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Research Article

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Nonlocal perturbations of the fractional Choquard equation

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Abstract: We study the equation

$$(-\Delta)^s u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u + \lambda(I_\beta * |u|^q)|u|^{q-2}u \quad \text{in } \mathbb{R}^N,$$

where $I_\gamma(x) = |x|^{-\gamma}$ for any $\gamma \in (0, N)$, $p, q > 0$, $\alpha, \beta \in (0, N)$, $N \geq 3$, and $\lambda \in \mathbb{R}$. First, the existence of groundstate solutions by using a minimization method on the associated Nehari manifold is obtained. Next, the existence of least energy sign-changing solutions is investigated by considering the Nehari nodal set.

Keywords: Choquard equation, fractional operator, nonlocal perturbation, groundstate solution, sign-changing solutions

MSC 2010: 35R11, 35J60, 35Q40, 35A15

1 Introduction

In this paper, we are concerned with the equation

$$(-\Delta)^s u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u + \lambda(I_\beta * |u|^q)|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $p, q > 0$, $\alpha, \beta \in (0, N)$, $N \geq 3$, and $\lambda \in \mathbb{R}$. Here I_γ stands for the Riesz potential of order γ defined as $I_\gamma = |x|^{\gamma-N}$ for any $\gamma \in (0, N)$.

The operator $(-\Delta)^s$ is the fractional Laplace operator of order $s \in (0, 1)$, and is defined as follows (see [6, 15]):

$$(-\Delta)^s u = C(N, s)\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}}\, dy,$$

where P.V. stands for the principal value of the integral and $C(N, s) > 0$ is a normalizing constant. The operator $(-\Delta)^s$ is referred to as the infinitesimal generator of the Levy stable diffusion process. The function $V \in C(\mathbb{R}^N)$ is required to satisfy one (or both) of the following conditions:

(V1) $\inf_{\mathbb{R}^N} V(x) \geq V_0 > 0$.

(V2) For all $M > 0$, the set $\{x \in \mathbb{R}^N : V(x) \leq M\}$ has finite Lebesgue measure.

Note that condition (V2) is weaker than $\lim_{|x|\to\infty} V(x) = \infty$, as for instance $V(x) = |x|^4 \sin^2|x|$ satisfies (V2) but has no limit as $|x| \to \infty$.

In the last few decades, problems involving the fractional Laplacian and nonlocal operators have received considerable attention. These kinds of problems arise in various applications such as continuum mechanics, phase transitions, population dynamics, optimization, finance, and many others.

The prototype model of (1.1) is the fractional Choquard equation

$$(-\Delta)^s u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

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studied by d'Avenia, Siciliano and Squassina in [5] in the case where V is a positive constant. They obtained the existence of groundstate and radially symmetric solutions with diverging norm and diverging energy levels.

The case of the standard Laplace operator in (1.2) has a long history in the literature. For $s = 1$, $V \equiv 1$ and $p = \alpha = 2$, equation (1.2) becomes the well-known Choquard or nonlinear Schrödinger–Newton equation

$$-\Delta u + u = (I_2 * u^2)u \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

Equation (1.3) for $N = 3$ was first introduced by Pekar [20] in 1954 in quantum mechanics. In 1996, Penrose [21, 22] used equation (1.3) in a different context as a model in self-gravitating matter (see also [11, 16]). Since then, the Choquard equation has been investigated in various settings and in many contexts (see, e.g., [1, 10, 14, 19]). For a most up to date reference on the study of the Choquard equation in a standard Laplace setting, the reader may consult [18].

For $s = \frac{1}{2}$, $V \equiv 1$, $p = \alpha = 2$, $N = 3$, and $\lambda = 0$, equation (1.1) becomes

$$(-\Delta)^{\frac{1}{2}} u + u = (I_2 * u^2)u \quad \text{in } \mathbb{R}^3,$$

and has been used to study the dynamics of pseudo-relativistic boson stars and their dynamical evolution (see [7–9, 12]).

In this paper, we shall be interested in the study of groundstate solutions and least energy sign-changing solutions to (1.1). To this aim, we denote by $\mathcal{D}^{2,s}(\mathbb{R}^N)$ the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the Gagliardo seminorm

$$[u]_{s,2} = \left[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}\,dx\,dy\right]^{\frac{1}{2}}.$$

Also, $H^s(\mathbb{R}^N)$ denotes the standard fractional Sobolev space defined as the set of $u \in \mathcal{D}^{2,s}(\mathbb{R}^N)$ satisfying $u \in L^2(\mathbb{R}^N)$ with the norm

$$\|u\|_{H^s} = \left[\int_{\mathbb{R}^N} |u|^2 + [u]_{s,2}^2\right]^{\frac{1}{2}}.$$

Let us define the functional space

$$X_V^s(\mathbb{R}^N) = \left\{u \in \mathcal{D}^{2,s}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 < \infty\right\}$$

endowed with the norm

$$\|u\|_{X_V^s} = \left[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)u^2\right]^{\frac{1}{2}}.$$

Throughout this paper, we shall assume that p and q satisfy

$$\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2s} \tag{1.4}$$

and

$$\frac{N+\beta}{N} < q < \frac{N+\beta}{N-2s}. \tag{1.5}$$

It is not difficult to see that (1.1) has a variational structure. Indeed, any solution of (1.1) is a critical point of the energy functional $\mathcal{E}_\lambda : X_V^s(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathcal{E}_\lambda(u) = \frac{1}{2}\|u\|_{X_V^s}^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p - \frac{\lambda}{2q}\int_{\mathbb{R}^N} (I_\beta * |u|^q)|u|^q.$$

A crucial tool to our approach is the Hardy–Littlewood–Sobolev inequality

$$\left|\int_{\mathbb{R}^N} (I_\gamma * u)v\right| \le C\|u\|_r\|v\|_t \tag{1.6}$$

for $\gamma \in (0, N)$, $u \in L^r(\mathbb{R}^N)$ and $v \in L^t(\mathbb{R}^N)$ such that

$$\frac{1}{r} + \frac{1}{t} = 1 + \frac{\gamma}{N}.$$

Using (1.4) and (1.5) together with the Hardy–Littlewood–Sobolev inequality (1.6), the energy functional $\mathcal{E}_\lambda$ is well defined, and moreover $\mathcal{E}_\lambda \in C^1(X_V^s)$.

We shall first be concerned with the existence of ground state solutions for equation (1.1) under the assumption that V satisfies (V1). This will be achieved by a minimization method on the Nehari manifold associated with $\mathcal{E}_\lambda$, which is defined as

$$\mathcal{N}_\lambda = \{u \in X_V^s(\mathbb{R}^N) \setminus \{0\} : \langle \mathcal{E}_\lambda'(u), u\rangle = 0\}.$$

The groundstate solutions will be obtained as minimizers of

$$m_\lambda = \inf_{u \in \mathcal{N}_\lambda} \mathcal{E}_\lambda(u).$$

Our main result in this sense is stated below.

