Research Article

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On the Cauchy problem of a degenerate parabolic-hyperbolic PDE with Lévy noise

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Abstract: In this article, we deal with the stochastic perturbation of degenerate parabolic partial differential equations (PDEs). The particular emphasis is on analyzing the effects of a multiplicative Lévy noise on such problems and on establishing a well-posedness theory by developing a suitable weak entropy solution framework. The proof of the existence of a solution is based on the vanishing viscosity technique. The uniqueness of the solution is settled by interpreting Kruzhkov's doubling technique in the presence of a noise.

Keywords: Stochastic PDEs, Lévy noise, degenerate parabolic equations, entropy solutions, Young measures

MSC 2010: 35K65 - 60H15

1 Introduction

Stochastic partial differential equations (SPDEs) often result from the efforts to model complex physical phenomena where uncertainty/randomness is inherent. A brief survey of relevant literature reveals the use of SPDEs in a wide variety of studies in areas that include physics, biology, engineering and finance. As examples we mention the studies of neural dynamics, the spread of infectious disease (a tracer or a passive population in a flow subject to possibly random external forces), Navier–Stokes-type flow under random forces, ferromagnetism under random influences (stochastic Landau–Lifschitz–Gilbert equation) and the modeling of *forward rate curve* in finance. A popular way to account for the randomness has been to add some *noise* to the deterministic models and, most often, the noise is assumed to be Gaussian/Brownian. However, the data collected through surveys/experiments often exhibit properties such as heavy-tailedness, which can not be adequately explained by a Brownian noise. This inadequacy has lately prompted a lot of interest in models with Lévy-type noise, which necessitates the development of a well-rounded mathematical theory for SPDEs driven by such type of noise. In this article, we embark on the well-posedness study of a class of nonlinear and degenerate parabolic SPDEs with a Lévy noise.

Let $(\Omega, P, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0})$ be a filtered probability space satisfying the usual hypothesis, i.e., $\{\mathcal{F}_t\}_{t\geq 0}$ is a right-continuous filtration such that $\mathcal{F}_0$ contains all the P-null subsets of $(\Omega, \mathcal{F})$. In addition, let $(E, \mathcal{E}, m)$ be a σ-finite measure space and let $N(dt, dz)$ be a Poisson random measure on $(E, \mathcal{E})$ with intensity measure $m(dz)$ with respect to the same stochastic basis. The existence and construction of such a general notion of Poisson random measure with a given intensity measure are detailed in [19]. We are interested in the Cauchy

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problem for a nonlinear degenerate parabolic stochastic PDE of the following type:

$$du(t,x) - \Delta\phi(u(t,x))\,dt - \operatorname{div}_x f(u(t,x))\,dt = \int_E \eta(x,u(t,x);z)\,\tilde{N}(dz,dt), \quad (t,x) \in \Pi_T, \tag{1.1}$$

with the initial condition

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^d,$$

where $\Pi_T = [0,T) \times \mathbb{R}^d$ with $T > 0$ fixed, $u(t,x)$ is the unknown random scalar-valued function, $F : \mathbb{R} \to \mathbb{R}^d$ is a given flux function and $\tilde{N}(dz,dt) = N(dz,dt) - m(dz)\,dt$ is the compensated Poisson random measure. Furthermore, $(x,u,z) \mapsto \eta(x,u;z)$ is a real-valued function defined on the domain $\mathbb{R}^d \times \mathbb{R} \times E$, and $\phi : \mathbb{R} \to \mathbb{R}$ is a given non-decreasing Lipschitz continuous function. The stochastic integral on the right-hand side of (1.1) is defined in the Lévy–Itô sense.

Remark 1.1. Since ϕ is a real-valued non-decreasing and Lipschitz continuous function, we know that the set $A = \{r \in \mathbb{R} : \phi'(r) = 0\}$ is not empty in general and hence the problem is called degenerate. Furthermore, A is not negligible either and the problem is strongly degenerate in the sense of [9].

Remark 1.2. The analysis of this paper remains valid if the noise in the right-hand side of (1.1) is of jump-diffusion type. In other words, the same analysis holds if we add the term $\sigma(x,u)\,dW_t$ in the right-hand side of (1.1), where W_t denotes a cylindrical Brownian motion. Moreover, we will carry out our analysis under the structural assumption that $E = \mathcal{O} \times \mathbb{R}^*$, where $\mathcal{O}$ is a subset of the Euclidean space. The measure m on E is defined as the product measure $\lambda \times \mu$, where λ is a Radon measure on $\mathcal{O}$ and μ is a so-called Lévy measure on $\mathbb{R}^*$. In such a case, the noise in the right-hand side would be called an impulsive white noise with the jump position intensity λ and the jump size intensity μ. We refer to [19] for more on Lévy sheet and related impulsive white noises.

Remark 1.3. In so far as the techniques are concerned, one anticipates to follow closely the methodology used in analyzing a version of (1.1) with a Brownian noise in [3]. However, there will be additional difficulties specific to the discontinuous character of the Lévy noise. Note that the solution of problem (1.1) is interpreted in the entropy sense as in [3, 9]. The entropy inequalities will have an Itô–Lévy correction term, and they will have non-localities resulting from the jump nature of the noise term in the right-hand side of (1.1). This non-locality tends to derail the stability analysis, and proving the uniqueness becomes a trickier affair. We are able to manage this added difficulty here and establish the uniqueness, but under a slightly restrictive assumption (iv) on the jump vector η (see Section 2.1).

Equation (1.1) becomes a multi-dimensional deterministic degenerate parabolic-hyperbolic equation if $\eta = 0$. It is well-documented in the literature that the *solution* has to be interpreted in the weak sense and one needs an entropy formulation to prove the well-posedness of the problem. We refer to [1, 4, 9, 11, 12, 20] and the references therein for more on the entropy solution theory for deterministic degenerate parabolic-hyperbolic equations.

1.1 Studies on degenerate parabolic-hyperbolic equations with a Brownian noise

The study of stochastic degenerate parabolic-hyperbolic equations has so far been limited to mainly equations with a Brownian noise. In particular, hyperbolic conservation laws with a Brownian noise are examples of such problems that have attracted the attention of many. The first documented development in this direction is [17], where Holden and Risebro established a result of existence of a path-wise weak solution (possibly non-unique) for one-dimensional balance laws via the splitting method. In a separate development, Khanin, Mazel and Sinai [15] published their celebrated work that described some statistical properties of Burgers equations with a noise. Kim [18] extended the Kruzhkov entropy formulation and established the well-posedness of the Cauchy problem for one-dimensional balance laws driven by an additive Brownian noise. The multi-dimensional analogue on bounded domains was studied by Vallet and Wittbold [21]. They established a result of well-posedness of the entropy solution with the theory of Young measures.

This approach is not applicable to the multiplicative noise case. This case was studied by many authors [2, 10, 14, 16]. In [16], Feng and Nualart found a way to recover the necessary information in the form of the *strong entropy* condition from the parabolic regularization, and they established a result of uniqueness of the strong entropy solution in the L^p-framework for the multi-dimensional case, but the existence of a solution was established only in the one-dimensional case. We also add here that Feng and Nualart [16] used an entropy formulation which is strong in time but weak in space, which in our view may give rise to problems when the solutions are not shown to have continuous sample paths. We refer to [7], where a few technical questions are raised on the strong in time formulation and remedial measures have been proposed. In [14], Debussche and Vovelle obtained the existence of a solution via a kinetic formulation, and Chen, Ding and Karlsen [10] used the BV solution framework. Bauzet, Vallet and Wittbold [2] established a result of well-posedness via the Young measure approach. The well-posedness of the problem to a multi-dimensional degenerate parabolic-hyperbolic stochastic problem has been studied by Debussche, Hofmanová and Vovelle [13] and Bauzet, Vallet and Wittbold [3]. The former adapted the notion of kinetic formulation and developed a well-posedness theory, while the latter revisited [1, 9, 11] and established the well-posedness of the entropy solution via the Young measure theory.

1.2 Relevant studies on problems with a Lévy noise

Over the last decade, there have been many contributions on the larger area of stochastic partial differential equations that are driven by a Lévy noise. A worthy reference on this subject is [19]. However, very little is available on the specific problem of degenerate parabolic problems with a Lévy noise such as (1.1). This article marks an important step in our quest to develop a comprehensive theory of stochastic degenerate parabolic equations that are driven by jump-diffusions. The relevant results in this context are made available recently and they are on conservation laws that are perturbed by a Lévy noise. In recent articles, Biswas, Karlsen and Majee [5] and Biswas, Koley and Majee [6] established the well-posedness of the entropy solution for multi-dimensional conservation laws with a Poisson noise via the Young measure approach. In [6], Biswas et al. developed a continuous dependence theory on nonlinearities within the BV solution setting.

Stochastic degenerate parabolic-hyperbolic equations are one of the most important classes of nonlinear stochastic PDEs. Nonlinearity and degeneracy are two main features on these equations and yield several striking phenomena. Therefore, this requires new mathematical ideas, approaches, and theories. It is well known that due to the presence of nonlinear flux terms, solutions to (1.1) are not smooth even for smooth initial data $u_0(x)$. Therefore, a solution must be interpreted in the weak sense. Before introducing the concept of weak solutions, we first recall the notion of predictable σ-field. By a predictable σ-field on $[0, T] \times \Omega$, denoted by $\mathcal{P}_T$, we mean the σ-field generated by the sets of the form $\{0\} \times A$ and $(s, t] \times B$ for any $A \in \mathcal{F}_0$, $B \in \mathcal{F}_s$, $0 < s, t \leq T$. The notion of a stochastic weak solution is defined as follows.

Definition 1.4 (Stochastic weak solution). We say that an $L^2(\mathbb{R}^d)$-valued $\{\mathcal{F}_t : t \geq 0\}$-predictable stochastic process $u(t) = u(t, x)$ is a weak solution to problem (1.1) provided the following conditions are satisfied:
(i) $u \in L^2(\Omega \times \Pi_T)$ and $\phi(u) \in L^2((0, T) \times \Omega; H^1(\mathbb{R}^d))$.
(ii) It holds

$$\frac{\partial}{\partial t}\Big[u - \int_0^t \int_E \eta(x, u(s, \cdot); z)\, \tilde{N}(dz, ds)\Big] \in L^2((0, T) \times \Omega; H^{-1}(\mathbb{R}^d))$$

in the sense of distribution.
(iii) For almost every $t \in [0, T]$ and P-a.s., the following variational formulation holds:

$$0 = \Big\langle \frac{\partial}{\partial t}\Big[u - \int_0^t \int_E \eta(x, u(s, \cdot); z)\, \tilde{N}(dz, ds)\Big], v \Big\rangle_{H^{-1}(\mathbb{R}^d), H^1(\mathbb{R}^d)} + \int_{\mathbb{R}^d} \{\nabla\phi(u(t, x)) + f(u(t, x))\}.\nabla v \, dx$$

for any $v \in H^1(\mathbb{R}^d)$.

However, it is well known that weak solutions may be discontinuous and are not uniquely determined by their initial data. Consequently, an admissibility criterion for the so-called *entropy solution* (see Section 2 for the definition of an entropy solution) must be imposed to single out the physically correct solution.

1.3 Goal of the study and outline of the paper

The case of a strongly degenerate stochastic problem driven by a Brownian noise is studied by Bauzet et al. [3]. In this article, drawing primary motivation from [3, 5, 9], we propose to establish a result of well-posedness of the entropy solution to a degenerate Cauchy problem (1.1) by using the vanishing viscosity method along with few a priori bounds.

The rest of the paper is organized as follows: We state the assumptions, details of the technical framework and the main results in Section 2. Section 3 is devoted to prove the existence of a weak solution for the viscous problem via an implicit time discretization scheme and to derive some a priori estimates for the sequence of viscous solutions. In Section 4, we establish first the uniqueness of the limit of the viscous solutions when the viscosity parameter goes to zero via the Young measure theory, and then we establish the existence of an entropy solution. The uniqueness of the entropy solution is presented in Section 5.

2 Technical framework and statements of the main results

Here and in the sequel, we denote by $N^2_\omega(0, T, L^2(\mathbb{R}^d))$ the space of predictable $L^2(\mathbb{R}^d)$-valued processes u such that

$$\mathbb{E}\Big[\int_{\Pi_T} |u|^2\,dt\,dx\Big] < +\infty.$$

Moreover, we use the letter C to denote various generic constants. There are situations where the constant may change from line to line, but the notation is kept unchanged so long as it does not impact the primary implication. We denote by c_ϕ and c_f the Lipschitz constants of ϕ and f, respectively. Also, we use $\langle\,\cdot\,,\cdot\,\rangle$ to denote the pairing between $H^1(\mathbb{R}^d)$ and $H^{-1}(\mathbb{R}^d)$.

2.1 Entropy inequalities

We begin this subsection with a formal derivation of the entropy inequalities à la Kruzhkov. Remember that we need to replace the traditional chain rule for deterministic calculus by the Itô–Lévy chain rule.

Definition 2.1 (Entropy flux triple). A triplet (β, ζ, ν) is called an entropy flux triple if $\beta \in C^2(\mathbb{R})$ is Lipschitz continuous and $\beta \geq 0$, $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_d) : \mathbb{R} \mapsto \mathbb{R}^d$ is a vector-valued function and $\nu : \mathbb{R} \mapsto \mathbb{R}$ is a scalar-valued function such that

$$\zeta'(r) = \beta'(r) f'(r) \quad \text{and} \quad \nu'(r) = \beta'(r)\phi'(r).$$

An entropy flux triple (β, ζ, ν) is called convex if $\beta''(s) \geq 0$.

For a small positive number $\varepsilon > 0$, assume that the parabolic perturbation

$$du(t,x) - \Delta\phi(u(t,x))\,dt = \operatorname{div}_x f(u(t,x))\,dt + \int_E \eta(x, u(t,x); z)\,\tilde{N}(dz, dt) + \varepsilon\Delta u(t,x)\,dt, \quad (t,x) \in \Pi_T$$

of (1.1) has a unique weak solution $u_\varepsilon(t,x)$. Note that this weak solution u_ε is in $L^2((0,T)\times\Omega; H^1(\mathbb{R}^d))$. Moreover, for the time being, we assume that it satisfies the initial condition in the sense of (A.2). This enables one to derive a weak version of the Itô–Lévy formula for the solutions to (1.1) as detailed in Theorem A.1 in Appendix A.

Let (β, ζ, ν) be an entropy flux triple. Given a non-negative test function $\psi \in C_c^{1,2}([0,\infty) \times \mathbb{R}^d)$, we apply the generalized version of the Itô–Lévy formula (cf. Appendix A) to have, for almost every $T > 0$,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \beta(u_\varepsilon(T,x))\psi(T,x)\,dx - \int_{\mathbb{R}^d} \beta(u_\varepsilon(0,x))\psi(0,x)\,dx \\
&\quad = \int_{\Pi_T} \beta(u_\varepsilon(t,x))\partial_t\psi(t,x)\,dx\,dt - \int_{\Pi_T} \nabla\psi(t,x)\cdot\zeta(u_\varepsilon(t,x))\,dx\,dt \\
&\qquad + \int_{\Pi_T}\int_E\int_0^1 \eta(x,u_\varepsilon(t,x);z)\beta'\big(u_\varepsilon(t,x)+\theta\eta(x,u_\varepsilon(t,x);z)\big)\psi(t,x)\,d\theta\,\tilde N(dz,dt)\,dx \\
&\qquad + \int_{\Pi_T}\int_E\int_0^1 (1-\theta)\eta^2(x,u_\varepsilon(t,x);z)\beta''\big(u_\varepsilon(t,x)+\theta\eta(x,u_\varepsilon(t,x);z)\big)\psi(t,x)\,d\theta\,m(dz)\,dx\,dt \\
&\qquad - \int_{\Pi_T} \big(\varepsilon\nabla_x\psi(t,x).\nabla_x\beta(u_\varepsilon(t,x)) + \varepsilon\beta''(u_\varepsilon(t,x))|\nabla_x u_\varepsilon(t,x)|^2\psi(t,x)\big)\,dx\,dt \\
&\qquad - \int_{\Pi_T} \phi'(u_\varepsilon(t,x))\beta''(u_\varepsilon(t,x))|\nabla u_\varepsilon(t,x)|^2\psi(t,x)\,dx\,dt + \int_{\Pi_T} \nu(u_\varepsilon(t,x))\Delta\psi(t,x)\,dx\,dt.
\end{aligned}
$$

Let G be the associated Kirchhoff function of ϕ, given by

$$G(x) = \int_0^x \sqrt{\phi'(r)}\,dr.$$

A simple calculation shows that

$$|\nabla G(u_\varepsilon(t,x))|^2 = \phi'(u_\varepsilon(t,x))|\nabla u_\varepsilon(t,x)|^2.$$

Since β and ψ are non-negative functions, we obtain

$$
\begin{aligned}
0 \le &\int_{\mathbb{R}^d} \beta(u_\varepsilon(0,x))\psi(0,x)\,dx + \int_{\Pi_T} \{\beta(u_\varepsilon(t,x))\partial_t\psi(t,x) - \nabla\psi(t,x)\cdot\zeta(u_\varepsilon(t,x))\}\,dx\,dt \\
&- \int_{\Pi_T} \beta''(u_\varepsilon(t,x))|\nabla G(u_\varepsilon(t,x))|^2\psi(t,x)\,dx\,dt + \int_{\Pi_T} \nu(u_\varepsilon(t,x))\Delta\psi(t,x)\,dx\,dt + \mathcal{O}(\varepsilon) \\
&+ \int_{\Pi_T}\int_E\int_0^1 \eta(x,u_\varepsilon(t,x);z)\beta'\big(u_\varepsilon(t,x)+\theta\eta(x,u_\varepsilon(t,x);z)\big)\psi(t,x)\,d\theta\,\tilde N(dz,dt)\,dx \\
&+ \int_{\Pi_T}\int_E\int_0^1 (1-\theta)\eta^2(x,u_\varepsilon(t,x);z)\beta''\big(u_\varepsilon(t,x)+\theta\eta(x,u_\varepsilon(t,x);z)\big)\psi(t,x)\,d\theta\,m(dz)\,dx\,dt.
\end{aligned}
$$

Clearly, the above inequality is stable under the limit $\varepsilon \to 0$ if the family $\{u_\varepsilon\}_{\varepsilon>0}$ has L^p_{loc}-type stability. Just as the deterministic equations, the above inequality provides us with the entropy condition. We now formally define the entropy solution.

Definition 2.2 (Stochastic entropy solution). A stochastic process $u \in N^2_\omega(0,T,L^2(\mathbb{R}^d))$ is called a stochastic entropy solution of (1.1) if the following conditions hold:

(i) For each $T > 0$,

$$G(u) \in L^2((0,T)\times\Omega; H^1(\mathbb{R}^d)) \quad \text{and} \quad \sup_{0\le t\le T} \mathbb{E}\big[\|u(t)\|_2^2\big] < \infty.$$

(ii) Given a non-negative test function $\psi \in C_c^{1,2}([0,\infty) \times \mathbb{R}^d)$ and a convex entropy flux triple (β, ζ, ν), the following inequality holds:

$$\begin{aligned}
&\int_{\Pi_T} \{\beta(u(t,x))\partial_t\psi(t,x) + \nu(u(t,x))\Delta\psi(t,x) - \nabla\psi(t,x)\cdot\zeta(u(t,x))\}\,dx\,dt \\
&\qquad + \int_{\Pi_T}\int_E\int_0^1 \eta(x,u(t,x);z)\beta'(u(t,x)+\theta\eta(x,u(t,x);z))\psi(t,x)\,d\theta\,\tilde{N}(dz,dt)\,dx \\
&\qquad + \int_{\Pi_T}\int_E\int_0^1 (1-\theta)\eta^2(x,u(t,x);z)\beta''(u(t,x)+\theta\eta(x,u(t,x);z))\psi(t,x)\,d\theta\,m(dz)\,dx\,dt \\
&\quad \geq \int_{\Pi_T} \beta''(u(t,x))|\nabla G(u(t,x))|^2\psi(t,x)\,dx\,dt - \int_{\mathbb{R}^d}\beta(u_0(x))\psi(0,x)\,dx, \quad P\text{-a.s.}
\end{aligned} \tag{2.1}$$

Remark 2.3. We point out that, by a classical separability argument, it is possible to choose a subset of Ω of P-full measure such that (2.1) holds on that subset for every admissible entropy triplet and test function.

