DE GRUYTER Adv. Nonlinear Anal. 2019; 8: 559–582

Research Article

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Boundary layers to a singularly perturbed Klein–Gordon–Maxwell–Proca system on a compact Riemannian manifold with boundary

https://doi.org/10.1515/anona-2017-0039
Received February 24, 2017; accepted April 25, 2017

Abstract: We study the semiclassical limit to a singularly perturbed nonlinear Klein–Gordon–Maxwell–Proca system, with Neumann boundary conditions, on a Riemannian manifold $\mathfrak{M}$ with boundary. We exhibit examples of manifolds, of arbitrary dimension, on which these systems have a solution which concentrates at a closed submanifold of the boundary of $\mathfrak{M}$, forming a positive layer, as the singular perturbation parameter goes to zero. Our results allow supercritical nonlinearities and apply, in particular, to bounded domains in $\mathbb{R}^N$. Similar results are obtained for the more classical electrostatic Klein–Gordon–Maxwell system with appropriate boundary conditions.

Keywords: Electrostatic Klein–Gordon–Maxwell–Proca system, semiclassical limit, boundary layer, Riemannian manifold with boundary, supercritical nonlinearity, Lyapunov–Schmidt reduction

MSC 2010: 35J60, 35J20, 35B40, 53C80, 58J32, 81V10

1 Introduction

On a compact smooth Riemannian manifold $(\mathfrak{M}, \mathfrak{g})$ with boundary, we consider the system

$$\begin{cases} -\varepsilon^2 \Delta_{\mathfrak{g}} \mathfrak{u} + \alpha(x)\mathfrak{u} = \mathfrak{u}^{p-1} + \omega^2 (q\mathfrak{v} - 1)^2 \mathfrak{u} & \text{on } \mathfrak{M}, \\ -\Delta_{\mathfrak{g}} \mathfrak{v} + \Lambda(\mathfrak{u})\mathfrak{v} = q\mathfrak{u}^2 & \text{on } \mathfrak{M}, \\ \dfrac{\partial \mathfrak{u}}{\partial \nu} = 0, \quad \dfrac{\partial \mathfrak{v}}{\partial \nu} = 0 \text{ or } \mathfrak{v} = 0 & \text{on } \partial\mathfrak{M}, \end{cases} \tag{1.1}$$

where $\Delta_{\mathfrak{g}} = \operatorname{div}_{\mathfrak{g}} \nabla_{\mathfrak{g}}$ is the Laplace–Beltrami operator (without a sign), $\varepsilon > 0$, $q > 0$, $\omega \in \mathbb{R}$, $\alpha \in \mathcal{C}^2(\mathfrak{M})$ is a real-valued function which satisfies $\alpha(x) > \omega^2$ on $\mathfrak{M}$, $p \in (2, \infty)$, and Λ is given by

$$\Lambda(\mathfrak{u}) = \begin{cases} 1 + q\mathfrak{u}^2 & \text{if } \dfrac{\partial \mathfrak{u}}{\partial \nu} = \dfrac{\partial \mathfrak{v}}{\partial \nu} = 0 \text{ on } \partial\mathfrak{M}, \\ q\mathfrak{u}^2 & \text{if } \dfrac{\partial \mathfrak{u}}{\partial \nu} = \mathfrak{v} = 0 \text{ on } \partial\mathfrak{M}. \end{cases}$$

We are interested in studying the semiclassical limit to this system, i.e., the existence of positive solutions and their asymptotic profile, as $\varepsilon \to 0$.

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Solutions to system (1.1) correspond to standing waves of an electrostatic Klein–Gordon–Maxwell (KGM) system if $\Lambda(\mathfrak{u}) = q\mathfrak{u}^2$, and of a Klein–Gordon–Maxwell–Proca (KGMP) system with Proca mass 1 if $\Lambda(\mathfrak{u}) = 1 + q\mathfrak{u}^2$. For the physical meaning of these systems, we refer to [3, 4, 25].

The seminal paper [3] by Benci and Fortunato attracted the attention of the mathematical community, and motivated much of the recent activity towards the study of this type of systems. For $\varepsilon = 1$, existence and nonexistence results for subcritical nonlinear terms have been obtained, e.g., in [1, 3, 6, 10, 13–15, 27] for systems in the entire space $\mathbb{R}^3$, or in a bounded domain in $\mathbb{R}^3$ with Dirichlet or Neumann boundary conditions. KGMP-systems on a closed (i.e., compact and without boundary) Riemannian manifold of dimension 3 or 4 have been recently investigated in [17, 24, 25] for subcritical or critical nonlinearities.

The existence and asymptotic behavior of semiclassical states in flat domains have been investigated, e.g., in [11, 12, 31]. In [11], D'Aprile and Wei constructed a family of positive radial solutions $(\mathfrak{u}_\varepsilon, \mathfrak{v}_e)$ to a KGM-system in a 3-dimensional ball, with Dirichlet boundary conditions, such that $\mathfrak{u}_\varepsilon$ concentrates around a sphere which lies in the interior of the ball. For compact manifolds of dimensions 2 and 3, with or without boundary, the existence and multiplicity of positive semiclassical states, such that $\mathfrak{u}_\varepsilon$ concentrates at a point, have been exhibited, e.g., in [20, 21, 23], for subcritical nonlinearities. The concentration at a positive-dimensional submanifold for a KGMP-system on closed manifolds of arbitrary dimension, and for nonlinearities which include supercritical ones, was recently exhibited in [7].

Our aim is to extend the results in [7, 8] to manifolds with boundary, i.e., we will establish the existence of positive semiclassical states $(\mathfrak{u}_\varepsilon, \mathfrak{v}_e)$ to system (1.1), on some compact Riemannian manifolds $\mathfrak{M}$ with boundary, such that $\mathfrak{u}_\varepsilon$ concentrates at a positive-dimensional submanifold as $\varepsilon \to 0$. Our results apply, in particular, to systems with supercritical nonlinearities in bounded smooth domains Ω of $\mathbb{R}^N$ of any dimension.

The Neumann boundary condition $\frac{\partial \mathfrak{v}}{\partial \nu} = 0$ on $\mathfrak{v}$ seems to be more meaningful from a physical point of view, as it gives a condition on the electric field on $\partial\mathfrak{M}$. However, if the Proca mass is 0, i.e., if $\Lambda(\mathfrak{u}) = q\mathfrak{u}^2$, and we set $\frac{\partial \mathfrak{v}}{\partial \nu} = 0$, then the second equation in system (1.1) admits the trivial solution $\mathfrak{v} = \frac{1}{q}$ and the first equation reduces to a Schrödinger equation, making the coupling effect unnoticeable. This is why we impose a Dirichlet boundary condition on $\mathfrak{v}$ when $\Lambda(\mathfrak{u}) = q\mathfrak{u}^2$.

The Neumann boundary condition $\frac{\partial \mathfrak{u}}{\partial \nu} = 0$ on $\mathfrak{u}$ produces an effect of the boundary of $\mathfrak{M}$ on the existence and concentration of solutions to system (1.1). In fact, the solutions that we obtain form a positive layer which concentrates around a submanifold of $\partial\mathfrak{M}$ as $\varepsilon \to 0$.

As in [7], our approach consists in reducing system (1.1) to a similar system, with the same power nonlinearity, on a manifold of lower dimension. Solutions to the new system which concentrate at a point will give rise to solutions to the original system concentrating at a positive-dimensional submanifold. This approach was introduced by Ruf and Srikanth in [29] and has been used, for instance, in [9, 28, 30]. We begin by describing some of the reductions that we will use.

1.1 Reducing the dimension of the system

Let (M, g) be a compact smooth n-dimensional Riemannian manifold with boundary, let $f : M \to (0, \infty)$ be a $\mathcal{C}^1$-function, and let (N, h) be a compact smooth Riemannian manifold without boundary of dimension $k \geq 1$. The *warped product* $M \times_{f^2} N$ is the cartesian product $M \times N$ endowed with the Riemannian metric $\mathfrak{g} := g + f^2 h$. It is a smooth Riemannian manifold of dimension $n + k$ with boundary $\partial M \times_{f^2} N$.

For example, if Θ is a bounded smooth domain in $\mathbb{R}^n$ whose closure is contained in $\mathbb{R}^{n-1} \times (0, \infty)$, $f(x_1, \ldots, x_n) = x_n$ and $\mathbb{S}^k$ is the standard k-sphere, then, up to isometry, the warped product $\Theta \times_{f^2} \mathbb{S}^k$ is

$$\Theta \times_{f^2} \mathbb{S}^k \equiv \{(y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}^{k+1} : (y, |z|) \in \Theta\},$$

which is a bounded smooth domain in $\mathbb{R}^{n+k}$.

Let $\pi_M : M \times_{f^2} N \to M$ be the projection, $\widehat{\alpha} \in \mathcal{C}^2(M)$ and $\alpha := \widehat{\alpha} \circ \pi_M$. A straightforward computation gives the following result; see, e.g., [16].

Proposition 1.1. *The functions $u_\varepsilon, v_\varepsilon : M \to \mathbb{R}$ solve the system*

$$\begin{cases} -\varepsilon^2 \operatorname{div}_g(f^k \nabla_g u) + f^k \widehat{a} u = f^k u^{p-1} + \omega^2 f^k (qv-1)^2 u & \text{on } M, \\ -\operatorname{div}_g(f^k \nabla_g v) + f^k \Lambda(u) v = q f^k u^2 & \text{on } M, \\ \dfrac{\partial u}{\partial \nu} = 0, \quad \dfrac{\partial v}{\partial \nu} = 0 \text{ or } v = 0 & \text{on } \partial M \end{cases} \tag{1.2}$$

if and only if the functions $\mathfrak{u}_\varepsilon := u_\varepsilon \circ \pi_M$, $\mathfrak{v}_\varepsilon := v_\varepsilon \circ \pi_M : M \times_{f^2} N \to \mathbb{R}$ *solve the system*

$$\begin{cases} -\varepsilon^2 \Delta_{\mathfrak{g}} \mathfrak{u} + \alpha \mathfrak{u} = \mathfrak{u}^{p-1} + \omega^2 (q\mathfrak{v}-1)^2 \mathfrak{u} & \text{on } M \times_{f^2} N, \\ -\Delta_{\mathfrak{g}} \mathfrak{v} + \Lambda(\mathfrak{u}) \mathfrak{v} = q \mathfrak{u}^2 & \text{on } M \times_{f^2} N, \\ \dfrac{\partial \mathfrak{u}}{\partial \nu} = 0, \quad \dfrac{\partial \mathfrak{v}}{\partial \nu} = 0 \text{ or } \mathfrak{v} = 0 & \text{on } \partial(M \times_{f^2} N). \end{cases} \tag{1.3}$$

We stress that the exponent p is the same in both systems. Since $k \geq 1$, we have that $2^*_{n+k} < 2^*_n$, where 2^*_d is the critical Sobolev exponent in dimension d, i.e., $2^*_d := \infty$ if $d = 2$ and $2^*_d := \frac{2d}{d-2}$ for $d > 2$. So, if $2^*_{n+k} \leq p < 2^*_n$, system (1.2) on M is subcritical, whereas system (1.3) on $M \times_{f^2} N$ is critical or supercritical. Moreover, if the solution u_ε of (1.2) concentrates at a point $x_0 \in M$ as $\varepsilon \to 0$, then the function $\mathfrak{u}_\varepsilon := u_\varepsilon \circ \pi_M$ concentrates at the submanifold $\pi_M^{-1}(x_0) \cong (N, f^2(x_0)h)$. Note also that $\mathfrak{u}_\varepsilon$ and $\mathfrak{v}_\varepsilon$ are positive if u_ε and v_ε are positive.

Another type of reduction is obtained from the Hopf maps. For $N = 2, 4, 8, 16$, we write $\mathbb{R}^N \equiv \mathbb{K} \times \mathbb{K}$, where $\mathbb{K}$ is either the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, or the Cayley numbers $\mathbb{O}$. The Hopf map $\mathfrak{h}_\mathbb{K}$ is defined by

$$\mathfrak{h}_\mathbb{K} : \mathbb{R}^{2 \dim \mathbb{K}} \equiv \mathbb{K} \times \mathbb{K} \to \mathbb{K} \times \mathbb{R} \equiv \mathbb{R}^{\dim \mathbb{K} + 1},$$
$$\mathfrak{h}_\mathbb{K}(z) := (2\overline{z_1} z_2, |z_1|^2 - |z_2|^2) \quad \text{for } z = (z_1, z_2) \in \mathbb{K} \times \mathbb{K}.$$

This map is horizontally conformal with dilation $\lambda(z) = 2|z|$. It is also invariant under the action of the units $S_\mathbb{K} := \{\zeta \in \mathbb{K} : |\zeta| = 1\}$, i.e., $\mathfrak{h}_\mathbb{K}(\zeta z) = \mathfrak{h}_\mathbb{K}(z)$ for all $\zeta \in S_\mathbb{K}$, $z \in \mathbb{K} \times \mathbb{K}$.

