)$ naturally means that $u(y,z) = u(S_1 y, S_2 z)$ for all isometries S_1 and S_2 of $\mathbb{R}^K$ and $\mathbb{R}^{N-K}$, respectively. Similarly for $u(x) = u(|x|)$. Note that $H_{\mathrm{r}} \subset H_K$ for every K, since $|x|^2 = |y|^2 + |z|^2$. The next lemma clarifies better the relation between the spaces H_K and H_{r}.

Lemma 2.1. *Let $1 \le K_1 < K_2 \le N-1$. Then $H_{K_1} \cap H_{K_2} = H_{\mathrm{r}}$.*

Proof. The proof is essentially an adaptation of the one of [21, Lemma 3.3]. For any $x \in \mathbb{R}^N$, we will denote by (y_1, z_1) its decomposition in $\mathbb{R}^{K_1} \times \mathbb{R}^{N-K_1}$, and by (y_2, z_2) its decomposition in $\mathbb{R}^{K_2} \times \mathbb{R}^{N-K_2}$. Let $u \in H_{K_1} \cap H_{K_2}$, and for every $s, t \ge 0$, define

$$\tilde{u}(s,t) := u(s, 0, \dots, 0, t).$$

Then we clearly have $u(x) = \tilde{u}(|y_1|, |z_1|) = \tilde{u}(|y_2|, |z_2|)$ for every $x = (y_1, z_1) = (y_2, z_2) \in \mathbb{R}^N$. Let $x = (y_1, z_1) \in \mathbb{R}^{K_1} \times \mathbb{R}^{N-K_1}$ and $x' = (y_2', z_2') \in \mathbb{R}^{K_2} \times \mathbb{R}^{N-K_2}$ be such that $|x| = |x'|$, i.e., $|y_1|^2 + |z_1|^2 = |y_2'|^2 + |z_2'|^2$. Suppose that $|y_1| \le |y_2'|$ and define $x'' \in \mathbb{R}^N$ by setting

$$x'' := \Big(|y_1|, 0, \dots, 0, \sqrt{|y_2'|^2 - |y_1|^2}, 0, \dots, 0, |z_2'|, 0, \dots, 0\Big),$$

where the first block of zeros has $K_1 - 1$ zeros, the second $K_2 - K_1 - 1$, and the third $N - K_2 - 1$. Then

$$u(x'') = \tilde{u}(|y_1''|, |z_1''|) = \tilde{u}\Big(|y_1|, \sqrt{|y_2'|^2 - |y_1|^2 + |z_2'|^2}\Big) = \tilde{u}(|y_1|, |z_1|)$$

and

$$u(x'') = \tilde{u}(|y_2''|, |z_2''|) = \tilde{u}\Big(\sqrt{|y_1|^2 + |y_2'|^2 - |y_1|^2}, |z_2'|\Big) = \tilde{u}(|y_2'|, |z_2'|),$$

which implies $\tilde{u}(|y_1|, |z_1|) = \tilde{u}(|y_2'|, |z_2'|)$, i.e., $u(x) = u(x')$. If $|y_1| > |y_2'|$, we repeat the argument with x'' defined by

$$x'' := \Big(|y_2'|, 0, \dots, 0, \sqrt{|y_1|^2 - |y_2'|^2}, 0, \dots, 0, |z_1|, 0, \dots, 0\Big)$$

(with the same lengths of blocks of zeros) and get the same result. Hence, $|x| = |x'|$ implies $u(x) = u(x')$, i.e., $u \in H_{\mathrm{r}}$. □

We modify the function f by setting $f(s) = 0$ for all $s < 0$ and, with a slight abuse of notation, we still denote by f the modified function. Then, by (f0), there exist $M_1, M_2 > 0$ such that

$$|f(s)| \le M_1 \min\{|s|^{p_1-1}, |s|^{p_2-1}\} \quad \text{and} \quad |F(s)| \le M_2 \min\{|s|^{p_1}, |s|^{p_2}\} \quad \text{for all } s \in \mathbb{R},$$

which, in particular, yields

$$|f(s)| \le M_1 |s|^{p-1} \quad \text{and} \quad |F(s)| \le M_2 |s|^p \quad \text{for all } p \in [p_1, p_2] \text{ and } s \in \mathbb{R}. \tag{2.1}$$

By the continuous embeddings $H^1_\alpha \hookrightarrow D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, one can check (see, for example, [20]) that condition (2.1) with $p = 2^*$ implies that the energy functional associated to the equation of (1.1), i.e.,

$$I(u) := \frac{1}{2}\|u\|_A^2 - \int_{\mathbb{R}^N} F(u)\,dx, \tag{2.2}$$

is of class C^1 on H^1_α and has Fréchet derivative $I'(u)$ at any $u \in H^1_\alpha$ given by

$$I'(u)v = (u,v)_A - \int_{\mathbb{R}^N} f(u)v\,dx \quad \text{for all } v \in H^1_\alpha. \tag{2.3}$$

This yields that the critical points of $I\colon H^1_\alpha \to \mathbb{R}$ satisfy (1.2). A standard argument shows that such critical points are nonnegative (see the proof of Theorem 1.1 in Section 5), and therefore we conclude that the nonzero critical points of I are weak solutions to problem (1.1).

Accordingly, our argument in proving Theorem 1.1 will be essentially the following. The existence of a critical point for the restriction $I_{|H_{\mathrm{r}}}$ readily follows from the results of [25]. By exploiting the compact embeddings of [2] and the results of [8] about Nemytskiĭ operators on the sum of Lebesgue spaces, we will show in Section 5 that $I_{|H_K}$ has a nonzero critical point u_K for every $2 \le K \le N-2$. Thanks to the classical Palais principle of symmetric criticality [23], all these critical points are also critical points of I, and thus weak solutions to (1.1). Hence, Theorem 1.1 is proved if we show that $u_K \notin H_{\mathrm{r}}$ for every K, which also implies $u_{K_1} \neq u_{K_2}$ for $K_1 \neq K_2$, by Lemma 2.1. This will be achieved by showing that the critical levels $I(u_K)$ are lower than all the nonzero critical levels of $I_{|H_{\mathrm{r}}}$. The starting points in proving this are the following lemmas.

Lemma 2.2. *For every $u \in H_{\mathrm{r}} \setminus \{0\}$, $u \ge 0$, there exists $t_u > 0$ such that $I(t_u u) = \max_{t \ge 0} I(tu)$.*

Proof. Since $u \ge 0$ and $u \neq 0$, we can fix $\delta > 0$ such that the set $\{x \in \mathbb{R}^N : u \ge \delta\}$ has positive measure. From assumptions (f1) and (f2), we deduce that there exists a constant $C > 0$ such that $F(s) \ge Cs^\theta$ for all $s \ge \delta$. Then, for every $t > 1$, one has

$$\begin{aligned}
\int_{\mathbb{R}^N} F(tu)\,dx &= \int_{\{x\in\mathbb{R}^N : tu\ge\delta\}} F(tu)\,dx + \int_{\{x\in\mathbb{R}^N : tu<\delta\}} F(tu)\,dx \\
&\ge \int_{\{x\in\mathbb{R}^N : tu\ge\delta\}} F(tu)\,dx \\
&\ge Ct^\theta \int_{\{x\in\mathbb{R}^N : tu\ge\delta\}} u^\theta\,dx \\
&\ge Ct^\theta \int_{\{x\in\mathbb{R}^N : u\ge\delta\}} u^\theta\,dx,
\end{aligned}$$

and therefore

$$I(tu) \le \frac{1}{2}t^2\|u\|_A^2 - Ct^\theta \int_{\{x\in\mathbb{R}^N : u\ge\delta\}} u^\theta\,dx \to -\infty \quad \text{as } t \to +\infty, \tag{2.4}$$

since $\theta > 2$. On the other hand, condition (2.1) with $p = 2^*$ and the continuous embeddings $H_{\mathrm{r}} \hookrightarrow H^1_\alpha \hookrightarrow D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ imply that there exists a constant $C > 0$ such that $I(tu) \ge \|u\|_A^2 \frac{t^2}{2} - C\|u\|_A^{2^*} t^{2^*}$ for all $t \ge 0$, which in turn yields $I(tu) > 0$ for all $t > 0$ small enough. Together with (2.4), this gives the result. □