The primary aim of this paper is to establish a result of existence and uniqueness of an entropy solution for the Cauchy problem (1.1) in accordance with Definition 2.2, and we do so under the following assumptions:

(i) $\phi : \mathbb{R} \to \mathbb{R}$ is a non-decreasing Lipschitz continuous function with $\phi(0) = 0$. Moreover, if η is not a constant function with respect to the space variable x, then $t \mapsto \sqrt{\phi'(t)}$ has a modulus of continuity ω_ϕ such that
$$\frac{\omega_\phi(r)}{r^{2/3}} \to 0 \quad \text{as } r \to 0.$$
(ii) $f = (f_1, f_2, \dots, f_d) : \mathbb{R} \to \mathbb{R}^d$ is a Lipschitz continuous function with $f_k(0) = 0$ for all $1 \leq k \leq d$.
(iii) The space E is of the form $\mathcal{O} \times \mathbb{R}^*$, and the Borel measure m on E has the form $\lambda \times \mu$, where λ is a Radon measure on $\mathcal{O}$ and μ is a so-called one-dimensional Lévy measure.
(iv) There exist positive constants $K > 0$, $\lambda^* \in (0,1)$ and $h_1(z) \in L^2(E,m)$ with $0 \leq h_1(z) \leq 1$ such that
$$|\eta(x,u;z) - \eta(y,v;z)| \leq (\lambda^*|u-v| + K|x-y|)h_1(z) \quad \text{for all } x,y \in \mathbb{R}^d,\ u,v \in \mathbb{R},\ z \in E.$$
(v) There exist a non-negative function $g \in L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $h_2(z) \in L^2(E,m)$ such that for all
$$(x,u,z) \in \mathbb{R}^d \times \mathbb{R} \times E$$
one has
$$|\eta(x,u;z)| \leq g(x)(1+|u|)h_2(z).$$

The above definition does not say anything explicitly about the way the entropy solution satisfies the initial condition. However, the initial condition is satisfied in a certain weak sense. Here we state the lemma whose proof follows the same line of argument as the one of [5, Lemma 2.3].

Lemma 2.4. *Any entropy solution $u(t,\cdot)$ of (1.1) satisfies the initial condition in the following sense: for every non-negative test function $\psi \in C_c^2(\mathbb{R}^d)$ such that* $\operatorname{supp}(\psi) = K$,
$$\lim_{h\to 0} \mathbb{E}\Big[\frac{1}{h}\int_0^h\int_K |u(t,x) - u_0(x)|\psi(x)\,dx\,dt\Big] = 0.$$

Next, we describe a special class of entropy functions that plays an important role in the sequel. Let $\beta : \mathbb{R} \to \mathbb{R}$ be a C^∞ and Lipschitz continuous function satisfying
$$\beta(0) = 0, \quad \beta(-r) = \beta(r), \quad \beta'' \geq 0$$
and
$$\beta'(r) \begin{cases} = -1 & \text{if } r \leq -1, \\ \in [-1,1] & \text{if } |r| < 1, \\ = +1 & \text{if } r \geq 1. \end{cases}$$

For any $\vartheta > 0$, define $\beta_\vartheta : \mathbb{R} \to \mathbb{R}$ by

$$\beta_\vartheta(r) = \vartheta\beta\Big(\frac{r}{\vartheta}\Big).$$

Then

$$|r| - M_1\vartheta \le \beta_\vartheta(r) \le |r| \quad \text{and} \quad |\beta_\vartheta''(r)| \le \frac{M_2}{\vartheta}\mathbf{1}_{|r|\le\vartheta},$$

where

$$M_1 = \sup_{|r|\le 1}\big||r| - \beta(r)\big|, \quad M_2 = \sup_{|r|\le 1}|\beta''(r)|.$$

By simply dropping ϑ, for $\beta = \beta_\vartheta$ we define

$$\phi^\beta(a,b) = \int_b^a \beta'(\sigma - b)\phi'(\sigma)\,d(\sigma), \qquad F_k^\beta(a,b) = \int_b^a \beta'(\sigma - b) f_k'(\sigma)\,d(\sigma),$$
$$F_k(a,b) = \operatorname{sign}(a-b)(f_k(a) - f_k(b)), \quad F(a,b) = (F_1(a,b), F_2(a,b), \dots, F_d(a,b)).$$

We conclude this section by stating the main results of this paper.

Theorem 2.5 (Existence). *Let assumptions* (i)–(v) *be true and that the $L^2(\mathbb{R}^d)$-valued $\mathcal{F}_0$-measurable random variable u_0 satisfies $\mathbb{E}[\|u_0\|_2^2] < \infty$. Then there exists an entropy solution of* (1.1) *in the sense of Definition 2.2.*

Theorem 2.6 (Uniqueness). *Let assumptions* (i)–(v) *be true and that the $L^2(\mathbb{R}^d)$-valued $\mathcal{F}_0$-measurable random variable u_0 satisfies $\mathbb{E}[\|u_0\|_2^2] < \infty$. Then the entropy solution of* (1.1) *is unique.*

Remark 2.7. In addition, if u_0 is in $L^p(\mathbb{R}^d)$ for $p \in [2,\infty)$, then it can be concluded that

$$u \in L^\infty(0,T,L^p(\Omega\times\mathbb{R}^d)).$$

Furthermore, if $u_0 \in L^\infty$ and there is $M > 0$ such that $\eta(x,u;z) = 0$ for $|u| > M$ and

$$M_1 = \sup_{x,|u|\le M,z} |\eta(x,u;z)| < \infty,$$

then $|u(t,x)| \le \max\{M + M_1, \|u_0\|_\infty\}$ for almost every $(t,x,\omega) \in \Pi_T \times \Omega$. We sketch a justification of this claim in Section 4.

3 Existence of a weak solution to the viscous problem

Just as in the case of the deterministic problem, here we also study the problem regularized by adding a small diffusion operator and derive some a priori bounds. Due to the nonlinear function ϕ and related degeneracy, one cannot expect a classical solution and instead seeks a weak solution.

3.1 Existence of a weak solution to the viscous problem

For a small parameter $\varepsilon > 0$, we consider the viscous approximation of (1.1) as

$$\begin{aligned} &du(t,x) - \Delta\phi(u(t,x))\,dt \\ &\quad = \operatorname{div}_x f(u(t,x))\,dt + \int_E \eta(x,u(t,x);z)\,\tilde{N}(dz,dt) + \varepsilon\Delta u(t,x)\,dt, \quad t > 0,\ x \in \mathbb{R}^d. \end{aligned} \tag{3.1}$$

In this subsection, we establish the existence of a weak solution for problem (3.1). To do this, we use an implicit time discretization scheme. Let $\Delta t = \frac{T}{N}$ for some positive integer $N \ge 1$. Further, set $t_n = n\Delta t$ for $n = 0, 1, 2, \dots, N$.

Define

$$\mathcal{N} = L^2(\Omega; H^1(\mathbb{R}^d)), \quad \mathcal{N}_n = \{\text{the } \mathcal{F}_{n\Delta t} \text{ measurable elements of } \mathcal{N}\},$$
$$\mathcal{H} = L^2(\Omega; L^2(\mathbb{R}^d)), \quad \mathcal{H}_n = \{\text{the } \mathcal{F}_{n\Delta t} \text{ measurable elements of } \mathcal{H}\}.$$

Proposition 3.1. *Assume that Δt is small. For any given $u_n \in \mathcal{H}_n$, there exists a unique $u_{n+1} \in \mathcal{N}_{n+1}$ with $\phi(u_{n+1}) \in \mathcal{N}_{n+1}$ such that P-a.s. for any $v \in H^1(\mathbb{R}^d)$ the following variational formula holds:*

$$\int_{\mathbb{R}^d} \big((u_{n+1} - u_n)v + \Delta t\{\nabla\phi(u_{n+1}) + \varepsilon\nabla u_{n+1} + f(u_{n+1})\} \cdot \nabla v\big)\, dx = \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \int_E \eta(x, u_n; z) v\, \tilde{N}(dz, ds)\, dx. \tag{3.2}$$

Before proving this proposition, we first state a key deterministic lemma, related to the weak solution of parabolic equations. We have the following lemma, a proof of which can be found in [8, p. 19].

Lemma 3.2. *Assume that Δt is small and $X \in L^2(\mathbb{R}^d)$. Then, for a fixed parameter $\varepsilon > 0$, the following holds:*

(i) *There exists a unique $u \in H^1(\mathbb{R}^d)$ with $\phi(u) \in H^1(\mathbb{R}^d)$ such that, for any $v \in H^1(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} \big(uv + \Delta t\{\nabla\phi(u) + \varepsilon\nabla u + f(u)\} \cdot \nabla v\big)\, dx = \int_{\mathbb{R}^d} Xv\, dx.$$

(ii) *There exists a constant $C = C(\Delta t) > 0$ such that the following a priori estimate holds:*

$$\|u\|^2_{L^2(\mathbb{R}^d)} + \|\phi(u)\|^2_{H^1(\mathbb{R}^d)} + \varepsilon\|\nabla u\|^2_{L^2(\mathbb{R}^d)} \le C\|X\|^2_{L^2(\mathbb{R}^d)}. \tag{3.3}$$

(iii) *The map $\Theta : X \in L^2(\mathbb{R}^d) \mapsto (u, \phi(u)) \in H^1(\mathbb{R}^d)^2$ is continuous.*

Proof of Proposition 3.1. Let $u_n \in \mathcal{N}_n$. Take

$$X = u_n + \int_{t_n}^{t_{n+1}} \int_E \eta(x, u_n; z)\, \tilde{N}(dz, ds).$$

Then, by assumption (v), we obtain

$$\mathbb{E}\big[\|X\|^2_{L^2(\mathbb{R}^d)}\big] \le \|u_n\|^2_{\mathcal{H}} + C\Delta t\big(\|g\|^2_{L^2(\mathbb{R}^d)} + \|u_n\|^2_{\mathcal{H}}\big).$$

This shows that $X \in L^2(\mathbb{R}^d)$ almost surely. Therefore, one can use Lemma 3.2 and conclude that for almost surely all $\omega \in \Omega$ there exists a unique $u(\omega)$ satisfying the variational equality (3.2). Moreover, by construction $X \in \mathcal{H}_{n+1}$. Thus, due to the continuity of Θ for the $\mathcal{F}_{(n+1)\Delta t}$ measurability and to the a priori estimate (3.3), we conclude that $u \in \mathcal{N}_{n+1}$ with $\phi(u) \in \mathcal{N}_{n+1}$. We denote this solution u by u_{n+1}. Hence the assertion of the proposition follows. □

3.1.1 A priori estimate

Note that $\int_{\mathbb{R}^d} f(v) \cdot \nabla v\, dx = 0$ for any $v \in \mathcal{D}(\mathbb{R}^d)$, and hence it holds true for any $v \in H^1(\mathbb{R}^d)$ by a density argument. We choose the test function $v = u_{n+1}$ in (3.2) to have

$$\begin{aligned}
&\int_{\mathbb{R}^d} (u_{n+1} - u_n)u_{n+1}\, dx + \Delta t \int_{\mathbb{R}^d} \phi'(u_{n+1})|\nabla u_{n+1}|^2\, dx + \varepsilon\Delta t \int_{\mathbb{R}^d} |\nabla u_{n+1}|^2\, dx \\
&\quad = \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \int_E \eta(x, u_n; z)\, \tilde{N}(dz, ds) u_{n+1}\, dx \\
&\quad \le \int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \int_E \eta(x, u_n; z) u_n\, \tilde{N}(dz, ds)\, dx + \frac{\alpha}{2}\|u_{n+1} - u_n\|^2_{L^2(\mathbb{R}^d)} \\
&\qquad + \frac{1}{2\alpha} \int_{\mathbb{R}^d} \Big(\int_{t_n}^{t_{n+1}} \int_E \eta(x, u_n; z)\, \tilde{N}(dz, ds)\Big)^2\, d \quad \text{for some } \alpha > 0.
\end{aligned}$$

Since

$$\int_{\mathbb{R}^d} |\nabla\phi(u)|^2\,dx = \int_{\mathbb{R}^d} |\phi'(u)\nabla u|^2\,dx \le c_\phi \int_{\mathbb{R}^d} \phi'(u)|\nabla u|^2\,dx,$$

we see that

$$\frac{\Delta t}{c_\phi}\|\nabla\phi(u)\|_{\mathcal{H}}^2 \le \Delta t\mathbb{E}\Big[\int_{\mathbb{R}^d} \phi'(u)|\nabla u|^2\,dx\Big]. \tag{3.4}$$

In view of assumption (v), inequality (3.4), the Itô–Lévy isometry and the fact that for any $a, b \in \mathbb{R}$ one has $(a-b)a = \frac{1}{2}(a^2 + (a-b)^2 - b^2)$, we obtain

$$\begin{aligned}\frac{1}{2}\big[\|u_{n+1}\|_{\mathcal{H}}^2 + \|u_{n+1}-u_n\|_{\mathcal{H}}^2 - \|u_n\|_{\mathcal{H}}^2\big] + \frac{\Delta t}{c_\phi}\|\nabla\phi(u_{n+1})\|_{\mathcal{H}}^2 + \varepsilon\Delta t\|\nabla u_{n+1}\|_{\mathcal{H}}^2 \\ \le \frac{\alpha}{2}\|u_{n+1}-u_n\|_{\mathcal{H}}^2 + \frac{C\Delta t}{2\alpha}(1+\|u_n\|_{\mathcal{H}}^2).\end{aligned}$$

Since $\alpha > 0$ is arbitrary, one can choose $\alpha > 0$ so that

$$\begin{aligned}\|u_n\|_{\mathcal{H}}^2 + \sum_{k=0}^{n-1}\|u_{k+1}-u_k\|_{\mathcal{H}}^2 + \frac{\Delta t}{c_\phi}\sum_{k=0}^{n-1}\|\nabla\phi(u_{k+1})\|_{\mathcal{H}}^2 + \varepsilon\Delta t\sum_{k=0}^{n-1}\|\nabla u_{k+1}\|_{\mathcal{H}}^2 \\ \le C_1 + C_2\Delta t\sum_{k=0}^{n-1}\|u_k\|_{\mathcal{H}}^2 \quad \text{for some constants } C_1, C_2 > 0.\end{aligned} \tag{3.5}$$

Thanks to the discrete Gronwall lemma, we can deduce from (3.5) that

$$\|u_n\|_{\mathcal{H}}^2 + \sum_{k=0}^{n-1}\|u_{k+1}-u_k\|_{\mathcal{H}}^2 + \frac{\Delta t}{c_\phi}\sum_{k=0}^{n-1}\|\nabla\phi(u_{k+1})\|_{\mathcal{H}}^2 + \varepsilon\Delta t\sum_{k=0}^{n-1}\|\nabla u_{k+1}\|_{\mathcal{H}}^2 \le C \tag{3.6}$$

For fixed $\Delta t = \frac{T}{N}$, we define

$$u^{\Delta t}(t) = \sum_{k=1}^{N} u_k \mathbf{1}_{[(k-1)\Delta t, k\Delta t)}(t), \quad \tilde{u}^{\Delta t}(t) = \sum_{k=1}^{N}\Big[\frac{u_k - u_{k-1}}{\Delta t}(t-(k-1)\Delta t) + u_{k-1}\Big]\mathbf{1}_{[(k-1)\Delta t, k\Delta t)}(t)$$

with $u^{\Delta t}(t) = u_0$ for $t < 0$. Similarly, we define

$$\tilde{B}^{\Delta t}(t) = \sum_{k=1}^{N}\Big[\frac{B_k - B_{k-1}}{\Delta t}[t-(k-1)\Delta t] + B_{k-1}\Big]\mathbf{1}_{[(k-1)\Delta t, k\Delta t)}(t),$$

where

$$B_n = \sum_{k=0}^{n-1}\int_{k\Delta t}^{(k+1)\Delta t}\int_E \eta(x, u_k; z)\,\tilde{N}(dz, ds) = \int_0^{n\Delta t}\int_E \eta(x, u^{\Delta t}(s-\Delta t); z)\,\tilde{N}(dz, ds).$$

A straightforward calculation shows that

$$\begin{aligned}&\|u^{\Delta t}\|_{L^\infty(0,T;\mathcal{H})} = \max_{k=1,2,\dots,N}\|u_k\|_{\mathcal{H}}, \quad \|\tilde{u}^{\Delta t}\|_{L^\infty(0,T;\mathcal{H})} = \max_{k=0,1,\dots,N}\|u_k\|_{\mathcal{H}}, \\ &\|u^{\Delta t} - \tilde{u}^{\Delta t}\|_{L^2(0,T;\mathcal{H})}^2 \le \Delta t\sum_{k=0}^{N-1}\|u_{k+1}-u_k\|_{\mathcal{H}}^2.\end{aligned}$$

Since ϕ is a Lipschitz continuous function with $\phi(0) = 0$, in view of the above definitions and the a priori estimate (3.6), we have the following proposition.

Proposition 3.3. *Assume that Δt is small. Then $u^{\Delta t}$ and $\tilde{u}^{\Delta t}$ are bounded sequences in $L^\infty(0, T; \mathcal{H})$, $\phi(u^{\Delta t})$ and $\sqrt{\epsilon}u^{\Delta t}$ are bounded sequences in $L^2(0, T; \mathcal{N})$, and $\|u^{\Delta t} - \tilde{u}^{\Delta t}\|_{L^2(0,T;\mathcal{H})}^2 \le C\Delta t$.*

Moreover, $u^{\Delta t} - u^{\Delta t}(\cdot - \Delta t) \to 0$ in $L^2(\Omega \times \Pi_T)$.

Next, we want to find some upper bounds for $\tilde{B}^{\Delta t}(t)$. Regarding this, we have the following proposition.

Proposition 3.4. *$\tilde{B}^{\Delta t}$ is a bounded sequence in $L^2(\Omega \times \Pi_T)$ and*

$$\Big\|\tilde{B}^{\Delta t}(\cdot) - \int_0^{\cdot}\int_E \eta(x, u^{\Delta t}(s-\Delta t); z)\,\tilde{N}(dz, ds)\Big\|^2_{L^2(\Omega\times\mathbb{R}^d)} \le C\Delta t.$$

Proof. First, we prove the boundedness of $\tilde{B}^{\Delta t}(t)$. By using the definition of $\tilde{B}^{\Delta t}(t)$, assumption (v) and the boundedness of $u^{\Delta t}$ in $L^\infty(0, T; \mathcal{H})$, we obtain

$$\begin{aligned}
\|\tilde{B}^{\Delta t}\|^2_{L^2(0,T;L^2(\Omega,L^2(\mathbb{R}^d)))} &\le \Delta t \sum_{k=0}^{N} \|B_k\|^2_{L^2(\Omega\times\mathbb{R}^d)} \\
&\le \Delta t \sum_{k=0}^{N} \mathbb{E}\Big[\Big|\int_{\mathbb{R}^d}\int_0^{k\Delta t}\int_E \eta(x, u^{\Delta t}(s-\Delta t); z)\,\tilde{N}(dz, ds)\,dx\Big|^2\Big] \\
&\le C\Delta t \sum_{k=0}^{N} \mathbb{E}\Big[\int_{\mathbb{R}^d}\int_0^{k\Delta t} g^2(x)(1 + |u^{\Delta t}(s-\Delta t)|^2)\,dx\,ds\Big] \\
&\le C(1 + \|u^{\Delta t}\|_{L^\infty(0,T;L^2(\Omega\times\mathbb{R}^d))}) \\
&< +\infty.
\end{aligned}$$

Thus, $\tilde{B}^{\Delta t}$ is a bounded sequence in $L^2(\Omega \times \Pi_T)$.