Theorem 1.1. *Assume $p > q > 1$, $\lambda > 0$, p, q satisfy* (1.4)–(1.5), *and V satisfies* (V1). *Then equation* (1.1) *has a ground state solution $u \in X_V^s(\mathbb{R}^N)$.*

Our approach relies on the analysis of the Palais–Smale sequences for $\mathcal{E}_\lambda|_{\mathcal{N}_\lambda}$. Using an idea from [3, 4], we show that any Palais–Smale sequence of $\mathcal{E}_\lambda|_{\mathcal{N}_\lambda}$ is either converging strongly to its weak limit or differs from it by a finite number of sequences which further are the translated solutions of (1.2). The novelty of our approach is that we shall rely on several nonlocal Brezis–Lieb results as we present in Section 2.

We now turn to the study of least energy sign-changing solutions of (1.1). In this setting, we require V to fulfill both conditions (V1) and (V2). By the result in [25, Lemma 2.1] (see also [23, 24]), the embedding $X_V^s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is compact for $q \in [2, 2_s^*)$, where $2_s^* = \frac{2N}{N-2s}$.

Our approach in the study of least energy sign-changing solutions of (1.1) is based on the minimization method on the Nehari nodal set defined as

$$\mathcal{M}_\lambda = \{u \in X_V^s(\mathbb{R}^N) : u^\pm \neq 0 \text{ and } \langle \mathcal{E}_\lambda'(u), u^\pm\rangle = 0\}.$$

The solutions will be obtained as minimizers for

$$c_\lambda = \inf_{u \in \mathcal{M}_\lambda} \mathcal{E}_\lambda(u).$$

In this situation, the problem is more delicate as some of the usual properties of the local nonlinear functional do not work. For instance, since

$$\begin{aligned}\langle \mathcal{E}_\lambda'(u), u^\pm\rangle = \|u^\pm\|_{X_V^s}^2 - \int_{\mathbb{R}^N} (I_\alpha * (u^\pm)^p)(u^\pm)^p - \lambda \int_{\mathbb{R}^N} (I_\beta * (u^\pm)^q)(u^\pm)^q - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^\pm(x)u^\mp(y) + u^\mp(x)u^\pm(y)}{|x-y|^{N+2s}}\,dx\,dy \\ - \int_{\mathbb{R}^N} (I_\alpha * (u^\pm)^p)(u^\mp)^p - \lambda \int_{\mathbb{R}^N} (I_\beta * (u^\pm)^q)(u^\mp)^q,\end{aligned}$$

we have in general that

$$\mathcal{E}_\lambda(u) \neq \mathcal{E}_\lambda(u^+) + \mathcal{E}_\lambda(u^-) \quad \text{and} \quad \langle \mathcal{E}_\lambda'(u), u^\pm\rangle \neq \langle \mathcal{E}_\lambda'(u^\pm), u^\pm\rangle.$$

Therefore, the standard local methods used to investigate the existence of sign-changing solutions do not apply immediately to our nonlocal setting.

Our second main result in this regard is stated below.

Theorem 1.2. *Assume $\lambda \in \mathbb{R}$, $(N-4s)_+ < \alpha, \beta < N$, $p > q > 2$ satisfy* (1.4) *and* (1.5), *and V satisfies* (V1) *and* (V2). *Then equation* (1.1) *has a least-energy sign-changing solution $u \in X_V^s(\mathbb{R}^N)$.*

The remaining of the paper is organized as follows: In Section 2, we collect some nonlocal versions of the Brezis–Lieb lemma, which will be crucial in investigating the groundstate solutions of (1.1). Further, Sections 3 and 4 contain the proofs of our main results.

2 Preliminary results

Lemma 2.1 ([13, Lemma 1.1], [17, Lemma 2.3]). *Let $r \in [2, 2_s^*]$. There exists a constant $C > 0$ such that for any $u \in X_V^s(\mathbb{R}^N)$ we have*

$$\int_{\mathbb{R}^N} |u|^r \le C\|u\|\Big(\sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u|^r\Big)^{1-\frac{2}{r}}.$$

Lemma 2.2 ([2, Proposition 4.7.12]). *Let $r \in (1, \infty)$. Assume (w_n) is a bounded sequence in $L^r(\mathbb{R}^N)$ that converges to w almost everywhere. Then $w_n \rightharpoonup w$ weakly in $L^r(\mathbb{R}^N)$.*

Lemma 2.3 (Local Brezis–Lieb lemma). *Let $r \in (1, \infty)$. Assume (w_n) is a bounded sequence in $L^r(\mathbb{R}^N)$ that converges to w almost everywhere. Then for every $q \in [1, r]$ we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \big||w_n|^q - |w_n - w|^q - |w|^q\big|^{\frac{r}{q}} = 0$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \big||w_n|^{q-1}w_n - |w_n - w|^{q-1}(w_n - w) - |w|^{q-1}w\big|^{\frac{r}{q}} = 0.$$

Proof. Fix $\varepsilon > 0$. Then there exists $C(\varepsilon) > 0$ such that for all $a, b \in \mathbb{R}$ we have

$$\big||a + b|^q - |a|^q\big|^{\frac{r}{q}} \le \varepsilon|a|^r + C(\varepsilon)|b|^r. \tag{2.1}$$

Using (2.1), we obtain

$$|f_{n,\varepsilon}| = \Big(\big||w_n|^q - |w_n - w|^q - |w^q|\big|^{\frac{r}{q}} - \varepsilon|w_n - w|^r\Big)^+ \le (1 + C(\varepsilon))|w|^r.$$

Now using the Lebesgue dominated convergence theorem, we have

$$\int_{\mathbb{R}^N} f_{n,\varepsilon} \to 0 \quad \text{as } n \to \infty.$$

Therefore, we get

$$\big||w_n|^q - |w_n - w|^q - |w|^q\big|^{\frac{r}{q}} \le f_{n,\varepsilon} + \varepsilon|w_n - w|^r,$$

which gives

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \big||w_n|^q - |w_n - w|^q - |w|^q\big|^{\frac{r}{q}} \le c\varepsilon,$$

where $c = \sup_n |w_n - w|_r^r < \infty$. Further, letting $\varepsilon \to 0$, we conclude the proof. □

Lemma 2.4 (Nonlocal Brezis–Lieb lemma [17, Lemma 2.4]). *Let $\alpha \in (0, N)$ and $p \in [1, \frac{2N}{N+\alpha})$. Assume (u_n) is a bounded sequence in $L^{2Np/(N+\alpha)}(\mathbb{R}^N)$ that converges almost everywhere to some $u : \mathbb{R}^N \to \mathbb{R}$. Then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \big|(I_\alpha * |u_n|^p)|u_n|^p - (I_\alpha * |u_n - u|^p)|u_n - u|^p - (I_\alpha * |u|^p)|u|^p\big| = 0.$$

Proof. For $n \in N$, we observe that

$$\begin{aligned}\int_{\mathbb{R}^N} [(I_\alpha * |u_n|^p)|u_n|^p - (I_\alpha * (|u_n - u|^p))(|u_n - u|^p)] &= \int_{\mathbb{R}^N} [I_\alpha * (|u_n|^p - |u_n - u|^p)](|u_n|^p - |u_n - u|^p) \\ &\quad + 2\int_{\mathbb{R}^N} [I_\alpha * (|u_n|^p - |u_n - u|^p)]|u_n - u|^p.\end{aligned} \tag{2.2}$$