According to Lemma 2.2, define

$$m_A := \inf_{\substack{u\in H_{\mathrm{r}}\setminus\{0\},\\ u\ge 0}} \max_{t\ge 0} I(tu).$$

Lemma 2.3. *Assume* (f3) *and let $u \in H_{\mathrm{r}} \setminus \{0\}$. If u is a critical point for I, then $I(u) \ge m_A$.*

Proof. As already observed, if u is a critical point for I, then u is nonnegative. We shall now prove that $I(u) = \max_{t\ge 0} I(tu)$, which obviously yields the result. For $t \ge 0$, define

$$g(t) := I(tu) = \frac{1}{2}t^2\|u\|_A^2 - \int_{\mathbb{R}^N} F(tu)\,dx.$$

As u is a critical point for I, we readily have that $t = 1$ is a critical point for g. Indeed, $g'(t) = I'(tu)u$, and thus $g'(1) = I'(u)u = 0$. We now show that, on the other hand, g has at most one critical point in $(0, +\infty)$. We have $g'(t) = 0$ if and only if $I'(tu)u = 0$, i.e.,

$$I'(tu)u = t\|u\|_A^2 - \int_{\mathbb{R}^N} f(tu)u\,dx = 0.$$

So, if $0 < t_1 < t_2$ are critical points for g, then one has

$$\|u\|_A^2 = \frac{1}{t_1} \int_{\mathbb{R}^N} f(t_1 u)u \, dx = \frac{1}{t_2} \int_{\mathbb{R}^N} f(t_2 u)u \, dx,$$

which implies

$$\int_{E_u} \left(\frac{f(t_2 u)}{t_2 u} - \frac{f(t_1 u)}{t_1 u} \right) u^2 \, dx = 0, \tag{2.5}$$

where $E_u := \{x \in \mathbb{R}^N : u > 0\}$. Since the integrand in (2.5) is nonnegative by assumption (f3), we have that $\frac{f(t_2 u)}{t_2 u} - \frac{f(t_1 u)}{t_1 u} = 0$ almost everywhere on E_u. Since E_u has positive measure (because $0 \neq u \geq 0$), this implies $t_1 = t_2$, again by assumption (f3). As a conclusion, according to Lemma 2.2, we deduce that $t_u = 1$ and the claim ensues. □

Lemma 2.4. *There exists $R > 0$ such that for every $1 \leq K \leq N - 1$, one has*

$$\inf_{u \in H_K, \|u\|_A \leq R} I(u) = 0 \quad \text{and} \quad \inf_{u \in H_K, \|u\|_A = R} I(u) > 0. \tag{2.6}$$

Proof. The claim readily follows from (2.1) with $p = 2^*$ and the continuous embeddings $H_K \hookrightarrow H_\alpha^1 \hookrightarrow D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, which imply that there exists a constant $C > 0$ such that $I(u) \geq \|u\|_A^2/2 - C\|u\|_A^{2^*}$ for all $u \in H_K$. □

In Section 4 we will see that $I_{|H_K}$ takes negative values by choosing a suitable $\overline{u}_K \in H_K$ such that $I(\overline{u}_K) < 0$. This implies $\|\overline{u}_K\|_A > R$, by (2.6), and therefore the functional $I_{|H_K}$ has a mountain-pass geometry. In Section 5 we will see that it also satisfies the Palais–Smale condition for $2 \leq K \leq N - 2$, and so it admits a (nonnegative) critical point u_K at the mountain-pass level

$$c_{A,K} := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) > 0, \quad \text{where } \Gamma := \{\gamma \in C([0, 1]; H_K) : \gamma(0) = 0, \ \gamma(1) = \overline{u}_K\}. \tag{2.7}$$

With a view to obtaining the separation inequality $c_{A,K} < m_A$, Sections 3 and 4 are devoted to estimating m_A and $c_{A,K}$.

3 Estimate of m_A

Let $N \geq 3$ and $\alpha, A > 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (f0)–(f2). This section is devoted to deriving the estimate of m_A given in Proposition 3.2 below, which relies on the following radial lemma (see also [6, Appendix] and [25, Lemmas 4 and 5] for similar results).

Lemma 3.1. *Every $u \in H_r$ satisfies*

$$|u(x)| \leq \frac{\sqrt{2/\sigma_N}}{A^{1/4}} \frac{\|u\|_A}{|x|^{(2N-2-\alpha)/4}} \quad \textit{almost everywhere in } \mathbb{R}^N$$

(recall that σ_N denotes the $(N-1)$-dimensional measure of the unit sphere of $\mathbb{R}^N$).

Proof. Let $u \in H_r$ and let $\tilde{u} : (0, +\infty) \to \mathbb{R}$ be continuous and such that $u(x) = \tilde{u}(|x|)$ for almost every $x \in \mathbb{R}^N$. Set

$$v(r) := r^{N-1-\alpha/2} \tilde{u}(r)^2 \quad \text{for all } r > 0.$$

By [6, Lemma 27], we have that $\tilde{u} \in W^{1,1}((a, b))$ for every $0 < a < b < +\infty$, whence $v \in W^{1,1}((a, b))$ and

$$v(b) - v(a) = \int_a^b v'(r) \, dr.$$

Moreover, for almost every $r \in (a, b)$, one has

$$v'(r) = \left(N - 1 - \frac{\alpha}{2}\right) r^{N-2-\alpha/2} \tilde{u}(r)^2 + 2r^{N-1-\alpha/2} \tilde{u}(r)\tilde{u}'(r). \tag{3.1}$$

If $\alpha < 2N - 2$, this implies $v'(r) \geq 2r^{N-1-\alpha/2}\tilde{u}(r)\tilde{u}'(r)$, and therefore

$$\begin{aligned} v(a) &\leq v(b) - \int_a^b v'(r)\,dr \\ &\leq v(b) - \int_a^b 2r^{N-1-\alpha/2}\tilde{u}(r)\tilde{u}'(r)\,dr \\ &\leq v(b) + 2\int_a^b r^{N-1-\alpha/2}|\tilde{u}(r)||\tilde{u}'(r)|\,dr \\ &= v(b) + 2\int_a^b r^{(N-1)/2}|\tilde{u}'(r)| r^{(N-1)/2-\alpha/2}|\tilde{u}(r)|\,dr \\ &\leq v(b) + 2\left(\int_0^{+\infty} r^{N-1}|\tilde{u}'(r)|^2\,dr\right)^{1/2}\left(\int_0^{+\infty} r^{N-1-\alpha}|\tilde{u}(r)|^2\,dr\right)^{1/2} \\ &\leq v(b) + \frac{2}{\sigma_N\sqrt{A}}\left(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)^{1/2}\left(\int_{\mathbb{R}^N}\frac{Au^2}{|x|^\alpha}\,dx\right)^{1/2} \\ &\leq v(b) + \frac{2}{\sigma_N\sqrt{A}}\|u\|_A^2. \end{aligned} \tag{3.2}$$