To prove the second part of the proposition, we see that for any $t \in [n\Delta t, (n+1)\Delta t)$,

$$\begin{aligned}
&\tilde{B}^{\Delta t}(t) - \int_0^t\int_E \eta(x, u^{\Delta t}(s-\Delta t); z)\,\tilde{N}(dz, ds) \\
&\quad = \frac{t - n\Delta t}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t}\int_E \eta(x, u^{\Delta t}(s-\Delta t); z)\,\tilde{N}(dz, ds) - \int_{n\Delta t}^{t}\int_E \eta(x, u^{\Delta t}(s-\Delta t); z)\,\tilde{N}(dz, ds) \\
&\quad = \frac{t - n\Delta t}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t}\int_E \eta(x, u_n; z)\,\tilde{N}(dz, ds) - \int_{n\Delta t}^{t}\int_E \eta(x, u_n; z)\,\tilde{N}(dz, ds).
\end{aligned}$$

Therefore, in view of (3.6) and assumption (v), we have

$$\begin{aligned}
&\Big\|\tilde{B}^{\Delta t}(t) - \int_0^t\int_E \eta(x, u^{\Delta t}(s-\Delta t); z)\,\tilde{N}(dz, ds)\Big\|^2_{L^2(\Omega\times\mathbb{R}^d)} \\
&\quad = \mathbb{E}\Big[\int_{\mathbb{R}^d}\Big|\frac{t - n\Delta t}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t}\int_E \eta(x, u_n; z)\,\tilde{N}(dz, ds) - \int_{n\Delta t}^{t}\int_E \eta(x, u_n; z)\,\tilde{N}(dz, ds)\Big|^2\,dx\Big] \\
&\quad \le 2\int_{\mathbb{R}^d}\mathbb{E}\Big[\Big(\frac{t - n\Delta t}{\Delta t}\Big)^2\int_{n\Delta t}^{(n+1)\Delta t}\int_E \eta^2(x, u_n; z)\,m(dz)\,ds + \int_{n\Delta t}^{t}\int_E \eta^2(x, u_n; z)\,m(dz)\,ds\Big]\,dx \\
&\quad \le C(1 + \|u_n\|^2_{\mathcal{H}})\Big[\frac{(t - n\Delta t)^2}{\Delta t} + (t - n\Delta t)\Big] \\
&\quad \le C\Delta t.
\end{aligned}$$

This completes the proof. □

3.1.2 Convergence of $u^{\Delta t}(t,x)$

Thanks to Proposition 3.3 and the Lipschitz property of f and ϕ, there exist u, ϕ_u and f_u such that (up to a subsequence)

$$\begin{cases} u^{\Delta t} \rightharpoonup^* u & \text{in } L^\infty(0,T;L^2(\Omega\times\mathbb{R}^d)),\\ u^{\Delta t} \rightharpoonup u & \text{in } L^2((0,T)\times\Omega;H^1(\mathbb{R}^d)) \text{ (for a fixed positive } \epsilon),\\ \phi(u^{\Delta t}) \rightharpoonup \phi_u & \text{in } L^2((0,T)\times\Omega;H^1(\mathbb{R}^d)),\\ F(u^{\Delta t}) \rightharpoonup f_u & \text{in } L^2((0,T)\times\Omega\times\mathbb{R}^d). \end{cases} \tag{3.7}$$

Next, we want to identify the weak limits ϕ_u and f_u. Note that, for any $v\in H^1(\mathbb{R}^d)$, we can rewrite (3.2) in terms of $u^{\Delta t}$, $\tilde{u}^{\Delta t}$ and $\tilde{B}^{\Delta t}$ as

$$\int_{\mathbb{R}^d}\Big(\frac{\partial}{\partial t}(\tilde{u}^{\Delta t}-\tilde{B}^{\Delta t})(t)v+\{\nabla\phi(u^{\Delta t}(t))+\varepsilon\nabla u^{\Delta t}(t)+f(u^{\Delta t}(t))\}\cdot\nabla v\Big)\,dx=0. \tag{3.8}$$

Lemma 3.5. *$\{u^{\Delta t}\}$ is a Cauchy sequence in $L^2(\Omega\times\Pi_T)$.*

Proof. Consider two positive integers N and M and denote $\Delta t=\frac{T}{N}$ and $\Delta s=\frac{T}{M}$. Then, for any $v\in H^1(\mathbb{R}^d)$, from (3.8) one gets

$$\begin{aligned}\int_{\mathbb{R}^d}\Big(\frac{\partial}{\partial t}[(\tilde{u}^{\Delta t}-\tilde{B}^{\Delta t})(t)-(\tilde{u}^{\Delta s}-\tilde{B}^{\Delta s})(t)]v+\Big\{\nabla(\phi(u^{\Delta t}(t))-\phi(u^{\Delta s}(t)))\\ +\varepsilon\nabla(u^{\Delta t}(t)-u^{\Delta s}(t))+\big(f(u^{\Delta t}(t))-f(u^{\Delta s}(t))\big)\Big\}\cdot\nabla v\Big)\,dx=0.\end{aligned} \tag{3.9}$$

Let $w=(\tilde{u}^{\Delta t}-\tilde{B}^{\Delta t})(t)-(\tilde{u}^{\Delta s}-\tilde{B}^{\Delta s})(t)$. Set $v=(I-\Delta)^{-1}w$ in (3.9); then one has

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\|v(t)\|^2_{H^1(\mathbb{R}^d)}+\int_{\mathbb{R}^d}(\phi(u^{\Delta t}(t))-\phi(u^{\Delta s}(t)))w\,dx-\int_{\mathbb{R}^d}(\phi(u^{\Delta t}(t))-\phi(u^{\Delta s}(t)))v\,dx\\ +\varepsilon\int_{\mathbb{R}^d}(u^{\Delta t}-u^{\Delta s})w\,dx-\varepsilon\int_{\mathbb{R}^d}(u^{\Delta t}-u^{\Delta s})v\,dx+\int_{\mathbb{R}^d}(f(u^{\Delta t})-f(u^{\Delta s}))\cdot\nabla v\,dx=0.\end{aligned} \tag{3.10}$$

Note that $w=(u^{\Delta t}-u^{\Delta s})-(\tilde{B}^{\Delta t}-\tilde{B}^{\Delta s})-(u^{\Delta t}-\tilde{u}^{\Delta t})+(u^{\Delta s}-\tilde{u}^{\Delta s})$.

Therefore,

$$\begin{aligned}&\int_{\mathbb{R}^d}(\phi(u^{\Delta t}(t))-\phi(u^{\Delta s}(t)))w\,dx\\ &\quad=\int_{\mathbb{R}^d}(\phi(u^{\Delta t}(t))-\phi(u^{\Delta s}(t)))(u^{\Delta t}-u^{\Delta s})\,dx-\int_{\mathbb{R}^d}(\phi(u^{\Delta t}(t))-\phi(u^{\Delta s}(t)))(\tilde{B}^{\Delta t}-\tilde{B}^{\Delta s})\,dx\\ &\qquad-\int_{\mathbb{R}^d}(\phi(u^{\Delta t}(t))-\phi(u^{\Delta s}(t)))\{(u^{\Delta t}-\tilde{u}^{\Delta t})+(\tilde{u}^{\Delta s}-u^{\Delta s})\}\,dx\\ &\quad\geq\int_{\mathbb{R}^d}(\phi(u^{\Delta t}(t))-\phi(u^{\Delta s}(t)))(u^{\Delta t}-u^{\Delta s})\,dx-\frac{1}{2c_\phi}\int_{\mathbb{R}^d}(\phi(u^{\Delta t}(t))-\phi(u^{\Delta s}(t)))^2\,dx\\ &\qquad-\frac{c_\phi}{2}\int_{\mathbb{R}^d}\{(\tilde{B}^{\Delta t}-\tilde{B}^{\Delta s})+(u^{\Delta t}-\tilde{u}^{\Delta t})+(\tilde{u}^{\Delta s}-u^{\Delta s})\}^2\,dx\\ &\quad\geq\frac{1}{2}\int_{\mathbb{R}^d}(\phi(u^{\Delta t}(t))-\phi(u^{\Delta s}(t)))(u^{\Delta t}-u^{\Delta s})\,dx\\ &\qquad-\frac{c_\phi}{2}\int_{\mathbb{R}^d}\{(\tilde{B}^{\Delta t}-\tilde{B}^{\Delta s})+(u^{\Delta t}-\tilde{u}^{\Delta t})+(\tilde{u}^{\Delta s}-u^{\Delta s})\}^2\,dx\\ &\quad\geq-\frac{c_\phi}{2}\int_{\mathbb{R}^d}\{(\tilde{B}^{\Delta t}-\tilde{B}^{\Delta s})+(u^{\Delta t}-\tilde{u}^{\Delta t})+(\tilde{u}^{\Delta s}-u^{\Delta s})\}^2\,dx\quad\text{(by (i))}.\end{aligned} \tag{3.11}$$

Similarly, we also have

$$\varepsilon \int_{\mathbb{R}^d} (u^{\Delta t} - u^{\Delta s}) w \, dx \geq \frac{\varepsilon}{2} \int_{\mathbb{R}^d} (u^{\Delta t} - u^{\Delta s})^2 \, dx - \frac{\varepsilon}{2} \int_{\mathbb{R}^d} \{(\tilde{B}^{\Delta t} - \tilde{B}^{\Delta s}) + (u^{\Delta t} - \tilde{u}^{\Delta t}) + (\tilde{u}^{\Delta s} - u^{\Delta s})\}^2 \, dx. \tag{3.12}$$

Combining (3.10), (3.11) and (3.12), we obtain

$$\begin{aligned}
&\frac{1}{2} \frac{\partial}{\partial t} \|v(t)\|^2_{H^1(\mathbb{R}^d)} + \frac{\varepsilon}{2} \int_{\mathbb{R}^d} (u^{\Delta t} - u^{\Delta s})^2 \, dx \\
&\quad \leq \frac{(c_\phi + \varepsilon)}{2} \int_{\mathbb{R}^d} \{(\tilde{B}^{\Delta t} - \tilde{B}^{\Delta s}) + (u^{\Delta t} - \tilde{u}^{\Delta t}) + (\tilde{u}^{\Delta s} - u^{\Delta s})\}^2 \, dx \\
&\qquad + \varepsilon \int_{\mathbb{R}^d} (u^{\Delta t} - u^{\Delta s}) v \, dx - \int_{\mathbb{R}^d} (F(u^{\Delta t}) - F(u^{\Delta s})) \cdot \nabla v \, dx + \int_{\mathbb{R}^d} (\phi(u^{\Delta t}(t)) - \phi(u^{\Delta s}(t))) v \, dx \\
&\quad \leq \frac{(c_\phi + \varepsilon)}{2} \int_{\mathbb{R}^d} \{(\tilde{B}^{\Delta t} - \tilde{B}^{\Delta s}) + (u^{\Delta t} - \tilde{u}^{\Delta t}) + (\tilde{u}^{\Delta s} - u^{\Delta s})\}^2 \, dx + \frac{\beta}{4} \int_{\mathbb{R}^d} |\nabla v|^2 \, dx \\
&\qquad + \frac{\varepsilon}{2\alpha} \int_{\mathbb{R}^d} (u^{\Delta t} - u^{\Delta s})^2 dx + \Big(\frac{\beta}{4} + \varepsilon \frac{\alpha}{2}\Big) \int_{\mathbb{R}^d} v^2 \, dx + \frac{(c_f)^2 + (c_\phi)^2}{\beta} \int_{\mathbb{R}^d} (u^{\Delta t} - u^{\Delta s})^2 \, dx
\end{aligned}$$

for some α and $\beta > 0$. Since $\alpha, \beta > 0$ are arbitrary, there exist some positive constants C_1, C_2 and C_3 such that

$$\begin{aligned}
&\mathbb{E}[\|v(t)\|^2_{H^1(\mathbb{R}^d)}] - C_1 \int_0^t \mathbb{E}[\|v(r)\|^2_{H^1(\mathbb{R}^d)}] \, dr + C_2 \int_0^t \mathbb{E}[\|u^{\Delta t} - u^{\Delta s}\|^2_{L^2(\mathbb{R}^d)}] \, dr \\
&\quad \leq C_3 \Big\{ \|u^{\Delta t} - \tilde{u}^{\Delta t}\|^2_{L^2(\Omega \times \Pi_T)} + \|u^{\Delta s} - \tilde{u}^{\Delta s}\|^2_{L^2(\Omega \times \Pi_T)} + \int_0^t \mathbb{E}[\|\tilde{B}^{\Delta t} - \tilde{B}^{\Delta s}\|^2_{L^2(\mathbb{R}^d)}] \, dr \Big\}.
\end{aligned} \tag{3.13}$$

In view of Proposition 3.3, we notice that

$$\|u^{\Delta t} - \tilde{u}^{\Delta t}\|^2_{L^2(\Omega \times \Pi_T)} + \|u^{\Delta s} - \tilde{u}^{\Delta s}\|^2_{L^2(\Omega \times \Pi_T)} \leq C(\Delta t + \Delta s). \tag{3.14}$$

In addition,

$$\begin{aligned}
\mathbb{E}[\|\tilde{B}^{\Delta t}(r) - \tilde{B}^{\Delta s}(r)\|^2_{L^2(\mathbb{R}^d)}] &\leq 3 \Big\| \tilde{B}^{\Delta t}(r) - \int_0^r \int_E \eta(x, u^{\Delta t}(\sigma - \Delta t); z) \, \tilde{N}(dz, d\sigma) \Big\|^2_{L^2(\Omega \times \mathbb{R}^d)} \\
&\quad + 3 \Big\| \tilde{B}^{\Delta s}(r) - \int_0^r \int_E \eta(x, u^{\Delta s}(\sigma - \Delta s); z) \, \tilde{N}(dz, d\sigma) \Big\|^2_{L^2(\Omega \times \mathbb{R}^d)} \\
&\quad + 3 \Big\| \int_0^r \int_E (\eta(x, u^{\Delta t}(\sigma - \Delta t); z) - \eta(x, u^{\Delta s}(\sigma - \Delta s); z)) \, \tilde{N}(dz, d\sigma) \Big\|^2_{L^2(\Omega \times \mathbb{R}^d)} \\
&\leq C(\Delta t + \Delta s) + C \int_0^r \mathbb{E}[\|u^{\Delta t}(\sigma - \Delta t) - u^{\Delta s}(\sigma - \Delta s)\|^2_{L^2(\mathbb{R}^d)}] \, d\sigma,
\end{aligned}$$

where we have used Proposition 3.4 and assumption (v). Thus, we get

$$\int_0^t \mathbb{E}[\|\tilde{B}^{\Delta t} - \tilde{B}^{\Delta s}\|^2_{L^2(\mathbb{R}^d)}] \, dr \leq C(\Delta t + \Delta s) + C \int_0^t \int_0^r \mathbb{E}[\|u^{\Delta t}(\sigma - \Delta t) - u^{\Delta s}(\sigma - \Delta s)\|^2_{L^2(\mathbb{R}^d)}] \, d\sigma \, dr. \tag{3.15}$$

We combine (3.14) and (3.15) in (3.13) and have

$$\begin{aligned}
\mathbb{E}[\|v(t)\|^2_{H^1(\mathbb{R}^d)}] - C_1 \int_0^t \mathbb{E}[\|v(r)\|^2_{H^1(\mathbb{R}^d)}]\,dr + C_2 \int_0^t \mathbb{E}[\|u^{\Delta t} - u^{\Delta s}\|^2_{L^2(\mathbb{R}^d)}]\,dr \\
\le C(\Delta t + \Delta s) + C \int_0^t \int_0^r \mathbb{E}[\|u^{\Delta t}(\sigma - \Delta t) - u^{\Delta s}(\sigma - \Delta s)\|^2_{L^2(\mathbb{R}^d)}]\,d\sigma\,dr \quad \text{(by Proposition 3.3)} \\
\le C(\Delta t + \Delta s) + C \int_0^t \int_0^r \mathbb{E}[\|u^{\Delta t} - u^{\Delta s}\|^2_{L^2(\mathbb{R}^d)}]\,d\sigma\,dr.
\end{aligned}$$

Hence, an application of Gronwall's lemma yields

$$\mathbb{E}[\|v(t)\|^2_{H^1(\mathbb{R}^d)}] + \int_0^t \mathbb{E}[\|u^{\Delta t} - u^{\Delta s}\|^2_{L^2(\mathbb{R}^d)}]\,dr \le C(\Delta t + \Delta s)e^{Ct}.$$

This implies that

$$\|u^{\Delta t} - u^{\Delta s}\|^2_{L^2(\Omega \times \Pi_T)} \le C(\Delta t + \Delta s)e^{CT},$$

i.e., $\{u^{\Delta t}\}$ is a Cauchy sequence in $L^2(\Omega \times \Pi_T)$. □

We are now in position to identify the weak limits ϕ_u and f_u. We have shown that $u^{\Delta t} \rightharpoonup u$ and that $u^{\Delta t}$ is a Cauchy sequence in $L^2(\Omega \times \Pi_T)$. Thanks to the Lipschitz continuity of ϕ and f, one can easily conclude that $\phi_u = \phi(u)$ and $f_u = f(u)$.

In view of the variational formula (3.8), one needs to show the boundedness of

$$\frac{\partial}{\partial t}(\tilde{u}^{\Delta t} - \tilde{B}^{\Delta t})$$

in $L^2(\Omega \times (0,T); H^{-1}(\mathbb{R}^d))$ and then identify its weak limit. To this end, we have the following lemma.

Lemma 3.6. *The sequence $\{\frac{\partial}{\partial t}(\tilde{u}^{\Delta t} - \tilde{B}^{\Delta t})(t)\}$ is bounded in $L^2(\Omega \times (0,T); H^{-1}(\mathbb{R}^d))$, and*

$$\frac{\partial}{\partial t}(\tilde{u}^{\Delta t} - \tilde{B}^{\Delta t}) \rightharpoonup \frac{\partial}{\partial t}\Big(u - \int_0^{\cdot} \int_E \eta(x, u; z)\,\tilde{N}(dz, ds)\Big) \quad \text{in } L^2(\Omega \times (0,T); H^{-1}(\mathbb{R}^d)),$$

where u is given by (3.7).

Proof. To prove the lemma, let $\Gamma = \Omega \times [0,T] \times E$, $\mathcal{G} = \mathcal{P}_T \times \mathcal{L}(E)$ and $\varsigma = P \otimes \ell_t \otimes m$, where $\mathcal{P}_T$ represents the predictable σ-algebra on $\Omega \times [0,T]$ and $\mathcal{L}(E)$ represents the Lebesgue σ- algebra on E.

The space $L^2((\Gamma, \mathcal{G}, \varsigma); \mathbb{R})$ represents then the space of the square integrable predictable integrands for the Itô–Lévy integral with respect to the compensated compound Poisson random measure $\tilde{N}(dz, dt)$. Moreover, the Itô–Lévy integral defines a linear operator from $L^2((\Gamma, \mathcal{G}, \varsigma); \mathbb{R})$ to $L^2((\Omega, \mathcal{F}_T); \mathbb{R})$ and it preserves the norm (see, for example, [19]).

Thanks to Propositions 3.3 and 3.5, $u^{\Delta t}(t - \Delta t)$ converges to u in $L^2(\Omega \times \Pi_T)$. Therefore, in view of Proposition 3.4, the Lipschitz property of η and the above discussion, we conclude that

$$\tilde{B}^{\Delta t} \to \int_0^{\cdot} \int_E \eta(x, u; z)\,\tilde{N}(dz, ds) \quad \text{in } L^2(\Omega \times \Pi_T).$$

Again, note that

$$\frac{\partial}{\partial t}(\tilde{u}^{\Delta t} - \tilde{B}^{\Delta t})(t) = \sum_{k=1}^{N} \frac{(u_k - u_{k-1}) - (B_k - B_{k-1})}{\Delta t} \mathbf{1}_{[(k-1)\Delta t, k\Delta t)}.$$

From (3.2), we see that for any $v \in H^1(\mathbb{R}^d)$,

$$\begin{aligned}\int_{\mathbb{R}^d}\Big(\frac{u_{n+1}-u_n}{\Delta t} - \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t}\int_E \eta(x,u_n;z)\,\tilde{N}(dz,ds)\Big)v\,dx &= -\int_{\mathbb{R}^d}\nabla\phi(u_{n+1})\cdot\nabla v\,dx - \varepsilon\int_{\mathbb{R}^d}\nabla u_{n+1}\cdot\nabla v\,dx - \int_{\mathbb{R}^d}F(u_{n+1})\cdot\nabla v\,dx\\ &\le \{\|\nabla\phi(u_{n+1})\|_{L^2(\mathbb{R}^d)} + \varepsilon\|\nabla u_{n+1}\|_{L^2(\mathbb{R}^d)} + c_f\|u_{n+1}\|_{L^2(\mathbb{R}^d)}\}\|v\|_{H^1(\mathbb{R}^d)},\end{aligned}$$

and hence

$$\begin{aligned}\sup_{v\in H^1(\mathbb{R}^d)\setminus\{0\}}\frac{\int_{\mathbb{R}^d}(\frac{u_{n+1}-u_n}{\Delta t} - \frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t}\int_E \eta(x,u_n;z)\,\tilde{N}(dz,ds))v\,dx}{\|v\|_{H^1(\mathbb{R}^d)}}\\ \le \|\nabla\phi(u_{n+1})\|_{L^2(\mathbb{R}^d)} + \varepsilon\|\nabla u_{n+1}\|_{L^2(\mathbb{R}^d)} + c_f\|u_{n+1}\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

This implies that $\frac{\partial}{\partial t}(\tilde{u}^{\Delta t} - \tilde{B}^{\Delta t})(t)$ is a bounded sequence in $L^2(\Omega\times(0,T);H^{-1}(\mathbb{R}^d))$.