Let Ω be a bounded smooth domain in $\mathbb{R}^{2 \dim \mathbb{K}} \setminus \{0\}$ such that $\zeta z \in \Omega$ for all $\zeta \in S_\mathbb{K}$, $z \in \Omega$. Then $\Theta := \mathfrak{h}_\mathbb{K}(\Omega)$ is a bounded smooth domain in $\mathbb{R}^{\dim \mathbb{K}+1} \setminus \{0\}$. The main property of Hopf maps, for our purposes, is that they locally preserve the Laplace operator up to a factor, i.e.,

$$\Delta(u \circ \mathfrak{h}_\mathbb{K}) = \lambda^2 [(\Delta u) \circ \mathfrak{h}_\mathbb{K}] \quad \text{in } \Omega \text{ for every } u \in \mathcal{C}^2(\Theta).$$

Such maps are called *harmonic morphisms*; see [2]. This property allows us to reduce system (1.1) on $\mathfrak{M} := \Omega$ to a system in Θ. Assume that $\alpha \in \mathcal{C}^2(\Omega)$ satisfies $\alpha(\zeta z) = \alpha(z)$ for all $\zeta \in S_\mathbb{K}$, $z \in \Omega$. Then the map $\widehat{a} : \Theta \to \mathbb{R}$ given by $\widehat{a}(x) := \alpha(\mathfrak{h}_\mathbb{K}^{-1}(x))$ is well defined and of class $\mathcal{C}^2$. Note that $\lambda^2(\mathfrak{h}_\mathbb{K}^{-1}(x)) = 4|x|$ for every $x \in \mathbb{R}^{\dim \mathbb{K}+1}$. The following proposition is an immediate consequence of these facts.

Proposition 1.2. *The functions $u_\varepsilon, v_\varepsilon : \Theta \to \mathbb{R}$ solve the system*

$$\begin{cases} -\varepsilon^2 \Delta u + \dfrac{\widehat{a}(x)}{4|x|} u = \dfrac{1}{4|x|} u^{p-1} + \dfrac{\omega^2}{4|x|} (qv-1)^2 u & \text{on } \Theta, \\ -\Delta v + \dfrac{1}{4|x|} \Lambda(u) v = \dfrac{q}{4|x|} u^2 & \text{on } \Theta, \\ \dfrac{\partial u}{\partial \nu} = 0, \quad \dfrac{\partial v}{\partial \nu} = 0 \text{ or } v = 0 & \text{on } \partial\Theta \end{cases} \tag{1.4}$$

if and only if the functions $\mathfrak{u}_\varepsilon := u_\varepsilon \circ \mathfrak{h}_\mathbb{K}$, $\mathfrak{v}_\varepsilon := v_\varepsilon \circ \mathfrak{h}_\mathbb{K} : \Omega \to \mathbb{R}$ *solve the system*

$$\begin{cases} -\varepsilon^2 \Delta \mathfrak{u} + \alpha(x) \mathfrak{u} = \mathfrak{u}^{p-1} + \omega^2 (q\mathfrak{v}-1)^2 \mathfrak{u} & \text{on } \Omega, \\ -\Delta \mathfrak{v} + \Lambda(\mathfrak{u}) \mathfrak{v} = q \mathfrak{u}^2 & \text{on } \Omega, \\ \dfrac{\partial \mathfrak{u}}{\partial \nu} = 0, \quad \dfrac{\partial \mathfrak{v}}{\partial \nu} = 0 \text{ or } \mathfrak{v} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Note again that, if $p \in [2^*_{2 \dim \mathbb{K}}, 2^*_{\dim \mathbb{K}+1})$, system (1.4) is subcritical, whereas system (1.5) is critical or supercritical. And if the functions u_ε concentrate at a point $\xi_0 \in \Theta$ as $\varepsilon \to 0$, then the functions $\mathfrak{u}_\varepsilon$ concentrate at the $(\dim \mathbb{K} - 1)$-dimensional sphere $\mathfrak{h}_\mathbb{K}^{-1}(\xi_0)$ in Ω.

Propositions 1.1 and 1.2 lead us to study the following problem.

1.2 The main results

Let (M, g) be a smooth compact Riemannian manifold with boundary of dimension $n = 2, 3, 4$. We consider the subcritical system

$$\begin{cases} -\varepsilon^2 \operatorname{div}_g(c(x)\nabla_g u) + a(x)u = b(x)u^{p-1} + b(x)\omega^2(qv-1)^2 u & \text{on } M, \\ -\operatorname{div}_g(c(x)\nabla_g v) + b(x)\Lambda(u)v = b(x)qu^2 & \text{on } M, \\ \dfrac{\partial u}{\partial \nu} = 0, \quad \dfrac{\partial v}{\partial \nu} = 0 \text{ or } v = 0 & \text{on } \partial M, \end{cases} \tag{1.6}$$

where $\varepsilon, q > 0$, $\omega \in \mathbb{R}$, $a, b, c \in \mathcal{C}^1(M)$ are strictly positive functions such that $a(x) > \omega^2 b(x)$ on M, and $p \in (2, 2_n^*)$. As before, $2_n^* := \infty$ if $n = 2$ and $2_n^* := \frac{2n}{n-2}$ if $n = 3, 4$.

Theorem 1.3. *Let $\mathcal{K} \subset \partial M$ be a nonempty $\mathcal{C}^1$-stable critical set for the function $\Gamma : \partial M \to \mathbb{R}$, which is given by*

$$\Gamma(\xi) := \frac{c(\xi)^{\frac{n}{2}}[a(\xi) - \omega^2 b(\xi)]^{\frac{p}{p-2} - \frac{n}{2}}}{b(\xi)^{\frac{2}{p-2}}}.$$

Then, for ε small enough, system (1.6) *has a positive solution $(u_\varepsilon, v_\varepsilon)$ such that u_ε concentrates at a point $\xi_0 \in \mathcal{K}$ as ε goes to zero.*

A $\mathcal{C}^1$-stable critical set is defined as follows.

Definition 1.4. Let $f \in \mathcal{C}^1(M, \mathbb{R})$. A subset $\mathcal{K}$ of M is called a $\mathcal{C}^1$-stable critical set of f if $\mathcal{K} \subset \{x \in M : \nabla_g f(x) = 0\}$ and if, for any $\mu > 0$, there exists $\delta > 0$ such that every function $h \in \mathcal{C}^1(M, \mathbb{R})$ which satisfies

$$\max_{\operatorname{dist}_g(x,\mathcal{K}) \le \mu} \big(|f(x) - h(x)| + |\nabla_g f(x) - \nabla_g h(x)|_g\big) \le \delta$$

has a critical point x_0 with $\operatorname{dist}_g(x_0, \mathcal{K}) \le \mu$. Here dist_g denotes the geodesic distance associated to the Riemannian metric g.

Theorem 1.3, together with Propositions 1.1 and 1.2, yields the existence of solutions to the KGMP (or the KGM) system (1.1), which concentrate at a submanifold for subcritical, critical and supercritical exponents. The following two results illustrate this fact.

We write the points in $\mathbb{R}^{n-1} \times (0, \infty)$ as $(\bar{y}, y_n)$ with $\bar{y} \in \mathbb{R}^{n-1}$ and $y_n \in (0, \infty)$.

Theorem 1.5. *Let Θ be a bounded smooth domain in $\mathbb{R}^n$ whose closure is contained in $\mathbb{R}^{n-1} \times (0, \infty)$ for $n = 2, 3, 4$, and let $\omega \in \mathbb{R}$ and $\widehat{\mathfrak{a}} \in \mathcal{C}^2(\Theta)$ be such that $\widehat{\mathfrak{a}} > \omega^2$. Let*

$$\mathfrak{M} := \{(\bar{y}, z) \in \mathbb{R}^{n-1} \times \mathbb{R}^{k+1} : (\bar{y}, |z|) \in \Theta\}$$

and $\mathfrak{a}(\bar{y}, z) := \widehat{\mathfrak{a}}(\bar{y}, |z|)$. If $\mathcal{K}$ is a nonempty $\mathcal{C}^1$-stable critical set for the function $\Gamma : \partial\Theta \to \mathbb{R}$ defined by

$$\Gamma(\bar{y}, y_n) := y_n^k [\widehat{\mathfrak{a}}(\bar{y}, y_n) - \omega^2]^{\frac{p}{p-2} - \frac{n}{2}},$$

then, for any $q > 0$, $p \in (2, 2_n^*)$ and ε small enough, system (1.1) *has a positive solution $(\mathfrak{u}_\varepsilon, \mathfrak{v}_\varepsilon)$ in $\mathfrak{M}$ such that, for some point $(\bar{\xi}, \xi_n) \in \mathcal{K}$, $\mathfrak{u}_\varepsilon$ concentrates at the k-dimensional sphere $\{(\bar{\xi}, z) \in \mathbb{R}^{n-1} \times \mathbb{R}^{k+1} : |z| = \xi_n\} \subset \partial\mathfrak{M}$ as $\varepsilon \to 0$.*

Proof. Set $M := \Theta$, $a := f^k \widehat{\mathfrak{a}}$ and $b := f^k =: c$ with $f(\bar{y}, y_n) := y_n$. Theorem 1.3 yields a positive solution $(u_\varepsilon, v_\varepsilon)$ to system (1.2) such that u_ε concentrates at a point $(\bar{\xi}, \xi_n) \in \mathcal{K}$ as $\varepsilon \to 0$. The result follows from Proposition 1.1. □

Theorem 1.6. *Let*

$$\mathfrak{M} := \{z \in \mathbb{C}^2 : 0 < r < |z| < R\}$$

and assume that $\mathfrak{a} \in \mathcal{C}^2(\mathfrak{M})$ satisfies $\mathfrak{a}(\zeta z) = \mathfrak{a}(z) > \omega^2$ for all $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, $z \in \mathfrak{M}$. If $\mathcal{K}$ is a nonempty $\mathcal{C}^1$-stable critical set for the function $\Gamma : \partial(\mathfrak{h}_{\mathbb{C}}(\mathfrak{M})) \to \mathbb{R}$ defined by

$$\Gamma(x) := \sqrt{2|x|}[\mathfrak{a}(\mathfrak{h}_{\mathbb{C}}^{-1}(x)) - \omega^2]^{\frac{p}{p-2} - \frac{3}{2}},$$

then, for any $q > 0$, $p \in (2, 6)$ and ε small enough, system (1.1) *has a positive solution* $(\mathfrak{u}_\varepsilon, \mathfrak{v}_\varepsilon)$ *in* $\mathfrak{M}$ *such that* $\mathfrak{u}_\varepsilon$ *concentrates at the circle* $\{\zeta z_0 : \zeta \in \mathbb{C},\ |\zeta| = 1\} \subset \partial\mathfrak{M}$*, for some* $z_0 \in \mathfrak{h}_{\mathbb{C}}^{-1}(\mathcal{K})$*, as* $\varepsilon \to 0$.

Proof. Set $M := \mathfrak{h}_{\mathbb{C}}(\mathfrak{M})$, $a(x) := \frac{\widehat{\alpha}(x)}{2|x|}$, $b(x) := \frac{1}{2|x|}$, and $c(x) := 1$ with $\widehat{\alpha}(x) := \alpha(\mathfrak{h}_{\mathbb{C}}^{-1}(x))$. Theorem 1.3 yields a positive solution $(u_\varepsilon, v_\varepsilon)$ to system (1.4) such that u_ε concentrates at a point $\xi_0 \in \mathcal{K}$ as $\varepsilon \to 0$. The result follows from Proposition 1.2. □

The rest of the paper is devoted to the proof of Theorem 1.3.

2 Preliminaries

2.1 Reducing system (1.6) to a single equation

In order to overcome the problems given by the competition between u and v, using an idea of Benci and Fortunato [3], we introduce the map $\Phi : H_g^1(M) \to H_g^1(M)$ which associates to each $u \in H_g^1(M)$ the solution $\Phi(u)$ to the problem

$$\begin{cases} -\operatorname{div}_g(c(x)\nabla_g[\Phi(u)]) + b(x)q^2u^2[\Phi(u)] = b(x)qu^2 & \text{in } M,\\ \Phi(u) = 0 & \text{on } \partial M \end{cases} \tag{2.1}$$

for system (1.6) with Dirichlet boundary conditions, or to the problem

$$\begin{cases} -\operatorname{div}_g(c(x)\nabla_g[\Phi(u)]) + b(x)(1 + q^2u^2)[\Phi(u)] = b(x)qu^2 & \text{in } M,\\ \dfrac{\partial[\Phi(u)]}{\partial\nu} = 0 & \text{on } \partial M \end{cases} \tag{2.2}$$

for system (1.6) with Neumann boundary conditions. It follows from standard variational arguments that Φ is well defined in $H_g^1(M)$. The proofs of the following two lemmas are contained in [17].