If $\alpha \geq 2N - 2$, then (3.1) gives $v'(r) \leq 2r^{N-1-\alpha/2}\tilde{u}(r)\tilde{u}'(r)$, and thus we have

$$v(b) \leq v(a) + \int_a^b 2r^{N-1-\alpha/2}\tilde{u}(r)\tilde{u}'(r)\,dr \leq v(a) + \frac{2}{\sigma_N\sqrt{A}}\|u\|_A^2, \tag{3.3}$$

as before. Now observe that there exist $0 < a_n \to 0$ and $b_n \to +\infty$ such that $v(a_n) \to 0$ and $v(b_n) \to 0$. Indeed, if $l := \liminf_{r\to 0^+} v(r) > 0$, then for every r smaller than some suitable $r_0 > 0$, one has $|\tilde{u}(r)| \geq \sqrt{l/2}\, r^{-(N-1-\alpha/2)/2}$, and therefore one of the following contradictions ensues:

$$\int_{\mathbb{R}^N} \frac{u^2}{|x|^\alpha}\,dx \geq \frac{l}{2}\int_{B_{r_0}} \frac{1}{|x|^{N-1+\alpha/2}}\,dx = +\infty \quad \text{if } \alpha \geq 2$$

or

$$\int_{\mathbb{R}^N} |u|^{2^*}\,dx \geq \left(\frac{l}{2}\right)^{2^*/2}\int_{B_{r_0}} \frac{1}{|x|^{\frac{N-1-\alpha/2}{N-2}N}}\,dx = +\infty \quad \text{if } \alpha \leq 2.$$

Similarly, if $\liminf_{r\to+\infty} v(r) > 0$, then one obtains

$$\int_{\mathbb{R}^N} \frac{u^2}{|x|^\alpha}\,dx = +\infty \quad \text{if } \alpha \leq 2 \qquad \text{and} \qquad \int_{\mathbb{R}^N} |u|^{2^*}\,dx = +\infty \quad \text{if } \alpha \geq 2.$$

Therefore, the claim follows by letting $n \to \infty$ in (3.2) with $a = r$ and $b = b_n$, and in (3.3) with $a = a_n$ and $b = r$. □

We can now prove our estimate for m_A.

Proposition 3.2. *Assume $0 < \alpha < 2N - 2$, $\alpha \neq 2$, and let $p = \max\{2_\alpha^*, p_1\}$ if $0 < \alpha < 2$, or $p = \min\{2_\alpha^*, p_2\}$ if $2 < \alpha < 2N - 2$. Then there exists a constant $C_0 > 0$, independent from A, such that*

$$m_A \geq C_0 A^{\frac{N-2}{\alpha-2}\frac{p-2^*}{p-2}}.$$

Proof. Let $u \in H_{\mathrm{r}} \setminus \{0\}$. By Lemma 3.1, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^{2_\alpha^*}\,dx &= \int_{\mathbb{R}^N} |u|^{2_\alpha^*-2}u^2\,dx\\ &\le \frac{(\frac{2}{\sigma_N})^{(2_\alpha^*-2)/2}}{A^{(2_\alpha^*-2)/4}}\|u\|_A^{2_\alpha^*-2}\int_{\mathbb{R}^N}\frac{u^2}{|x|^{[(2N-2-\alpha)/4](2_\alpha^*-2)}}\,dx\\ &= \frac{(\frac{2}{\sigma_N})^{2\alpha/(2N-2-\alpha)}}{A^{\alpha/(2N-2-\alpha)}}\frac{\|u\|_A^{2_\alpha^*-2}}{A}\int_{\mathbb{R}^N}\frac{Au^2}{|x|^\alpha}\,dx\\ &\le \frac{(\frac{2}{\sigma_N})^{2\alpha/(2N-2-\alpha)}}{A^{(2N-2)/(2N-2-\alpha)}}\|u\|_A^{2_\alpha^*},\end{aligned}$$

since $2_\alpha^* - 2 = 4\alpha/(2N-2-\alpha)$. On the other hand, one has

$$\int_{\mathbb{R}^N} |u|^{2^*}\,dx \le S_N^{2^*}\Big(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\Big)^{2^*/2} \le S_N^{2^*}\|u\|_A^{2^*},$$

where S_N denotes the Sobolev constant in dimension N. Then, in either case $p = \max\{2_\alpha^*, p_1\} < 2^*$ or $p = \min\{2_\alpha^*, p_2\} > 2^*$, we can argue by interpolation: there exists $\lambda \in [0,1)$ such that $p = \lambda 2^* + (1-\lambda)2_\alpha^*$, and by Hölder inequality, we get

$$\int_{\mathbb{R}^N}|u|^p\,dx = \int_{\mathbb{R}^N}|u|^{\lambda 2^*+(1-\lambda)2_\alpha^*}\,dx \le \Big(\int_{\mathbb{R}^N}|u|^{2^*}\,dx\Big)^\lambda\Big(\int_{\mathbb{R}^N}|u|^{2_\alpha^*}\,dx\Big)^{1-\lambda} \le C\frac{\|u\|_A^p}{A^{(2N-2)(1-\lambda)/(2N-2-\alpha)}},$$

where $C := S_N^{\lambda 2^*}(\frac{2}{\sigma_N})^{2\alpha(1-\lambda)/(2N-2-\alpha)}$ only depends on N, α, p. Recalling condition (2.1), this implies

$$\Big|\int_{\mathbb{R}^N} F(u)\,dx\Big| \le M_2\int_{\mathbb{R}^N}|u|^p\,dx \le M_2 C\frac{\|u\|_A^p}{A^{(2N-2)(1-\lambda)/(2N-2-\alpha)}},$$

and therefore

$$I(u) \ge \frac{1}{2}\|u\|_A^2 - a\|u\|_A^p,$$

where we set $a = M_2 C A^{-(2N-2)(1-\lambda)/(2N-2-\alpha)}$ for brevity. Hence,

$$I(tu) \ge \frac{1}{2}t^2\|u\|_A^2 - at^p\|u\|_A^p =: g_u(t) \quad \text{for every } t \ge 0.$$

The function $g_u\colon [0,+\infty) \to \mathbb{R}$ attains its maximum in $t_u := \frac{(ap)^{-1/(p-2)}}{\|u\|_A}$ and since

$$1-\lambda = \frac{p-2^*}{2_\alpha^*-2^*} = \frac{(p-2^*)(2N-2-\alpha)(N-2)}{4(\alpha-2)(N-1)},$$

one computes

$$\begin{aligned}g_u(t_u) &= \Big(\frac{1}{ap}\Big)^{2/(p-2)}\Big(\frac{1}{2}-\frac{1}{p}\Big)\\ &= \frac{p-2}{2p^{p/(p-2)}}\Big(\frac{1}{a}\Big)^{2/(p-2)}\\ &= \frac{p-2}{2p^{p/(p-2)}}\Big(\frac{A^{(2N-2)(1-\lambda)/(2N-2-\alpha)}}{M_2 C}\Big)^{2/(p-2)}\\ &= \frac{p-2}{2p^{p/(p-2)}}\frac{A^{\frac{N-2}{\alpha-2}\frac{p-2^*}{p-2}}}{(M_2C)^{2/(p-2)}}.\end{aligned}$$