To prove the second part of the lemma, we recall that

$$\tilde{B}^{\Delta t} \rightharpoonup \int_0^{\cdot}\int_E \eta(x,u;z)\,\tilde{N}(dz,ds) \quad\text{and}\quad \tilde{u}^{\Delta t} \rightharpoonup u$$

in $L^2(\Omega\times\Pi_T)$. In view of the first part of this lemma, one can conclude that, up to a subsequence,

$$\frac{\partial}{\partial t}(\tilde{u}^{\Delta t} - \tilde{B}^{\Delta t}) \rightharpoonup \frac{\partial}{\partial t}\Big(u - \int_0^{\cdot}\int_E \eta(x,u;z)\,\tilde{N}(dz,ds)\Big) \quad \text{in } L^2(\Omega\times(0,T);H^{-1}(\mathbb{R}^d)).$$

This completes the proof. □

3.1.3 Existence of a weak solution

As we have emphasized, our aim is to prove the existence of a weak solution to the viscous problem. For this, it is required to pass to the limit as $\Delta t \to 0$. To this end, let us choose $\alpha \in L^2(0,T)$ and $\beta \in L^2(\Omega)$. Then, in view of the variational formula (3.8), we obtain

$$\begin{aligned}\int_{\Omega\times(0,T)}\Big\langle\frac{\partial}{\partial t}(\tilde{u}^{\Delta t} - \tilde{B}^{\Delta t}), v\Big\rangle\alpha\beta\,dt\,dP + \varepsilon\int_{\Omega\times\Pi_T}\nabla u^{\Delta t}\cdot\nabla v\alpha\beta\,dx\,dt\,dP\\ + \int_{\Omega\times\Pi_T}\nabla\phi(u^{\Delta t})\cdot\nabla v\alpha\beta\,dx\,dt\,dP + \int_{\Omega\times\Pi_T} f(u^{\Delta t})\cdot\nabla v\alpha\beta\,dx\,dt\,dP = 0.\end{aligned}$$

We make use of (3.7) and Lemmas 3.5 and 3.6 to pass to the limit as $\Delta t \to 0$ in the above variational formulation, and then arrive at

$$\begin{aligned}\int_{\Omega\times(0,T)}\Big\langle\frac{\partial}{\partial t}\Big(u - \int_0^t\int_E \eta(x,u;z)\,\tilde{N}(dz,ds)\Big), v\Big\rangle\alpha\beta\,dt\,dP + \varepsilon\int_{\Omega\times\Pi_T}\nabla u\cdot\nabla v\alpha\beta\,dx\,dt\,dP\\ + \int_{\Omega\times\Pi_T}\{\nabla\phi(u)\cdot\nabla v + f(u)\cdot\nabla v\}\alpha\beta\,dx\,dt\,dP = 0.\end{aligned}\tag{3.16}$$

Since $H^1(\mathbb{R}^d)$ is a separable Hilbert space, the above formulation (3.16) yields almost surely in Ω,

$$\Big\langle\frac{\partial}{\partial t}\Big(u - \int_0^t\int_E \eta(x,u;z)\,\tilde{N}(dz,ds)\Big), v\Big\rangle + \int_{\mathbb{R}^d}(\varepsilon\nabla u + \nabla\phi(u) + f(u))\cdot\nabla v\,dx = 0$$

for almost every $t \in [0,T]$.

This proves that u is a weak solution of (3.1). For every $\phi \in H^1(\mathbb{R}^d)$, it is easily seen that

$$\mathbb{E}\Big|\frac{1}{\delta}\int_0^\delta \int_{\mathbb{R}^d} (u(s,x) - u_0(x))\psi(x)\,dx\,ds\Big| \le C\Delta t \quad \text{if } \delta < \Delta t.$$

Therefore,

$$\limsup_{\delta\downarrow 0} \mathbb{E}\Big|\frac{1}{\delta}\int_0^\delta \int_{\mathbb{R}^d} (u(s,x) - u_0(x))\psi(x)\,dx\,ds\Big| \le C\Delta t \quad \text{for very } \delta < \Delta t.$$

Now, by letting $\Delta t \downarrow 0$, we have

$$\lim_{\delta\downarrow 0} \frac{1}{\delta}\int_0^\delta \int_{\mathbb{R}^d} u(s,x)\psi(x)\,dx\,ds = \int_{\mathbb{R}^d} u_0(x)\psi(x)\,dx \quad P\text{-almost surely.}$$

3.2 A priori bounds for the viscous solutions

Note that for a fixed $\varepsilon > 0$ there exists a weak solution, denoted as $u_\varepsilon \in H^1(\mathbb{R}^d)$, which satisfies the following variational formulation: P-almost surely in Ω, and for almost every $t \in (0,T)$,

$$\begin{aligned}\Big\langle \frac{\partial}{\partial t}\Big[u_\varepsilon - \int_0^t \int_E \eta(\cdot, u_\varepsilon(s,\cdot); z)\,\tilde N(dz,ds)\Big], v\Big\rangle + \int_{\mathbb{R}^d} \nabla\phi(u_\varepsilon(t,x))\cdot\nabla v\,dx \\ + \int_{\mathbb{R}^d} \{f(u_\varepsilon(t,x)) + \varepsilon\nabla u_\varepsilon(t,x)\}\cdot\nabla v\,dx = 0\end{aligned}$$

for any $v \in H^1(\mathbb{R}^d)$. Let $\beta(u) = u^2$. Applying the Itô–Lévy formula (cf. Theorem A.1) to $\beta(u)$, one gets that for any $t > 0$,

$$\begin{aligned}&\int_{\mathbb{R}^d} u_\varepsilon^2(t)\,dx + 2\int_0^t \int_{\mathbb{R}^d} [\varepsilon + \phi'(u_\varepsilon(s))]|\nabla u_\varepsilon|^2\,ds + 2\int_{\mathbb{R}^d} f(u_\varepsilon)\cdot\nabla u_\varepsilon\,dx \\ &\quad= \int_{\mathbb{R}^d} u_\varepsilon^2(0)\,dx + 2\int_0^t \int_E \int_{\mathbb{R}^d} \int_0^1 \eta(x, u_\varepsilon(s,x); z)(u_\varepsilon + \theta\eta(x,u_\varepsilon;z))\,d\theta\,dx\,\tilde N(dz,ds) \\ &\qquad + \int_0^t \int_E \int_{\mathbb{R}^d} \eta^2(x, u_\varepsilon(s,x); z)\,dx\,m(dz)\,ds.\end{aligned}$$

Note that $\int_{\mathbb{R}^d} f(u_\varepsilon)\cdot\nabla u_\varepsilon\,dx = 0$ as $u_\varepsilon \in H^1(\mathbb{R}^d)$. Taking the expectation, we obtain

$$\mathbb{E}[\|u_\varepsilon(t)\|_2^2] + \varepsilon\int_0^t \mathbb{E}[\|\nabla u_\varepsilon\|_2^2]\,ds + \int_0^t \mathbb{E}[\|\nabla G(u_\varepsilon(s))\|_2^2]\,ds \le \mathbb{E}[\|u_\varepsilon(0)\|_2^2 + Ct\|g\|_{L^2(\mathbb{R}^d)}^2] + C\int_0^t \mathbb{E}[\|u_\varepsilon(s)\|_2^2]\,ds.$$

An application of Gronwall's inequality yields

$$\sup_{0\le t\le T} \mathbb{E}[\|u_\varepsilon(t)\|_2^2] + \varepsilon\int_0^T \mathbb{E}[\|\nabla u_\varepsilon(s)\|_2^2]\,ds + \int_0^T \mathbb{E}[\|\nabla G(u_\varepsilon(s))\|_2^2]\,ds \le C.$$

The achieved result can be summarized by the following theorem.

Theorem 3.7. *For any $\varepsilon > 0$, there exists a weak solution u_ε to problem* (3.1). *Moreover, it satisfies the following estimate:*

$$\sup_{0\le t\le T} \mathbb{E}[\|u_\varepsilon(t)\|_2^2] + \varepsilon \int_0^T \mathbb{E}[\|\nabla u_\varepsilon(s)\|_2^2]\,ds + \int_0^T \mathbb{E}[\|\nabla G(u_\varepsilon(s))\|_2^2]\,ds \le C, \tag{3.17}$$

where G is an associated Kirchhoff's function of ϕ, defined by $G(x) = \int_0^x \sqrt{\phi'(r)}\,dr$.

Remark 3.8. Let us remark that since any solution to (3.1) is an entropy solution, the solution u_ε in unique.

4 Existence of an entropy solution

In this section, we will establish the existence of an entropy solution. In view of the a priori estimates given in (3.17), we can apply [5, Lemmas 4.2 and 4.3] (see also [2]) and show the existence of a Young measure-valued limit process solution $u(t, x, \alpha)$, $\alpha \in (0, 1)$ associated with the sequence $\{u_\varepsilon(t, x)\}_{\varepsilon>0}$.

The basic strategy in this case is to apply the Young measure technique and to adapt Kruzhkov's doubling variable method in the presence of a noise to viscous solutions with two different parameters, and then to send the viscosity parameters to zero. One needs a version of the classical L^1 contraction principle (for conservation laws) to get the uniqueness of the Young measure-valued limit and to show that the Young measure-valued limit process is independent of the additional (dummy) variable. Hence, it will imply the point-wise convergence of the sequence of viscous solutions.

4.1 Uniqueness of the Young measure-valued limit process

To do this, we follow the same line of argument as in [3] for the degenerate parabolic part and [5] for the Lévy noise. For the convenience of the reader, we have chosen to provide detailed proofs of a few crucial technical lemmas and the rest is referred to appropriate resources. Bauzet et al. [2] and Biswas et al. [5] used the fact that $\Delta u_\varepsilon \in L^2(\Omega \times \Pi_T)$. Note that, in this case, $u_\varepsilon \in H^1(\mathbb{R}^d)$. Therefore, we need to regularize u_ε by convolution. Let $\{\tau_\kappa\}$ be a sequence of mollifiers in $\mathbb{R}^d$. Since u_ε is a solution to problem (3.1), as shown in the proof of Theorem A.1, $u_\varepsilon * \tau_\kappa$ is a solution to the problem

$$\begin{aligned}(u_\varepsilon * \tau_\kappa) - \int_0^t \Delta(\phi(u_\varepsilon) * \tau_\kappa)\,ds &= \int_0^t \operatorname{div}_x(f(u_\varepsilon) * \tau_\kappa)\,ds + \int_0^t \int_E (\eta(x, u_\varepsilon; z) * \tau_\kappa)\,\tilde N(dz, ds)\\ &\quad + \int_0^t \varepsilon\Delta(u_\varepsilon * \tau_\kappa(t, x))\,ds \quad \text{a.e. } t > 0,\ x \in \mathbb{R}^d.\end{aligned} \tag{4.1}$$

Note that $\Delta(u_\varepsilon * \tau_\kappa) \in L^2(\Omega \times \Pi_T)$ for fixed $\varepsilon > 0$.

Let ρ and ϱ be standard non-negative mollifiers on $\mathbb{R}$ and $\mathbb{R}^d$, respectively, such that $\operatorname{supp}(\rho) \subset [-1, 0]$ and $\operatorname{supp}(\varrho) = \overline{B}_1(0)$. We define

$$\rho_{\delta_0}(r) = \frac{1}{\delta_0}\rho\Big(\frac{r}{\delta_0}\Big) \quad \text{and} \quad \varrho_\delta(x) = \frac{1}{\delta^d}\varrho\Big(\frac{x}{\delta}\Big),$$

where δ and δ_0 are two positive constants. Given a non-negative test function $\psi \in C_c^{1,2}([0, \infty) \times \mathbb{R}^d)$ and two positive constants δ and δ_0, define

$$\varphi_{\delta,\delta_0}(t, x, s, y) = \rho_{\delta_0}(t - s)\varrho_\delta(x - y)\psi(s, y).$$

Clearly $\rho_{\delta_0}(t - s) \neq 0$ only if $s - \delta_0 \le t \le s$, and hence $\varphi_{\delta,\delta_0}(t, x; s, y) = 0$ outside $s - \delta_0 \le t \le s$.

Let $u_\theta(t, x)$ be the weak solution to the viscous problem (3.1) with parameter $\theta > 0$ and initial condition $u_\theta(0, x) = v_0(x)$. Moreover, let J be the standard symmetric non-negative mollifier on $\mathbb{R}$ with support in $[-1, 1]$

and $J_l(r) = \frac{1}{l}J(\frac{r}{l})$ for $l > 0$. We now simply write down the Itô–Lévy formula for $u_\theta(t,x)$ against the convex entropy triple $(\beta(\cdot - k), F^\beta(\cdot, k), \phi^\beta(\cdot, k))$, multiply by $J_l(u_\varepsilon * \tau_\kappa(s,y) - k)$ for $k \in \mathbb{R}$ and then integrate with respect to s, y and k. Taking the expectation of the resulting expressions, we have

$$
\begin{aligned}
\theta\mathbb{E}\Big[&\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}} \beta''(u_\theta(t,x)-k)|\nabla u_\theta(t,x)|^2 \varphi_{\delta,\delta_0}(t,x,s,y) J_l(u_\varepsilon * \tau_\kappa(s,y)-k)\,dk\,dx\,dt\,dy\,ds\Big]\\
&+\mathbb{E}\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}} \beta''(u_\theta(t,x)-k)|\nabla G(u_\theta(t,x))|^2 \varphi_{\delta,\delta_0}(t,x,s,y) J_l(u_\varepsilon * \tau_\kappa(s,y)-k)\,dk\,dx\,dt\,dy\,ds\Big]\\
\le{}& \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_{\mathbb{R}} \beta(v_0(x)-k)\varphi_{\delta,\delta_0}(0,x,s,y) J_l(u_\varepsilon * \tau_\kappa(s,y)-k)\,dk\,dx\,dy\,ds\Big]\\
&+\mathbb{E}\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}} \beta(u_\theta(t,x)-k)\partial_t\varphi_{\delta,\delta_0}(t,x,s,y) J_l(u_\varepsilon * \tau_\kappa(s,y)-k)\,dk\,dx\,dt\,dy\,ds\Big]\\
&+\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}}\int_0^T\int_E\int_{\mathbb{R}^d}\int_0^1 \eta(x,u_\theta(t,x);z)\beta'(u_\theta(t,x)+\tau\eta(x,u_\theta(t,x);z)-k)\\
&\qquad\times\varphi_{\delta,\delta_0}(t,x,s,y)\,d\tau\,dx\,\tilde{N}(dz,dt) J_l(u_\varepsilon * \tau_\kappa(s,y)-k)\,dk\,dy\,ds\Big]\\
&+\mathbb{E}\Big[\int_{\Pi_T}\int_0^T\int_E\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_0^1 (1-\tau)\eta^2(x,u_\theta(t,x);z)\beta''(u_\theta(t,x)+\tau\eta(x,u_\theta(t,x);z)-k)\\
&\qquad\times\varphi_{\delta,\delta_0}(t,x,s,y) J_l(u_\varepsilon * \tau_\kappa(s,y)-k)\,d\tau\,dk\,dx\,m(dz)\,dt\,dy\,ds\Big]\\
&-\mathbb{E}\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}} F^\beta(u_\theta(t,x),k)\nabla_x\varrho_\delta(x-y)\psi(s,y)\rho_{\delta_0}(t-s) J_l(u_\varepsilon * \tau_\kappa(s,y)-k)\,dk\,dx\,dt\,dy\,ds\Big]\\
&+\mathbb{E}\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}} \phi^\beta(u_\theta(t,x),k)\Delta_x\varrho_\delta(x-y)\psi(s,y)\rho_{\delta_0}(t-s) J_l(u_\varepsilon * \tau_\kappa(s,y)-k)\,dk\,dx\,dt\,dy\,ds\Big]\\
&-\theta\mathbb{E}\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}} \beta'(u_\theta(t,x)-k)\nabla_x u_\theta(t,x)\cdot\nabla_x\varphi_{\delta,\delta_0}(t,x,s,y) J_l(u_\varepsilon * \tau_\kappa(s,y)-k)\,dk\,dx\,dt\,dy\,ds\Big],
\end{aligned}
$$

i.e.,

$$
I_{0,1} + I_{0,2} \le I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7. \tag{4.2}
$$

We now apply the Itô–Lévy formula to (4.1) and obtain

$$
\begin{aligned}
\varepsilon\mathbb{E}\Big[&\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}} \beta''(u_\varepsilon * \tau_\kappa - k)|\nabla(u_\varepsilon * \tau_\kappa)|^2 \varphi_{\delta,\delta_0} J_l(u_\theta(t,x)-k)\,dk\,dx\,dt\,dy\,ds\Big]\\
&+\mathbb{E}\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}} \beta''(u_\varepsilon * \tau_\kappa - k)\nabla(\phi(u_\varepsilon) * \tau_\kappa)\cdot\nabla(u_\varepsilon * \tau_\kappa)\varphi_{\delta,\delta_0} J_l(u_\theta(t,x)-k)\,dk\,dx\,dt\,dy\,ds\Big]\\
\le{}& \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_{\mathbb{R}} \beta(u_\varepsilon * \tau_\kappa(0,y)-k)\varphi_{\delta,\delta_0}(t,x,0,y) J_l(u_\theta(t,x)-k)\,dk\,dx\,dy\,dt\Big]\\
&+\mathbb{E}\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}} \beta(u_\varepsilon * \tau_\kappa(s,y)-k)\partial_s\phi_{\delta,\delta_0} J_l(u_\theta(t,x)-k)\,dk\,dy\,ds\,dx\,dt\Big]\\
&+\mathbb{E}\Big[\int_{\Pi_T}\int_0^T\int_E\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_0^1 (\eta(y,u_\varepsilon;z) * \tau_\kappa)\beta'(u_\varepsilon * \tau_\kappa + \theta(\eta(y,u_\varepsilon;z) * \tau_\kappa)-k)\\
&\qquad\times\varphi_{\delta,\delta_0} J_l(u_\theta(t,x)-k)\,d\theta\,dy\,dk\,\tilde{N}(dz,ds)\,dx\,dt\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbb{E}\Big[\int_{\Pi_T}\int_0^T\int_E\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_0^1(1-\theta)(\eta(y,u_\varepsilon;z)*\tau_\kappa)^2\beta''(u_\varepsilon*\tau_\kappa+\theta(\eta(y,u_\varepsilon;z)*\tau_\kappa)-k)\\
&\qquad\times\varphi_{\delta,\delta_0}J_l(u_\theta(t,x)-k)\,d\theta\,dy\,dk\,m(dz)\,ds\,dx\,dt\Big]\\
&-\mathbb{E}\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}}\beta'(u_\varepsilon*\tau_\kappa-k)\nabla(\phi(u_\varepsilon)*\tau_\kappa)\nabla_y\varphi_{\delta,\delta_0}J_l(u_\theta(t,x)-k)\,dk\,dx\,dt\,dy\,ds\Big]\\
&-\mathbb{E}\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}}\beta'(u_\varepsilon*\tau_\kappa-k)(F(u_\varepsilon)*\tau_\kappa)\nabla_y\varphi_{\delta,\delta_0}J_l(u_\theta(t,x)-k)\,dk\,dx\,dt\,dy\,ds\Big]\\
&-\mathbb{E}\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}}\beta''(u_\varepsilon*\tau_\kappa-k)(F(u_\varepsilon)*\tau_\kappa)\nabla_y(u_\varepsilon*\tau_\kappa)\varphi_{\delta,\delta_0}J_l(u_\theta(t,x)-k)\,dk\,dx\,dt\,dy\,ds\Big]\\
&-\varepsilon\mathbb{E}\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}}\beta'(u_\varepsilon*\tau_\kappa(s,y)-k)\nabla_y(u_\varepsilon*\tau_\kappa)\cdot\nabla_y\varphi_{\delta,\delta_0}J_l(u_\theta(t,x)-k)\,dk\,dy\,ds\,dx\,dt\Big],
\end{aligned}
$$

i.e.,

$$
J_{0,1}+J_{0,2}\le J_1+J_2+J_3+J_4+J_5+J_6+J_7+J_8. \tag{4.3}
$$

We now add (4.2) and (4.3) and look for the passage to the limit with respect to the different approximation parameters. Note that $I_{0,1}$ and $J_{0,1}$ are both non-negative terms, they are the left-hand sides of inequalities (4.2) and (4.3), respectively. Hence we can omit these terms. Let us consider the expressions $I_{0,2}$ and $J_{0,2}$. Recall that

$$
I_{0,2}=\mathbb{E}\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}}\beta''(u_\theta(t,x)-k)|\nabla G(u_\theta(t,x))|^2\varphi_{\delta,\delta_0}J_l(u_\varepsilon*\tau_\kappa(s,y)-k)\,dk\,dx\,dt\,dy\,ds\Big]
$$

and

$$
J_{0,2}=\mathbb{E}\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}}\beta''(u_\varepsilon*\tau_\kappa-k)\nabla(\phi(u_\varepsilon)*\tau_\kappa)\cdot\nabla(u_\varepsilon*\tau_\kappa)\varphi_{\delta,\delta_0}J_l(u_\theta(t,x)-k)\,dk\,dx\,dt\,dy\,ds\Big].
$$

By using the properties of Lebesgue points, convolutions and approximations by mollifications, one is able to pass to the limit in $I_{0,2}$ and $J_{0,2}$, and conclude the following lemma.