Lemma 2.1. *The map $\Phi : H_g^1(M) \to H_g^1(M)$ is of class $\mathcal{C}^1$ and its differential $\Phi'(u)[h] = V_u[h]$ at $u \in H_g^1(M)$ is the map defined by*

$$-\operatorname{div}_g\big(c(x)\nabla_g[V_u[h]]\big) + b(x)q^2u^2[V_u[h]] = 2b(x)qu(1 - q\Phi(u))h$$

for all $h \in H_g^1(M)$, in case of Dirichlet boundary conditions, or by

$$-\operatorname{div}_g\big(c(x)\nabla_g[V_u[h]]\big) + b(x)(1 + q^2u^2)[V_u[h]] = 2b(x)qu(1 - q\Phi(u))h,$$

for all $h \in H_g^1(M)$, in case of Neumann boundary conditions. Moreover,

$$0 \le \Phi(u) \le \frac{1}{q} \quad \text{and} \quad 0 \le \Phi'(u)[u] \le \frac{2}{q}.$$

Lemma 2.2. *The function $\Theta : H_g^1(M) \to \mathbb{R}$ given by*

$$\Theta(u) = \frac{1}{2}\int_M b(x)(1 - q\Phi(u))u^2\, d\mu_g$$

is of class $\mathcal{C}^1$, and its differential is given by

$$\Theta'(u)[h] = \int_M b(x)(1 - q\Phi(u))^2uh\, d\mu_g$$

for any $u, h \in H_g^1(M)$.

Now, we introduce the functionals $I_\varepsilon, J_\varepsilon, G_\varepsilon : H_g^1(M) \to \mathbb{R}$ given by

$$I_\varepsilon(u) := J_\varepsilon(u) + \frac{\omega^2}{2}G_\varepsilon(u), \tag{2.3}$$

where

$$J_\varepsilon(u) := \frac{1}{2\varepsilon^n}\int_M [\varepsilon^2 c(x)|\nabla_g u|^2 + d(x)u^2]\,d\mu_g - \frac{1}{p\varepsilon^n}\int_M b(x)(u^+)^p\,d\mu_g$$

with $d(x) := a(x) - \omega^2 b(x)$, and

$$G_\varepsilon(u) := \frac{q}{\varepsilon^n}\int_M b(x)\Phi(u)u^2\,d\mu_g.$$

From Lemma 2.2 we deduce that

$$\frac{1}{2}G'_\varepsilon(u)[\varphi] = \frac{1}{\varepsilon^n}\int_M b(x)[2q\Phi(u) - q^2\Phi^2(u)]u\varphi\,d\mu_g,$$

so

$$I'_\varepsilon(u)\varphi = \frac{1}{\varepsilon^n}\int_M [\varepsilon^2 c(x)\nabla_g u\nabla_g\varphi + a(x)u\varphi - b(x)(u^+)^{p-1}\varphi - b(x)\omega^2(1 - q\Phi(u))^2 u\varphi\,d\mu_g].$$

Therefore, if u is a critical point of the functional I_ε, we have that

$$-\varepsilon^2\,\mathrm{div}_g(c(x)\nabla_g u) + d(x)u + \omega^2 qb(x)\Phi(u)(2 - q\Phi(u))u = b(x)(u^+)^{p-1}, \tag{2.4}$$

with $d(x) := a(x) - \omega^2 b(x)$. In particular, if $u \neq 0$, by the maximum principle and regularity arguments we have that $u > 0$. Thus, the pair $(u, \Phi(u))$ is a positive solution to system (1.6).

This reduces solving system (1.6) to finding a solution $u_\varepsilon \in H^1_g(M)$ to the single equation (2.4).

Some useful estimates involving the function Φ are contained in the appendix.

2.2 The approximate solution

We shall obtain a solution u_ε to equation (2.4) using the Lyapunov–Schmidt reduction method. It will be an approximation to a function $W_{\varepsilon,\xi}$, which we introduce next.

If (M, g) is an n-dimensional compact smooth Riemannian manifold with boundary, its boundary ∂M is a closed smooth Riemannian manifold of dimension $n-1$, possibly not connected. We fix $R > 0$, smaller than the injectivity radius of ∂M, such that for each point $x \in M$ with $\mathrm{dist}_g(x, \partial M) < R$ there exists a unique $\bar{x} \in \partial M$ for which $\mathrm{dist}_g(x, \bar{x}) = \mathrm{dist}_g(x, \partial M)$, where dist_g denotes the geodesic distance in (M, g). For $\xi \in \partial M$, we set

$$Q_\xi := \{x \in M : \mathrm{dist}_g(x, \partial M) = \mathrm{dist}_g(x, \bar{x}) < R,\ \bar{x} \in \partial M,\ \mathrm{dist}_g(\xi, \bar{x}) < R\}.$$

We write each point $x \in Q_\xi$ in *Fermi coordinates* $(y_1, \dots, y_n)$ *at* ξ, i.e., $(y_1, \dots, y_{n-1})$ are normal coordinates for $\bar{x}$ on ∂M at the point ξ, and $y_n = \mathrm{dist}_g(x, \bar{x})$ is the geodesic distance from x to ∂M. We write $\psi^\partial_\xi : D^+ \to Q_\xi$ for the chart whose inverse is given by $(\psi^\partial_\xi)^{-1}(x) := (y_1, \dots, y_n)$, defined on

$$D^+ := B^{n-1}_R(0) \times [0, R), \quad \text{where } B^{n-1}_R(0) := \{\bar{y} \in \mathbb{R}^{n-1} : |\bar{y}| < R\}.$$

The second fundamental form $\mathrm{II}(X, Y)$ of two vector fields X and Y on ∂M is the component of $\nabla_X Y$ which is normal to ∂M, where ∇ is the covariant derivative operator in the ambient manifold M. In Fermi coordinates at q it is given by a matrix $(h_{ij})_{i,j=1,\dots,n-1}$. One has the well-known formulas

$$g^{ij}(y) = \delta_{ij} + 2h_{ij}y_n + O(|y|^2) \quad \text{for } i, j = 1, \dots, n-1, \tag{2.5}$$

$$g^{in}(y) = \delta_{in}, \tag{2.6}$$

$$\sqrt{|g|}(y) = 1 - (n-1)Hy_n + O(|y|^2), \tag{2.7}$$

where $y = (y_1, \dots, y_n)$ are the Fermi coordinates, $|g|$ is the determinant of $g = (g_{ij})$, g^{ij} are the coefficients of the inverse of (g_{ij}), and $H = \frac{1}{n-1}\sum_{i=1}^{n-1} h_{ii}$; see [5, 18, 19]. Abusing notation, we shall write $(h_{ij})_{i,j=1,\dots,n}$ for the matrix which coincides with the second fundamental form for $i, j = 1, \dots, n-1$ and has $h_{i,n} = h_{n,j} = 0$ for $i, j = 1, \dots, n$.

Set $d(x) := a(x) - \omega^2 b(x)$. By assumption, this function is positive on M. Given $\xi \in \partial M$, we consider the unique positive radial solution $\bar{V} = \bar{V}^\xi$ to the equation

$$- c(\xi)\Delta\bar{V} + d(\xi)\bar{V} = b(\xi)\bar{V}^{p-1} \quad \text{in } \mathbb{R}^n. \tag{2.8}$$

By direct computation, one sees that

$$\bar{V}^\xi(y) = \Big(\frac{d(\xi)}{b(\xi)}\Big)^{\frac{1}{p-2}} U\Big(\sqrt{\frac{d(\xi)}{c(\xi)}}y\Big),$$

where U is the unique positive radial solution of

$$-\Delta U + U = U^{p-1} \text{ in } \mathbb{R}^n.$$

In the following, we set

$$\gamma(\xi) := \Big(\frac{d(\xi)}{b(\xi)}\Big)^{\frac{1}{p-2}} \quad \text{and} \quad A(\xi) := \frac{d(\xi)}{c(\xi)},$$

so

$$\bar{V}^\xi(y) = \gamma(\xi)U\big(\sqrt{A(\xi)}y\big).$$

The restriction $V^\xi(y) := \bar{V}^\xi|_{\mathbb{R}^n_+}$ of $\bar{V}^\xi$ to the half-space $\mathbb{R}^n_+ := \{y_n \geq 0\}$ solves the Neumann problem

$$\begin{cases} -c(\xi)\Delta V + d(\xi)V = b(\xi)V^{p-1} & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial V}{\partial y_n} = 0 & \text{on } \{y_n = 0\}. \end{cases}$$

For $\xi \in \partial M$ and $\varepsilon > 0$, set $V^\xi_\varepsilon(y) := V^\xi(\frac{y}{\varepsilon})$. We define the functions $W_{\varepsilon,\xi} \in \mathcal{C}^\infty(M)$ by

$$W_{\varepsilon,\xi}(x) := \begin{cases} V^\xi_\varepsilon((\psi^\partial_\xi)^{-1}(x))\chi((\psi^\partial_\xi)^{-1}(x)), & x \in Q_\xi, \\ 0 & \text{elsewhere}. \end{cases} \tag{2.9}$$

Here the function χ is a fixed cut-off function of the form $\chi(\bar{y}, y_n) := \tilde{\chi}(|\bar{y}|)\tilde{\chi}(y_n)$ for $(\bar{y}, y_n) \in D^+$, where $\tilde{\chi} : \mathbb{R}_+ \to [0, 1]$ is a smooth function such that $\chi(s) \equiv 1$ for $0 \leq s \leq \frac{R}{2}$, $\chi(s) \equiv 0$ for $s \geq R$ and $|\tilde{\chi}'(s)| \leq \frac{1}{R}$.

Remark 2.3. The following limits hold uniformly with respect to $\xi \in \partial M$,

$$\begin{aligned} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^n}|W_{\varepsilon,\xi}|^p_{p,g} &\leq C|U|^p_p, \quad p \geq 2, \\ \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^n}|\varepsilon^2\nabla_g W_{\varepsilon,\xi}|^2_{2,g} &\leq C|\nabla U|^2_2, \end{aligned}$$

where the constant C does not depend on ξ.

It is well known that the space of solutions to the linearized problem

$$\begin{cases} -\Delta\varphi + \varphi = (p-1)(V^\xi)^{p-2}\varphi & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial\varphi}{\partial y_n} = 0 & \text{on } \{y_n = 0\}, \end{cases}$$

is generated by the functions $\varphi^i := \frac{\partial V^\xi}{\partial y_i}$ for $i = 1, \dots, n-1$. The corresponding local functions on the manifold M are given by

$$Z^i_{\varepsilon,\xi}(x) := \begin{cases} \varphi^i_\varepsilon((\psi^\partial_\xi)^{-1}(x))\chi((\psi^\partial_\xi)^{-1}(x)), & x \in Q_\xi, \\ 0 & \text{elsewhere}, \end{cases} \tag{2.10}$$

where $\varphi^i_\varepsilon(y) := \varphi^i(\frac{y}{\varepsilon})$ and χ is as above.