Hence,

$$\max_{t\ge0} I(tu) \ge \max_{t\ge0} g_u(t) = g_u(t_u) = C_0 A^{\frac{N-2}{\alpha-2}\frac{p-2^*}{p-2}},$$

with the definition of C_0 being obvious. Since $u \in H_{\mathrm{r}} \setminus \{0\}$ is arbitrary, we conclude

$$m_A = \inf_{\substack{u\in H_{\mathrm{r}}\setminus\{0\},\\ u\ge0}}\max_{t\ge0} I(tu) \ge \inf_{u\in H_{\mathrm{r}}\setminus\{0\}}\max_{t\ge0} I(tu) \ge C_0 A^{\frac{N-2}{\alpha-2}\frac{p-2^*}{p-2}},$$

and the proof is complete. □

Remark 3.3. If p is as in Proposition 3.2, it is easy to check that

$$\frac{N-2}{\alpha-2}\frac{p-2^*}{p-2}=\begin{cases}\min\left\{\dfrac{N-1}{\alpha},\dfrac{N-2}{2-\alpha}\dfrac{2^*-p_1}{p_1-2}\right\} & \text{if } 0<\alpha<2,\\ \min\left\{\dfrac{N-1}{\alpha},\dfrac{N-2}{\alpha-2}\dfrac{p_2-2^*}{p_2-2}\right\} & \text{if } 2<\alpha<2N-2.\end{cases}$$

4 Estimate of $c_{A,K}$

Let $N\geq 3$, $2\leq K\leq N-2$ and $\alpha>0$, $\alpha\neq 2$. Let $f\colon\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying (f0)–(f2). In this section we define a suitable $\overline{u}_K\in H_K$ such that $I(\overline{u}_K)<0$ and estimate the corresponding mountain-pass level (2.7).

In defining $\overline{u}_K$, we will use the following construction of positive H_K functions, which is inspired by [10]. Denote by $\phi\colon D\to\mathbb{R}^2\setminus\{0\}$ the change to polar coordinates in $\mathbb{R}^2\setminus\{0\}$, namely, $\phi(\rho,\theta)=(\rho\cos\theta,\rho\sin\theta)$ for all $(\rho,\theta)\in D:=(0,+\infty)\times[0,2\pi)$. Define

$$E:=\left(\frac14,\frac34\right)\times\left(\frac{\pi}{6},\frac{\pi}{3}\right)$$

and take any $\psi\colon\mathbb{R}^2\to\mathbb{R}$ such that $\psi\in C_c^\infty(E)$ and $\psi>0$. For $0<\varepsilon<1$ and $(\rho,\theta)\in\mathbb{R}^2$, define

$$\psi_\varepsilon(\rho,\theta):=\psi\Big(\rho^{1/\varepsilon},\frac{\theta}{\varepsilon}\Big),$$

in such a way that $\psi_\varepsilon\in C_c^\infty(E_\varepsilon)$, where

$$E_\varepsilon:=\Big\{(\rho,\theta)\in\mathbb{R}^2:\Big(\rho^{1/\varepsilon},\frac{\theta}{\varepsilon}\Big)\in E\Big\}=\Big\{(\rho,\theta)\in\mathbb{R}^2:\Big(\frac14\Big)^\varepsilon<\rho<\Big(\frac34\Big)^\varepsilon,\ \frac{\pi\varepsilon}{6}<\theta<\frac{\pi\varepsilon}{3}\Big\}.$$

Finally, define

$$v_\varepsilon(y,z):=\psi_\varepsilon(\phi^{-1}(|y|,|z|))\quad\text{for } x=(y,z)\in(\mathbb{R}^K\times\mathbb{R}^{N-K})\setminus\{0\},\qquad v_\varepsilon(0):=0.$$

Then $v_\varepsilon\in C_c^\infty(\Omega_\varepsilon)\cap H_K$, where $\Omega_\varepsilon:=\{(y,z)\in\mathbb{R}^K\times\mathbb{R}^{N-K}:(|y|,|z|)\in\phi(E_\varepsilon)\}$.

For future reference, we now compute the relevant integrals of v_ε. By means of spherical coordinates in $\mathbb{R}^K$ and $\mathbb{R}^{N-K}$, one has

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{v_\varepsilon^2}{|x|^\alpha}\,dx&=\int_{\Omega_\varepsilon}\frac{\psi_\varepsilon(\phi^{-1}(|y|,|z|))^2}{|x|^\alpha}\,dx\\&=\sigma_K\sigma_{N-K}\int_{\phi(E_\varepsilon)}\frac{\psi_\varepsilon(\phi^{-1}(s,t))^2}{(s^2+t^2)^{\alpha/2}}s^{K-1}t^{N-K-1}\,ds\,dt\\&=\sigma_K\sigma_{N-K}\int_{E_\varepsilon}\frac{\psi_\varepsilon(\rho,\theta)^2}{\rho^{\alpha-N+1}}H(\theta)\,d\rho\,d\theta\\&=\sigma_K\sigma_{N-K}\int_{E_\varepsilon}\frac{\psi(\rho^{1/\varepsilon},\theta/\varepsilon)^2}{\rho^{\alpha-N+1}}H(\theta)\,d\rho\,d\theta,\end{aligned}$$

where $H(\theta):=(\cos\theta)^{K-1}(\sin\theta)^{N-K-1}$, and by the change of variables

$$r=\rho^{1/\varepsilon},\quad \varphi=\frac{\theta}{\varepsilon},$$

one obtains

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{v_\varepsilon^2}{|x|^\alpha}\,dx&=\sigma_K\sigma_{N-K}\int_E\frac{\psi(r,\varphi)^2}{r^{(\alpha-N+1)\varepsilon}}H(\varepsilon\varphi)\varepsilon^2r^{\varepsilon-1}\,dr\,d\varphi\\&=\sigma_K\sigma_{N-K}\varepsilon^2\int_E\frac{\psi(r,\varphi)^2}{r^{(\alpha-N)\varepsilon+1}}H(\varepsilon\varphi)\,dr\,d\varphi.\end{aligned}\tag{4.1}$$