Lemma 4.1. *It holds that*

$$
\lim_{l\to0}\lim_{\kappa\to0}\lim_{\delta_0\to0}I_{0,2}=\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\beta''(u_\theta(t,x)-u_\varepsilon(t,y))|\nabla G(u_\theta(t,x))|^2\psi(t,y)\varrho_\delta(x-y)\,dx\,dt\,dy\Big]
$$

and

$$
\lim_{l\to0}\lim_{\kappa\to0}\lim_{\delta_0\to0}J_{0,2}=\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\beta''(u_\varepsilon(s,y)-u_\theta(s,x))\nabla\phi(u_\varepsilon(s,y))\cdot\nabla u_\varepsilon(s,y)\times\psi(s,y)\varrho_\delta(x-y)\,dx\,dy\,ds\Big].
$$

Lemma 4.2. *It follows that*

$$
\begin{aligned}
\limsup_{\theta\to0}\limsup_{\varepsilon\to0}\lim_{l\to0}\lim_{\kappa\to0}\lim_{\delta_0\to0}(I_{0,2}+J_{0,2})\ge 2\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_0^1\int_{\tilde u(t,x,y)}^{u(t,y,\alpha)}\Big(\int_{s=\sigma}^{\tilde u(t,x,y)}\beta''(\sigma-s)\sqrt{\phi'(s)}\,ds\Big)\sqrt{\phi'(\sigma)}\,d\sigma\\
\times\operatorname{div}_y\nabla_x[\psi(t,y)\varrho_\delta(x-y)]\,dy\,d\alpha\,dx\,dy\,dt\Big].
\end{aligned} \tag{4.4}
$$

Proof. In view of Lemma 4.1, we see that

$$
\begin{aligned}
&\lim_{l\to0}\lim_{\kappa\to0}\lim_{\delta_0\to0}(I_{0,2}+J_{0,2})\\
&\quad=\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\beta''(u_\varepsilon(t,y)-u_\theta(t,x))(|\nabla_y G(u_\varepsilon(t,y))|^2+|\nabla_x G(u_\theta(t,x))|^2)\psi(t,y)\varrho_\delta(x-y)\,dx\,dy\,dt\Big].
\end{aligned}
$$

Let $u(t,y,\alpha)$ and $\tilde{u}(t,x,y)$ be the Young measure-valued narrow limits associated with the sequences $\{u_\varepsilon(t,y)\}_{\varepsilon>0}$ and $\{u_\theta(t,x)\}_{\theta>0}$, respectively. With these at hand, one can use a similar argument as in [3, Lemma 3.4] and arrive at the conclusion that (4.4) holds. □

We consider the terms (I_1+J_1) originating from the initial conditions. Note that $I_1=0$ as $\operatorname{supp}\rho_{\delta_0}\subset[-\delta_0,0)$. Under a slight modification of the same line of arguments as in [5], we arrive at the following lemma.

Lemma 4.3. *It holds that*

$$\lim_{\theta\to0}\lim_{\varepsilon\to0}\lim_{l\to0}\lim_{\kappa\to0}\lim_{\delta_0\to0}(I_1+J_1)=\mathbb{E}\Big[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\beta_\vartheta(v_0(x)-u_0(y))\psi(0,y)\varrho_\delta(x-y)\,dx\,dy\Big]$$

and

$$\lim_{(\vartheta,\delta)\to(0,0)}\mathbb{E}\Big[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\beta_\vartheta(v_0(x)-u_0(y))\psi(0,y)\varrho_\delta(x-y)\,dx\,dy\Big]=\mathbb{E}\Big[\int_{\mathbb{R}^d}|v_0(x)-u_0(x)|\psi(0,x)\,dx\Big].$$

We now turn our attention to (I_2+J_2). Note that $\partial_t\rho_{\delta_0}(t-s)=-\partial_s\rho_{\delta_0}(t-s)$ and that β and J_l are even functions. A simple calculation gives

$$I_2+J_2=\mathbb{E}\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}}\beta(u_\varepsilon*\tau_\kappa(s,y)-k)\partial_s\psi(s,y)\rho_{\delta_0}(t-s)\varrho_\delta(x-y)\times J_l(u_\theta(t,x)-k)\,dk\,dy\,ds\,dx\,dt\Big].$$

One can now pass to the limit in (I_2+J_2) and have the following conclusion.

Lemma 4.4. *It holds that*

$$\begin{aligned}&\lim_{\theta\to0}\lim_{\varepsilon\to0}\lim_{l\to0}\lim_{\kappa\to0}\lim_{\delta_0\to0}(I_2+J_2)\\&\quad=\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_0^1\beta(u(s,y,\alpha)-\tilde{u}(s,x,y))\partial_s\psi(s,y)\varrho_\delta(x-y)\,d\gamma\,d\alpha\,dy\,dx\,ds\Big]\end{aligned}$$

and

$$\begin{aligned}&\lim_{(\vartheta,\delta)\to(0,0)}\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_0^1\beta(u(s,y,\alpha)-\tilde{u}(s,x,y))\partial_s\psi(s,y)\varrho_\delta(x-y)\,d\gamma\,d\alpha\,dy\,dx\,ds\Big]\\&\quad=\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^d}\int_0^1\int_0^1|u(s,y,\alpha)-\tilde{u}(s,y,\gamma)|\partial_s\psi(s,y)\,d\gamma\,d\alpha\,dy\,ds\Big].\end{aligned}$$

In regard to I_6 and J_5, we have the following lemma.

Lemma 4.5. *It holds that*

$$\lim_{\theta\downarrow0}\lim_{\varepsilon\downarrow0}\lim_{l\downarrow0}\lim_{\kappa\downarrow0}\lim_{\delta_0\downarrow0}I_6=\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_0^1\phi^\beta(\tilde{u}(s,x,\gamma),u(s,y,\alpha))\Delta_x\varrho_\delta(x-y)\psi(s,y)\,d\gamma\,d\alpha\,dx\,dy\,ds\Big]$$

and

$$\lim_{\theta\downarrow0}\lim_{\varepsilon\downarrow0}\lim_{l\downarrow0}\lim_{\kappa\downarrow0}\lim_{\delta_0\downarrow0}J_5=\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_{[0,1]^2}\phi^\beta(u(s,y,\alpha),\tilde{u}(s,x,\gamma))\Delta_y[\psi(s,y)\varrho_\delta(x-y)]\,d\gamma\,d\alpha\,dx\,dy\,ds\Big].$$

Proof. Let us consider the passage to the limits in I_6. To do this, we define

$$\begin{aligned}\mathcal{B}_1=\;&\mathbb{E}\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}}\phi^\beta(u_\theta(t,x),k)\Delta_x\varrho_\delta(x-y)\psi(s,y)\rho_{\delta_0}(t-s)J_l(u_\varepsilon*\tau_\kappa(s,y)-k)\,dk\,dx\,dt\,dy\,ds\Big]\\&-\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\phi^\beta(u_\theta(s,x),k)\Delta_x\varrho_\delta(x-y)\psi(s,y)J_l(u_\varepsilon*\tau_\kappa(s,y)-k)\,dk\,dx\,dy\,ds\Big].\end{aligned}$$

Note that, for all $a, b, c \in \mathbb{R}$,

$$|\phi^\beta(a,b) - \phi^\beta(c,b)| \le C|c-a|. \tag{4.5}$$

By using (4.5), we have

$$\begin{aligned}
\mathcal{B}_1 &= \mathbb{E}\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}} \phi^\beta(u_\theta(t,x), u_\varepsilon * \tau_\kappa(s,y) - k)\Delta_x\varrho_\delta(x-y)\psi(s,y)\rho_{\delta_0}(t-s)J_l(k)\,dk\,dx\,dt\,dy\,ds\Big] \\
&\quad - \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_{\mathbb{R}} \phi^\beta(u_\theta(s,x), u_\varepsilon * \tau_\kappa(s,y) - k)\Delta_x\varrho_\delta(x-y)\psi(s,y)J_l(k)\,dk\,dx\,dy\,ds\Big] \\
&= \mathbb{E}\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}} (\phi^\beta(u_\theta(t,x), u_\varepsilon * \tau_\kappa(s,y) - k) - \phi^\beta(u_\theta(s,x), u_\varepsilon * \tau_\kappa(s,y) - k))\Delta_x\varrho_\delta(x-y) \\
&\qquad\qquad \times \psi(s,y)\rho_{\delta_0}(t-s)J_l(k)\,dk\,dx\,dt\,dy\,ds\Big] \\
&\quad - \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_{\mathbb{R}} \phi^\beta(u_\theta(s,x), u_\varepsilon * \tau_\kappa(s,y) - k)\psi(s,y)\Big(1 - \int_0^T \rho_{\delta_0}(t-s)\,dt\Big) \\
&\qquad\qquad \times \Delta_x\varrho_\delta(x-y)J_l(k)\,dk\,dx\,dy\,ds\Big].
\end{aligned}$$

Then

$$\begin{aligned}
|\mathcal{B}_1| &\le C\mathbb{E}\Big[\int_{s=\delta_0}^T\int_{\Pi_T}\int_{\mathbb{R}^d}\int_{\mathbb{R}} |u_\theta(t,x) - u_\theta(s,x)||\Delta_x\varrho_\delta(x-y)|\psi(s,y)\rho_{\delta_0}(t-s)J_l(k)\,dk\,dx\,dt\,dy\,ds\Big] + \mathcal{O}(\delta_0) \\
&\le C\mathbb{E}\Big[\int_{s=\delta_0}^T\int_0^T\int_{K_\delta} |u_\theta(t,x) - u_\theta(s,x)|\rho_{\delta_0}(t-s)\,dx\,dt\,ds\Big] + \mathcal{O}(\delta_0) \\
&\le C\mathbb{E}\Big[\int_{r=0}^1\int_0^T\int_{K_\delta} |u_\theta(t+\delta_0 r,x) - u_\theta(t,x)|\rho(-r)\,dx\,dt\,dr\Big] + \mathcal{O}(\delta_0),
\end{aligned}$$

where $K_\delta \subset \mathbb{R}^d$ is a compact set depending on ψ and δ.

Note that

$$\lim_{\delta_0\downarrow 0}\int_0^T\int_{K_\delta} |u_\theta(t+\delta_0 r,x) - u_\theta(t,x)|\,dx\,dt \to 0$$

almost surely for all $r \in [0,1]$. Therefore, by the dominated convergence theorem, we have

$$\lim_{\delta_0\to 0} I_6 = \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_{\mathbb{R}} \phi^\beta(u_\theta(s,x), k)\Delta_x\varrho_\delta(x-y)\psi(s,y)J_l(u_\varepsilon * \tau_\kappa(s,y) - k)\,dk\,dx\,dy\,ds\Big].$$

Moreover, one can use the property of convolution to conclude

$$\lim_{\kappa\to 0}\lim_{\delta_0\to 0} I_6 = \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_{\mathbb{R}} \phi^\beta(v(s,x), k)\Delta_x\varrho_\delta(x-y)\psi(s,y)J_l(u_\varepsilon(s,y) - k)\,dk\,dx\,dy\,ds\Big].$$

Passage to the limit as $l \to 0$: Let

$$\begin{aligned}
\mathcal{B}_2 &:= \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_{\mathbb{R}} \phi^\beta(u_\theta(s,x), k)\Delta_x\varrho_\delta(x-y)\psi(s,y)J_l(u_\varepsilon(s,y) - k)\,dk\,dx\,dy\,ds\Big] \\
&\quad - \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d} \phi^\beta(u_\theta(s,x), u_\varepsilon(s,y))\Delta_x\varrho_\delta(x-y)\psi(s,y)\,dx\,dy\,ds\Big] \\
&= \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_{\mathbb{R}} (\phi^\beta(u_\theta(s,x), k) - \phi^\beta(u_\theta(s,x), u_\varepsilon(s,y)))\Delta_x\varrho_\delta(x-y)\psi(s,y) \\
&\qquad\qquad \times J_l(u_\varepsilon(s,y) - k)\,dk\,dx\,dy\,ds\Big].
\end{aligned}$$

Note that for all $a, b, c \in \mathbb{R}$,

$$|\phi^\beta(a,b) - \phi^\beta(a,c)| \le C(1 + |a-b|)|b-c|. \tag{4.6}$$

Therefore, by (4.6) we have

$$\begin{aligned}|\mathcal{B}_2| &\le C\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_{\mathbb{R}}(1 + |u_\theta(s,x) - k|)|u_\varepsilon(s,y) - k||\Delta_x\varrho_\delta(x-y)|\psi(s,y) \times J_l(u_\varepsilon(s,y) - k)\,dk\,dx\,dy\,ds\Big]\\ &\le Cl\Big\{1 + \big(\sup_{\theta>0}\sup_{t>0}\mathbb{E}[\|u_\theta(t)\|_2^2]\big)^{\frac12} + \big(\sup_{\varepsilon>0}\sup_{t>0}\mathbb{E}[\|u_\varepsilon(t)\|_2^2]\big)^{\frac12}\Big\} \to 0 \quad \text{as } l \to 0.\end{aligned}$$

One can justify the passage to the limit as $\varepsilon \to 0$ and $\theta \to 0$ in the sense of Young measures as in [2, 5] and conclude

$$\begin{aligned}&\lim_{\theta\to0}\lim_{\varepsilon\to0}\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\phi^\beta(u_\theta(s,x), u_\varepsilon(s,y))\Delta_x\varrho_\delta(x-y)\psi(s,y)\,dx\,dy\,ds\Big]\\ &\quad = \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_0^1\phi^\beta(\tilde{u}(s,x,\gamma), u(s,y,\alpha))\Delta_x\varrho_\delta(x-y)\psi(s,y)\,d\gamma\,d\alpha\,dx\,dy\,ds\Big].\end{aligned}$$

This proves the first part of the lemma.

To prove the second part, let us recall that

$$J_5 := -\mathbb{E}\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}}\beta'(u_\varepsilon * \tau_\kappa - k)\nabla(\phi(u_\varepsilon) * \tau_\kappa)\nabla_y\varphi_{\delta,\delta_0}J_l(u_\theta(t,x) - k)\,dk\,dx\,dt\,dy\,ds\Big].$$

A classical property of Lebesgue points and convolution yields

$$\lim_{\kappa\to0}\lim_{\delta_0\to0}J_5 = -\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\beta'(u_\varepsilon(s,y) - k)\nabla\phi(u_\varepsilon(s,y))\cdot\nabla_y[\psi(s,y)\varrho_\delta(x-y)] \times J_l(u_\theta(s,x) - k)\,dk\,dx\,dy\,ds\Big].$$

Recall that $\phi^\beta(a,b) = \int_b^a \beta'(s-b)\phi'(s)\,ds$. Making use of Green's type formula along with the Young measure theory and keeping in mind that $u(s,y,\alpha)$ and $\tilde{u}(s,x,\gamma)$ are Young measure-valued limit processes associated with the sequences $\{u_\varepsilon(s,y)\}_{\varepsilon>0}$ and $\{u_\theta(s,x)\}_{\theta>0}$, respectively, we arrive at the following conclusion:

$$\lim_{\theta\downarrow0}\lim_{\varepsilon\downarrow0}\lim_{l\downarrow0}\lim_{\kappa\to0}\lim_{\delta_0\to0}J_5 = \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_0^1\phi^\beta(u(s,y,\alpha), \tilde{u}(s,x,\gamma))\Delta_y[\psi(s,y)\varrho_\delta(x-y)]\,d\gamma\,d\alpha\,dx\,dy\,ds\Big].$$

This completes the proof of the lemma. □

Next, we want to pass to the limits in $(J_6 + J_7)$ and I_5, respectively. A slight modification of an argument used in [2, 3, 5] yields the following lemma.

Lemma 4.6. *It holds that*

$$\begin{aligned}\lim_{l\to0}\lim_{\kappa\to0}\lim_{\delta_0\to0}(J_6 + J_7) &= -\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}F^\beta(u_\varepsilon(s,y), u_\theta(s,x))\cdot\nabla_y[\psi(s,y)\varrho_\delta(x-y)]\,dx\,dy\,ds\Big]\\ &\xrightarrow{\varepsilon\to0} -\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1 F^\beta(u(s,y,\alpha), u_\theta(s,x))\cdot\nabla_y[\psi(s,y)\varrho_\delta(x-y)]\,d\alpha\,dx\,dy\,ds\Big]\\ &\xrightarrow{\theta\to0} -\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_0^1 F^\beta(u(s,y,\alpha), \tilde{u}(s,x,\gamma))\cdot\nabla_y[\psi(s,y)\varrho_\delta(x-y)]\,d\gamma\,d\alpha\,dx\,dy\,ds\Big]\end{aligned}$$

and

$$\lim_{\theta\to0}\lim_{\varepsilon\to0}\lim_{l\to0}\lim_{\kappa\to0}\lim_{\delta_0\to0}I_5 = -\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_0^1 F^\beta(\tilde{u}(s,x,\gamma), u(s,y,\alpha))\cdot\nabla_x[\varrho_\delta(x-y)\psi(s,y)]\,d\gamma\,d\alpha\,dx\,dy\,ds\Big].$$

In view of the above, we now want to pass to the limit as $(\vartheta, \delta) \to (0,0)$. We follow a line of argument similar to the proof of the second part of [5, Lemmas 5.7 and 5.8], and arrive at the following lemma.

Lemma 4.7. *Assume that $\vartheta \to 0$, $\delta \to 0$ and $\frac{\vartheta}{\delta} \to 0$. Then*

$$\lim_{\vartheta\to 0}\lim_{\delta\to 0}\lim_{\frac{\vartheta}{\delta}\to 0}\lim_{\theta\downarrow 0}\lim_{\varepsilon\downarrow 0}\lim_{l\to 0}\lim_{\kappa\to 0}\lim_{\delta_0\to 0}[(J_6+J_7)+I_5]$$
$$= -\mathbb{E}\Big[\int_{\Pi_T}\int_0^1\int_0^1 F(u(s,y,\alpha),\tilde{u}(s,y,\gamma))\cdot\nabla_y\psi(s,y)\,dy\,d\alpha\,dy\,ds\Big].$$

We now focus on $I_7 + J_8$ and establish the following lemma.