2.3 Proof of Theorem 1.3

As before, we set $d(x) := a(x) - \omega^2 b(x) > 0$. We denote by H_ε the space $H_g^1(M)$ equipped with the scalar product

$$\langle u, v\rangle_\varepsilon := \frac{1}{\varepsilon^n}\int_M \varepsilon^2 c(x)\nabla_g u \nabla_g v + d(x)uv\, d\mu_g$$

and the norm $\|u\|_\varepsilon = \langle u, u\rangle_\varepsilon^{1/2}$. Similarly, we write L_ε^p for the space $L_g^p(M)$ endowed with the norm

$$|u|_{\varepsilon,p} = \frac{1}{\varepsilon^n}\Big(\int_M |u|^p\, d\mu_g\Big)^{\frac{1}{p}}.$$

For any $p \in [2, 2_n^*)$, the embedding $i_\varepsilon : H_\varepsilon \hookrightarrow L_{\varepsilon,p}$ is compact and there is a positive constant C, independent of ε, such that $|u|_{\varepsilon,p} \le C\|u\|_\varepsilon$. The adjoint operator $i_\varepsilon^* : L_{\varepsilon,p'} \hookrightarrow H_\varepsilon$, $p' := \frac{p}{p-1}$, is defined by

$$\begin{aligned} u = i_\varepsilon^*(v) &\iff \langle u, \varphi\rangle_\varepsilon = \frac{1}{\varepsilon^n}\int_M v\varphi\, d\mu_g \quad \text{for all } \varphi \in H_g^1(M) \\ &\iff -\varepsilon^2 \operatorname{div}_g(c(x)\nabla_g u) + d(x)u = v. \end{aligned}$$

Note that, for some positive constant C independent of ε,

$$\|i_\varepsilon^*(v)\|_\varepsilon \le C|v|_{p',\varepsilon} \quad \text{for all } v \in L_{\varepsilon,p'}. \tag{2.11}$$

Using the adjoint operator, we can rewrite equation (2.4) as

$$u = i_\varepsilon^*(b(x)f(u) + \omega^2 b(x)g(u)),$$

where

$$f(u) := (u^+)^{p-1}; \quad g(u) := [q^2\Phi^2(u) - 2q\Phi(u)]u.$$

For $\xi \in \partial M$ and $\varepsilon > 0$, let

$$K_{\varepsilon,\xi} := \operatorname{Span}\{Z_{\varepsilon,\xi}^1, \dots, Z_{\varepsilon,\xi}^{n-1}\},$$

where the $Z_{\varepsilon,\xi}^i$ are the functions defined in (2.10). This is an $(n-1)$-dimensional subspace of H_ε. We denote its orthogonal complement with respect to $\langle\cdot,\cdot\rangle_\varepsilon$ by

$$K_{\varepsilon,\xi}^\perp := \{u \in H_\varepsilon : \langle u, Z_{\varepsilon,\xi}^i\rangle_\varepsilon = 0\}.$$

We look for a solution to equation (2.4) of the form $W_{\varepsilon,\xi} + \phi$ with $\phi \in K_{\varepsilon,\xi}^\perp$. Thus, $W_{\varepsilon,\xi} + \phi$ solves the equations

$$\Pi_{\varepsilon,\xi}^\perp\big(W_{\varepsilon,\xi} + \phi - i_\varepsilon^*[b(x)f(W_{\varepsilon,\xi} + \phi) + \omega^2 b(x)g(W_{\varepsilon,\xi} + \phi)]\big) = 0, \tag{2.12}$$

$$\Pi_{\varepsilon,\xi}\big(W_{\varepsilon,\xi} + \phi - i_\varepsilon^*[b(x)f(W_{\varepsilon,\xi} + \phi) + \omega^2 b(x)g(W_{\varepsilon,\xi} + \phi)]\big) = 0,$$

where $\Pi_{\varepsilon,\xi} : H_\varepsilon \to K_{\varepsilon,\xi}$ and $\Pi_{\varepsilon,\xi}^\perp : H_\varepsilon \to K_{\varepsilon,\xi}^\perp$ are the orthogonal projections onto $K_{\varepsilon,\xi}$ and $K_{\varepsilon,\xi}^\perp$, respectively.

The first step in the proof of Theorem 1.3 is to solve equation (2.12). To this end, we define the linear operator $L_{\varepsilon,\xi} : K_{\varepsilon,\xi}^\perp \to K_{\varepsilon,\xi}^\perp$ by

$$L_{\varepsilon,\xi}(\phi) := \Pi_{\varepsilon,\xi}^\perp(\phi - i_\varepsilon^*[b(x)f'(W_{\varepsilon,\xi})\phi]). \tag{2.13}$$

Lemma 3.1 yields the invertibility of $L_{\varepsilon,\xi}$. Then we will use a contraction mapping argument to solve equation (2.12). In Section 3, we will prove the following result.

Proposition 2.4. *There exist $\varepsilon_0 > 0$ and $C > 0$ such that, for any $\xi \in \partial M$ and any $\varepsilon \in (0, \varepsilon_0)$, there is a unique $\phi = \phi_{\varepsilon,\xi}$ which solves equation* (2.12). *This function satisfies*

$$\|\phi_{\varepsilon,\xi}\|_\varepsilon \le C\varepsilon.$$

Moreover, $\xi \to \phi_{\varepsilon,\xi}$ is a $\mathcal{C}^1$-map.

Now, for each $\varepsilon \in (0, \varepsilon_0)$, we introduce the reduced energy $\widetilde{I}_\varepsilon : \partial M \to \mathbb{R}$, defined by

$$\widetilde{I}_\varepsilon(\xi) := I_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}),$$

where I_ε is the functional defined in (2.3), whose critical points are the solutions to equation (2.4). It is easy to verify that ξ_ε is a critical point of $\widetilde{I}_\varepsilon$ if and only if the function $u_\varepsilon = W_{\varepsilon,\xi_\varepsilon} + \phi_{\varepsilon,\xi_\varepsilon}$ is a weak solution to problem (2.4).

In Section 4, we will compute the asymptotic expansion of the reduced functional $\widetilde{I}_\varepsilon$ with respect to the parameter ε. We will show that

$$\widetilde{I}_\varepsilon(\xi) = \kappa \frac{c(\xi)^{\frac{n}{2}} d(\xi)^{\frac{p}{p-2} - \frac{n}{2}}}{b(\xi)^{\frac{2}{p-2}}} + o(1)$$

$\mathcal{C}^1$-uniformly with respect to $\xi \in \partial M$ as $\varepsilon \to 0$, where

$$\kappa := \Big(\frac{1}{2} - \frac{1}{p}\Big) \int_{\mathbb{R}^n_+} U^p \, dz.$$

If $\mathcal{K}$ is a nonempty $\mathcal{C}^1$-stable critical set for the function Γ, then, by Definition 1.4, there exists a critical point $\xi_\varepsilon \in \partial M$ of $\widetilde{I}_\varepsilon$ such that $\operatorname{dist}_g(\xi_\varepsilon, \mathcal{K}) \to 0$ as $\varepsilon \to 0$. Consequently, $u_\varepsilon = W_{\varepsilon,\xi_\varepsilon} + \phi_{\varepsilon,\xi_\varepsilon}$ is a solution of (2.4), and Theorem 1.3 is proved.

3 The finite-dimensional reduction

In this section, we prove Proposition 2.4. Using the linear operator $L_{\varepsilon,\xi} : K^\perp_{\varepsilon,\xi} \to K^\perp_{\varepsilon,\xi}$ introduced in (2.13), equation (2.12) can be rewritten as

$$L_{\varepsilon,\xi}(\phi) = N_{\varepsilon,\xi}(\phi) + R_{\varepsilon,\xi} + S_{\varepsilon,\xi}(\phi),$$

where

$$\begin{aligned}
N_{\varepsilon,\xi}(\phi) &:= \Pi^\perp_{\varepsilon,\xi}(i^*_\varepsilon[b(x)(f(W_{\varepsilon,\xi} + \phi) - f(W_{\varepsilon,\xi}) - f'(W_{\varepsilon,\xi})\phi)]),\\
R_{\varepsilon,\xi} &:= \Pi^\perp_{\varepsilon,\xi}(i^*_\varepsilon[b(x)f(W_{\varepsilon,\xi})] - W_{\varepsilon,\xi}),\\
S_{\varepsilon,\xi}(\phi) &:= \omega^2 \Pi^\perp_{\varepsilon,\xi}(i^*_\varepsilon[b(x)(q^2\Phi^2(W_{\varepsilon,\xi} + \phi) - 2q\Phi(W_{\varepsilon,\xi} + \phi))(W_{\varepsilon,\xi} + \phi)]).
\end{aligned}$$

We refer to [26, Proposition 3.1], [7, Lemma 4.1] or [22, Lemma 10] for the proof of the following lemma.

Lemma 3.1. *There exist ε_0 and $C > 0$ such that, for any $\xi \in \partial M$ and $\varepsilon \in (0, \varepsilon_0)$,*

$$\|L_{\varepsilon,\xi}\|_\varepsilon \geq C\|\phi\|_\varepsilon \quad \textit{for every } \phi \in K^\perp_{\varepsilon,\xi}.$$

We now estimate the remainder term $R_{\varepsilon,\xi}$.

Lemma 3.2. *There exists $\varepsilon_0 > 0$ such that, for any $\xi \in \partial M$ and $\varepsilon \in (0, \varepsilon_0)$, one has*

$$\|R_{\varepsilon,\xi}\|_\varepsilon = o(\varepsilon).$$

Proof. Let $G_{\varepsilon,\xi}$ be the function such that $W_{\varepsilon,\xi} = i^*_\varepsilon(b(x)G_{\varepsilon,\xi})$, i.e.,

$$-\varepsilon^2 \operatorname{div}_g(c(x)\nabla_g W_{\varepsilon,\xi}) + d(x)W_{\varepsilon,\xi} = b(x)G_{\varepsilon,\xi}.$$

Then, for $x \in Q_\xi$ and its Fermi coordinates $y := (\psi^\partial_\xi)^{-1}(x)$, setting $\tilde{c}(y) := c(x)$, $\tilde{d}(y) := d(x)$ and $\tilde{b}(y) := b(x)$, we have

$$\begin{aligned}
b(x)G_{\varepsilon,\xi}(x) &= \tilde{d}(y)V^\xi_\varepsilon(y)\chi(y) - \frac{\varepsilon^2}{\sqrt{|g(y)|}} \frac{\partial}{\partial y_j}\Big[\sqrt{|g(y)|}g^{ij}(y)\tilde{c}(y)\frac{\partial}{\partial y_i}(V^\xi_\varepsilon(y)\chi(y))\Big]\\
&= \tilde{d}(y)V^\xi_\varepsilon(y)\chi(y) - \varepsilon^2 g^{ij}(y)\frac{\partial}{\partial y_j}\Big[\tilde{c}(y)\frac{\partial}{\partial y_i}(V^\xi_\varepsilon(y)\chi(y))\Big]\\
&\quad - \frac{\varepsilon^2}{\sqrt{|g(y)|}} \frac{\partial}{\partial y_j}\Big[\sqrt{|g(y)|}g^{ij}(y)\Big]\tilde{c}(y)\frac{\partial}{\partial y_i}(V^\xi_\varepsilon(y)\chi(y))
\end{aligned}$$

$$= \tilde{d}(y)V_\varepsilon^\xi(y)\chi(y) - \varepsilon^2\frac{\partial}{\partial y_i}\Big[\tilde{c}(y)\frac{\partial}{\partial y_i}V_\varepsilon^\xi(y)\Big]\chi(y) - \varepsilon^2\tilde{c}(y)\frac{\partial}{\partial y_i}V_\varepsilon^\xi(y)\frac{\partial}{\partial y_i}\chi(y)$$
$$- \varepsilon^2\frac{\partial}{\partial y_i}\Big[\tilde{c}(y)V_\varepsilon^\xi(y)\frac{\partial}{\partial y_i}\chi(y)\Big] - \varepsilon^2(g^{ij}(y)-\delta_{ij})\frac{\partial}{\partial y_j}\Big[\tilde{c}(y)\frac{\partial}{\partial y_i}(V_\varepsilon^\xi(y)\chi(y))\Big]$$
$$- \frac{\varepsilon^2}{\sqrt{|g(y)|}}\frac{\partial}{\partial y_j}[|g(y)|^{\frac{1}{2}}g^{ij}(y)]\tilde{c}(y)\frac{\partial}{\partial y_i}(V_\varepsilon^\xi(y)\chi(y)).$$

Moreover, by (2.8), we have

$$\tilde{d}(y)V_\varepsilon^\xi(y)\chi(y) - \varepsilon^2\frac{\partial}{\partial y_i}\Big[\tilde{c}(y)\frac{\partial}{\partial y_i}(V_\varepsilon^\xi(y))\Big]\chi(y)$$
$$= d(\xi)V_\varepsilon^\xi(y)\chi(y) - \varepsilon^2 c(\xi)\Delta V_\varepsilon^\xi(y)\chi(y) + [\tilde{d}(y)-d(\xi)]V_\varepsilon^\xi(y)\chi(y) - \varepsilon^2\frac{\partial}{\partial y_i}\Big[(\tilde{c}(y)-c(\xi))\frac{\partial}{\partial y_i}(V_\varepsilon^\xi(y))\Big]\chi(y)$$
$$= \Big(b(\xi)(V_\varepsilon^\xi(y))^{p-1} + [\tilde{d}(y)-d(\xi)]V_\varepsilon^\xi(y) - \varepsilon^2\frac{\partial}{\partial y_i}\Big[(\tilde{c}(y)-c(\xi))\frac{\partial}{\partial y_i}(V_\varepsilon^\xi(y))\Big]\Big)\chi(y).$$

From the definition of $R_{\varepsilon,\xi}$ we obtain

$$\|R_{\varepsilon,\xi}\|_\varepsilon \le \|i_\varepsilon^*(b(x)f(W_{\varepsilon,\xi})) - W_{\varepsilon,\xi}\|_\varepsilon = \big\|i_\varepsilon^*(b(x)[f(W_{\varepsilon,\xi}) - G_{\varepsilon,\xi}])\big\|_\varepsilon.$$