Using Lemma 2.3 with $q = p$, $r = \frac{2Np}{N+\alpha}$, we have $|u_n - u|^p - |u_n|^p \to |u|^p$ strongly in $L^{2N/(N+\alpha)}(\mathbb{R}^N)$, and by Lemma 2.2 we get $|u_n - u|^p \rightharpoonup 0$ weakly in $L^{2N/(N+\alpha)}(\mathbb{R}^N)$. Also, by the Hardy–Littlewood–Sobolev inequality (1.6) we obtain

$$I_\alpha * (|u_n - u|^p - |u_n|^p) \to I_\alpha * |u|^p \quad \text{in } L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N).$$

Using all the above arguments and passing to the limit in (2.2), we conclude the proof. □

Lemma 2.5. *Let $\alpha \in (0, N)$ and $p \in [1, \frac{2N}{N+\alpha})$. Assume (u_n) is a bounded sequence in $L^{2Np/(N+\alpha)}(\mathbb{R}^N)$ that converges almost everywhere to u. Then for any $h \in L^{2Np/(N+\alpha)}(\mathbb{R}^N)$ we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^{p-2}u_n h = \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^{p-2}uh.$$

Proof. By using $h = h^+ - h^-$, it is enough to prove our lemma for $h \geq 0$. Denote $v_n = u_n - u$ and observe that

$$\begin{aligned}\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^{p-2}u_n h &= \int_{\mathbb{R}^N} [I_\alpha * (|u_n|^p - |v_n|^p)](|u_n|^{p-2}u_n h - |v_n|^{p-2}v_n h) \\ &\quad + \int_{\mathbb{R}^N} [I_\alpha * (|u_n|^p - |v_n|^p)]|v_n|^{p-2}v_n h + \int_{\mathbb{R}^N} [I_\alpha * (|u_n|^{p-2}u_n h - |v_n|^{p-2}v_n h)]|v_n|^p \\ &\quad + \int_{\mathbb{R}^N} (I_\alpha * |v_n|^p)|v_n|^{p-2}v_n h.\end{aligned} \tag{2.3}$$

Apply Lemma 2.3 with $q = p$ and $r = \frac{2Np}{N+\alpha}$ by taking $(w_n, w) = (u_n, u)$ and then $(w_n, w) = (u_n h^{1/p}, uh^{1/p})$, respectively. We find

$$\begin{cases} |u_n|^p - |v_n|^p \to |u|^p, \\ |u_n|^{p-2}u_n h - |v_n|^{p-2}v_n h \to |u|^{p-2}uh \end{cases} \quad \text{strongly in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N).$$

Using now the Hardy–Littlewood–Sobolev inequality, we obtain

$$\begin{cases} I_\alpha * (|u_n|^p - |v_n|^p) \to I_\alpha * |u|^p, \\ I_\alpha * (|u_n|^{p-2}u_n h - |v_n|^{p-2}v_n h) \to I_\alpha * (|u|^{p-2}uh) \end{cases} \quad \text{strongly in } L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N). \tag{2.4}$$

Also, by Lemma 2.2 we have

$$|u_n|^{p-2}u_n h \rightharpoonup |u|^{p-2}uh, \quad |v_n|^p \rightharpoonup 0, \quad |v_n|^{p-2}v_n h \rightharpoonup 0 \qquad \text{weakly in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N). \tag{2.5}$$

Combining (2.4)–(2.5), we find

$$\begin{cases} \lim\limits_{n\to\infty} \displaystyle\int_{\mathbb{R}^N} [I_\alpha * (|u_n|^p - |v_n|^p)](|u_n|^{p-2}u_n h - |v_n|^{p-2}v_n h) = \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^{p-2}uh, \\ \lim\limits_{n\to\infty} \displaystyle\int_{\mathbb{R}^N} [I_\alpha * (|u_n|^p - |v_n|^p)]|v_n|^{p-2}v_n h = 0, \\ \lim\limits_{n\to\infty} \displaystyle\int_{\mathbb{R}^N} [I_\alpha * (|u_n|^{p-2}u_n h - |v_n|^{p-2}v_n h)]|v_n|^p = 0. \end{cases} \tag{2.6}$$

By Hölder's inequality and the Hardy–Littlewood–Sobolev inequality, we have

$$\left| \int_{\mathbb{R}^N} (I_\alpha * |v_n|^p)|v_n|^{p-2}v_n h \right| \leq \|v_n\|^p_{\frac{2Np}{N+\alpha}} \||v_n|^{p-1}h\|_{\frac{2N}{N+\alpha}} \leq C\||v_n|^{p-1}h\|_{\frac{2N}{N+\alpha}}. \tag{2.7}$$

On the other hand, by Lemma 2.2 we have $v_n^{2N(p-1)/(N+\alpha)} \rightharpoonup 0$ weakly in $L^{p/(p-1)}(\mathbb{R}^N)$, so

$$\||v_n|^{p-1}h\|_{\frac{2N}{N+\alpha}} = \left(\int_{\mathbb{R}^N} |v_n|^{\frac{2N(p-1)}{N+\alpha}} |h|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \to 0.$$

Thus, from (2.7) we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |v_n|^p)|v_n|^{p-2}v_n h = 0. \tag{2.8}$$

Passing to the limit in (2.3), from (2.6) and (2.8) we reach the conclusion. □

3 Proof of Theorem 1.1

In this section, we discuss the existence of groundstate solutions to (1.1) under the assumption $\lambda > 0$. For $u, v \in X_V^s(\mathbb{R}^N)$, we have

$$\langle \mathcal{E}_\lambda'(u), v\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)uv - \int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^{p-1}v - \lambda\int_{\mathbb{R}^N}(I_\beta * |u|^q)|u|^{q-1}v.$$

So, for $t > 0$ we have

$$\langle \mathcal{E}_\lambda'(tu), tu\rangle = t^2\|u\|^2_{X_V^s} - t^{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p - \lambda t^{2q}\int_{\mathbb{R}^N}(I_\beta * |u|^q)|u|^q.$$

Since $p > q > 1$, the equation

$$\langle \mathcal{E}_\lambda'(tu), tu\rangle = 0$$

has a unique positive solution $t = t(u)$, and the corresponding element $tu \in \mathcal{N}_\lambda$ is called the *projection of u* on $\mathcal{N}_\lambda$. The following result presents the main properties of the Nehari manifold $\mathcal{N}_\lambda$, which we use in this paper.