Lemma 4.8. *For fixed $\delta > 0$ and β, it holds that*

$$\limsup_{(\theta,\varepsilon,l,\kappa,\delta_0,)\to 0} |I_7 + J_8| = 0.$$

Proof. Note that

$$\begin{aligned}
|J_8| &\le \varepsilon\|\beta'\|_\infty\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}|\nabla_y(u_\varepsilon*\tau_\kappa(s,y))|\big|\nabla_y[\psi(s,y)\varrho_\delta(x-y)]\big|\,dy\,dx\,ds\Big]\\
&\le \varepsilon\|\beta'\|_\infty\mathbb{E}\Big[\int_{|y|\le K}\int_0^T\int_{\mathbb{R}^d}|\nabla_y(u_\varepsilon*\tau_\kappa(t,y))|\big|\nabla_y[\psi(t,y)\varrho_\delta(x-y)]\big|\,dx\,dt\,dy\Big]\\
&\le C(\beta)\varepsilon^{\frac12}\Big(\mathbb{E}\Big[\int_{\Pi_T}\varepsilon|\nabla_y(u_\varepsilon*\tau_\kappa(t,y))|^2\,dy\,dt\Big]\Big)^{\frac12}\Big(\mathbb{E}\Big[\int_K\int_{\Pi_T}\big|\nabla_y[\psi(t,y)\varrho_\delta(x-y)]\big|^2\,dx\,dt\,dy\Big]\Big)^{\frac12}\\
&\le C(\beta,\psi,\delta)\varepsilon^{\frac12}\Big(\sup_{\varepsilon>0}\mathbb{E}\Big[|\varepsilon\int_0^T\int_{\mathbb{R}^d}|\nabla_y u_\varepsilon(t,y)|^2\,dy\,dt|\Big]\Big)^{\frac12} \le C(\beta,\psi,\delta)\varepsilon^{\frac12},
\end{aligned}$$

where the second line follows by the Cauchy–Schwarz inequality. Similarly, we have $|I_7| \le C(\beta,\psi,\delta)\theta^{1/2}$. This completes the proof. □

Next we consider the stochastic terms $I_3 + J_3$. To this end, we cite [5] and assert that for two constants $T_1, T_2 \ge 0$ with $T_1 < T_2$,

$$\mathbb{E}\Big[X_{T_1}\int_{T_1}^{T_2}\int_E J(t,z)\,\tilde{N}(dz,dt)\Big] = 0, \tag{4.7}$$

where J is a predictable integrand and X is an adapted process.

For each $\beta \in C^\infty(\mathbb{R})$ with $\beta', \beta'' \in C_b(\mathbb{R})$ and each non-negative $\varphi \in C_c^\infty(\Pi_\infty \times \Pi_\infty)$, we define

$$M[\beta,\varphi](s;y,v) := \int_0^T\int_E\int_{\mathbb{R}^d}\{\beta(u_\theta(r,x)+\eta(x,u_\theta(r,x);z)-v)-\beta(u_\theta(r,x)-v)\}\times\varphi(r,x,s,y)\,dx\,\tilde{N}(dz,dr),$$

where $0 \le s \le T$, $(y,v) \in \mathbb{R}^d \times \mathbb{R}$. Furthermore, we extend the process $u_\varepsilon * \tau_\kappa(\cdot, y)$ for negative time simply by $u_\varepsilon * \tau_\kappa(s,y) = u_\varepsilon(0,\cdot) * \tau_k(y)$ if $s < 0$. With this convention, it follows from (4.7) that

$$\mathbb{E}\Big[\int_{\mathbb{R}_v}\int_{\Pi_T} M[\beta,\varphi_{\delta,\delta_0}](s;y,v)J_l(u_\varepsilon(s-\delta_0,y)-v)\,dy\,ds\,dv\Big] = 0,$$

and hence we have $J_3 = 0$ and

$$I_3 = \mathbb{E}\Big[\int_{\mathbb{R}}\int_{\Pi_T} M[\beta,\varphi_{\delta,\delta_0}](s;y,v)(J_l(u_\varepsilon*\tau_\kappa(s,y)-v)-J_l(u_\varepsilon*\tau_\kappa(s-\delta_0,y)-v))\,dy\,ds\,dv\Big]. \tag{4.8}$$

Our aim is to pass to the limit into the stochastic terms $I_3 + J_3$. This requires the following moment estimate of $M[\beta, \varphi_{\delta,\delta_0}]$, a proof of which can be found in [5].

Lemma 4.9. *Let $\gamma \in C^\infty(\mathbb{R})$ be a function such that $\gamma' \in C_c^\infty(\mathbb{R})$ and let p be a positive integer of the form $p = 2^k$ for some $k \in \mathbb{N}$. If $p \geq d + 3$, then there exists a constant $C = C(\gamma', \psi, \delta)$ such that*

$$\sup_{0\leq s\leq T} (\mathbb{E}[\|M[\gamma, \varphi_{\delta,\delta_0}](s; \cdot, \cdot)\|^2_{L^\infty(\mathbb{R}^d\times\mathbb{R})}]) \leq \frac{C(\gamma', \psi, \delta)}{\delta_0^{2(p-1)/p}},$$

and the following identities hold:

$$\begin{aligned} \partial_v M[\gamma, \varphi](s; y, v) &= M[-\gamma', \varphi](s; y, v), \\ \partial_{y_k} M[\gamma, \varphi](s; y, v) &= M[\gamma, \partial_{y_k}\varphi](s; y, v). \end{aligned}$$

Lemma 4.10. *It holds that*

$$\begin{aligned} \lim_{l\to0}\lim_{\kappa\to0}\lim_{\delta_0\to0} I_3 = \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_E \Big\{&\beta(u_\theta(r, x) + \eta(x, u_\theta(r, x); z) - u_\varepsilon(r, y) - \eta(y, u_\varepsilon(r, y); z)) \\ &- \beta(u_\theta(r, x) - u_\varepsilon(r, y) - \eta(y, u_\varepsilon(r, y); z)) + \beta(u_\theta(r, x) - u_\varepsilon(r, y)) \\ &- \beta(u_\theta(r, x) + \eta(x, u_\theta(r, x); z) - u_\varepsilon(r, y))\Big\}\psi(r, y)\varrho_\delta(x - y)\, m(dz)\, dx\, dy\, dr\Big]. \end{aligned}$$

Proof. Note that $u_\varepsilon * \tau_\kappa(\cdot, y)$ satisfies for all $y \in \mathbb{R}^d$,

$$\begin{aligned} du_\varepsilon * \tau_\kappa(s, y) = \operatorname{div}(f(u_\varepsilon) * \tau_\kappa(s, y))\, ds + \Delta(\phi(u_\varepsilon) * \tau_\kappa(s, y))\, ds \\ + \varepsilon\Delta u_\varepsilon * \tau_\kappa(s, y)\, ds + \int_E (\eta(\cdot, u_\varepsilon; z) * \tau_\kappa(s, y))\, \tilde{N}(dz, ds). \end{aligned}$$

Now apply the Itô–Lévy formula on $J_l(u_\varepsilon * \tau_\kappa(s, y) - v)$ to get

$$\begin{aligned} &J_l(u_\varepsilon * \tau_\kappa(s, y) - v) - J_l(u_\varepsilon * \tau_\kappa(s - \delta_0, y) - v) \\ &= \int_{s-\delta_0}^{s} J_l'(u_\varepsilon * \tau_\kappa(\sigma, y) - v)(\operatorname{div}(f(u_\varepsilon) * \tau_\kappa(\sigma, y)) + \varepsilon\Delta(u_\varepsilon * \tau_\kappa(\sigma, y)) + \Delta(\phi(u_\varepsilon) * \tau_\kappa(\sigma, y)))\, d\sigma \\ &\quad + \int_{s-\delta_0}^{s}\int_E \Big(J_l(u_\varepsilon * \tau_\kappa(\sigma, y) + (\eta(\cdot, u_\varepsilon; z) * \tau_\kappa(\sigma, y)) - v) - J_l(u_\varepsilon * \tau_\kappa(\sigma, y) - v)\Big)\, \tilde{N}(dz, d\sigma) \\ &\quad + \int_{s-\delta_0}^{s}\int_E\int_{\lambda=0}^{1} (1 - \lambda)J_l''(u_\varepsilon * \tau_\kappa(\sigma, y) - v + \lambda(\eta(\cdot, u_\varepsilon; z) * \tau_\kappa(\sigma, y))) \\ &\qquad\qquad \times (\eta(\cdot, u_\varepsilon; z) * \tau_\kappa(\sigma, y))^2\, d\lambda\, m(dz)\, d\sigma \\ &= -\frac{\partial}{\partial v}\int_{s-\delta_0}^{s} (\operatorname{div}(f(u_\varepsilon) * \tau_\kappa(\sigma, y)) + \varepsilon\Delta(u_\varepsilon * \tau_\kappa(\sigma, y)) + \Delta(\phi(u_\varepsilon) * \tau_\kappa(\sigma, y)))J_l(u_\varepsilon * \tau_\kappa(\sigma, y) - v)\, d\sigma \\ &\quad + \int_{s-\delta_0}^{s}\int_E \Big(J_l(u_\varepsilon * \tau_\kappa(\sigma, y) + (\eta(\cdot, u_\varepsilon; z) * \tau_\kappa(\sigma, y)) - v) - J_l(u_\varepsilon * \tau_\kappa(\sigma, y) - v)\Big)\, \tilde{N}(dz, d\sigma) \\ &\quad + \int_{s-\delta_0}^{s}\int_E\int_{\lambda=0}^{1} (1 - \lambda)J_l''(u_\varepsilon * \tau_\kappa(\sigma, y) - v + \lambda(\eta(\cdot, u_\varepsilon; z) * \tau_\kappa(\sigma, y))) \\ &\qquad\qquad \times (\eta(\cdot, u_\varepsilon; z) * \tau_\kappa(\sigma, y))^2\, d\lambda\, m(dz)\, d\sigma. \end{aligned}$$

Therefore, from (4.8) and Lemma 4.9 we have

$$
\begin{aligned}
I_3 = &-\mathbb{E}\Big[\int_{\mathbb{R}}\int_{\Pi_T} M[\beta',\varphi_{\delta,\delta_0}](s;y,v)\Big(\int_{s-\delta_0}^{s} J_l(u_\varepsilon * \tau_\kappa(\sigma,y)-v)\operatorname{div}(f(u_\varepsilon)*\tau_\kappa(\sigma,y))\,d\sigma\Big)\,ds\,dy\,dv\Big]\\
&-\mathbb{E}\Big[\int_{\mathbb{R}}\int_{\Pi_T} M[\beta',\varphi_{\delta,\delta_0}](s;y,v)\Big(\int_{s-\delta_0}^{s} J_l(u_\varepsilon * \tau_\kappa(\sigma,y)-v)\varepsilon\Delta(u_\varepsilon*\tau_\kappa(\sigma,y))\,d\sigma\Big)\,ds\,dy\,dv\Big]\\
&-\mathbb{E}\Big[\int_{\mathbb{R}}\int_{\Pi_T} M[\beta',\varphi_{\delta,\delta_0}](s;y,v)\Big(\int_{s-\delta_0}^{s} J_l(u_\varepsilon * \tau_\kappa(\sigma,y)-v)\Delta(\phi(u_\varepsilon)*\tau_\kappa(\sigma,y))\,d\sigma\Big)\,ds\,dy\,dv\Big]\\
&+\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}}\int_{s-\delta_0}^{s}\int_{\mathbb{R}^d}\int_E \big(\beta(u_\theta(r,x)+\eta(x,u_\theta(r,x);z)-v)-\beta(u_\theta(r,x)-v)\big)\\
&\qquad\times\big(J_l(u_\varepsilon*\tau_\kappa(r,y)+\eta(\cdot,u_\varepsilon;z)*\tau_\kappa(r,y)-v)-J_l(u_\varepsilon*\tau_\kappa(r,y)-v)\big)\\
&\qquad\times\rho_{\delta_0}(r-s)\psi(s,y)\varrho_\delta(x-y)\,m(dz)\,dx\,dr\,dv\,dy\,ds\Big]\\
&+\mathbb{E}\Big[\int_{\mathbb{R}}\int_{\Pi_T} M[\beta,\phi_{\delta,\delta_0}](s;y,v)\Big\{\int_{s-\delta_0}^{s}\int_E\int_{\lambda=0}^{1} J_l''(u_\varepsilon*\tau_\kappa(\sigma,y)-v+\lambda(\eta(\cdot,u_\varepsilon;z)*\tau_\kappa(\sigma,y)))\\
&\qquad\times(1-\lambda)(\eta(\cdot,u_\varepsilon;z)*\tau_\kappa(\sigma,y))^2\,d\lambda\,m(dz)\,d\sigma\Big\}\,dy\,ds\,dv\Big]\\
\equiv{}& A_1^{\kappa,l,\varepsilon}(\delta,\delta_0)+A_2^{\kappa,l,\varepsilon}(\delta,\delta_0)+A_3^{\kappa,l,\varepsilon}(\delta,\delta_0)+B^{\varepsilon,l,\kappa}+A_4^{\kappa,l,\varepsilon}(\delta,\delta_0).
\end{aligned}
$$

Note that $\Delta(u_\varepsilon * \tau_\kappa)\in L^2(\Omega\times\Pi_T)$ for fixed κ and ε. One can use Young's inequality for convolution and replace u_ε by $u_\varepsilon * \tau_\kappa$ to adapt the same line of argument as in [5] and conclude

$$
A_1^{\kappa,l,\varepsilon}(\delta,\delta_0)\to 0,\quad A_2^{\kappa,l,\varepsilon}(\delta,\delta_0)\to 0,\quad A_3^{\kappa,l,\varepsilon}(\delta,\delta_0)\to 0,\quad A_4^{\kappa,l,\varepsilon}(\delta,\delta_0)\to 0\qquad \text{as } \delta_0\to 0.
$$

Again, it is routine to pass to the limit in $B^{\varepsilon,l,\kappa}$ and arrive at the conclusion that

$$
\begin{aligned}
\lim_{l\to0}\lim_{\kappa\to0}\lim_{\delta_0\to0} B^{\varepsilon,l,\kappa} = \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_E &\Big\{\beta(u_\theta(r,x)+\eta(x,u_\theta(r,x);z)-u_\varepsilon(r,y)-\eta(y,u_\varepsilon(r,y);z))\\
&-\beta(u_\theta(r,x)-u_\varepsilon(r,y)-\eta(y,u_\varepsilon(r,y);z))+\beta(u_\theta(r,x)-u_\varepsilon(r,y))\\
&-\beta(u_\theta(r,x)+\eta(x,u_\theta(r,x);z)-u_\varepsilon(r,y))\Big\}\psi(r,y)\varrho_\delta(x-y)\,m(dz)\,dx\,dy\,dr\Big].
\end{aligned}
$$

This completes the proof. □

Let us consider the additional terms I_4+J_4. A line of arguments similar to the ones in [2, 3, 5] and classical properties of convolution yield the following lemma.

Lemma 4.11. *It holds that*

$$
\begin{aligned}
\lim_{l\to0}\lim_{\kappa\to0}\lim_{\delta_0\to0} J_4 = \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_E\int_{\lambda=0}^{1} &(1-\lambda)\beta''(u_\varepsilon(s,y)-u_\theta(s,x)+\lambda\eta(y,u_\varepsilon(s,y);z))\\
&\times\eta^2(y,u_\varepsilon(s,y);z)\psi(s,y)\varrho_\delta(x-y)\,d\lambda\,m(dz)\,dx\,dy\,ds\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{l\to0}\lim_{\kappa\to0}\lim_{\delta_0\to0} I_4 = \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_E\int_{\lambda=0}^{1} &(1-\lambda)\beta''(u_\theta(s,x)-u_\varepsilon(s,y)+\lambda\eta(x,u_\theta(s,x);z))\\
&\times\eta^2(x,u_\theta(s,x);z)\psi(s,x)\varrho_\delta(x-y)\,d\lambda\,m(dz)\,dx\,dy\,ds\Big].
\end{aligned}
$$

Now we add these terms (cf. Lemma 4.11) with the terms resulting from Lemma 4.10 and have the following lemma.

Lemma 4.12. *Assume that $\vartheta \to 0^+$, $\delta \to 0^+$ and $\vartheta^{-1}\delta^2 \to 0^+$. Then*

$$\limsup_{\vartheta\to 0^+,\,\delta\to 0^+,\,\vartheta^{-1}\delta^2\to 0^+} \limsup_{\theta,\varepsilon\to 0}\Big[\lim_{l\to 0}\lim_{\kappa\to 0}\lim_{\delta_0\to 0}((I_3+J_3)+(I_4+J_4))\Big]=0.$$

Proof. In view of Lemmas 4.10 and 4.11, we see that

$$\begin{aligned}
&\lim_{l\to 0}\lim_{\kappa\to 0}\lim_{\delta_0\to 0}((I_3+J_3)+(I_4+J_4))\\
&\quad= \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\Big(\int_E\Big\{\beta(u_\theta(t,x)-u_\varepsilon(t,y)+\eta(x,u_\theta(t,x);z)-\eta(y,u_\varepsilon(t,y);z))\\
&\qquad\qquad -(\eta(x,u_\theta(t,x);z)-\eta(y,u_\varepsilon(t,y);z))\beta'(u_\theta(t,x)-u_\varepsilon(t,y))\\
&\qquad\qquad -\beta(u_\theta(t,x)-u_\varepsilon(t,y))\Big\}\,m(dz)\Big)\psi(t,y)\varrho_\delta(x-y)\,dx\,dy\,dt\Big]\\
&\quad= \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\Big(\int_E\int_{\tau=0}^1 b^2(1-\tau)\beta''(a+\tau b)\,d\tau\,m(dz)\Big)\psi(t,y)\varrho_\delta(x-y)\,dx\,dy\,dt\Big],
\end{aligned}\tag{4.9}$$

where

$$a=u_\theta(t,x)-u_\varepsilon(t,y)\quad\text{and}\quad b=\eta(x,u_\theta(t,x);z)-\eta(y,u_\varepsilon(t,y);z).$$

By using arguments similar to the ones used in the proof of [5, Lemma 5.11], we arrive at

$$\lim_{l\to 0}\lim_{\kappa\to 0}\lim_{\delta_0\to 0}((I_3+J_3)+(I_4+J_4))\le C_1(\vartheta+\vartheta^{-1}\delta^2)T,$$

where the constant C_1 depends only on ψ and is in particular independent of ε. We now let $\vartheta\to 0$, $\delta\to 0$ and $\vartheta^{-1}\delta^2\to 0$, yielding

$$\limsup_{\vartheta\to 0,\delta\to 0,\vartheta^{-1}\delta^2\to 0}\limsup_{\theta,\varepsilon\to 0}\Big[\lim_{l\to 0}\lim_{\kappa\to 0}\lim_{\delta_0\to 0}((I_3+J_3)+(I_4+J_4))\Big]\le 0.$$

This concludes the proof as

$$\lim_{l\to 0}\lim_{\kappa\to 0}\lim_{\delta_0\to 0}((I_3+J_3)+(I_4+J_4))\ge 0,$$

thanks to (4.9). □

We now turn our attention back to the terms which are originating from Lemmas 4.2 and 4.5. To this end, define

$$\begin{aligned}
\mathcal{H} := -2\mathbb{E}\Big[&\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_0^1\int_{\tilde u(s,x,y)}^{u(s,y,\alpha)}\Big(\int_{r=\sigma}^{\tilde u(s,x,y)}\beta''(\sigma-r)\sqrt{\phi'(r)}\,dr\Big)\sqrt{\phi'(\sigma)}\,d\sigma\\
&\times \operatorname{div}_y\nabla_x[\psi(s,y)\varrho_\delta(x-y)]\,d\gamma\,d\alpha\,dx\,dy\,ds\Big]\\
&+\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_0^1\phi^\beta(\tilde u(s,x,\gamma),u(s,y,\alpha))\Delta_x\varrho_\delta(x-y)\psi(s,y)\,d\gamma\,d\alpha\,dx\,dy\,ds\Big]\\
&+\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_0^1\phi^\beta(u(s,y,\alpha),\tilde u(s,x,\gamma))\Delta_y[\psi(s,y)\varrho_\delta(x-y)]\,d\gamma\,d\alpha\,dx\,dy\,ds\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_0^1 \Big\{2I_\beta(\tilde u(s,x,y), u(s,y,\alpha)) + \phi^\beta(\tilde u(s,x,y), u(s,y,\alpha)) \\
&\qquad\qquad + \phi^\beta(u(s,y,\alpha), \tilde u(s,x,y))\Big\}\psi(s,y)\Delta_y\varrho_\delta(x-y)\,dy\,d\alpha\,dx\,dy\,ds\Big] \\
&\quad + \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_0^1 \big(2I_\beta(u(s,y,\alpha), \tilde u(s,x,y)) + 2\phi^\beta(\tilde u(s,x,y), u(s,y,\alpha))\big) \\
&\qquad\qquad \times \nabla_y\psi(s,y)\cdot\nabla_y\varrho_\delta(x-y)\,dy\,d\alpha\,dx\,dy\,ds\Big] \\
&\quad + \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_0^1 \phi^\beta(u(s,y,\alpha), \tilde u(s,x,y))\Delta_y\psi(s,y)\varrho_\delta(x-y)\,dy\,d\alpha\,dx\,dy\,ds\Big] \\
&\equiv \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3,
\end{aligned}
$$

where

$$
I_\beta(a,b) = \int_a^b\int_\mu^a \beta''(\mu-\sigma)\sqrt{\phi'(\sigma)}\,d\sigma\sqrt{\phi'(\mu)}\,d\mu \quad \text{for any } a, b \in \mathbb{R}.
$$

Our aim is to pass to the limit in $\mathcal{H}$ as $(\vartheta,\delta)\to(0,0)$. For this, we need some a priori estimates on $I_\beta(a,b)$. Here we state the required lemma whose proof can be found in [3].