Similarly (recall that $F(0) = 0$),

$$\begin{aligned}\int_{\mathbb{R}^N} F(v_\varepsilon)\,dx &= \int_{\Omega_\varepsilon} F(\psi_\varepsilon(\phi^{-1}(|y|,|z|)))\,dx\\ &= \sigma_K\sigma_{N-K}\int_{\phi(E_\varepsilon)} F(\psi_\varepsilon(\phi^{-1}(s,t)))s^{K-1}t^{N-K-1}\,ds\,dt\\ &= \sigma_K\sigma_{N-K}\int_{E_\varepsilon} F(\psi_\varepsilon(\rho,\theta))\rho^{N-1}H(\theta)\,d\rho\,d\theta\\ &= \sigma_K\sigma_{N-K}\varepsilon^2\int_E F(\psi(r,\varphi))r^{N\varepsilon-1}H(\varepsilon\varphi)\,dr\,d\varphi \end{aligned}\tag{4.2}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2\,dx &= \int_{\Omega_\varepsilon} \big|\nabla\psi_\varepsilon(\phi^{-1}(|y|,|z|))\cdot J_{\phi^{-1}}(|y|,|z|)\big|^2\,dx\\ &= \sigma_K\sigma_{N-K}\int_{\phi(E_\varepsilon)} \big|\nabla\psi_\varepsilon(\phi^{-1}(s,t))\cdot J_{\phi^{-1}}(s,t)\big|^2 s^{K-1}t^{N-K-1}\,ds\,dt\\ &= \sigma_K\sigma_{N-K}\int_{E_\varepsilon} |\nabla\psi_\varepsilon(\rho,\theta)\cdot J_\phi^{-1}(\rho,\theta)|^2\rho^{N-1}H(\theta)\,d\rho\,d\theta\\ &= \sigma_K\sigma_{N-K}\int_{E_\varepsilon}\Big(\frac{\partial\psi_\varepsilon}{\partial\rho}(\rho,\theta)^2+\frac{1}{\rho^2}\frac{\partial\psi_\varepsilon}{\partial\theta}(\rho,\theta)^2\Big)\rho^{N-1}H(\theta)\,d\rho\,d\theta\\ &= \sigma_K\sigma_{N-K}\int_{E_\varepsilon}\frac{1}{\varepsilon^2}\Big(\rho^{2/\varepsilon}\frac{\partial\psi}{\partial r}\Big(\rho^{1/\varepsilon},\frac{\theta}{\varepsilon}\Big)^2+\frac{\partial\psi}{\partial\varphi}\Big(\rho^{1/\varepsilon},\frac{\theta}{\varepsilon}\Big)^2\Big)\rho^{N-3}H(\theta)\,d\rho\,d\theta\\ &= \sigma_K\sigma_{N-K}\int_E\Big(\psi_r(r,\varphi)^2+\frac{1}{r^2}\psi_\varphi(r,\varphi)^2\Big)r^{(N-2)\varepsilon+1}H(\varepsilon\varphi)\,dr\,d\varphi, \end{aligned}\tag{4.3}$$

where we set $\psi_r = \frac{\partial\psi}{\partial r}$ and $\psi_\varphi = \frac{\partial\psi}{\partial\varphi}$ for brevity.

Lemma 4.1. *The mapping $w_A := v_{A^{-1/2}} \in H_K$, $A > 1$, is such that*

$$\lim_{A\to+\infty}\frac{\|w_A\|_A^2}{\int_{\mathbb{R}^N} F(w_A)\,dx} = +\infty.$$

Proof. According to the previous computations, for $\varepsilon = A^{-1/2} < 1$, we have

$$\begin{aligned}\frac{\|v_\varepsilon\|_A^2}{\int_{\mathbb{R}^N} F(v_\varepsilon)\,dx} &= \frac{\int_{\mathbb{R}^N}|\nabla v_\varepsilon|^2\,dx + A\int_{\mathbb{R}^N}\frac{v_\varepsilon^2}{|x|^\alpha}\,dx}{\int_{\mathbb{R}^N} F(v_\varepsilon)\,dx}\\ &= \frac{\int_E\big((\psi_r^2+\frac{1}{r^2}\psi_\varphi^2)r^{(N-2)\varepsilon+1}+A\varepsilon^2\psi^2r^{(N-\alpha)\varepsilon-1}\big)H(\varepsilon\varphi)\,dr\,d\varphi}{\varepsilon^2\int_E F(\psi)r^{N\varepsilon-1}H(\varepsilon\varphi)\,dr\,d\varphi}\\ &= A\frac{\int_E\big((\psi_r^2+\frac{1}{r^2}\psi_\varphi^2)r^{(N-2)\varepsilon+1}+\psi^2r^{(N-\alpha)\varepsilon-1}\big)H(\varepsilon\varphi)\,dr\,d\varphi}{\int_E F(\psi)r^{N\varepsilon-1}H(\varepsilon\varphi)\,dr\,d\varphi}. \end{aligned}\tag{4.4}$$

In the integration set E, one has $\varepsilon\frac{\pi}{6} < \varepsilon\varphi < \varepsilon\frac{\pi}{3}$, and thus, for $\varepsilon > 0$ small enough (i.e., $A > 1$ large enough), we get that $\varepsilon\frac{\varphi}{2} < \sin\varepsilon\varphi < \varepsilon\varphi$ and $\frac{1}{2} < \cos\varepsilon\varphi < 1$. Hence, there exist two constants $\overline{C}_1, \overline{C}_2 > 0$ such that

$$\overline{C}_1\varepsilon^{N-K-1} < H(\varepsilon\varphi) < \overline{C}_2\varepsilon^{N-K-1}.$$

Similarly, since $\frac{1}{4} < r < \frac{3}{4}$ in E, all the terms $r^{(N-2)\varepsilon+1}$, $r^{(N-\alpha)\varepsilon-1}$ and $r^{N\varepsilon-1}$ are bounded, and bounded away from zero by positive constants independent of $\varepsilon \in (0,1)$ (i.e., of $A > 1$), say $\overline{C}_3$ and $\overline{C}_4$, respectively. Using these bounds in (4.4), we have

$$\frac{\|v_\varepsilon\|_A^2}{\int_{\mathbb{R}^N} F(v_\varepsilon)\,dx} \geq A\frac{\overline{C}_1 \int_E \big((\psi_r^2 + \frac{1}{r^2}\psi_\varphi^2) r^{(N-2)\varepsilon+1} + \psi^2 r^{(N-\alpha)\varepsilon-1}\big)\,dr\,d\varphi}{\overline{C}_2 \int_E F r^{N\varepsilon-1}\,dr\,d\varphi}$$
$$\geq A\frac{\overline{C}_1\overline{C}_4 \int_E (\psi_r^2 + \frac{1}{r^2}\psi_\varphi^2 + \psi^2)\,dr\,d\varphi}{\overline{C}_2\overline{C}_3 \int_E F(\psi)\,dr\,d\varphi}.$$

The last ratio is positive and independent of A, whence the claim follows. □

According to Lemma 4.1, we fix $A_0 > 1$ so that

$$\frac{\|w_A\|_A^2}{\int_{\mathbb{R}^N} F(w_A)\,dx} > 1 \quad \text{for every } A > A_0. \tag{4.5}$$

We now distinguish the cases $0 < \alpha < 2$ and $\alpha > 2$.

Proposition 4.2. *Assume* (f4) *and* $0 < \alpha < 2$. *Let* $A > A_0$ *and define* $\overline{u}_K \in H_K$ *by setting*

$$\overline{u}_K(x) := w_A\Big(\frac{x}{\lambda}\Big), \quad \text{with } \lambda := \frac{\|w_A\|_A^{2/\alpha}}{(\int_{\mathbb{R}^N} F(w_A)\,dx)^{1/\alpha}}.$$

Then $I(\overline{u}_K) < 0$ *and the corresponding mountain-pass level* (2.7) *satisfies*

$$c_{A,K} \leq C_1 A^{(K-1)/2+N(1/\alpha-1/2)},$$

where the constant $C_1 > 0$ *does not depend on* A.