Lemma 3.1. (i) *$\mathcal{E}_\lambda|_{\mathcal{N}_\lambda}$ is bounded from below by a positive constant.*
(ii) *Any critical point u of $\mathcal{E}_\lambda|_{\mathcal{N}_\lambda}$ is a free critical point.*

Proof. (i) By using the continuous embeddings $X_V^s(\mathbb{R}^N) \hookrightarrow L^{2Np/(N+\alpha)}(\mathbb{R}^N)$ and $X_V^s(\mathbb{R}^N) \hookrightarrow L^{2Nq/(N+\beta)}(\mathbb{R}^N)$ together with the Hardy–Littlewood–Sobolev inequality, for any $u \in \mathcal{N}_\lambda$ we have

$$\begin{aligned} 0 = \langle \mathcal{E}_\lambda'(u), u\rangle &= \|u\|^2_{X_V^s} - \int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p - \lambda\int_{\mathbb{R}^N}(I_\beta * |u|^q)|u|^q \\ &\geq \|u\|^2_{X_V^s} - C\|u\|^{2p}_{X_V^s} - C\lambda\|u\|^{2q}_{X_V^s}. \end{aligned}$$

Therefore, there exists $C_0 > 0$ such that

$$\|u\|_{X_V^s} \geq C_0 > 0 \quad \text{for all } u \in \mathcal{N}_\lambda. \tag{3.1}$$

Using the above fact, we have

$$\begin{aligned} \mathcal{E}_\lambda(u) &= \mathcal{E}_\lambda(u) - \frac{1}{2q}\langle \mathcal{E}_\lambda'(u), u\rangle \\ &= \Big(\frac12 - \frac{1}{2q}\Big)\|u\|^2_{X_V^s} + \Big(\frac{1}{2q} - \frac{1}{2p}\Big)\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p \\ &\geq \Big(\frac12 - \frac{1}{2q}\Big)\|u\|^2_{X_V^s} \\ &\geq \Big(\frac12 - \frac{1}{2q}\Big)C_0^2 > 0. \end{aligned}$$

(ii) Let $\mathcal{L}(u) = \langle \mathcal{E}_\lambda'(u), u\rangle$ for $u \in X_V^s(\mathbb{R}^N)$. Now, for $u \in \mathcal{N}_\lambda$, from (3.1) we get

$$\begin{aligned} \langle \mathcal{L}'(u), u\rangle &= 2\|u\|^2 - 2p\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p - 2q\lambda\int_{\mathbb{R}^N}(I_\beta * |u|^q)|u|^q \\ &= 2(1-q)\|u\|^2_{X_V^s} - 2(p-q)\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p \\ &\leq -2(q-1)\|u\|^2_{X_V^s} \\ &< -2(q-1)C_0. \end{aligned}$$

Assuming that $u \in \mathcal{N}_\lambda$ is a critical point of $\mathcal{E}_\lambda|_{\mathcal{N}_\lambda}$ and using the Lagrange multiplier theorem, there exists $\mu \in \mathbb{R}$ such that $\mathcal{E}_\lambda'(u) = \mu\mathcal{L}'(u)$. In particular, $\langle \mathcal{E}_\lambda'(u), u\rangle = \mu\langle \mathcal{L}'(u), u\rangle$. As $\langle \mathcal{L}'(u), u\rangle < 0$, this implies $\mu = 0$, so $\mathcal{E}_\lambda'(u) = 0$. □

3.1 Compactness

Define

$$\mathcal{J} : X_V^s(\mathbb{R}^N) \to \mathbb{R}, \quad \mathcal{J}(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p.$$

For all $\phi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\langle \mathcal{J}'(u), \phi\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)u\phi - \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^{p-1}\phi$$

and

$$\langle \mathcal{J}'(u), u\rangle = \|u\|_{X_V^s}^2 - \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p.$$

Also, consider the Nehari manifold associated with $\mathcal{J}$ as

$$\mathcal{N}_{\mathcal{J}} = \{u \in X_V^s(\mathbb{R}^N) \setminus \{0\} : \langle \mathcal{J}'(u), u\rangle = 0\},$$

and let

$$m_{\mathcal{J}} = \inf_{u\in\mathcal{N}_{\mathcal{J}}} \mathcal{J}(u).$$

Lemma 3.2. *Let $(u_n) \subset \mathcal{N}_{\mathcal{J}}$ be a (PS) sequence of $\mathcal{E}_\lambda|_{\mathcal{N}_\lambda}$, that is, $(\mathcal{E}_\lambda(u_n))$ is bounded and $\mathcal{E}'_\lambda|_{\mathcal{N}_\lambda}(u_n) \to 0$ strongly in $X_V^{-s}(\mathbb{R}^N)$. Then there exists a solution $u \in X_V^s(\mathbb{R}^N)$ of (1.1) such that, by replacing (u_n) with a subsequence, one of the following alternatives holds:*

(i) *$u_n \to u$ strongly in $X_V^s(\mathbb{R}^N)$.*

(ii) *$u_n \rightharpoonup u$ weakly in $X_V^s(\mathbb{R}^N)$, and there exists a positive integer $k \geq 1$ and k functions $u_1, u_2, \ldots, u_k \in X_V^s(\mathbb{R}^N)$, which are nontrivial weak solutions to (1.2), and k sequences of points $(z_{n,1}), (z_{n,2}), \ldots, (z_{n,k}) \subset \mathbb{R}^N$ such that the following conditions hold:*
   (a) *$|z_{n,j}| \to \infty$ and $|z_{n,j} - z_{n,i}| \to \infty$ if $i \neq j$, $n \to \infty$;*
   (b) *$u_n - \sum_{j=1}^k u_j(\cdot + z_{n,j}) \to u$ in $X_V^s(\mathbb{R}^N)$;*
   (c) *$\mathcal{E}_\lambda(u_n) \to \mathcal{E}_\lambda(u) + \sum_{j=1}^k \mathcal{J}(u_j)$.*

Proof. Since (u_n) is bounded in $X_V^s(\mathbb{R}^N)$, there exists $u \in X_V^s(\mathbb{R}^N)$ such that, up to a subsequence, we have

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } X_V^s(\mathbb{R}^N), \\ u_n \rightharpoonup u & \text{weakly in } L^r(\mathbb{R}^N),\ 2 \leq r \leq 2_s^*, \\ u_n \to u & \text{a.e. in } \mathbb{R}^N. \end{cases} \tag{3.2}$$

By using (3.2) and Lemma 2.5, it follows that $\mathcal{E}'_\lambda(u) = 0$, so $u \in X_V^s(\mathbb{R}^N)$ is a solution of (1.1). Further, if $u_n \to u$ strongly in $X_V^s(\mathbb{R}^N)$, then Lemma 3.2 (i) holds.

Now, assume that (u_n) does not converge strongly to u in $X_V^s(\mathbb{R}^N)$, and set $w_{n,1} = u_n - u$. Then $(w_{n,1})$ converges weakly to zero in $X_V^s(\mathbb{R}^N)$, and

$$\|u_n\|_{X_V^s}^2 = \|u\|_{X_V^s}^2 + \|w_{n,1}\|_{X_V^s}^2 + o(1). \tag{3.3}$$

By Lemma 2.4, we have

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p = \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p + \int_{\mathbb{R}^N} (I_\alpha * |w_{n,1}|^p)|w_{n,1}|^p + o(1). \tag{3.4}$$

Using (3.3) and (3.4), we get

$$\mathcal{E}_\lambda(u_n) = \mathcal{E}_\lambda(u) + \mathcal{J}(w_{n,1}) + o(1). \tag{3.5}$$

Further, for any $h \in X_V^s(\mathbb{R}^N)$, by Lemma 2.5 we have

$$\langle \mathcal{J}'(w_{n,1}), h\rangle = o(1).$$

From Lemma 2.4 we deduce that

$$0 = \langle \mathcal{E}_\lambda'(u_n), u_n\rangle = \langle \mathcal{E}_\lambda'(u), u\rangle + \langle \mathcal{J}'(w_{n,1}), w_{n,1}\rangle + o(1) = \langle \mathcal{J}'(w_{n,1}), w_{n,1}\rangle + o(1).$$

This implies

$$\langle \mathcal{J}'(w_{n,1}), w_{n,1}\rangle = o(1). \tag{3.6}$$

We need the following auxiliary result.