Lemma 4.13. *The following holds:*

$$
I_\beta(a,b) = I_\beta(b,a) \quad \textit{and} \quad I_\beta(a,b) = -\frac{1}{2}\int_a^b\int_a^b \beta''(\sigma-\mu)\sqrt{\phi'(\mu)}\sqrt{\phi'(\sigma)}\,d\mu\,d\sigma,
$$

$$
2I_\beta(a,b) + \phi^\beta(a,b) + \phi^\beta(b,a) = \frac{1}{2}\int_a^b\int_a^b \beta''(\mu-\sigma)\big[\sqrt{\phi'(\mu)} - \sqrt{\phi'(\sigma)}\big]^2\,d\mu\,d\sigma.
$$

Moreover, if $\sqrt{\phi'}$ has a modulus of continuity ω_ϕ, then

$$
2I_\beta(a,b) + \phi^\beta(a,b) + \phi^\beta(b,a) \le C|b-a|\,|\omega_\phi(|\vartheta|)|^2
$$

and

$$
2I_\beta(a,b) + \phi^\beta(b,a) \le C|b-a|\,|\omega_\phi(|\vartheta|)|^2 + C\min\{2\vartheta, |b-a|\}.
$$

We now shift our focus back to the expression $\mathcal{H}$ and prove the following lemma.

Lemma 4.14. *It holds*

$$
\lim_{(\vartheta,\delta)\to(0,0)} \mathcal{H} = \mathbb{E}\Big[\int_{\Pi_T}\int_0^1\int_0^1 |\phi(u(s,y,\alpha)) - \phi(\tilde u(s,y,\gamma))|\Delta_y\psi(s,y)\,dy\,d\alpha\,dy\,ds\Big].
$$

Proof. Let ω_ϕ be a modulus of continuity of $\sqrt{\phi'}$. Then, thanks to Lemma 4.13, we obtain

$$
\begin{aligned}
|\mathcal{H}_1| &\le C\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_0^1 |\omega_\phi(|\vartheta|)|^2|u(s,y,\alpha) - \tilde u(s,x,y)|\psi(s,y)|\Delta_y\varrho_\delta(x-y)|\,dy\,d\alpha\,dx\,dy\,ds\Big] \\
&\le C(\psi)\frac{|\omega_\phi(|\vartheta|)|^2}{\delta^2}
\end{aligned}
$$

and

$$|\mathcal{H}_2| \le C\mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d}\int_0^1\int_0^1 |\omega_\phi(|\vartheta|)|^2 |u(s,y,\alpha)-\tilde{u}(s,x,\gamma)|\,|\nabla_y\psi(s,y)|\,|\nabla_y\varrho_\delta(x-y)|\,d\gamma\,d\alpha\,dx\,dy\,ds\Big]$$
$$\quad + \mathbb{E}\Big[\int_{\Pi_T}\int_{\mathbb{R}^d} C\vartheta|\nabla_y\psi(s,y)|\,|\nabla_y\varrho_\delta(x-y)|\,dx\,dy\,ds\Big]$$
$$\le C(\psi)\frac{|\omega_\phi(|\vartheta|)|^2}{\delta} + C\frac{\vartheta}{\delta}.$$

Hence, we have

$$|\mathcal{H}_1| + |\mathcal{H}_2| \le C(\psi)\frac{|\omega_\phi(|\vartheta|)|^2}{\delta^2} + C(\psi)\frac{|\omega_\phi(|\vartheta|)|^2}{\delta} + C\frac{\vartheta}{\delta}.$$

Put $\delta = \vartheta^{2/3}$. Then, by our assumption (i), we see that

$$\lim_{(\vartheta,\delta)\to(0,0)} (\mathcal{H}_1 + \mathcal{H}_2) = 0.$$

To conclude the proof, it is now required to show

$$\lim_{(\vartheta,\delta)\to(0,0)} \mathcal{H}_3 = \mathbb{E}\Big[\int_{\Pi_T}\int_0^1\int_0^1 |\phi(u(s,y,\alpha)) - \phi(u(s,y,\gamma))|\Delta_y\psi(s,y)\,d\gamma\,d\alpha\,dy\,ds\Big],$$

which follows easily from the fact that $(a,b)\mapsto|\phi(a)-\phi(b)|$ is Lipschitz continuous and that

$$\big|\phi^{\beta_\vartheta}(a,b) - |\phi(a)-\phi(b)|\big| \le C\vartheta \quad \text{for any } a, b \in \mathbb{R}. \qquad \square$$

The following proposition combines all of the above results.

Proposition 4.15. *Let $\tilde{u}(t,x,\gamma)$ and $u(t,x,\alpha)$ be the predictable process with initial data $v(0,x)$ and $u(0,x)$, respectively, which have been extracted out of Young measure-valued sub-sequential limits of the sequences $\{u_\theta(t,x)\}_{\theta>0}$ and $\{u_\varepsilon(t,x)\}_{\varepsilon>0}$, respectively. Then, for any non-negative $H^1([0,\infty)\times\mathbb{R}^d)$ function $\psi(t,x)$ with compact support, the following inequality holds:*

$$0 \le \mathbb{E}\Big[\int_{\mathbb{R}^d} |v_0(x)-u_0(x)|\psi(0,x)\,dx\Big] + \mathbb{E}\Big[\int_{\Pi_T}\int_0^1\int_0^1 |\tilde{u}(t,x,\gamma)-u(t,x,\alpha)|\partial_t\psi(t,x)\,d\gamma\,d\alpha\,dx\,dt\Big]$$
$$- \mathbb{E}\Big[\int_{\Pi_T}\int_0^1\int_0^1 F(u(t,x,\alpha),\tilde{u}(t,x,\gamma))\cdot\nabla_x\psi(t,x)\,d\gamma\,d\alpha\,dx\,dt\Big]$$
$$- \mathbb{E}\Big[\int_{\Pi_T}\nabla\Big(\int_0^1\int_0^1 |\phi(u(t,x,\alpha)) - \phi(\tilde{u}(t,x,\gamma))|\,d\gamma\,d\alpha\Big)\cdot\nabla_x\psi(t,x)\,dx\,dt\Big]. \tag{4.10}$$

Proof. First we add (4.2) and (4.3) and then pass to the limit $\lim_{\varepsilon\downarrow 0}\lim_{l\to 0}\lim_{\kappa\to 0}\lim_{\delta_0\downarrow 0}$. Invoking Lemmas 4.2, 4.3, 4.4, 4.5, 4.6, 4.8 and 4.12, we put $\delta = \vartheta^{2/3}$ in the resulting expression and then let $\vartheta\to 0$ with the second parts of Lemmas 4.3 and 4.4. Keeping in mind Lemmas 4.7, 4.12 and 4.14, we conclude that (4.10) holds for any non-negative test function $\psi \in C_c^2([0,\infty)\times\mathbb{R}^d)$. It now follows by routine approximation arguments that (4.10) holds for any ψ with compact support such that $\psi\in H^1([0,\infty)\times\mathbb{R}^d)$. This completes the proof. $\square$

Remark 4.16. Note that the same proposition holds without assuming the existence of a modulus of continuity for ϕ' if η is not a function of x. Indeed, it is possible to pass to the limit first in the parameter ϑ, then δ, in Lemmas 4.3, 4.4, 4.7 and 4.12. Thus, if one assumes that η is not a function of the space variable x, it is also possible to pass to the limit first in the parameter ϑ, then δ, in Lemma 4.12 since one would have

$$\lim_{l\to 0}\lim_{\kappa\to 0}\lim_{\delta_0\to 0}((I_3+J_3)+(I_4+J_4)) \le C_1\vartheta T$$

in its proof. Then the result holds following the first situation on [3, p. 523].

Our aim is to show the uniqueness of $u(t,x,\alpha)$ and $\tilde{u}(t,x,\gamma)$. To do this, we follow the ideas of [1, 3], and define for each $n \in \mathbb{N}$,

$$\phi_n(x) = \begin{cases} 1 & \text{if } |x| \le n, \\ \dfrac{n^a}{|x|^a} & \text{if } |x| > n, \end{cases}$$

where $a = \frac{d}{2} + \tilde{\varepsilon}$ in which $\tilde{\varepsilon} > 0$ could be chosen in such a way that $\phi_n \in L^2(\mathbb{R}^d)$. Also, for each $h > 0$ and fixed $t \ge 0$, we define

$$\psi_h^t(s) = \begin{cases} 1 & \text{if } s \le t, \\ 1 - \dfrac{s-t}{h} & \text{if } t \le s \le t+h, \\ 0 & \text{if } s > t+h. \end{cases}$$

A straightforward calculation revels that

$$\begin{aligned} \nabla\phi_n(x) &= -a\frac{\phi_n(x)}{|x|}\frac{x}{|x|}\mathbf{1}_{|x|>n} \in L^2(\mathbb{R}^d)^d, \\ \Delta\phi_n(x) &= a(2+2\tilde{\varepsilon}-a)\frac{\phi_n(x)}{|x|^2} \in L^2(\{|x|>n\}). \end{aligned}$$

Clearly, (4.10) holds with $\psi(s,x) = \phi_n(x)\psi_h^t(s)$. Thus, for a.e $t \ge 0$, we obtain

$$\begin{aligned} \frac{1}{h}\int_t^{t+h} &\mathbb{E}\Big[\int_{\mathbb{R}^d}\int_0^1\int_0^1 |u(s,x,\alpha)-\tilde{u}(s,x,\gamma)|\phi_n(x)\,d\gamma\,d\alpha\,dx\Big]\,ds \\ &\le \mathbb{E}\Big[\int_0^T\int_{\{|x|>n\}}\int_0^1\int_0^1 |\phi(u(s,x,\alpha))-\phi(\tilde{u}(s,x,\gamma))|\Delta\phi_n(x)\psi_h^t(s)\,d\gamma\,d\alpha\,dx\,ds\Big] \\ &\quad - \mathbb{E}\Big[\int_0^T\int_{\{|x|>n\}}\int_0^1\int_0^1 F(u(s,x,\alpha),\tilde{u}(s,x,\gamma))\cdot\nabla\phi_n(x)\psi_h^t(s)\,d\gamma\,d\alpha\,dx\,ds\Big] \\ &\quad - \mathbb{E}\Big[\int_0^T \psi_h^t(s)\Big(\int_{\partial\{|x|>n\}}\int_0^1\int_0^1 |\phi(u(s,x,\alpha))-\phi(\tilde{u}(s,x,\gamma))|\nabla\phi_n(x)\cdot\tilde{n}\,d\gamma\,d\alpha\,dx\Big)\,ds\Big] \\ &\quad + \mathbb{E}\Big[\int_{\mathbb{R}^d}|v_0(x)-u_0(x)|\phi_n(x)\,dx\Big]. \end{aligned} \tag{4.11}$$

Since $\nabla\phi_n(x)\cdot\tilde{n} = \frac{a}{n} > 0$ on the set $\partial\{|x|>n\}$, we have from (4.11),

$$\begin{aligned} \frac{1}{h}\int_t^{t+h} &\mathbb{E}\Big[\int_{\mathbb{R}^d}\int_0^1\int_0^1 |u(s,x,\alpha)-\tilde{u}(s,x,\gamma)|\phi_n(x)\,d\gamma\,d\alpha\,dx\Big]\,ds \\ &\le \mathbb{E}\Big[\int_0^T\int_{\{|x|>n\}}\int_0^1\int_0^1 a(2+2\tilde{\varepsilon}-a)\frac{\phi_n(x)}{|x|^2}|\phi(u(s,x,\alpha))-\phi(\tilde{u}(s,x,\gamma))|\psi_h^t(s)\,d\gamma\,d\alpha\,dx\,ds\Big] \\ &\quad + \mathbb{E}\Big[\int_0^T\int_{\{|x|>n\}}\int_0^1\int_0^1 a\frac{\phi_n(x)}{|x|}F(u(s,x,\alpha),\tilde{u}(s,x,\gamma))\cdot\frac{x}{|x|}\psi_h^t(s)\,d\gamma\,d\alpha\,dx\,ds\Big] \\ &\quad + \mathbb{E}\Big[\int_{\mathbb{R}^d}|v_0(x)-u_0(x)|\phi_n(x)\,dx\Big]. \end{aligned} \tag{4.12}$$

Note that $|F(a,b)| \le c_f|a-b|$ for any $a, b \in \mathbb{R}$. Since ϕ is a Lipschitz continuous function and $n \ge 1$, inequality (4.12) gives

$$\frac{1}{h}\int_t^{t+h} \mathbb{E}\Big[\int_{\mathbb{R}^d}\int_0^1\int_0^1 |u(s,x,\alpha)-\tilde{u}(s,x,y)|\phi_n(x)\,dy\,d\alpha\,dx\Big]\,ds$$
$$\le C\mathbb{E}\Big[\int_{\Pi_T}\int_{[0,1]^2} |u(s,x,\alpha)-\tilde{u}(s,x,y)|\phi_n(x)\psi_h^t(s)\,dy\,d\alpha\,dx\,ds\Big] + \mathbb{E}\Big[\int_{\mathbb{R}^d} |v_0(x)-u_0(x)|\phi_n(x)\,dx\Big].$$

Now passing to the limit as $h \to 0$, and then using a weaker version of Gronwall's inequality, we obtain for a.e. $t > 0$,

$$\mathbb{E}\Big[\int_{\mathbb{R}^d}\int_0^1\int_0^1 |u(t,x,\alpha)-\tilde{u}(t,x,y)|\phi_n(x)\,dy\,d\alpha\,dx\Big] \le e^{CT}\mathbb{E}\Big[\int_{\mathbb{R}^d} |v_0(x)-u_0(x)|\phi_n(x)\,dx\Big].$$

Thus, if we assume that $v_0(x) = u_0(x)$, then we arrive at the conclusion

$$\mathbb{E}\Big[\int_{\mathbb{R}^d}\int_0^1\int_0^1 |u(t,x,\alpha)-\tilde{u}(t,x,y)|\phi_n(x)\,dy\,d\alpha\,dx\Big] = 0,$$

which says that for almost all $\omega \in \Omega$, a.e. $(t,x) \in (0,T] \times \mathbb{R}^d$ and a.e. $(\alpha, y) \in [0,1]^2$, we have the equation $u(t,x,\alpha) = \tilde{u}(t,x,y)$. On the other hand, we conclude that the whole sequence of viscous approximation converges weakly in $L^2(\Omega \times \Pi_T)$. Since the limit process is independent of the additional (dummy) variable, the viscous approximation converges strongly in $L^p(\Omega \times (0,T); L^p(\Theta))$ for any $p < 2$ and any bounded open set $\Theta \subset \mathbb{R}^d$.

4.2 Existence of an entropy solution

In this subsection, using the strong convergence of the sequence of viscous solutions and the a priori bounds (3.17), we establish the existence of an entropy solution to problem (1.1).

Fix a non-negative test function $\psi \in C_c^\infty([0,\infty) \times \mathbb{R}^d)$, $B \in \mathcal{F}_T$ and a convex entropy flux triple (β, ζ, ν). Now apply the Itô–Lévy formula (3.1) and conclude

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_B \int_{\Pi_T} \beta''(u_\varepsilon(t,x))|\nabla G(u_\varepsilon(t,x))|^2\psi(t,x)\,dx\,dt\Big] \\
&\quad\le \mathbb{E}\Big[\mathbf{1}_B \int_{\mathbb{R}^d} \beta(u_\varepsilon(0,x))\psi(0,x)\,dx\Big] - \varepsilon\mathbb{E}\Big[\mathbf{1}_B \int_{\Pi_T} \beta'(u_\varepsilon(t,x))\nabla u_\varepsilon(t,x)\cdot\nabla\psi(t,x)\,dx\,dt\Big] \\
&\qquad + \mathbb{E}\Big[\mathbf{1}_B \int_{\Pi_T} (\beta(u_\varepsilon(t,x))\partial_t\psi(t,x) + \nu(u_\varepsilon(t,x))\Delta\psi(t,x) - \nabla\psi(t,x)\cdot\zeta(u_\varepsilon(t,x)))\,dx\,dt\Big] \\
&\qquad + \mathbb{E}\Big[\mathbf{1}_B \int_0^T\int_E\int_{\mathbb{R}^d}\int_0^1 \eta(x,u_\varepsilon(t,x);z)\beta'(u_\varepsilon(t,x)+\theta\eta(x,u_\varepsilon(t,x);z))\psi(t,x)\,d\theta\,dx\,\tilde{N}(dz,dt)\Big] \\
&\qquad + \mathbb{E}\Big[\mathbf{1}_B \int_0^T\int_E\int_{\mathbb{R}^d}\int_0^1 (1-\theta)\eta^2(x,u_\varepsilon(t,x);z)\beta''(u_\varepsilon(t,x)+\theta\eta(x,u_\varepsilon(t,x);z)) \\
&\qquad\qquad \times \psi(t,x)\,d\theta\,dx\,m(dz)\,dt\Big].
\end{aligned} \tag{4.13}$$

Let the predictable process $u(t,x)$ be the pointwise limit of $u_\varepsilon(t,x)$ for a.e. $(t,x) \in (0,T) \times \mathbb{R}^d$ almost surely. One can now pass to the limit in (4.13) (same argument as in [5]) except the first term. The pointwise limit of $u_\varepsilon(t,x)$ is not enough to pass to the limit in the first term of the inequality because u_ε is in a gradient term.

For this, we proceed as follows: Fix $v \in L^2(\Omega \times \Pi_T)$. Define

$$f_\varepsilon = \sqrt{\beta''(u_\varepsilon(t,x))\psi(t,x)\mathbf{1}_B} \quad \text{and} \quad g_\varepsilon = \nabla G(u_\varepsilon(t,x)).$$

Note that f_ε is uniformly bounded and $g_\varepsilon \rightharpoonup g = \nabla G(u(t,x))$ in $L^2(\Omega \times \Pi_T)$. Also, f_ε converges to f pointwise (up to a subsequence), where

$$f = \sqrt{\beta''(u(t,x))\psi(t,x)\mathbf{1}_B}.$$

Since

$$|f_\varepsilon v| \le \sqrt{\|\beta''\|_\infty \psi(t,x)}|v(t,x)|$$

and the right-hand side is L^2 integrable, one can apply the dominated convergence theorem to conclude $f_\varepsilon v \to fv$ in $L^2(\Omega \times \Pi_T)$. Moreover, we have $f_\varepsilon g_\varepsilon \rightharpoonup fg$ in $L^2(\Omega \times \Pi_T)$, and therefore, by Fatou's lemma for weak convergences,

$$\begin{aligned}\mathbb{E}\Big[\mathbf{1}_B \int_{\Pi_T} \beta''(u(t,x))|\nabla G(u(t,x))|^2\psi(t,x)\,dx\,dt\Big] \\ \le \liminf_{\varepsilon\downarrow 0} \mathbb{E}\Big[\mathbf{1}_B \int_{\Pi_T} \beta''(u_\varepsilon(t,x))|\nabla G(u_\varepsilon(t,x))|^2\psi(t,x)\,dx\,dt\Big].\end{aligned}$$

Thus, we can pass to the limit in (4.13) as $\varepsilon \to 0$ and arrive at following inequality:

$$\begin{aligned}&\mathbb{E}\Big[\mathbf{1}_B \int_{\Pi_T} \beta''(u(t,x))|\nabla G(u(t,x))|^2\psi(t,x)\,dx\,dt\Big] - \mathbb{E}\Big[\mathbf{1}_B \int_{\mathbb{R}^d} \beta(u_0(x))\psi(0,x)\,dx\Big] \\ &\le \mathbb{E}\Big[\mathbf{1}_B \int_{\Pi_T} \Big(\beta(u(t,x))\partial_t\psi(t,x) + \nu(u(t,x))\Delta\psi(t,x) - \nabla\psi(t,x)\cdot\zeta(u_\varepsilon(t,x))\Big)\,dx\,dt\Big] \\ &\quad + \mathbb{E}\Big[\mathbf{1}_B \int_0^T\int_E\int_{\mathbb{R}^d}\int_0^1 \eta(x,u(t,x);z)\beta'(u(t,x)+\theta\eta(x,u(t,x);z))\psi(t,x)\,d\theta\,dx\,\tilde{N}(dz,dt)\Big] \\ &\quad + \mathbb{E}\Big[\mathbf{1}_B \int_0^T\int_E\int_{\mathbb{R}^d}\int_0^1 (1-\theta)\eta^2(x,u(t,x);z)\beta''(u(t,x)+\theta\eta(x,u(t,x);z)) \\ &\qquad \times \psi(t,x)\,d\theta\,dx\,m(dz)\,dt\Big]. \end{aligned} \tag{4.14}$$

We are now in a position to prove the result of existence of an entropy solution for the original problem (1.1).