Using (2.11), we estimate the right-hand side by

$$\int_M \big|b(x)W_{\varepsilon,\xi}^{p-1} - b(x)G_{\varepsilon,\xi}\big|^{p'}\,d\mu_g \le C\int_{B(0,r)} \tilde{b}(y)^{p'}\big|(V_\varepsilon^\xi(y))^{p-1} - G_{\varepsilon,\xi}(\psi_\xi^\partial(y))\big|^{p'}\,dy$$
$$\le C\int_{D^+}|\tilde{b}(y)-b(\xi)|^{p'}|V_\varepsilon^\xi(y)|^p\,dy + C\int_{B(0,r)}|\tilde{d}(y)-d(\xi)|^{p'}|V_\varepsilon^\xi(y)|^p\,dy$$
$$+ C\varepsilon^2\int_{D^+}\Big|\frac{\partial}{\partial y_i}\Big[(\tilde{c}(y)-c(\xi))\frac{\partial}{\partial y_i}(V_\varepsilon^\xi(y))\Big]\Big|^{p'}\,dy$$
$$+ C\varepsilon^2\int_{D^+}|g^{ij}(y)-\delta_{ij}|^{p'}\Big|\frac{\partial}{\partial v_j}\Big[\tilde{c}(y)\frac{\partial}{\partial y_i}(V_\varepsilon^\xi(y))\Big]\Big|^{p'}\,dy$$
$$+ C\varepsilon^2\int_{D^+}\frac{1}{|g(y)|^{\frac{p'}{2}}}\frac{\partial}{\partial y_j}\big|[|g(y)|^{\frac{1}{2}}g^{ij}(y)]\big|^{p'}\Big|\tilde{c}(y)\frac{\partial}{\partial y_i}(V_\varepsilon^\xi(y))\Big|^{p'}\,dy.$$

By the usual change of variables $y = \varepsilon z$, we can easily estimate almost all terms in the previous equation. The only term needing more attention is

$$I_1 = C\varepsilon^2\int_{D^+}|g^{ij}(y)-\delta_{ij}|^{p'}\Big|\frac{\partial}{\partial y_j}\Big[\tilde{c}(y)\frac{\partial}{\partial y_i}(V_\varepsilon^\xi(y))\Big]\Big|^{p'}\,dy.$$

We have

$$I_1 \le C\varepsilon^2\int_{\mathbb{R}_+^n}|g^{ij}(\varepsilon z)-\delta_{ij}|^{p'}\Big|\frac{1}{\varepsilon^2}\frac{\partial^2}{\partial z_i^2}(V^\xi(z))\Big|^{p'}\varepsilon^n\,dz + o(\varepsilon^{p'}),$$

and by (2.5) we get

$$\|R_{\varepsilon,\xi}\|_\varepsilon = O\Big(\varepsilon^{\frac{2+p'+n-2p'}{p'}}\Big) = O\Big(\varepsilon^{\frac{2+n}{p'}-1}\Big) = o(\varepsilon)$$

since $p > 2$ and $n \ge 2$, so $\frac{2+n}{p'} > 2$. □

Lemma 3.3. *There exist $\varepsilon_0 > 0$ and $C > 0$ such that, for any $\xi \in \partial M$, $\varepsilon \in (0,\varepsilon_0)$ and $r > 0$, we have that*

$$\|S_{\varepsilon,\xi}(\phi)\|_\varepsilon \le C\varepsilon \tag{3.1}$$

and

$$\|S_{\varepsilon,\xi}(\phi_1) - S_{\varepsilon,\xi}(\phi_2)\|_\varepsilon \le \ell_\varepsilon\|\phi_1-\phi_2\|_\varepsilon \tag{3.2}$$

for $\phi,\phi_1,\phi_2 \in \{v \in H_\varepsilon : \|v\|_\varepsilon \le r\varepsilon\}$, with $\ell_\varepsilon \to 0$ as $\varepsilon \to 0$.

Proof. Let us prove (3.1). From the definition of i^* and (2.11) it follows that

$$\begin{aligned}\|S_{\varepsilon,\xi}(\phi)\|_\varepsilon &\le C|\Phi^2(W_{\varepsilon,\xi}+\phi)(W_{\varepsilon,\xi}+\phi)|_{\varepsilon,p'} + C|\Phi(W_{\varepsilon,\xi}+\phi)(W_{\varepsilon,\xi}+\phi)|_{\varepsilon,p'}\\ &\le C|\Phi(W_{\varepsilon,\xi}+\phi)(W_{\varepsilon,\xi}+\phi)|_{\varepsilon,p'}\end{aligned}$$

since $0 < \Phi(u) < \frac{1}{q}$. Hence, for some $t > 2$ if $n = 2$ or for $t = 2_n^*$ if $n = 3, 4$, we have that

$$\begin{aligned}\|S_{\varepsilon,\xi}(\phi)\|_\varepsilon &\le C\frac{1}{\varepsilon^{n/p'}}\Big(\int_M |\Phi(W_{\varepsilon,\xi}+\phi)|^t\Big)^{\frac{1}{t}}\Big(|W_{\varepsilon,\xi}+\phi|^{\frac{tp'}{t-p'}}\Big)^{\frac{t-p'}{tp'}}\\ &\le C\frac{1}{\varepsilon^{n/p'}}\|\Phi(W_{\varepsilon,\xi}+\phi)\|_{H_g^1}|W_{\varepsilon,\xi}+\phi|_{\frac{tp'}{t-p'},g}\\ &\le C\frac{1}{\varepsilon^{n/p'}}\|\Phi(W_{\varepsilon,\xi}+\phi)\|_{H_g^1}\varepsilon^{n\frac{t-p'}{tp'}}|W_{\varepsilon,\xi}+\phi|_{\varepsilon,\frac{tp'}{t-p'}}\\ &\le C\varepsilon^{-\frac{n}{t}}\|\Phi(W_{\varepsilon,\xi}+\phi)\|_{H_g^1}(1+\|\phi\|_\varepsilon)\end{aligned}$$

by Remark 2.3. Now, for $n = 2$, by (5.2) we have that

$$\|S_{\varepsilon,\xi}(\phi)\|_\varepsilon \le C\varepsilon^{\beta-\frac{2}{t}},$$

and we can chose $t > 2$ sufficiently large and $\beta < 2$ sufficiently close to 2 to prove the claim. On the other hand, for $n = 3, 4$, recalling that $t = 2_n^*$ and using (5.4), we have

$$\|S_{\varepsilon,\xi}(\phi)\|_\varepsilon \le C\varepsilon^{-\frac{n-2}{2}}\varepsilon^{\frac{n+2}{2}} = C\varepsilon^2.$$

In every case, $\|S_{\varepsilon,\xi}(\phi)\|_\varepsilon \le C\varepsilon$, and we have proved (3.1).

Let us prove (3.2). From (2.11), since $0 < \Phi(u) < \frac{1}{q}$, it follows that

$$\begin{aligned}&\|S_{\varepsilon,\xi}(\phi_1) - S_{\varepsilon,\xi}(\phi_2)\|_\varepsilon\\ &\quad\le C|\Phi^2(W_{\varepsilon,\xi}+\phi_1)(W_{\varepsilon,\xi}+\phi_1) - \Phi^2(W_{\varepsilon,\xi}+\phi_2)(W_{\varepsilon,\xi}+\phi_1)|_{\varepsilon,p'}\\ &\qquad + C|\Phi(W_{\varepsilon,\xi}+\phi_1)(W_{\varepsilon,\xi}+\phi_1) - \Phi(W_{\varepsilon,\xi}+\phi_2)(W_{\varepsilon,\xi}+\phi_1)|_{\varepsilon,p'}\\ &\quad\le C|\Phi(W_{\varepsilon,\xi}+\phi_1)(W_{\varepsilon,\xi}+\phi_1) - \Phi(W_{\varepsilon,\xi}+\phi_2)(W_{\varepsilon,\xi}+\phi_1)|_{\varepsilon,p'}\\ &\quad= C\big|[\Phi(W_{\varepsilon,\xi}+\phi_1) - \Phi(W_{\varepsilon,\xi}+\phi_2)]W_{\varepsilon,\xi}\big|_{\varepsilon,p'} + C|\Phi(W_{\varepsilon,\xi}+\phi_2)(\phi_1-\phi_2)|_{\varepsilon,p'} =: I_1 + I_2.\end{aligned}$$

In the light of Remark 2.3, for some $\theta \in (0, 1)$ we have that

$$\begin{aligned}I_1^{p'} &= \frac{C}{\varepsilon^n}\int_M |\Phi'(W_{\varepsilon,\xi}+\theta\phi_1+(1-\theta)\phi_2)(\phi_1-\phi_2)|^{p'}|W_{\varepsilon,\xi}+\phi_1|^{p'}\\ &\le \frac{C}{\varepsilon^n}\Big(\int_M |\Phi'(W_{\varepsilon,\xi}+\theta\phi_1+(1-\theta)\phi_2)(\phi_1-\phi_2)|^t\Big)^{\frac{p'}{t}}\Big(\int_M |W_{\varepsilon,\xi}+\phi_1|^{\frac{p't}{t-p'}}\Big)^{\frac{t-p'}{t}}\\ &= \frac{C}{\varepsilon^n}\|\Phi'(W_{\varepsilon,\xi}+\theta\phi_1+(1-\theta)\phi_2)(\phi_1-\phi_2)\|_{H_g^1}^{p'}\varepsilon^{n\frac{t-p'}{t}}|W_{\varepsilon,\xi}+\phi_1|^{p'}_{\varepsilon,\frac{p't}{t-p'}}\\ &\le C\varepsilon^{-\frac{np'}{t}}\|\Phi'(W_{\varepsilon,\xi}+\theta\phi_1+(1-\theta)\phi_2)(\phi_1-\phi_2)\|_{H_g^1}^{p'}.\end{aligned}$$

Here, as before, $t > 2$ for $n = 2$ and $t = 2_n^*$ for $n = 3, 4$. Notice that, since $p' < 2$, we have

$$\frac{p'2^*}{2^*-p'} < 2^*.$$

By direct computation, one sees that $\|u\|_{H_g^1} \le \varepsilon^{(n-2)/2}\|u\|_\varepsilon$ for $n = 2, 3, 4$. Thus, in case $n = 2$, from Lemma 5.2 we obtain that

$$I_1 \le C\varepsilon^{-\frac{2}{t}}(\varepsilon^\beta + \|\phi_1\|_\varepsilon + \|\phi_2\|_\varepsilon)\|\phi_1-\phi_2\|_\varepsilon \le C\varepsilon^{\beta-\frac{2}{t}}\|\phi_1-\phi_2\|_\varepsilon$$

and, choosing t sufficiently large, we conclude that $I_1 \le \ell_\varepsilon\|\phi_1-\phi_2\|_\varepsilon$ with $\ell_\varepsilon \to 0$. For $n = 3, 4$, again by

Lemma 5.2, we have

$$\begin{aligned}I_1 &\le C\varepsilon^{-\frac{n}{2^*}}(\varepsilon^2+\varepsilon^{\frac{n-2}{2}}(\|\phi_1\|_\varepsilon+\|\phi_2\|_\varepsilon))\varepsilon^{\frac{n-2}{2}}\|\phi_1-\phi_2\|_\varepsilon\\ &\le C(\varepsilon^2+\varepsilon^{\frac{n-2}{2}}(\|\phi_1\|_\varepsilon+\|\phi_2\|_\varepsilon))\|\phi_1-\phi_2\|_\varepsilon,\end{aligned}$$

and since $\|\phi_1\|_\varepsilon+\|\phi_2\|_\varepsilon\le C\varepsilon$, we have again that $I_1\le \ell_\varepsilon\|\phi_1-\phi_2\|_\varepsilon$ with $\ell_\varepsilon\to 0$. To estimate I_2 we proceed in a similar way, obtaining

$$\begin{aligned}I_2^{p'} &= \frac{C}{\varepsilon^n}\int_M |\Phi(W_{\varepsilon,\xi}+\phi_2)|^{p'}|\phi_1-\phi_2|^{p'}\\ &\le \frac{C}{\varepsilon^n}\Big(\int_M |\Phi(W_{\varepsilon,\xi}+\phi_2)|^t\Big)^{\frac{p'}{t}}\Big(\int_M |\phi_1-\phi_2|^{\frac{p't}{t-p'}}\Big)^{\frac{t-p'}{t}}\\ &\le C\varepsilon^{-\frac{np'}{t}}\|\Phi(W_{\varepsilon,\xi}+\phi_2)\|_{H_g^1}^{p'}\|\phi_1-\phi_2\|_\varepsilon^{p'}.\end{aligned}$$