Lemma 3.3. *Define*

$$\delta := \limsup_{n\to\infty}\Big(\sup_{w\in\mathbb{R}^N}\int_{B_1(z)} |w_{n,1}|^{\frac{2Np}{N+\alpha}}\Big).$$

Then $\delta > 0$.

Proof. Assume by contradiction $\delta = 0$. By Lemma 2.1, we deduce that $w_{n,1} \to 0$ strongly in $L^{2Np/(N+\alpha)}(\mathbb{R}^N)$. Then by the Hardy–Littlewood–Sobolev inequality we get

$$\int_{\mathbb{R}^N} (I_\alpha * |w_{n,1}|^p)|w_{n,1}|^p = o(1).$$

Using this fact together with (3.6), we get $w_{n,1} \to 0$ strongly in $X_V^s(\mathbb{R}^N)$. This is a contradiction. Hence, $\delta > 0$. □

Now, we return to the proof of Lemma 3.2. Since $\delta > 0$, we may find $z_{n,1} \in \mathbb{R}^N$ such that

$$\int_{B_1(z_{n,1})} |w_{n,1}|^{\frac{2Np}{N+\alpha}} > \frac{\delta}{2}. \tag{3.7}$$

Consider the sequence $(w_{n,1}(\cdot + z_{n,1}))$. Then there exists $u_1 \in X_V^s(\mathbb{R}^N)$ such that, up to a subsequence, we have

$$\begin{aligned} w_{n,1}(\cdot + z_{n,1}) &\rightharpoonup u_1 \quad \text{weakly in } X_V^s(\mathbb{R}^N),\\ w_{n,1}(\cdot + z_{n,1}) &\to u_1 \quad \text{strongly in } L_{\mathrm{loc}}^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N),\\ w_{n,1}(\cdot + z_{n,1}) &\to u_1 \quad \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

Next, passing to the limit in (3.7), we get

$$\int_{B_1(0)} |u_1|^{\frac{2Np}{N+\alpha}} \geq \frac{\delta}{2},$$

therefore $u_1 \not\equiv 0$. Since $(w_{n,1})$ converges weakly to zero in $X_V^s(\mathbb{R}^N)$, it follows that $(z_{n,1})$ is unbounded. Thus, passing to a subsequence, we may assume that $|z_{n,1}| \to \infty$. By (3.6), we deduce that $\mathcal{J}'(u_1) = 0$, so u_1 is a nontrivial solution of (1.2).

Further, define

$$w_{n,2}(x) = w_{n,1}(x) - u_1(x - z_{n,1}).$$

Similarly to before, we have

$$\|w_{n,1}\|^2 = \|u_1\|^2 + \|w_{n,2}\|^2 + o(1).$$

Then, using Lemma 2.4, we deduce that

$$\int_{\mathbb{R}^N} (I_\alpha * |w_{n,1}|^p)|w_{n,1}|^p = \int_{\mathbb{R}^N} (I_\alpha * |u_1|^p)|u_1|^p + \int_{\mathbb{R}^N} (I_\alpha * |w_{n,2}|^p)|w_{n,2}|^p + o(1).$$

Hence,

$$\mathcal{J}(w_{n,1}) = \mathcal{J}(u_1) + \mathcal{J}(w_{n,2}) + o(1).$$

So, by (3.5) one can get

$$\mathcal{E}_\lambda(u_n) = \mathcal{E}_\lambda(u) + \mathcal{J}(u_1) + \mathcal{J}(w_{n,2}) + o(1).$$

Using the above techniques, we also obtain

$$\langle \mathcal{J}'(w_{n,2}), h\rangle = o(1) \quad \text{for any } h \in X_V^s(\mathbb{R}^N)$$

and

$$\langle \mathcal{J}'(w_{n,2}), w_{n,2}\rangle = o(1).$$

Now, if $(w_{n,2})$ converges strongly to zero, then we finish the proof by taking $k = 1$ in the statement of Lemma 3.2. If $w_{n,2} \rightharpoonup 0$ weakly and not strongly in $X_V^s(\mathbb{R}^N)$, then we iterate the process. In k steps one could find a set of sequences $(z_{n,j}) \subset \mathbb{R}^N$, $1 \le j \le k$, with

$$|z_{n,j}| \to \infty \quad \text{and} \quad |z_{n,i} - z_{n,j}| \to \infty \qquad \text{as } n \to \infty,\ i \ne j,$$

and k nontrivial solutions $u_1, u_2, \dots, u_k \in X_V^s(\mathbb{R}^N)$ of (1.2) such that, denoting

$$w_{n,j}(x) := w_{n,j-1}(x) - u_{j-1}(x - z_{n,j-1}), \quad 2 \le j \le k,$$

we have

$$w_{n,j}(x + z_{n,j}) \rightharpoonup u_j \quad \text{weakly in } X_V^s(\mathbb{R}^N)$$

and

$$\mathcal{E}_\lambda(u_n) = \mathcal{E}_\lambda(u) + \sum_{j=1}^{k} \mathcal{J}(u_j) + \mathcal{J}(w_{n,k}) + o(1).$$

As $\mathcal{E}_\lambda(u_n)$ is bounded and $\mathcal{J}(u_j) \ge m_{\mathcal{J}}$, we can iterate the process only a finite number of times, which concludes our proof. □

Corollary 3.4. *For $c \in (0, m_{\mathcal{J}})$, any $(\mathrm{PS})_c$ sequence of $\mathcal{E}_\lambda|_{\mathcal{N}_\lambda}$ is relatively compact.*

Proof. Assume (u_n) is a $(\mathrm{PS})_c$ sequence of $\mathcal{E}_\lambda|_{\mathcal{N}_\lambda}$. From Lemma 3.2 we have $\mathcal{J}(u_j) \ge m_{\mathcal{J}}$, and hence it follows that, up to a subsequence, $u_n \to u$ strongly in $X_V^s(\mathbb{R}^N)$ and u is a solution of (1.1). □

In order to finish the proof of Theorem 1.1 we need the following result.

Lemma 3.5. $m_\lambda < m_{\mathcal{J}}$.