Proof. Since $A > A_0$, one has $\lambda > 1$. Then an obvious change of variables yields

$$\begin{aligned} I(\overline{u}_K) &= \frac{\lambda^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla w_A|^2\,dx + \frac{\lambda^{N-\alpha}}{2}\int_{\mathbb{R}^N} \frac{A}{|x|^\alpha} w_A^2\,dx - \lambda^N \int_{\mathbb{R}^N} F(w_A)\,dx \\ &\leq \frac{\lambda^{N-\alpha}}{2}\Big(\int_{\mathbb{R}^N} |\nabla w_A|^2\,dx + \int_{\mathbb{R}^N} \frac{A}{|x|^\alpha} w_A^2\,dx\Big) - \lambda^N \int_{\mathbb{R}^N} F(w_A)\,dx \\ &= \frac{\lambda^N}{2}\Big(\lambda^{-\alpha}\|w_A\|_A^2 - 2\int_{\mathbb{R}^N} F(w_A)\,dx\Big) \\ &= -\frac{\lambda^N}{2}\int_{\mathbb{R}^N} F(w_A)\,dx < 0, \end{aligned}$$

where the last inequality follows from assumption (f2), since $w_A > 0$ almost everywhere. In order to estimate $c_{A,K}$, consider the straight path $\gamma(t) := t\overline{u}_K$, $t \in [0,1]$. Clearly, $c_{A,K} \leq \max_{t\in[0,1]} I(\gamma(t))$. Thanks to assumption (f4), which implies $F(ts) \geq t^\mu F(s)$ for all $s > 0$ and $t \in [0,1]$, we have

$$\begin{aligned} I(\gamma(t)) &= \frac{1}{2}t^2\|\overline{u}_K\|_A^2 - \int_{\mathbb{R}^N} F(t\overline{u}_K)\,dx \\ &\leq \frac{1}{2}t^2\|\overline{u}_K\|_A^2 - t^\mu \int_{\mathbb{R}^N} F(\overline{u}_K)\,dx \\ &= \frac{1}{2}t^2 a - t^\mu b, \end{aligned}$$

where we set $a := \|\overline{u}_K\|_A^2$ and $b := \int_{\mathbb{R}^N} F(\overline{u}_K)\,dx$ for brevity. The function $g(t) := \frac{1}{2}t^2 a - t^\mu b$ reaches its maximum in $t = (\frac{a}{b\mu})^{1/(\mu-2)}$, and so we get

$$I(\gamma(t)) \leq g\Big(\Big(\frac{a}{b\mu}\Big)^{1/(\mu-2)}\Big) = a\Big(\frac{a}{b\mu}\Big)^{2/(\mu-2)}\Big(\frac{1}{2} - \frac{1}{\mu}\Big).$$

Hence, setting $m := (\frac{1}{\mu})^{2/(\mu-2)}(\frac{1}{2}-\frac{1}{\mu})$ for brevity and recalling that $\lambda > 1$, we obtain

$$\begin{aligned} c_{A,K} &\le m \frac{\|\overline{u}_K\|_A^{2\mu/(\mu-2)}}{(\int_{\mathbb{R}^N} F(\overline{u}_K)\,dx)^{2/(\mu-2)}} \\ &= m \frac{(\lambda^{N-2}\int_{\mathbb{R}^N}|\nabla w_A|^2\,dx + \lambda^{N-\alpha}\int_{\mathbb{R}^N} A|x|^{-\alpha} w_A^2\,dx)^{\mu/(\mu-2)}}{(\lambda^N \int_{\mathbb{R}^N} F(w_A)\,dx)^{2/(\mu-2)}} \\ &\le m \frac{\lambda^{\mu(N-\alpha)/(\mu-2)}(\int_{\mathbb{R}^N}|\nabla w_A|^2\,dx + \int_{\mathbb{R}^N} A|x|^{-\alpha} w_A^2\,dx)^{\mu/(\mu-2)}}{\lambda^{2N/(\mu-2)}(\int_{\mathbb{R}^N} F(w_A)\,dx)^{2/(\mu-2)}} \\ &= m\lambda^{\frac{\mu(N-\alpha)-2N}{\mu-2}} \frac{\|w_A\|_A^{2\mu/(\mu-2)}}{(\int_{\mathbb{R}^N} F(w_A)\,dx)^{2/(\mu-2)}}. \end{aligned} \tag{4.6}$$

Inserting the definition of λ into (4.6), we get

$$c_{A,K} \le m \frac{\|w_A\|_A^{\frac{2\mu}{\mu-2}+\frac{2}{\alpha}\frac{\mu(N-\alpha)-2N}{\mu-2}}}{(\int_{\mathbb{R}^N} F(w_A)\,dx)^{\frac{2}{(\mu-2)}+\frac{1}{\alpha}\frac{\mu(N-\alpha)-2N}{\mu-2}}} = m \frac{\|w_A\|_A^{2N/\alpha}}{(\int_{\mathbb{R}^N} F(w_A)\,dx)^{(N-\alpha)/\alpha}}$$

and therefore, using computations (4.1)–(4.3) with $\varepsilon = A^{-1/2}$, we have

$$\overline{C}_1\varepsilon^{N-K-1} < H(\varepsilon\varphi) < \overline{C}_2\varepsilon^{N-K-1}$$

and

$$c_{A,K} \le m\sigma_K\sigma_{N-K} \frac{\big(\int_E((\psi_r^2 + \frac{1}{r^2}\psi_\varphi^2)r^{(N-2)\varepsilon+1} + \psi^2 r^{(N-\alpha)\varepsilon-1})H(\varepsilon\varphi)\,dr\,d\varphi\big)^{N/\alpha}}{\varepsilon^{2(N-\alpha)/\alpha}(\int_E F(\psi)r^{N\varepsilon-1}H(\varepsilon\varphi)\,dr\,d\varphi)^{(N-\alpha)/\alpha}}.$$

As in the proof of Lemma 4.1, we take four constants $\overline{C}_1,\dots,\overline{C}_4 > 0$ independent of A such that for every $(r,\varphi) \in E$, one has $\overline{C}_1\varepsilon^{N-K-1} < H(\varepsilon\varphi) < \overline{C}_2\varepsilon^{N-K-1}$ and the terms $r^{(N-2)\varepsilon+1}$, $r^{(N-\alpha)\varepsilon-1}$ and $r^{N\varepsilon-1}$ are bounded, and bounded away from zero by $\overline{C}_3$ and $\overline{C}_4$, respectively. Hence, we conclude

$$\begin{aligned} c_{A,K} &\le m\sigma_K\sigma_{N-K} \frac{\big(\overline{C}_2\overline{C}_3\int_E((\psi_r^2 + \frac{1}{r^2}\psi_\varphi^2) + \psi^2 r)\varepsilon^{N-K-1}\,dr\,d\varphi\big)^{N/\alpha}}{\varepsilon^{2(N-\alpha)/\alpha}(\overline{C}_1\overline{C}_4\int_E F(\psi)\varepsilon^{N-K-1}\,dr\,d\varphi)^{(N-\alpha)/\alpha}} \\ &= C\frac{\varepsilon^{(N-K-1)N/\alpha}\big(\int_E((\psi_r^2 + \frac{1}{r^2}\psi_\varphi^2) + \psi^2 r)\,dr\,d\varphi\big)^{N/\alpha}}{\varepsilon^{(N-K+1)(N-\alpha)/\alpha}(\int_E F(\psi)\,dr\,d\varphi)^{(N-\alpha)/\alpha}} \\ &= CA^{(K-1)/2+N(1/\alpha-1/2)}\frac{\big(\int_E((\psi_r^2 + \frac{1}{r^2}\psi_\varphi^2) + \psi^2 r)\,dr\,d\varphi\big)^{N/\alpha}}{(\int_E F(\psi)\,dr\,d\varphi)^{(N-\alpha)/\alpha}}, \end{aligned}$$

with the definition of the constant C being obvious. As the last ratio does not depend on A, the conclusion ensues. □

Proposition 4.3. *Assume* (f4) *and* $\alpha > 2$. *Let* $A > A_0$ *and define* $\overline{u} \in H_K$ *by setting*

$$\overline{u}_K(x) := w_A\Big(\frac{x}{\lambda}\Big), \quad \textit{with } \lambda := \frac{\|w_A\|_A}{(\int_{\mathbb{R}^N} F(w_A)\,dx)^{1/2}}.$$

Then $I(\overline{u}_K) < 0$ *and the corresponding mountain-pass level* (2.7) *satisfies*

$$c_{A,K} \le C_2 A^{(K-1)/2},$$

where the constant $C_2 > 0$ *does not depend on* A.