For $n=2$, we have by (5.2) that

$$I_2\le C\varepsilon^{-\frac{2}{t}}\varepsilon^\beta(1+\|\phi_2\|_\varepsilon)\|\phi_1-\phi_2\|_\varepsilon\le C\varepsilon^{\beta-\frac{2}{t}}\|\phi_1-\phi_2\|_\varepsilon$$

and, since t may be chosen arbitrarily large, we have $\varepsilon^{\beta-2/t}\to 0$. For $n=3,4$, again by (5.2) we conclude that

$$I_2\le C\varepsilon^{-\frac{n}{2^*}}\varepsilon^2(1+\|\phi_2\|_\varepsilon)\|\phi_1-\phi_2\|_\varepsilon\le C\varepsilon^2\|\phi_1-\phi_2\|_\varepsilon,$$

so $I_2\le \ell_\varepsilon\|\phi_1-\phi_2\|_\varepsilon$ with $\ell_\varepsilon\to 0$. Collecting the estimates for I_1 and I_2, we get (3.2). □

Sketch of the proof of Proposition 2.4. Since, by Lemma 3.1, $L_{\varepsilon,\xi}$ is invertible, the map

$$T_{\varepsilon,\xi}: K_{\varepsilon,\xi}^\perp\to K_{\varepsilon,\xi}^\perp,\quad T_{\varepsilon,\xi}(\phi):=L_{\varepsilon,\xi}^{-1}(N_{\varepsilon,\xi}(\phi)+R_{\varepsilon,\xi}+S_{\varepsilon,\xi}(\phi))$$

is well defined. As

$$\|T_{\varepsilon,\xi}(\phi)\|_\varepsilon\le C(\|N_{\varepsilon,\xi}(\phi)\|_\varepsilon+\|S_{\varepsilon,\xi}(\phi)\|_\varepsilon+\|R_{\varepsilon,\xi}\|_\varepsilon)$$

and

$$\|T_{\varepsilon,\xi}(\phi_1)-T_{\varepsilon,\xi}(\phi_2)\|_\varepsilon\le C\|N_{\varepsilon,\xi}(\phi_1)-N_{\varepsilon,\xi}(\phi_2)\|_\varepsilon+C\|S_{\varepsilon,\xi}(\phi_1)-S_{\varepsilon,\xi}(\phi_2)\|_\varepsilon,$$

we deduce from Lemmas 3.2 and 3.3 that $T_{\varepsilon,\xi}$ is a contraction in the ball centered at 0 with radius $C\varepsilon$ in $K_{\varepsilon,\xi}^\perp$ for a suitable constant C. Then $T_{\varepsilon,\xi}$ has a unique fixed point. The proof that the map $\xi\to\phi_{\varepsilon,\xi}$ is a $\mathcal{C}^1$-map uses the implicit function theorem. This part of the proof is standard. □

4 The reduced energy

In this section, we obtain the expansion of the functional $\widetilde{I}_\varepsilon(\xi)$ with respect to ε. Recall the notation introduced in Section 2.2.

Lemma 4.1. *The expression*

$$\widetilde{I}_\varepsilon(\xi)=I_\varepsilon(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi})=I_\varepsilon(W_{\varepsilon,\xi})+o(1)=J_\varepsilon(W_{\varepsilon,\xi})+\frac{\omega^2}{2}G_\varepsilon(W_{\varepsilon,\xi})+o(1)$$

holds true $\mathcal{C}^0$-uniformly with respect to ξ as ε goes to zero. Moreover, setting $\xi(\bar z):=\exp_\xi(\bar z)$ for $\bar z\in B_R^{n-1}(0)$, we have that

$$\begin{aligned}\Big(\frac{\partial}{\partial\bar z_h}\widetilde{I}_\varepsilon(\xi(\bar z))\Big)\Big|_{\bar z=0} &= \Big(\frac{\partial}{\partial\bar z_h}I_\varepsilon(W_{\varepsilon,\xi(\bar z)}+\phi_{\varepsilon,\xi(\bar z)})\Big)\Big|_{\bar z=0}\\ &= \Big(\frac{\partial}{\partial\bar z_h}I_\varepsilon(W_{\varepsilon,\xi(\bar z)})\Big)\Big|_{\bar z=0}+o(1)\\ &= \Big(\frac{\partial}{\partial\bar z_h}J_\varepsilon(W_{\varepsilon,\xi(\bar z)})\Big)\Big|_{\bar z=0}+\frac{\omega^2}{2}\Big(\frac{\partial}{\partial\bar z_h}G_\varepsilon(W_{\varepsilon,\xi(\bar z)})\Big)\Big|_{\bar z=0}+o(1)\end{aligned}$$

$\mathcal{C}^0$-uniformly with respect to ξ as ε goes to zero, for every $h=1,\dots,n-1$.

Proof. As in [7, Lemma 5.1], we obtain the estimates

$$J_\varepsilon(W_{\varepsilon,\xi(\overline{z})} + \phi_{\varepsilon,\xi(\overline{z})}) - J_\varepsilon(W_{\varepsilon,\xi(\overline{z})}) = o(1),$$
$$(J'_\varepsilon(W_{\varepsilon,\xi(\overline{z})} + \phi_{\varepsilon,\xi(\overline{z})}) - J'_\varepsilon(W_{\varepsilon,\xi(\overline{z})}))\Big[\Big(\frac{\partial}{\partial \overline{z}_h} W_{\varepsilon,\xi(\overline{z})}\Big)\Big|_{\overline{z}=0}\Big] = o(1).$$

To complete the proof we need the following estimates:

$$G_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - G_\varepsilon(W_{\varepsilon,\xi}) = o(1), \tag{4.1}$$
$$[G'_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - G'_\varepsilon(W_{\varepsilon,\xi})]\Big[\Big(\frac{\partial}{\partial \overline{z}_h} W_{\varepsilon,\xi(\overline{z})}\Big)\Big|_{\overline{z}=0}\Big] = o(1), \tag{4.2}$$
$$\Big(J'_\varepsilon(W_{\varepsilon,\xi(\overline{z})} + \phi_{\varepsilon,\xi(\overline{z})}) + \frac{\omega^2}{2} G'_\varepsilon(W_{\varepsilon,\xi(\overline{z})} + \phi_{\varepsilon,\xi(\overline{z})})\Big)\Big[\frac{\partial}{\partial \overline{z}_h}\phi_{\varepsilon,\xi(\overline{z})}\Big] = o(1). \tag{4.3}$$

The proof of (4.1), (4.2) and (4.3) is technical and it is postponed to the appendix. With these estimates, one can prove the claim following the argument of [7, Lemma 5.1]. □

Lemma 4.2. *The estimate*

$$J_\varepsilon(W_{\varepsilon,\xi}) = \Big(\frac{1}{2} - \frac{1}{p}\Big)\frac{c(\xi)^{\frac{n}{2}} d(\xi)^{\frac{p}{p-2}-\frac{n}{2}}}{b(\xi)^{\frac{2}{p-2}}} \int_{\mathbb{R}^n_+} U^p\, dz + O(\varepsilon)$$

holds true $\mathcal{C}^1$-uniformly with respect to $\xi \in \partial M$.

Proof. For $y \in D^+$, setting $\tilde{c}(y) := c(x)$, $\tilde{d}(y) := d(x)$ and $\tilde{b}(y) := b(x)$ with $x := \psi^\partial_\xi(y) \in Q_\xi$, we have

$$\begin{aligned} J_\varepsilon(W_{\varepsilon,\xi}) = {}& \frac{\varepsilon^2}{2\varepsilon^n} \int_{D^+} \tilde{c}(y) \sum_{i,j=1}^n g^{ij}(y) \frac{\partial(V^\xi_\varepsilon(y)\chi(y))}{\partial y_i} \frac{\partial(V^\xi_\varepsilon(y)\chi(y))}{\partial y_j} |g(y)|^{\frac{1}{2}}\, dy \\ & + \frac{1}{2\varepsilon^n} \int_{D^+} \tilde{d}(y)(V^\xi_\varepsilon(y)\chi(y))^2 |g(y)|^{\frac{1}{2}}\, dy - \frac{1}{p\varepsilon^n} \int_{D^+} \tilde{b}(y)(V^\xi_\varepsilon(y)\chi(y))^p |g(y)|^{\frac{1}{2}}\, dy. \end{aligned}$$

Using the change of variables $y = \varepsilon\zeta$, from the expansions (2.5), (2.6) and (2.7) we immediately obtain

$$J_\varepsilon(W_{\varepsilon,\xi}) = \frac{1}{2}\int_{\mathbb{R}^n_+} c(\xi)|\nabla V^\xi(\zeta)|^2 + d(\xi)(V^\xi(\zeta))^2\, d\zeta - \frac{1}{p}\int_{\mathbb{R}^n_+} b(\xi)(V^\xi(\zeta))^p\, d\zeta + O(\varepsilon).$$

From the definitions of V^ξ and U we get

$$\begin{aligned} J_\varepsilon(W_{\varepsilon,\xi}) &= \frac{c(\xi)^{\frac{n}{2}} d(\xi)^{\frac{p}{p-2}-\frac{n}{2}}}{b(\xi)^{\frac{2}{p-2}}}\Big[\frac{1}{2}\int_{\mathbb{R}^n_+} (|\nabla U|^2 + U^2) - \frac{1}{p}\int_{\mathbb{R}^n_+} U^p\, d\zeta\Big] + O(\varepsilon) \\ &= \Big(\frac{1}{2} - \frac{1}{p}\Big)\frac{c(\xi)^{\frac{n}{2}} d(\xi)^{\frac{p}{p-2}-\frac{n}{2}}}{b(\xi)^{\frac{2}{p-2}}} \int_{\mathbb{R}^n_+} U^p\, d\zeta + O(\varepsilon) \end{aligned}$$

$\mathcal{C}^0$-uniformly with respect to $\xi \in \partial M$. For the sake of readability, the $\mathcal{C}^1$-convergence is postponed to the appendix, where a proof is given in full detail. □

Lemma 4.3. *The expression*

$$I_\varepsilon(W_{\varepsilon,\xi}) = \Big(\frac{1}{2} - \frac{1}{p}\Big)\frac{c(\xi)^{\frac{n}{2}} d(\xi)^{\frac{p}{p-2}-\frac{n}{2}}}{b(\xi)^{\frac{2}{p-2}}} \int_{\mathbb{R}^n_+} U^p\, dz + o(1)$$

holds true $\mathcal{C}^1$-uniformly with respect to $\xi \in \partial M$.

Proof. In Lemma 4.2 we proved that

$$J_\varepsilon(W_{\varepsilon,\xi}) = \Big(\frac{1}{2} - \frac{1}{p}\Big)\frac{c(\xi)^{\frac{n}{2}} d(\xi)^{\frac{p}{p-2}-\frac{n}{2}}}{b(\xi)^{\frac{2}{p-2}}} \int_{\mathbb{R}^n} U^p\, dz + O(\varepsilon).$$

It is enough to show now that $G_\varepsilon(W_{\varepsilon,\xi}) = o(1)$ holds true $\mathcal{C}^1$-uniformly with respect to $\xi \in \partial M$. For the $\mathcal{C}^0$-convergence, by Remark 2.3 and since $\|\Phi(W_{\varepsilon,\xi})\|_\varepsilon \le C\varepsilon$, we have that

$$\begin{aligned}|G_\varepsilon(W_{\varepsilon,\xi})| &\le \frac{C}{\varepsilon^n}\Big|\int\limits_M \Phi(W_{\varepsilon,\xi})W_{\varepsilon,\xi}^2\,d\mu_g\Big|\\ &\le \frac{C}{\varepsilon^n}|\Phi(W_{\varepsilon,\xi})|_{2,g}|W_{\varepsilon,\xi}|_{4,g}^2\\ &\le C|\Phi(W_{\varepsilon,\xi})|_{\varepsilon,2}|W_{\varepsilon,\xi}|_{\varepsilon,4}^2\\ &\le C\|\Phi(W_{\varepsilon,\xi})\|_\varepsilon \le C\varepsilon.\end{aligned}$$