Proof. Let $Q \in X_V^s(\mathbb{R}^N)$ be a groundstate solution of (1.2); we know that such a groundstate exists, and for this we refer the reader to [5]. Denote by tQ the projection of Q on $\mathcal{N}_\lambda$, that is, $t = t(Q) > 0$ is the unique real number such that $tQ \in \mathcal{N}_\lambda$. Set

$$A(Q) = \int_{\mathbb{R}^N} (I_\alpha * |Q|^p)|Q|^p, \quad B(Q) = \lambda \int_{\mathbb{R}^N} (I_\beta * |Q|^p)|Q|^p.$$

As $Q \in \mathcal{N}_{\mathcal{J}}$ and $tQ \in \mathcal{N}_\lambda$, we get

$$\|Q\|^2 = A(Q)$$

and

$$t^2\|Q\|^2 = t^{2p}A(Q) + t^{2q}B(Q).$$

From the above equalities one can easily deduce that $t < 1$. Therefore, we have

$$\begin{aligned}
m_\lambda \le \mathcal{E}_\lambda(tQ) &= \frac{1}{2}t^2\|Q\|^2 - \frac{1}{2p}t^{2p}A(Q) - \frac{1}{2q}t^{2q}B(Q)\\
&= \Big(\frac{t^2}{2} - \frac{t^{2p}}{2p}\Big)\|Q\|^2 - \frac{1}{2q}(t^2\|Q\|^2 - t^{2p}A(Q))\\
&= t^2\Big(\frac{1}{2} - \frac{1}{2q}\Big)\|Q\|^2 + t^{2p}\Big(\frac{1}{2q} - \frac{1}{2p}\Big)\|Q\|^2\\
&< \Big(\frac{1}{2} - \frac{1}{2q}\Big)\|Q\|^2 + \Big(\frac{1}{2q} - \frac{1}{2p}\Big)\|Q\|^2\\
&< \Big(\frac{1}{2} - \frac{1}{2p}\Big)\|Q\|^2 = \mathcal{J}(Q) = m_{\mathcal{J}},
\end{aligned}$$

as desired. □

Further, using the Ekeland variational principle, for any $n \geq 1$ there exists $(u_n) \in \mathcal{N}_\lambda$ such that

$$\begin{aligned} \mathcal{E}_\lambda(u_n) &\leq m_\lambda + \frac{1}{n} && \text{for all } n \geq 1, \\ \mathcal{E}_\lambda(u_n) &\leq \mathcal{E}_\lambda(v) + \frac{1}{n}\|v - u_n\| && \text{for all } v \in \mathcal{N}_\lambda,\ n \geq 1. \end{aligned}$$

Now, one can easily deduce that $(u_n) \in \mathcal{N}_\lambda$ is a $(\mathrm{PS})_{m_\lambda}$ sequence for $\mathcal{E}_\lambda$ on $\mathcal{N}_\lambda$. Further, using Lemma 3.5 and Corollary 3.4, we obtain that, up to a subsequence, (u_n) converges strongly to some $u \in X_V^s(\mathbb{R}^N)$ which is a groundstate of $\mathcal{E}_\lambda$.

4 Proof of Theorem 1.2

In this section, we discuss the existence of a least energy sign-changing solution of (1.1).

Lemma 4.1. *Assume $p > q > 2$ and $\lambda \in \mathbb{R}$. Then for any $u \in X_V^s(\mathbb{R}^N)$ and $u^\pm \neq 0$ there exists a unique pair $(\tau_0, \theta_0) \in (0,\infty) \times (0,\infty)$ such that $\tau_0 u^+ + \theta_0 u^- \in \mathcal{M}_\lambda$. Furthermore, if $u \in \mathcal{M}_\lambda$, then for all $\tau, \theta \geq 0$ we have $\mathcal{E}_\lambda(u) \geq \mathcal{E}_\lambda(\tau u^+ + \theta u^-)$.*

Proof. We shall follow an idea developed in [26]. Denote

$$\begin{aligned} a_1 &= \|u^+\|^2_{X_V^s}, & b_1 &= \|u^-\|^2_{X_V^s}, \\ a_2 &= \int_{\mathbb{R}^N} (I_\alpha * |u^+|^p)|u^+|^p, & b_2 &= \int_{\mathbb{R}^N} (I_\beta * |u^+|^q)|u^+|^q, \\ a_3 &= \int_{\mathbb{R}^N} (I_\alpha * |u^-|^p)|u^-|^p, & b_3 &= \int_{\mathbb{R}^N} (I_\beta * |u^-|^q)|u^-|^q, \\ a_4 &= \int_{\mathbb{R}^N} (I_\alpha * |u^+|^p)|u^-|^p, & b_4 &= \int_{\mathbb{R}^N} (I_\beta * |u^+|^q)|u^-|^q, \\ A &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}}\,dx\,dy. \end{aligned}$$

Let us define the function $\Phi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by

$$\begin{aligned} \Phi(\tau,\theta) &= \mathcal{E}_\lambda\big(\tau^{\frac{1}{2p}} u^+ + \theta^{\frac{1}{2p}} u^-\big) \\ &= \frac{\tau^{\frac{1}{p}}}{2} a_1 + \frac{\theta^{\frac{1}{p}}}{2} b_1 - \lambda\frac{\tau^{\frac{q}{p}}}{2q} b_2 - \lambda\frac{\theta^{\frac{q}{p}}}{2q} b_3 - \lambda\frac{\tau^{\frac{q}{2p}}\theta^{\frac{q}{2p}}}{2q} b_4 - \frac{\tau}{2p} a_2 - \frac{\theta}{2p} a_3 - \frac{\tau^{\frac{1}{2}}\theta^{\frac{1}{2}}}{2p} a_4 - \tau^{\frac{1}{2p}}\theta^{\frac{1}{2p}} A. \end{aligned}$$

Note that Φ is strictly concave. Therefore, Φ has at most one maximum point. Also

$$\lim_{\tau\to\infty} \Phi(\tau,\theta) = -\infty \quad \text{for all } \theta \geq 0 \qquad \text{and} \qquad \lim_{\theta\to\infty} \Phi(\tau,\theta) = -\infty \quad \text{for all } \tau \geq 0, \tag{4.1}$$

and it is easy to check that

$$\lim_{\tau\searrow 0} \frac{\partial\Phi}{\partial\tau}(\tau,\theta) = \infty \quad \text{for all } \theta > 0 \qquad \text{and} \qquad \lim_{\theta\searrow 0} \frac{\partial\Phi}{\partial\theta}(\tau,\theta) = \infty \quad \text{for all } \tau > 0. \tag{4.2}$$

Hence, (4.1) and (4.2) rule out the possibility of achieving a maximum at the boundary. Therefore, Φ has exactly one maximum point $(\tau_0, \theta_0) \in (0,\infty) \times (0,\infty)$. □

Lemma 4.2. *The energy level $c_\lambda > 0$ is achieved by some $v \in \mathcal{M}_\lambda$.*

Proof. Let $(u_n) \subset \mathcal{M}_\lambda$ be a minimizing sequence for c_λ. Note that

$$\begin{aligned}\mathcal{E}_\lambda(u_n) &= \mathcal{E}_\lambda(u_n) - \frac{1}{2q}\langle \mathcal{E}'_\lambda(u_n), u_n\rangle \\ &= \Big(\frac{1}{2} - \frac{1}{2q}\Big)\|u_n\|^2_{X^s_V} + \Big(\frac{1}{2q} - \frac{1}{2p}\Big)\int_{\mathbb{R}^N} (I_\alpha * |u|^q)|u|^q \\ &\geq \Big(\frac{1}{2} - \frac{1}{2q}\Big)\|u_n\|^2_{X^s_V} \\ &\geq C_1\|u_n\|^2_{X^s_V},\end{aligned}$$

where $C_1 > 0$ is a positive constant. Therefore, for some constant $C_2 > 0$ we have

$$\|u_n\|^2_{X^s_V} \leq C_2\mathcal{E}_\lambda(u_n) \leq M,$$

which implies that (u_n) is bounded in $X^s_V(\mathbb{R}^N)$. So, (u_n^+) and (u_n^-) are also bounded in $X^s_V(\mathbb{R}^N)$, and, by passing to a subsequence, there exists $u^+, u^- \in H^s(\mathbb{R}^N)$ such that

$$u_n^+ \rightharpoonup u^+ \quad \text{and} \quad u_n^- \rightharpoonup u^- \qquad \text{weakly in } X^s_V(\mathbb{R}^N).$$