Proof. The proof is very similar to the one of Proposition 4.2, so we omit here some computational details. As $\alpha > 2$, we have

$$I(\overline{u}_K) \le \frac{\lambda^{N-2}}{2}\Big(\int_{\mathbb{R}^N}|\nabla w_A|^2\,dx + \int_{\mathbb{R}^N}\frac{A}{|x|^\alpha}w_A^2\,dx\Big) - \lambda^N\int_{\mathbb{R}^N}F(w_A)\,dx = -\frac{\lambda^N}{2}\int_{\mathbb{R}^N}F(w_A)\,dx < 0$$

and

$$\begin{aligned}\max_{t\in[0,1]} I(t\overline{u}_K) &\le m\frac{\|\overline{u}_K\|_A^{2\mu/(\mu-2)}}{(\int_{\mathbb{R}^N} F(\overline{u}_K)\,dx)^{2/(\mu-2)}}\\ &= m\frac{(\lambda^{N-2}\int_{\mathbb{R}^N}|\nabla w_A|^2\,dx + \lambda^{N-\alpha}\int_{\mathbb{R}^N} A|x|^{-\alpha}w_A^2\,dx)^{\mu/(\mu-2)}}{(\lambda^N\int_{\mathbb{R}^N} F(w_A)\,dx)^{2/(\mu-2)}}\\ &\le m\frac{\lambda^{\mu(N-2)/(\mu-2)}(\int_{\mathbb{R}^N}|\nabla w_A|^2\,dx + \int_{\mathbb{R}^N} A|x|^{-\alpha}w_A^2\,dx)^{\mu/(\mu-2)}}{\lambda^{2N/(\mu-2)}(\int_{\mathbb{R}^N} F(w_A)\,dx)^{2/(\mu-2)}}\\ &= m\lambda^{\frac{\mu(N-2)-2N}{\mu-2}}\frac{\|w_A\|_A^{2\mu/(\mu-2)}}{(\int_{\mathbb{R}^N} F(w_A)\,dx)^{2/(\mu-2)}}.\end{aligned}$$

Recalling the definition of $c_{A,K}$ and inserting the one of λ, we get

$$c_{A,K} \le m\frac{\|w_A\|_A^{\frac{2\mu}{\mu-2}+\frac{\mu(N-2)-2N}{\mu-2}}}{(\int_{\mathbb{R}^N} F(w_A)\,dx)^{\frac{2}{\mu-2}+\frac{1}{2}\frac{\mu(N-2)-2N}{\mu-2}}} = m\frac{\|w_A\|_A^N}{(\int_{\mathbb{R}^N} F(w_A)\,dx)^{(N-2)/2}},$$

and therefore, using computations (4.1)–(4.3) with $\varepsilon = A^{-1/2}$, we have

$$\begin{aligned}c_{A,K} &\le m\sigma_K\sigma_{N-K}\frac{(\int_E((\psi_r^2+\frac{1}{r^2}\psi_\varphi^2)r^{(N-2)\varepsilon+1}+\psi^2 r^{(N-\alpha)\varepsilon-1})H(\varepsilon\varphi)\,dr\,d\varphi)^{N/2}}{\varepsilon^{N-2}(\int_E F(\psi)r^{N\varepsilon-1}H(\varepsilon\varphi)\,dr\,d\varphi)^{(N-2)/2}}\\ &\le C\frac{(\int_E((\psi_r^2+\frac{1}{r^2}\psi_\varphi^2)+\psi^2 r)\varepsilon^{N-K-1}\,dr\,d\varphi)^{N/2}}{\varepsilon^{N-2}(\int_E F(\psi)\varepsilon^{N-K-1}\,dr\,d\varphi)^{(N-2)/2}}\\ &= CA^{(K-1)/2}\frac{(\int_E((\psi_r^2+\frac{1}{r^2}\psi_\varphi^2)+\psi^2 r)\,dr\,d\varphi)^{N/2}}{(\int_E F(\psi)\,dr\,d\varphi)^{(N-2)/2}},\end{aligned}$$

where $C > 0$ is a suitable constant independent of A. This concludes the proof. □

5 Proof of Theorem 1.1

This section is entirely devoted to the proof of Theorem 1.1, so we assume all the hypotheses of the theorem. The proof will be achieved through some lemmas.

Let K be any integer such that $2 \le K \le N-2$. Assume $A > A_K$ (where A_K is defined by (4.5)) and consider the mountain-pass level $c_{A,K}$ defined by (2.7), with $\overline{u}_K \in H_K$ given by Lemma 4.2 if $\alpha \in (\frac{2}{N-1}, 2)$, or Lemma 4.3 if $\alpha \in (2, 2N-2)$. We are going to show that $c_{A,K}$ is a critical level for the energy functional I defined in (2.2). To this end, we will make use of the sum space

$$L^{p_1} + L^{p_2} := \{u_1 + u_2 : u_1 \in L^{p_1}(\mathbb{R}^N),\ u_2 \in L^{p_2}(\mathbb{R}^N)\}.$$

We recall from [8] that such a space can be characterized as the set of measurable mappings $u\colon \mathbb{R}^N \to \mathbb{R}$ for which there exists a measurable set $E \subseteq \mathbb{R}^N$ such that $u \in L^{p_1}(E) \cap L^{p_2}(\mathbb{R}^N \setminus E)$ (see [8, Proposition 2.3]). This is a Banach space with respect to the norm

$$\|u\|_{L^{p_1}+L^{p_2}} := \inf_{u_1+u_2=u} \max\{\|u_1\|_{L^{p_1}(\mathbb{R}^N)}, \|u_2\|_{L^{p_2}(\mathbb{R}^N)}\}$$

(see [8, Corollary 2.11]), and the continuous embedding $L^p(\mathbb{R}^N) \hookrightarrow L^{p_1} + L^{p_2}$ holds for all $p \in [p_1, p_2]$ (see [8, Proposition 2.17]), in particular, for $p = 2^*$. Moreover, for every $u \in L^{p_1} + L^{p_2}$ and every $\varphi \in L^{p_1'}(\mathbb{R}^N) \cap L^{p_2'}(\mathbb{R}^N)$, one has

$$\int_{\mathbb{R}^N} |u\varphi|\,dx \le \|u\|_{L^{p_1}+L^{p_2}}(\|\varphi\|_{L^{p_1'}(\mathbb{R}^N)} + \|\varphi\|_{L^{p_2'}(\mathbb{R}^N)}), \tag{5.1}$$

where $p_i' = p_i/(p_i - 1)$ is the Hölder conjugate exponent of p_i (see [8, Lemma 2.9]).