For the $\mathcal{C}^1$-convergence, we estimate

$$\begin{aligned}\Big|\frac{\partial}{\partial \bar z_h}G_\varepsilon(W_{\varepsilon,\xi(\bar z)})|_{\bar z=0}\Big| &\le \Big|\frac{C}{\varepsilon^n}\frac{\partial}{\partial \bar z_h}\int\limits_M \Phi(W_{\varepsilon,\xi(\bar z)})W_{\varepsilon,\xi(\bar z)}^2|_{\bar z=0}\,d\mu_g\Big|\\ &\le \Big|\frac{C}{\varepsilon^n}\int\limits_M \Phi(W_{\varepsilon,\xi(\bar z)})2W_{\varepsilon,\xi(h)}\Big(\frac{\partial}{\partial \bar z_h}W_{\varepsilon,\xi(\bar z)}\Big)\Big|_{\bar z=0}\,d\mu_g\Big|\\ &\qquad + \Big|\frac{C}{\varepsilon^n}\int\limits_M W_{\varepsilon,\xi(h)}^2\Phi'(W_{\varepsilon,\xi(\bar z)})\Big[\frac{\partial}{\partial \bar z_h}W_{\varepsilon,\xi(\bar z)}|_{\bar z=0}\Big]\,d\mu_g\Big|\\ &=: I_1 + I_2.\end{aligned}$$

Now, by Remark 2.3 and since

$$\Big\|\frac{\partial}{\partial \bar z_h}W_{\varepsilon,\xi(\bar z)}|_{\bar z=0}\Big\|_\varepsilon = O\Big(\frac{1}{\varepsilon}\Big),$$

we have

$$\begin{aligned}I_1 &\le \frac{C}{\varepsilon^n}|\Phi(W_{\varepsilon,\xi(\bar z)})|_{3,g}|W_{\varepsilon,\xi(\bar z)}|_{3,g}\Big|\frac{\partial}{\partial \bar z_h}W_{\varepsilon,\xi(\bar z)}|_{\bar z=0}\Big|_{3,g}\\ &\le C\frac{\varepsilon^{\frac{2}{3}n}}{\varepsilon^n}\|\Phi(W_{\varepsilon,\xi(\bar z)})\|_{H_g^1}|W_{\varepsilon,\xi(\bar z)}|_{\varepsilon,3}\Big\|\frac{\partial}{\partial \bar z_h}W_{\varepsilon,\xi(\bar z)}|_{\bar z=0}\Big\|_\varepsilon\\ &\le C\varepsilon^{-\frac{n}{3}-1}\|\Phi(W_{\varepsilon,\xi(\bar z)})\|_{H_g^1}.\end{aligned}$$

From Lemma 5.1, choosing $\frac{5}{3} < \beta < 2$ if $n = 2$, we get $I_1 \le C\varepsilon^{\beta-2/3-1} = o(1)$, and for $n = 3, 4$ we get

$$I_1 \le C\varepsilon^{\frac{n+2}{2}-\frac{n}{3}-1} = \varepsilon^{\frac{n}{6}} = o(1).$$

Using Remark 2.3 and choosing $\frac{2n}{n+2} < t < 2$, we obtain

$$\begin{aligned}I_2 &\le \frac{C}{\varepsilon^n}|W_{\varepsilon,\xi(\bar z)}|_{2t,g}^2\Big|\Phi'(W_{\varepsilon,\xi(\bar z)})\Big[\frac{\partial}{\partial \bar z_h}W_{\varepsilon,\xi(\bar z)}|_{\bar z=0}\Big]\Big|_{t',g}\\ &\le C\frac{\varepsilon^{\frac{n}{t}}}{\varepsilon^n}|W_{\varepsilon,\xi(\bar z)}|_{\varepsilon,2t}^2\Big\|\Phi'(W_{\varepsilon,\xi(\bar z)})\Big[\frac{\partial}{\partial \bar z_h}W_{\varepsilon,\xi(\bar z)}|_{\bar z=0}\Big]\Big\|_{H_g^1}\\ &\le C\varepsilon^{\frac{n}{t}-n}\Big\|\Phi'(W_{\varepsilon,\xi(\bar z)})\Big[\frac{\partial}{\partial \bar z_h}W_{\varepsilon,\xi(\bar z)}|_{\bar z=0}\Big]\Big\|_{H_g^1}.\end{aligned}$$

Finally, using Lemma 5.2 and noting that $\|u\|_{H_g^1} \le C\varepsilon^{(n-2)/2}\|u\|_\varepsilon$, for $n = 2$ and $3 - \frac{2}{t} < \beta < 2$ we have

$$I_2 \le C\varepsilon^{\frac{2}{t}-2}\varepsilon^\beta\Big\|\frac{\partial}{\partial \bar z_h}W_{\varepsilon,\xi(\bar z)}|_{\bar z=0}\Big\|_\varepsilon \le C\varepsilon^{\beta+\frac{2}{t}-3} = o(1),$$

while for $n = 3, 4$ we get

$$I_2 \le C\varepsilon^{\frac{n}{t}-n}\varepsilon^2\varepsilon^{\frac{n-2}{2}}\Big\|\frac{\partial}{\partial \bar z_h}W_{\varepsilon,\xi(\bar z)}|_{\bar z=0}\Big\|_\varepsilon \le C\varepsilon^{\frac{n}{t}-\frac{n}{2}} = o(1)$$

since $t < 2$. □

5 Appendix

We collect a series of technical results that were used previously.

5.1 Key estimates for the function Φ

Lemma 5.1. *For $\varepsilon > 0$, $\xi \in \partial M$ and $\varphi \in H_g^1(M)$, we have the following estimates:*
For $n = 2$ and $1 < \beta < 2$, we have

$$\|\Phi(W_{\varepsilon,\xi} + \varphi)\|_{H_g^1} \le C_1(\varepsilon^\beta + \|\varphi\|_{H_g^1}^2), \tag{5.1}$$

$$\|\Phi(W_{\varepsilon,\xi} + \varphi)\|_{H_g^1} \le C_1\varepsilon^\beta(1 + \|\varphi\|_\varepsilon^2), \tag{5.2}$$

and for $n = 3, 4$ we have

$$\|\Phi(W_{\varepsilon,\xi} + \varphi)\|_{H_g^1} \le C_1(\varepsilon^{\frac{n+2}{2}} + \|\varphi\|_{H_g^1}^2), \tag{5.3}$$

$$\|\Phi(W_{\varepsilon,\xi} + \varphi)\|_{H_g^1} \le C_1\varepsilon^{\frac{n+2}{2}}(1 + \|\varphi\|_\varepsilon^2), \tag{5.4}$$

where the constant C_1 does not depend on ε, ξ and φ.

Proof. To simplify the notation we set $v := \Phi(W_{\varepsilon,\xi} + \varphi)$. By (2.1) or (2.2) we have

$$\begin{aligned}
\|v\|_{H_g^1}^2 &\le C\int_M c(x)|\nabla_g v|^2 + b(x)q^2(W_{\varepsilon,\xi} + \varphi)^2v^2 \\
&= Cq\int_M b(x)(W_{\varepsilon,\xi} + \varphi)^2 v \\
&\le C\Big(\int_M v^t\Big)^{\frac{1}{t}}\Big(\int_M (W_{\varepsilon,\xi} + \varphi)^{2t'}\Big)^{\frac{1}{t'}} \\
&\le C\|v\|_{H_g^1}\|W_{\varepsilon,\xi} + \varphi\|_{L_g^{2t'}}^2 \\
&\le C\|v\|_{H_g^1}(\|W_{\varepsilon,\xi}\|_{L_g^{2t'}}^2 + \|\varphi\|_{L_g^{2t'}}^2),
\end{aligned}$$

where $t = 2_n^*$ for $n = 3, 4$ and $t \ge 2$ for $n = 2$. We recall (see Remark 2.3) that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^n}|W_{\varepsilon,\xi}|_q^q \le C|U|_q^q \quad \text{uniformly with respect to } \xi \in \partial M.$$

Thus, we have

$$\|v\|_{H_g^1} \le C_1(\varepsilon^{\frac{n}{t'}} + |\varphi|_{2t',g}^2) \le C_1(\varepsilon^{\frac{n}{t'}} + \|\varphi\|_{H_g^1}^2). \tag{5.5}$$

Notice that for $n = 2$, since $t \ge 2$, we have that $1 \le \frac{2}{t'} < 2$, while for $n = 3, 4$ we have $t' = \frac{2n}{n+2}$, which proves (5.1) and (5.3). In the light of (5.5), we also obtain that

$$\|v\|_{H_g^1} \le C_1(\varepsilon^{\frac{n}{t'}} + |\varphi|_{2t',g}^2) \le C_1\varepsilon^{\frac{n}{t'}}(1 + |\varphi|_{2t',\varepsilon}^2) \le C_1\varepsilon^{\frac{n}{t'}}(1 + \|\varphi\|_\varepsilon^2),$$

which proves the other two inequalities (5.2) and (5.4). □

Lemma 5.2. *For $\varepsilon > 0$, $\xi \in \partial M$ and $h, k \in H_g^1(M)$, we have the following estimates:*
For $n = 2$ and $\beta \in (0, 2)$, we have

$$\|\Phi'(W_{\varepsilon,\xi} + k)[h]\|_{H_g^1} \le C\|h\|_{H_g^1}(\varepsilon^\beta + \|k\|_{H_g^1}),$$

and for $n = 3, 4$ we have

$$\|\Phi'(W_{\varepsilon,\xi} + k)[h]\|_{H_g^1} \le C\|h\|_{H_g^1}(\varepsilon^2 + \|k\|_{H_g^1}),$$

where the constant C does not depend on ε, ξ, h and k.

Proof. From Lemma 2.1 we obtain

$$\begin{aligned}\|\Phi'(W_{\varepsilon,\xi}+k)[h]\|^2_{H^1_g} &= 2q\int_M b(x)(W_{\varepsilon,\xi}+k)(1-q\Phi(W_{\varepsilon,\xi}+k))h\Phi'(W_{\varepsilon,\xi}+k)[h]\\ &\quad - q^2\int_M b(x)(W_{\varepsilon,\xi}+k)^2(\Phi'(W_{\varepsilon,\xi}+k)[h])^2\\ &\le C\int_M W_{\varepsilon,\xi}|h||\Phi'(W_{\varepsilon,\xi}+k)[h]| + \int_M |k||h||\Phi'(W_{\varepsilon,\xi}+k)[h]|.\end{aligned}$$

We call the last two integrals I_1 and I_2, respectively, and we estimate each of them separately. We have, by Remark 2.3, that

$$I_2 \le \|k\|_{L^3_g}\|h\|_{L^3_g}\|\Phi'(W_{\varepsilon,\xi}+k)[h]\|_{L^3_g} \le \|k\|_{H^1_g}\|h\|_{H^1_g}\|\Phi'\|_{H^1_g}$$

and

$$I_1 \le \|\Phi'(W_{\varepsilon,\xi}+k)[h]\|_{L^t_g}\|h\|_{L^t_g}\|W_{\varepsilon,\xi}\|_{L_g^{\frac{t}{t-2}}} \le \varepsilon^{n\frac{t-2}{t}}\|\Phi'\|_{H^1_g}\|h\|_{H^1_g},$$

where $t = 2^*_n$ for $n = 3, 4$ and $t > 2$ for $n = 2$. □

5.2 Change of coordinates along ∂M

For $\xi \in \partial M$, we consider the chart $\psi^\partial_\xi : D^+ \to Q_\xi$, introduced in Section 2.2, whose inverse $(\psi^\partial_\xi)^{-1}(x) = y$ expresses a point $x \in Q_\xi \subset M$ in Fermi coordinates $y = (y_1, \dots, y_n)$ around ξ.