Since p, $q > 2$ satisfy (1.4) and (1.5), we deduce that the embeddings $X^s_V(\mathbb{R}^N) \hookrightarrow L^{2Np/(N+\alpha)}(\mathbb{R}^N)$ and $X^s_V(\mathbb{R}^N) \hookrightarrow L^{2Nq/(N+\beta)}(\mathbb{R}^N)$ are compact. Thus,

$$u_n^\pm \to u^\pm \quad \text{strongly in } L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N) \cap L^{\frac{2Nq}{N+\beta}}(\mathbb{R}^N). \tag{4.3}$$

Moreover, by the Hardy–Littlewood–Sobolev inequality, we estimate

$$\begin{aligned}C(\|u_n^\pm\|^2_{L^{\frac{2Np}{N+\alpha}}} + \|u_n^\pm\|^2_{L^{\frac{2Nq}{N+\beta}}}) \leq \|u_n^\pm\|^2_{X^s_V} &= \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n^\pm|^p + |\lambda| \int_{\mathbb{R}^N} (I_\beta * |u_n|^q)|u_n^\pm|^q \\ &\leq C(\|u_n^\pm\|^p_{L^{\frac{2Np}{N+\alpha}}} + \|u_n^\pm\|^p_{L^{\frac{2Np}{N+\alpha}}}) \\ &\leq C(\|u_n^\pm\|^2_{L^{\frac{2Np}{N+\alpha}}} + \|u_n^\pm\|^2_{L^{\frac{2Nq}{N+\beta}}})(\|u_n^\pm\|^{p-2}_{L^{\frac{2Np}{N+\alpha}}} + \|u_n^\pm\|^{q-2}_{L^{\frac{2Nq}{N+\beta}}}).\end{aligned}$$

Since $u_n^\pm \neq 0$, we can deduce

$$\|u_n^\pm\|^{p-2}_{L^{\frac{2Np}{N+\alpha}}} + \|u_n^\pm\|^{q-2}_{L^{\frac{2Nq}{N+\beta}}} \geq C > 0 \quad \text{for all } n \geq 1. \tag{4.4}$$

Hence, by (4.3) and (4.4) it follows that $u^\pm \neq 0$. Further, using (4.3) and the Hardy–Littlewood–Sobolev inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^N} (I_\alpha * |u_n^\pm|^p)|u_n^\pm|^p &\to \int_{\mathbb{R}^N} (I_\alpha * |u^\pm|^p)|u^\pm|^p, \\ \int_{\mathbb{R}^N} (I_\alpha * |u_n^+|^p)|u_n^-|^p &\to \int_{\mathbb{R}^N} (I_\alpha * |u^+|^p)|u^-|^p, \\ \int_{\mathbb{R}^N} (I_\beta * |u_n^\pm|^q)|u_n^\pm|^q &\to \int_{\mathbb{R}^N} (I_\beta * |u^\pm|^q)|u^\pm|^q, \\ \int_{\mathbb{R}^N} (I_\beta * |u_n^+|^q)|u_n^-|^q &\to \int_{\mathbb{R}^N} (I_\beta * |u^+|^q)|u^-|^q.\end{aligned}$$

By Lemma 4.1, there exists a unique pair (τ_0, θ_0) such that $\tau_0 u^+ + \theta_0 u^- \in \mathcal{M}_\lambda$. By the weakly lower semicontinuity of the norm $\|\cdot\|_{X^s_V}$, we deduce that

$$\begin{aligned}c_\lambda \leq \mathcal{E}_\lambda(\tau_0 u^+ + \theta_0 u^-) &\leq \liminf_{n\to\infty} \mathcal{E}_\lambda(\tau_0 u^+ + \theta_0 u^-) \\ &\leq \limsup_{n\to\infty} \mathcal{E}_\lambda(\tau_0 u^+ + \theta_0 u^-) \\ &\leq \lim_{n\to\infty} \mathcal{E}_\lambda(u_n) \\ &= c_\lambda.\end{aligned}$$

Letting now $v = \tau_0 u^+ + \theta_0 u^- \in \mathcal{M}_\lambda$, we finish the proof. ☐

Lemma 4.3. *$v = \tau_0 u^+ + \theta_0 u^- \in \mathcal{M}_\lambda$ is a critical point of $\mathcal{E}_\lambda : X_V^s(\mathbb{R}^N) \to \mathbb{R}$, that is,*

$$\mathcal{E}'_\lambda(v) = 0.$$

Proof. Assume by contradiction that v is not a critical point of $\mathcal{E}_\lambda$. Then there exists $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that

$$\langle \mathcal{E}'_\lambda(v), \varphi \rangle = -2.$$

Since $\mathcal{E}_\lambda$ is continuous and differentiable, there exists $r > 0$ small such that

$$\langle \mathcal{E}'_\lambda(\tau u^+ + \theta u^- + \varepsilon \bar{v}), \bar{v} \rangle \le -1 \quad \text{if } (\tau - \tau_0)^2 + (\theta - \theta_0)^2 \le r^2 \text{ and } 0 \le \varepsilon \le r. \tag{4.5}$$

Let D be the open disc in $\mathbb{R}^2$ of radius $r > 0$ centered at (τ_0, θ_0). We define a continuous function $\psi : D \to [0, 1]$ as

$$\psi(\tau, \theta) = \begin{cases} 1 & \text{if } (\tau - \tau_0)^2 + (\theta - \theta_0)^2 \le \frac{r^2}{16}, \\ 0 & \text{if } (\tau - \tau_0)^2 + (\theta - \theta_0)^2 \ge \frac{r^2}{4}. \end{cases}$$

Further, we define a continuous map $S : D \to X_V^s(\mathbb{R}^N)$ as

$$S(\tau, \theta) = \tau u^+ + \theta u^- + r\psi(\tau, \theta)\bar{v} \quad \text{for all } (\tau, \theta) \in D$$

and $L : D \to \mathbb{R}^2$ as

$$L(\tau, \theta) = (\langle \mathcal{E}'_\lambda(S(\tau, \theta)), S(\tau, \theta)^+ \rangle, \langle \mathcal{E}'_\lambda(S(\tau, \theta)), S(\tau, \theta)^- \rangle) \quad \text{for all } (\tau, \theta) \in D.$$

Since the mapping $u \mapsto u^+$ is continuous in $X_V^s(\mathbb{R}^N)$, it follows that L is continuous. If $(\tau - \tau_0)^2 + (\theta - \theta_0)^2 = r^2$, that is, if we are on the boundary of D, then $\psi = 0$ by definition. Then $S(\tau, \theta) = \tau u^+ + \theta u^-$ and, using Lemma 4.1, we get

$$L(\tau, \theta) = (\langle \mathcal{E}'_\lambda(\tau u^+ + \theta u^-), (\tau u^+ + \theta u^-)^+ \rangle, \langle \mathcal{E}'_\lambda(\tau u^+ + \theta u^-), (\tau u^+ + \theta u^-)^- \rangle) \ne 0 \quad \text{on } \partial D.$$