Proof of the theorem 2.5. The uniform moment estimate (3.17) together with a general version of Fatou's lemma give

$$\sup_{0\le t\le T} \mathbb{E}[\|u(t,\cdot)\|_2^2] < \infty \quad \text{and} \quad \|\nabla G(u)\|^2_{L^2(\Omega\times\Pi_T)} < \infty.$$

For any $0 \le \psi \in C_c^\infty([0,\infty)\times\mathbb{R}^d)$ and any given convex entropy flux triple (β,ζ,ν), inequality (4.14) holds for every $B \in \mathcal{F}_T$. Hence, the inequality

$$\begin{aligned}&\int_{\mathbb{R}^d} \beta(u_0(x))\psi(0,x)\,dx + \int_{\Pi_T} \beta(u(t,x))\partial_t\psi(t,x)\,dx\,dt + \int_{\Pi_T} \nu(u(t,x))\Delta\psi(t,x)\,dx\,dt - \int_{\Pi_T} \nabla\psi(t,x)\cdot\zeta(u(t,x))\,dx\,dt \\ &\quad + \int_0^T\int_E\int_{\mathbb{R}^d}\int_0^1 \eta(x,u(t,x);z)\beta'(u(t,x)+\theta\eta(x,u(t,x);z))\psi(t,x)\,d\theta\,dx\,\tilde{N}(dz,dt) \\ &\quad + \int_E\int_{\Pi_T}\int_0^1 (1-\theta)\eta^2(x,u(t,x);z)\beta''(u(t,x)+\theta\eta(x,u(t,x);z))\psi(t,x)\,d\theta\,dx\,dt\,m(dz) \\ &\ge \int_{\Pi_T} \beta''(u(t,x))|\nabla G(u(t,x))|^2\psi(t,x)\,dx\,dt\end{aligned}$$

holds P-almost surely. This shows that $u(t,x)$ is an entropy solution of (1.1) in the sense of Definition 2.2. This completes the proof. □

We now close this section with a sketch of the justification of our claim in Remark 2.7. To see this, let h_δ denote a smooth even convex approximation of $|\cdot|^p$ defined for positive x by the following: h_δ vanishes at 0, and uniquely recovered from its second-order derivative defined as $h_\delta''(x) = x^{p-2}$ if $x \in [0, \frac{1}{\delta}]$ and $1/\delta^{p-2}$ if $x > \frac{1}{\delta}$. It holds that $0 \le h_\delta(x) \nearrow h(x) = K_p|x|^p$, and there exists C_p such that $0 \le h_\delta''(x) \le C_p h(x)$. Furthermore, it is easily seen that

$$h_\delta''(x+y) \le \tilde{C}_p(h_\delta''(x) + h_\delta''(y)).$$

Note that the weak Itô–Lévy formula in Theorem A.1 makes sense for $\beta = h_\delta$ as h_δ'' is bounded. This enables us to write, for almost every $t > 0$,

$$\begin{aligned}
&E\int_{\mathbb{R}^d} h_\delta(u_\epsilon)\,dx - E\int_{\mathbb{R}^d} h_\delta(u_0)\,dx + E\int_0^t\int_{\mathbb{R}^d} (\phi'(u_\epsilon)+\epsilon)h_\delta''(u_\epsilon)|\nabla u_\epsilon|^2\,dx\,dt\\
&\quad= E\int_0^t\int_E\int_{\mathbb{R}^d} \big(h_\delta(u_\epsilon + \eta(x,u_\epsilon;z)) - h_\delta(u_\epsilon) - \eta(x,u_\epsilon;z)h_\delta'(u_\epsilon)\big)\,dx\,m(dz)\,dt.\\
&\quad= E\int_0^t\int_E\int_{\mathbb{R}^d}\int_0^1 (1-\theta)(\eta(x,u_\epsilon;z))^2 h_\delta''(u_\epsilon + \theta\eta(x,u_\epsilon;z))\,d\theta\,dx\,m(dz)\,ds.
\end{aligned}$$

We can now use the properties of h_δ and the assumptions on η to arrive at

$$\begin{aligned}
E\int_{\mathbb{R}^d} h_\delta(u_\epsilon)\,dx &\le E\int_{\mathbb{R}^d} h_\delta(u_0)\,dx + C_\eta E\int_0^t\int_{\mathbb{R}^d} (1+u_\epsilon^2)h_\delta''(u_\epsilon)\,ds\\
&\le E\int_{\mathbb{R}^d} |u_0|^p\,dx + K_\eta E\int_0^t\int_{\mathbb{R}^d} h_\delta(u_\epsilon)\,ds,
\end{aligned}$$

and, by a weak Gronwall inequality,

$$E\int_{\mathbb{R}^d} h_\delta(u_\epsilon)\,dx \le e^{K_\eta t}E\int_{\mathbb{R}^d} |u_0|^p\,dx$$

for almost all t. This implies

$$E\int_{\mathbb{R}^d} |u_\epsilon|^p\,dx \le e^{C_\eta t}E\int_{\mathbb{R}^d} |u_0|^p\,dx$$

by the monotone convergence theorem. The solution u will inherit the same property by Fatou's lemma.

If u_0 is bounded and $\eta(x,u;z) = 0$ for $|u| \ge M$, with given M, then consider the non-negative regular convex function $x \mapsto h(x) = [(x+K)^-]^2 + [(x-K)^+]^2$, where $K = \max(M + M_1, \|u_0\|_\infty)$. Since $h(u_0) = 0$ and h vanishes where η is active, Itô's formula yields

$$E\int_{\mathbb{R}^d} |h(u_\epsilon)|\,dx = 0$$

and u_ϵ is uniformly bounded by K. Again, the solution u will inherit the same property by passing to the limit.

5 Uniqueness of the entropy solution

To prove the uniqueness of the entropy solution, we compare any entropy solution to the viscous solution via Kruzhkov's doubling variables method and then pass to the limit as the viscosity parameter goes to zero.

We have already shown that the limit of the sequence of viscous solutions serves to prove the existence of an entropy solution to the underlying problem. Now, let $v(t,x)$ be any entropy solution and let $u_\varepsilon(t,x)$ be a viscous solution for problem (3.1). Then one can use exactly the same arguments as in Section 4, and end up with the following equality:

$$\mathbb{E}\Big[\int_{\mathbb{R}^d}\int_0^1 |u(t,x,\alpha)-v(t,x)|\phi_n(x)\,d\alpha\,dx\Big]=0.$$

This implies that, for almost every $t\in[0,\infty)$, one has $v(t,x)=u(t,x,\alpha)$ for almost every $x\in\mathbb{R}^d$ and $(\omega,\alpha)\in\Omega\times(0,1)$. In other words, this proves the uniqueness of the entropy solution.

A Weak Itô–Lévy formula

Let u be an $H^1(\mathbb{R}^d)$-valued $\mathcal{F}_t$-predictable process and assume that it is a weak solution to the SPDE

$$\begin{aligned}&du(t,x)-\Delta\phi(u(t,x))\,dt\\&\qquad=\operatorname{div}_x f(u(t,x))\,dt+\int_E \eta(x,u(t,x);z)\,\tilde N(dz,dt)+\varepsilon\Delta u(t,x)\,dt,\quad t>0,\ x\in\mathbb{R}^d.\end{aligned}\tag{A.1}$$

In addition, in view of (3.7), we further assume that $u\in L^2((0,T)\times\Omega;H^1(\mathbb{R}^d))$. Moreover, u satisfies the initial condition $u_0\in L^2(\mathbb{R}^d)$ in the following sense: P -almost surely

$$\lim_{h\to 0}\frac{1}{h}\int_0^h\int_{\mathbb{R}^d} u(t,x)\phi(x)\,dx=\int_{\mathbb{R}^d} u_0(x)\phi(x)\,dx\tag{A.2}$$

for every $\phi\in C_c^\infty(\mathbb{R}^d)$. We have the following weak version of the Itô–Lévy formula for $u(t,\cdot)$.

Theorem A.1. *Let assumptions* (i)–(v) *hold and let $u(t,\cdot)$ be an $H^1(\mathbb{R}^d)$-valued weak solution of equation* (A.1), *as described in Section 3.1.3, which satisfies equation* (A.2). *Then for every entropy triplet (β,ζ,ν) and $\psi\in C_c^{1,2}([0,\infty)\times\mathbb{R}^d)$, it holds P-almost surely that*

$$\begin{aligned}&\int_{\mathbb{R}^d}\beta(u(T,x))\psi(T,x)\,dx-\int_{\mathbb{R}^d}\beta(u(0,x))\psi(0,x)\,dx\\&\quad=\int_{\Pi_T}\beta(u(t,x))\partial_t\psi(t,x)\,dx\,dt-\int_{\Pi_T}\nabla\psi(t,x)\cdot\zeta(u(t,x))\,dx\,dt\\&\qquad+\int_{\Pi_T}\int_E\int_0^1\eta(x,u(t,x);z)\beta'(u(t,x)+\theta\eta(x,u(t,x);z))\psi(t,x)\,d\theta\,\tilde N(dz,dt)\,dx\\&\qquad+\int_{\Pi_T}\int_E\int_0^1(1-\theta)\eta^2(x,u(t,x);z)\beta''(u(t,x)+\theta\eta(x,u(t,x);z))\psi(t,x)\,d\theta\,m(dz)\,dx\,dt\\&\qquad-\int_{\Pi_T}\big(\varepsilon\nabla\psi(t,x).\nabla_x\beta(u(t,x))+\varepsilon\beta''(u(t,x))|\nabla_x u(t,x)|^2\psi(t,x)\big)\,dx\,dt\\&\qquad-\int_{\Pi_T}\phi'(u(t,x))\beta''(u(t,x))|\nabla u(t,x)|^2\psi(t,x)\,dx\,dt+\int_{\Pi_T}\nu(u(t,x))\Delta\psi(t,x)\,dx\,dt\end{aligned}$$

for almost every $T>0$.

Proof. Let $\{\tau_k\}$ be a standard sequence of mollifiers on $\mathbb{R}^d$. Then for every $\rho(\cdot) \in C_c^1((0,T))$ we have that

$$-\int_0^T u(s,\cdot) * \tau_k \rho'(s)\,ds = \int_0^T \rho(s)\Delta(\phi(u(s,\cdot)) * \tau_k)\,ds + \int_0^T \rho(s)\operatorname{div}_x(f(u) * \tau_\kappa)\,ds$$
$$+ \int_0^T\int_E \rho(s)(\eta(x,u,z) * \tau_k)\,\tilde{N}(\,dz,\,ds) + \epsilon\int_0^T \Delta(u * \tau_k(s,x))\rho(s)\,ds \tag{A.3}$$

holds P-almost surely. For every $n \in \mathbb{N}$, define

$$\rho_n^t(s) = \begin{cases} ns & \text{if } 0 \le s \le \frac{1}{n}, \\ 1 & \text{if } \frac{1}{n} \le s < t, \\ 1 - n(s-t) & \text{if } t + \frac{1}{n} > s \ge t, \\ 0 & \text{elsewhere.} \end{cases}$$

It follows by standard approximation arguments that (A.3) is still valid if we replace $\rho(\cdot)$ by $\rho_n^t(\cdot)$. Afterwards, we invoke the right continuity of the stochastic integral and standard facts related to Lebesgue points of Banach space valued functions to pass to the limit $n \to \infty$ and conclude that, for almost all $t > 0$,

$$u * \tau_k(t,\cdot) - u_0 * \rho_k = \int_0^t \Delta(\phi(u(s,\cdot)) * \tau_k)\,ds + \int_0^t \operatorname{div}_x(f(u) * \tau_\kappa)\,ds$$
$$+ \int_0^t\int_E (\eta(x,u,z) * \tau_k)\,\tilde{N}(dz,ds) + \epsilon\int_0^t \Delta(u * \tau_k(s,x))\,ds$$

holds P-almost surely. Above, we have used that the weak solution satisfies the initial condition in the sense of (A.2). Let β be the entropy function mentioned in the statement and let ψ be the test function specified. Now we apply the Itô–Lévy chain rule to $\beta(u * \tau_k(t,\cdot))$ to have, for almost every $t > 0$,

$$\beta(u * \tau_k(t,\cdot)) = \beta(u_0 * \rho_k) + \int_0^t \beta'(u * \tau_k(t,\cdot))\Delta(\phi(u(s,\cdot)) * \tau_k)\,ds$$
$$+ \int_0^t \beta'(u * \tau_k(t,\cdot))\operatorname{div}_x(f(u) * \tau_\kappa)\,ds + \epsilon\int_0^t \beta'(u * \tau_k(t,\cdot))\Delta(u * \tau_k(s,x))\,ds$$
$$+ \int_0^t\int_E (\beta(u * \tau_k + \eta(x,u,z) * \tau_k) - \beta(u * \tau))\,\tilde{N}(dz,ds)$$
$$+ \int_0^t\int_E (\beta(u * \tau_k + \eta(x,u,z) * \tau_k) - \beta(u * \tau) - \eta(x,u,z) * \tau_k\beta'(u * \tau_k))\,m(dz)\,dt$$

P-almost surely. We now apply the Itô–Lévy product rule to $\beta(u * \tau_k)\psi(t,x)$ and integrate with respect to x to obtain for almost every $T > 0$,

$$\int_{\mathbb{R}^d} \beta(u * \tau_k(T,x))\psi(T,x)\,dx = \int_{\mathbb{R}^d} \beta(u_0 * \rho_k)\psi(0,x)\,dx + \int_0^T\int_{\mathbb{R}^d} \beta(u * \tau_k)\partial_s\psi(s,x)\,dx\,ds$$
$$+ \int_0^T\int_{\mathbb{R}^d} \beta'(u * \tau_k(s,\cdot))\Delta(\phi(u(s,\cdot)) * \tau_k)\psi(s,x)\,dx\,ds$$
$$+ \int_0^T\int_{\mathbb{R}^d} \beta'(u * \tau_k(s,\cdot))\operatorname{div}_x(f(u) * \tau_\kappa)\psi(s,x)\,dx\,ds$$

$$
\begin{aligned}
&+\epsilon\int_0^T\int_{R^d}\psi(s,x)\beta'(u*\tau_k(s,\cdot))\Delta(u*\tau_k(s,x))\,dx\,ds\\
&+\int_0^T\int_{\mathbb{R}^d}\int_E\psi(s,x)(\beta(u*\tau_k+\eta(x,u,z)*\tau_k)-\beta(u*\tau))\,dx\,\tilde{N}(dz,ds)\\
&+\int_0^T\int_{\mathbb{R}^d}\int_E\psi(s,x)\Big(\beta(u*\tau_k+\eta(x,u,z)*\tau_k)-\beta(u*\tau)-\eta(x,u,z)*\tau_k\beta'(u*\tau_k)\Big)\,m(dz)\,dx\,ds
\end{aligned}
\tag{A.4}
$$

almost surely. Note that $u*\tau_k(T,\cdot)\to u(T,\cdot)$ and $u_0*\tau_k\to u_0$ in $L^2(\Omega\times\mathbb{R}^d)$ as $k\to 0$. Therefore, by the Lipschitz continuity of β, we have

$$\int_{\mathbb{R}^d}\beta(u*\tau_k(T,x))\psi(T,x)\,dx\to\int_{\mathbb{R}^d}\beta(u(T,x))\psi(T,x)\,dx$$

and

$$\int_{\mathbb{R}^d}\beta(u_0*\rho_k)\psi(0,x)\,dx\to\int_{\mathbb{R}^d}\beta(u_0)\psi(0,x)\,dx$$

in $L^2(\Omega)$. By a similar reasoning,

$$\int_0^T\int_{\mathbb{R}^d}\beta(u*\tau_k)\partial_s\psi(s,x)\,dx\,ds\to\int_0^T\int_{\mathbb{R}^d}\beta(u)\partial_s\psi(s,x)\,dx\,ds\quad\text{as } k\to 0.$$

Furthermore, note that

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}^d}\beta'(u*\tau_k(t,\cdot))\Delta(\phi(u(s,\cdot))*\tau_k)\psi(s,x)\,ds\,dx\\
&\qquad=-\int_0^T\int_{\mathbb{R}^d}\nabla_x(\psi(t,x)\beta'(u*\tau_k(t,x))).(\nabla\phi(u())*\tau_k)(s,x))\,dx\,ds
\end{aligned}
$$

and $\nabla u,\nabla\phi(u)\in L^2(0,T;L^2(\Omega\times\mathbb{R}^d))$. Therefore,

$$\nabla_x(\psi(t,x)\beta'(u*\tau_k(t,x)))\to\nabla_x(\psi(t,x)\beta'(u(t,x)))$$

and $\nabla\phi(u)*\tau_k\to\nabla\phi(u)$ in $L^2(0,T;L^2(\Omega\times\mathbb{R}^d))$ as $k\to 0$. Therefore,

$$\int_0^T\int_{\mathbb{R}^d}\beta'(u*\tau_k(t,\cdot))\Delta(\phi(u(s,\cdot))*\tau_k)\psi(s,x)\,ds\,dx\to-\int_0^T\int_{\mathbb{R}^d}\nabla_x(\psi(t,x)\beta'(u(t,x))).(\nabla\phi(u(s,x)))\,dx\,ds$$

in $L^1(\Omega)$ as $k\to 0$. By the same reasoning,

$$\int_{\Pi_T}\psi(s,x)\beta'(u*\tau_k(s,x))\Delta(u*\tau_k(s,x))\,dx\,ds\to-\int_{\Pi_T}\nabla_x(\psi(s,x)\beta'(u(s,x))).\nabla(u(s,x))\,dx\,ds$$

in $L^1(\Omega)$ as $k\to 0$.

Also, it may be recalled that $\operatorname{div}_x f(u)\in L^2(0,T;L^2(\Omega\times\mathbb{R}^d))$ and $\beta(u*\tau_k)\to\beta(u)$ in $L^2(0,T;L^2(\Omega\times\mathbb{R}^d))$ as $\kappa\to 0$. Therefore,

$$\int_0^T\int_{\mathbb{R}^d}\beta'(u*\tau_k(t,\cdot))\operatorname{div}_x(f(u)*\tau_\kappa)\psi(s,x)\,dx\,ds\to\int_0^T\int_{\mathbb{R}^d}\beta'(u)\operatorname{div}_x(f(u))\psi(s,x)\,dx\,ds$$

as $k \to 0$ in $L^1(\Omega)$. To this end, we denote

$$I_k(s,z) = \int_{\mathbb{R}^d} \psi(s,x)\big(\beta(u * \tau_k + \eta(x,u,z) * \tau_k) - \beta(u * \tau)\big)\,dx,$$

$$I(s,z) = \int_{\mathbb{R}^d} \psi(s,x)\big(\beta(u + \eta(x,u,z)) - \beta(u * \tau)\big)\,dx.$$

It follows from straightforward computations that

$$\int_0^T \int_E |I_k(s,z) - I(s,z)|^2\, m(dz)\,ds \to 0 \quad \text{as } k \to 0.$$

Therefore, we can invoke the Itô–Lévy isometry and pass to the limit as $k \to 0$, in the martingale term in (A.4). This completes the validation of the passage to the limit as $k \to 0$ in every term of (A.4). The assertion is now concluded by simply letting $k \to 0$ in (A.4) and rearranging the terms. □

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