Lemma 5.1. *$c_{A,K}$ is a critical level for the functional $I_{|H_K}$.*

Proof. Thanks to Lemma 2.4 (note that $I(\overline{u}_K) < 0$ implies $\|\overline{u}_K\|_A > R$), the claim follows from the mountain pass theorem [1] if we show that $I_{|H_K}$ satisfies the Palais–Smale condition. Using the compact embeddings of [2] and the results of [8] about Nemytskiĭ operators on $L^{p_1} + L^{p_2}$, this is a standard proof but we still give some details for the sake of completeness. Let $\{u_n\}$ be a sequence in H_K such that $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$ in the dual space of H_K. Then, recalling (2.2) and (2.3), we have

$$\frac{1}{2}\|u_n\|_A^2 - \int_{\mathbb{R}^N} F(u_n)\,dx = O(1) \quad \text{and} \quad \|u_n\|_A^2 - \int_{\mathbb{R}^N} f(u_n)u_n\,dx = o(1)\|u_n\|,$$

and so assumption (f1) implies

$$\frac{1}{2}\|u_n\|_A^2 + O(1) = \int_{\mathbb{R}^N} F(u_n)\,dx \le \frac{1}{\theta}\int_{\mathbb{R}^N} f(u_n)u_n\,dx = \frac{1}{\theta}\|u_n\|_A^2 + o(1)\|u_n\|.$$

This yields that $\{\|u_n\|_A\}$ is bounded, since $\theta > 2$. On the other hand, thanks to the fact that $p_1 < 2^* < p_2$, the space H_K is compactly embedded into $L^{p_1} + L^{p_2}$, since so is the subspace of $D^{1,2}(\mathbb{R}^N)$ made up of the mappings with the same symmetries of H_K (see [2, Theorem A.1]). Hence, there exists $u \in H_K$ such that, up to a subsequence, we have $u_n \rightharpoonup u$ in H_K and $u_n \to u$ in $L^{p_1} + L^{p_2}$. This implies that $\{f(u_n)\}$ is bounded in both $L^{p_1'}(\mathbb{R}^N)$ and $L^{p_2'}(\mathbb{R}^N)$, since assumption (f0) ensures that the operator $v \mapsto f(v)$ is continuous from $L^{p_1} + L^{p_2}$ into $L^{p_1'}(\mathbb{R}^N) \cap L^{p_2'}(\mathbb{R}^N)$ (see [8, Corollary 3.7]). Then, by (5.1), we get

$$\begin{aligned}\Big|\int_{\mathbb{R}^N} f(u_n)(u_n - u)\,dx\Big| &\le \int_{\mathbb{R}^N} |f(u_n)||u_n - u|\,dx \\ &\le \|u_n - u\|_{L^{p_1}+L^{p_2}}\big(\|f(u_n)\|_{L^{p_1'}(\mathbb{R}^N)} + \|f(u_n)\|_{L^{p_2'}(\mathbb{R}^N)}\big) \\ &\le (\text{const.})\|u_n - u\|_{L^{p_1}+L^{p_2}} = o(1),\end{aligned}$$

and therefore

$$\begin{aligned}\|u_n - u\|_A^2 &= (u_n, u_n - u)_A - (u, u_n - u)_A \\ &= I'(u_n)(u_n - u) + \int_{\mathbb{R}^N} f(u_n)(u_n - u)\,dx - (u, u_n - u)_A = o(1),\end{aligned}$$

where $(u, u_n - u)_A = o(1)$, since $u_n \rightharpoonup u$ in H_K, and $I'(u_n)(u_n - u) = o(1)$ because $I'(u_n) \to 0$ in the dual space of H_K and $\{u_n - u\}$ is bounded in H_K. This completes the proof. □

The next lemma clarifies why our separation of $c_{A,K}$ and m_A needs assumption (1.5) and the lower bound $\alpha > \frac{2}{N-1}$. Recall the definition (1.4) of $\nu = \nu_{N,\alpha,p_1,p_2}$.

Lemma 5.2. *For every $\alpha \in (\frac{2}{N-1}, 2N-2)$, $\alpha \neq 2$, we have $\nu \ge 1$.*

Proof. Assume $\frac{2}{N-1} < \alpha < 2$. Since $\alpha > \frac{2}{N-1}$, we have

$$2\frac{N-1}{\alpha} - 2N\Big(\frac{1}{\alpha} - \frac{1}{2}\Big) = N - \frac{2}{\alpha} > 1.$$

On the other hand, by easy computations, condition

$$2\frac{N-2}{2-\alpha}\frac{2^* - p_1}{p_1 - 2} - 2N\Big(\frac{1}{\alpha} - \frac{1}{2}\Big) > 1$$

turns out to be equivalent to the first inequality of assumption (1.5). This proves that

$$2\min\Big\{\frac{N-1}{\alpha}, \frac{N-2}{2-\alpha}\frac{2^* - p_1}{p_1 - 2}\Big\} - 2N\Big(\frac{1}{\alpha} - \frac{1}{2}\Big) > 1,$$

which means

$$\left\lceil 2\min\left\{\frac{N-1}{\alpha}, \frac{N-2}{2-\alpha}\frac{2^*-p_1}{p_1-2}\right\} - 2N\left(\frac{1}{\alpha}-\frac{1}{2}\right)\right\rceil \geq 2,$$

and thus $\nu \geq 1$. Similarly, if $2 < \alpha < 2N-2$, we readily have $\frac{2(N-1)}{\alpha} > 1$, and condition

$$2\frac{N-2}{\alpha-2}\frac{p_2-2^*}{p_2-2} > 1$$

turns out to be equivalent to the second inequality of (1.5). This proves again that $\nu \geq 1$. □

Proof of Theorem 1.1. On the one hand, the restriction $I_{|H_{\mathrm{r}}}$ has a critical point $u_{\mathrm{r}} \neq 0$ thanks to the results of [25], since (f0) ensures that one can find $p \in [p_1, p_2]$ such that $|f(u)| \leq (\mathrm{const.})u^{p-1}$ (cf. (2.1)) and (1.3) holds. On the other hand, according to Lemma 5.2, there are $\nu \geq 1$ integers K (precisely $K = 2, \dots, \nu+1$) such that

$$\frac{K-1}{2} + N\left(\frac{1}{\alpha}-\frac{1}{2}\right) < \min\left\{\frac{N-1}{\alpha}, \frac{N-2}{2-\alpha}\frac{2^*-p_1}{p_1-2}\right\} \quad \text{if } \frac{2}{N-1} < \alpha < 2$$

and

$$\frac{K-1}{2} < \min\left\{\frac{N-1}{\alpha}, \frac{N-2}{\alpha-2}\frac{p_2-2^*}{p_2-2}\right\} \quad \text{if } 2 < \alpha < 2N-2.$$

Let K be any of such integers. By Remark 3.3 and Propositions 3.2, 4.2 and 4.3, there exists $A_* > A_K$ such that

$$c_{A,K} < m_A \quad \text{for every } A > A_*. \tag{5.2}$$

Then, by Lemma 5.1, there exists $u_K \in H_K$ such that $I(u_K) = c_{A,K}$ and $I'_{|H_K}(u_K) = 0$, where $u_K \neq 0$, since $c_{A,K} > 0$ and $I(0) = 0$. Both u_{r} and u_K are also critical points for the functional $I\colon H^1_\alpha \to \mathbb{R}$, by the Palais principle of symmetric criticality [23]. Moreover, it easy to check that they are nonnegative: test $I'(u_K)$ with the negative part $u_K^- \in H^1_\alpha$ of u_K and use the fact that $f(s) = 0$ for $s < 0$ to get $I'(u_K)u_K^- = -\|u_K^-\|_A^2 = 0$; the same applies for u_{r}. Therefore, u_{r} and u_K are weak solutions to problem (1.1). Finally, u_K is not radial, because otherwise Lemma 2.3 would imply $c_{A,K} = I(u_K) \geq m_A$, which is false by (5.2). This also implies $u_{K_1} \neq u_{K_2}$ for $K_1 \neq K_2$, thanks to Lemma 2.1. □

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