Therefore, the Brouwer degree is well defined and $\deg(L, \operatorname{int}(D), (0, 0)) = 1$. Then there exists $(\tau_1, \theta_1) \in \operatorname{int}(D)$ such that $L(\tau_1, \theta_1) = (0, 0)$. Thus, $S(\tau_1, \theta_1) \in \mathcal{M}_\lambda$ and, using the definition of c_λ, we get

$$\mathcal{E}_\lambda(S(\tau_1, \theta_1)) \ge c_\lambda. \tag{4.6}$$

Using equation (4.5), we deduce that

$$\begin{aligned} \mathcal{E}_\lambda(S(\tau_1, \theta_1)) &= \mathcal{E}_\lambda(\tau_1 u^+ + \theta_1 u^-) + \int_0^1 \frac{d}{dt} \mathcal{E}_\lambda(\tau_1 u^+ + \theta_1 u^- + rt\psi(\tau_1, \theta_1)\bar{v})\, dt \\ &= \mathcal{E}_\lambda(\tau_1 u^+ + \theta_1 u^-) + \int_0^1 \langle \mathcal{E}'_\lambda(\tau_1 u^+ + \theta_1 u^- + rt\psi(\tau_1, \theta_1)\bar{v}), r\psi(\tau_1, \theta_1)\bar{v} \rangle\, dt \\ &= \mathcal{E}_\lambda(\tau_1 u^+ + \theta_1 u^-) - r\psi(\tau_1, \theta_1). \end{aligned}$$

If $(\tau_1, \theta_1) = (\tau_0, \theta_0)$, then $\psi(\tau_1, \theta_1) = 1$ by definition and we deduce that

$$\mathcal{E}_\lambda(S(\tau_1, \theta_1)) \le \mathcal{E}_\lambda(\tau_1 u^+ + \theta_1 u^-) - r \le c_\lambda - r < c_\lambda.$$

If $(\tau_1, \theta_1) \ne (\tau_0, \theta_0)$, then, using Lemma 4.1, we have

$$\mathcal{E}_\lambda(\tau_1 u^+ + \theta_1 u^-) < \mathcal{E}_\lambda(\tau_0 u^+ + \theta_0 u^-) = c_\lambda.$$

This yields

$$\mathcal{E}_\lambda(S(\tau_1, \theta_1)) \le \mathcal{E}_\lambda(\tau_1 u^+ + \theta_1 u^-) < c_\lambda,$$

which is a contradiction to equation (4.6). □

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References

[1] C. O. Alves, G. M. Figueiredo and M. Yang, Existence of solutions for a nonlinear Choquard equation with potential vanishing at infinity, *Adv. Nonlinear Anal.* **5** (2016), no. 4, 331–345.
[2] V. I. Bogachev, *Measure Theory. Vol. I, II*, Springer, Berlin, 2007.
[3] G. Cerami and R. Molle, Positive bound state solutions for some Schrödinger–Poisson systems, *Nonlinearity* **29** (2016), no. 10, 3103–3119.
[4] G. Cerami and G. Vaira, Positive solutions for some non-autonomous Schrödinger–Poisson systems, *J. Differential Equations* **248** (2010), no. 3, 521–543.
[5] P. d'Avenia, G. Siciliano and M. Squassina, On fractional Choquard equations, *Math. Models Methods Appl. Sci.* **25** (2015), no. 8, 1447–1476.
[6] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.* **136** (2012), no. 5, 521–573.
[7] A. Elgart and B. Schlein, Mean field dynamics of boson stars, *Comm. Pure Appl. Math.* **60** (2007), no. 4, 500–545.
[8] J. Fröhlich, B. L. G. Jonsson and E. Lenzmann, Boson stars as solitary waves, *Comm. Math. Phys.* **274** (2007), no. 1, 1–30.
[9] J. Fröhlich, B. L. G. Jonsson and E. Lenzmann, Effective dynamics for boson stars, *Nonlinearity* **20** (2007), no. 5, 1031–1075.
[10] M. Ghergu and S. D. Taliaferro, Pointwise bounds and blow-up for Choquard–Pekar inequalities at an isolated singularity, *J. Differential Equations* **261** (2016), no. 1, 189–217.
[11] K. R. W. Jones, Newtonian quantum gravity, *Australian J. Phys.* **48** (1995), 1055–1081.
[12] E. Lenzmann, Uniqueness of ground states for pseudorelativistic Hartree equations, *Anal. PDE* **2** (2009), 1–27.
[13] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. II, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1** (1984), no. 4, 223–283.
[14] L. Ma and L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation, *Arch. Ration. Mech. Anal.* **195** (2010), no. 2, 455–467.
[15] G. Molica Bisci, V. D. Radulescu and R. Servadei, *Variational Methods for Nonlocal Fractional Problems*, Encyclopedia Math. Appl. 162, Cambridge University Press, Cambridge, 2016.
[16] I. M. Moroz, R. Penrose and P. Tod, Spherically-symmetric solutions of the Schrödinger–Newton equations, *Classical Quantum Gravity* **15** (1998), no. 9, 2733–2742.
[17] V. Moroz and J. Van Schaftingen, Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics, *J. Funct. Anal.* **265** (2013), no. 2, 153–184.
[18] V. Moroz and J. Van Schaftingen, A guide to the Choquard equation, *J. Fixed Point Theory Appl.* **19** (2017), no. 1, 773–813.
[19] T. Mukherjee and K. Sreenadh, Positive solutions for nonlinear Choquard equation with singular nonlinearity, *Complex Var. Elliptic Equ.* **62** (2017), no. 8, 1044–1071.
[20] S. Pekar, *Untersuchung über die Elektronentheorie der Kristalle*, Akademie Verlag, Berlin, 1954.
[21] R. Penrose, On gravity's role in quantum state reduction, *Gen. Relativity Gravitation* **28** (1996), no. 5, 581–600.
[22] R. Penrose, Quantum computation, entanglement and state reduction, *R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci.* **356** (1998), no. 1743, 1927–1939.
[23] P. Pucci, M. Xiang and B. Zhang, Multiple solutions for nonhomogeneous Schrödinger–Kirchhoff type equations involving the fractional p-Laplacian in $\mathbb{R}^N$, *Calc. Var. Partial Differential Equations* **54** (2015), no. 3, 2785–2806.
[24] P. Pucci, M. Xiang and B. Zhang, Existence and multiplicity of entire solutions for fractional p-Kirchhoff equations, *Adv. Nonlinear Anal.* **5** (2016), no. 1, 27–55.
[25] C. E. Torres Ledesma, Existence and symmetry result for fractional p-Laplacian in $\mathbb{R}^n$, *Commun. Pure Appl. Anal.* **16** (2017), no. 1, 99–113.
[26] J. Van Schaftingen and J. Xia, Choquard equations under confining external potentials, *NoDEA Nonlinear Differential Equations Appl.* **24** (2017), no. 1, Paper No. 1.