For $\bar z \in B^{n-1}_R(0)$ and $x \in Q_\xi \cap Q_{\exp_\xi(\bar z)}$, we consider the change of coordinates map

$$\mathcal{E}(\bar z, x) = (\psi^\partial_{\exp_\xi(\bar z)})^{-1}(x) = (\psi^\partial_{\exp_\xi(\bar z)})^{-1}\psi^\partial_\xi(y) = \tilde{\mathcal{E}}(\bar z, y). \tag{5.6}$$

Since $y_n = \operatorname{dist}_g(x, \partial M)$, writing $y = (\bar y, y_n)$ with $\bar y \in \mathbb{R}^{n-1}$ and $y_n \in [0, \infty)$, we have that

$$\tilde{\mathcal{E}}(\bar z, \bar y, y_n) = (\exp^{-1}_{\exp_\xi(\bar z)}\exp_\xi(\bar y), y_n). \tag{5.7}$$

Lemma 5.3. *The derivatives of $\mathcal{E}$ at $(0, \xi)$ are given by*

$$\begin{aligned}\frac{\partial\mathcal{E}_k}{\partial y_h}(0,\xi) &= \frac{\partial\tilde{\mathcal{E}}_k}{\partial y_h}(0,0) = -\delta_{hk} \quad \textit{for } h = 1,\dots,n-1,\ k = 1,\dots,n,\\ \frac{\partial^2\mathcal{E}_k}{\partial\eta_j\partial y_h}(0,\xi) &= \frac{\partial^2\tilde{\mathcal{E}}_k}{\partial\eta_j\partial y_h}(0,0) = 0 \quad \textit{for } h = 1,\dots,n-1,\ j,k = 1,\dots,n.\end{aligned}$$

Proof. This follows from [26, Lemma 6.4] by using the expression (5.7). □

For $\bar z \in B^{n-1}_R(0)$, we set $\xi(\bar z) := \exp_\xi(\bar z) \in \partial M$. The function $W_{\varepsilon,\xi(\bar z)}$, defined in (2.9), can now be written as

$$\begin{aligned}W_{\varepsilon,\xi(\bar z)}(x) &= \gamma(\xi(\bar z))U_\varepsilon\Big(\sqrt{A(\xi(\bar z))}(\psi^\partial_{\xi(\bar z)})^{-1}(x)\Big)\chi((\psi^\partial_{\xi(\bar z)})^{-1}(x))\\ &= \tilde\gamma(\bar z)U_\varepsilon\Big(\sqrt{\tilde A(\bar z)}\mathcal{E}(\bar z, x)\Big)\chi(\mathcal{E}(\bar z, x)),\end{aligned}$$

where $\tilde A(\bar z) := A(\exp_\xi(\bar z))$ and $\tilde\gamma(\bar z) := \gamma(\exp_\xi(\bar z))$. Thus, we have

$$\begin{aligned}\frac{\partial}{\partial\bar z_s}W_{\varepsilon,\xi(\bar z)}|_{\bar z=0} &= \Big(\frac{\partial}{\partial\bar z_s}\tilde\gamma(\bar z)|_{\bar z=0}\Big)U\Big(\frac{1}{\varepsilon}\sqrt{\tilde A(0)}\mathcal{E}(0,x)\Big)\chi(\mathcal{E}(0,x))\\ &\quad + \tilde\gamma(0)U\Big(\frac{1}{\varepsilon}\sqrt{\tilde A(0)}\mathcal{E}(0,x)\Big)\frac{\partial}{\partial\bar z_s}\chi(\mathcal{E}(\bar z,x))|_{\bar z=0}\\ &\quad + \tilde\gamma(0)\chi(\mathcal{E}(0,x))\frac{\partial}{\partial\bar z_s}U\Big(\frac{1}{\varepsilon}\sqrt{\tilde A(\bar z)}\mathcal{E}(\bar z,x)\Big)\Big|_{\bar z=0}.\end{aligned}$$

If $x := \psi^{\partial}_{\xi}(\varepsilon y)$, $\xi := \xi(0)$, then $\mathcal{E}(0,x) = \varepsilon y$, and we have

$$\begin{aligned}\frac{\partial}{\partial \bar z_s} W_{\varepsilon,\xi(\bar z)}|_{\bar z=0} &= \Big(\frac{\partial}{\partial \bar z_s}\tilde\gamma(\bar z)|_{\bar z=0}\Big)U\Big(\sqrt{\tilde A(0)}y\Big)\chi(\varepsilon y) + \tilde\gamma(0)U\Big(\sqrt{\tilde A(0)}y\Big)\frac{\partial\chi}{\partial\eta_k}(\varepsilon y)\frac{\partial}{\partial \bar z_s}\mathcal{E}_k(\bar z,\psi^{\partial}_{\xi_0}(\varepsilon y))|_{\bar z=0}\\ &\quad + \tilde\gamma(0)\chi(\varepsilon y)\frac{\sqrt{\tilde A(0)}}{\varepsilon}\frac{\partial U}{\partial \eta_k}\Big(\sqrt{\tilde A(0)}y\Big)\frac{\partial}{\partial \bar z_s}\mathcal{E}_k(\bar z,\psi^{\partial}_{\xi_0}(\varepsilon y))|_{\bar z=0}\\ &\quad + \tilde\gamma(0)\chi(\varepsilon y)\frac{\partial U}{\partial \eta_k}\Big(\sqrt{\tilde A(0)}y\Big)\frac{\partial}{\partial \bar z_s}\sqrt{\tilde A(\bar z)}\,\Big|_{\bar z=0} y_k, \end{aligned} \tag{5.8}$$

where $\frac{\partial f}{\partial \eta_k}(\cdot)$ denotes the derivative of the function f with respect to its k-th variable.

5.3 The pending proofs in Section 4

Conclusion of the proof of Lemma 4.1. To finish the proof of this lemma we need to prove (4.1), (4.2) and (4.3).

Proof of (4.1). For some $\theta \in [0,1]$, we have

$$\begin{aligned} G_\varepsilon(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi}) - G_\varepsilon(W_{\varepsilon,\xi}) &= \frac{1}{\varepsilon^n}\int_M b(x)[\Phi(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi})(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi})^2 - \Phi(W_{\varepsilon,\xi})(W_{\varepsilon,\xi})^2]\\ &= \frac{1}{\varepsilon^n}\int_M b(x)\Phi'(W_{\varepsilon,\xi}+\theta\phi_{\varepsilon,\xi})[\phi_{\varepsilon,\xi}](W_{\varepsilon,\xi})^2\\ &\quad + \frac{1}{\varepsilon^n}\int_M b(x)\Phi(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi})[2\phi_{\varepsilon,\xi}W_{\varepsilon,\xi}+\phi^2_{\varepsilon,\xi}]. \end{aligned}$$

Since $\|\phi_{\varepsilon,\xi}\|_\varepsilon \le C\varepsilon$ and $0<\Phi(u)<\frac{1}{q}$, from Remark 2.3 we obtain

$$\begin{aligned} |G_\varepsilon(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi}) - G_\varepsilon(W_{\varepsilon,\xi})| &\le \frac{C}{\varepsilon^n}|\Phi'(W_{\varepsilon,\xi}+\theta\phi_{\varepsilon,\xi})[\phi_{\varepsilon,\xi}]|_{2,g}|\Phi_{\varepsilon,\xi}|^2_{4,g} + \frac{C}{\varepsilon^n}|\phi_{\varepsilon,\xi}|_{2,g}(|W_{\varepsilon,\xi}|_{2,g}+|\phi_{\varepsilon,\xi}|_{2,g})\\ &\le C\frac{\varepsilon^{n/2}}{\varepsilon^n}\|\Phi'(W_{\varepsilon,\xi}+\theta\phi_{\varepsilon,\xi})[\phi_{\varepsilon,\xi}]\|_{H^1_g}\|\Phi_{\varepsilon,\xi}\|^2_\varepsilon + C\|\Phi_{\varepsilon,\xi}\|_\varepsilon(\|W_{\varepsilon,\xi}\|_\varepsilon + \|\Phi_{\varepsilon,\xi}\|_\varepsilon)\\ &\le \frac{\varepsilon^2}{\varepsilon^{n/2}}\|\Phi'(W_{\varepsilon,\xi}+\theta\phi_{\varepsilon,\xi})[\phi_{\varepsilon,\xi}]\|_{H^1_g} + C\varepsilon(1+\varepsilon). \end{aligned}$$

Using Lemma 5.2, we conclude that

$$|G_\varepsilon(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi}) - G_\varepsilon(W_{\varepsilon,\xi})| \le C(\varepsilon^{3-\frac{n}{2}}+\varepsilon) \le C\varepsilon.$$

Proof of (4.2). Recall that $\xi(\bar z) := \exp_\xi(\bar z)$ for $\bar z \in B^{n-1}_R(0)$. Since $0<\Phi(u)<\frac{1}{q}$, for some $\theta\in[0,1]$ we have

$$\begin{aligned} &\Big|[G'_\varepsilon(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi}) - G'_\varepsilon(W_{\varepsilon,\xi})]\Big[\Big(\frac{\partial}{\partial \bar z_h}W_{\varepsilon,\xi(\bar z)}\Big)\Big|_{\bar z=0}\Big]\Big|\\ &\le \Big|\frac{C}{\varepsilon^n}\int_M [\Phi(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi}) - \Phi(W_{\varepsilon,\xi})]W_{\varepsilon,\xi}\Big(\frac{\partial}{\partial \bar z_h}W_{\varepsilon,\xi(\bar z)}\Big)\Big|_{\bar z=0}\Big|\\ &\quad + \Big|\frac{C}{\varepsilon^n}\int_M [q\Phi^2(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi}) - q\Phi^2(W_{\varepsilon,\xi})]W_{\varepsilon,\xi}\Big(\frac{\partial}{\partial \bar z_h}W_{\varepsilon,\xi(\bar z)}\Big)\Big|_{\bar z=0}\Big|\\ &\quad + \Big|\frac{C}{\varepsilon^n}\int_M [\Phi(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi}) - q\Phi^2(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi})]\phi_{\varepsilon,\xi}\Big(\frac{\partial}{\partial \bar z_h}W_{\varepsilon,\xi(\bar z)}\Big)\Big|_{\bar z=0}\Big|\\ &\le \Big|\frac{C}{\varepsilon^n}\int_M [\Phi(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi}) - \Phi(W_{\varepsilon,\xi})]W_{\varepsilon,\xi}\Big(\frac{\partial}{\partial \bar z_h}W_{\varepsilon,\xi(\bar z)}\Big)\Big|_{\bar z=0}\Big|\\ &\quad + \Big|\frac{C}{\varepsilon^n}\int_M \Phi(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi})\phi_{\varepsilon,\xi}\Big(\frac{\partial}{\partial \bar z_h}W_{\varepsilon,\xi(\bar z)}\Big)\Big|_{\bar z=0}\Big| \end{aligned}$$

$$
\begin{aligned}
&\le \Big|\frac{C}{\varepsilon^n}\int_M \Phi'(W_{\varepsilon,\xi}+\theta\phi_{\varepsilon,\xi})(\phi_{\varepsilon,\xi})W_{\varepsilon,\xi}\Big(\frac{\partial}{\partial \bar z_h}W_{\varepsilon,\xi(\bar z)}\Big)\Big|_{\bar z=0}\Big| \\
&\qquad + \Big|\frac{C}{\varepsilon^n}\int_M \Phi'(W_{\varepsilon,\xi}+\theta\phi_{\varepsilon,\xi})(\phi_{\varepsilon,\xi})\phi_{\varepsilon,\xi}\Big(\frac{\partial}{\partial \bar z_h}W_{\varepsilon,\xi(\bar z)}\Big)\Big|_{\bar z=0}\Big| \\
&\qquad + \Big|\frac{C}{\varepsilon^n}\int_M \Phi(W_{\varepsilon,\xi})\phi_{\varepsilon,\xi}\Big(\frac{\partial}{\partial \bar z_h}W_{\varepsilon,\xi(\bar z)}\Big)\Big|_{\bar z=0}\Big| \\
&=: I_1 + I_2 + I_3.
\end{aligned}
$$

From (5.8) and a straightforward computation we derive that

$$
\Big|\Big(\frac{\partial}{\partial \bar z_h}W_{\varepsilon,\xi(\bar z)}\Big)\Big|_{\bar z=0}\Big|_{\varepsilon,3} \le \Big(\int_{\mathbb{R}^n}\Big[\sum_{k=1}^n\Big|\frac{1}{\varepsilon}\frac{\partial U}{\partial y_k}(y)\Big|\Big]^3\,dy\Big)^{\frac13} = O\Big(\frac{1}{\varepsilon}\Big).
$$

Now, recalling that $\|\phi_{\varepsilon,\xi(\bar z)}\|_\varepsilon \le C\varepsilon$ and that $\|u\|_{H^1_g} \le C\varepsilon^{(n-2)/2}\|u\|_\varepsilon$, from Remark 2.3 and Lemma 5.2 we get that

$$
\begin{aligned}
I_1 &\le C\frac{\varepsilon^{\frac{2n}{3}}}{\varepsilon^n}\Big(\int_M|\Phi'(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi})(\phi_{\varepsilon,\xi})|^3\Big)^{\frac13}\Big(\frac{1}{\varepsilon^n}\int_M W^3_{\varepsilon,\xi}\Big)^{\frac13}\Big(\frac{1}{\varepsilon^n}\int_M\Big|\Big(\frac{\partial}{\partial \bar z_h}W_{\varepsilon,\xi(\bar z)}\Big)\Big|_{\bar z=0}\Big|^3\Big)^{\frac13} \\
&\le C\varepsilon^{-\frac{n}{3}-1}\|\Phi'(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi})(\phi_{\varepsilon,\xi})\|_{H^1_g} \\
&\le C\varepsilon^{-\frac{n}{3}-1}\|\phi_{\varepsilon,\xi}\|^2_{H^1_g} \\
&\le C\varepsilon^{-\frac{n}{3}-1}\varepsilon^n = o(1).
\end{aligned}
$$

The term I_2 can be estimated in the same way, while for I_3 we have

$$
\begin{aligned}
I_3 &\le C\frac{\varepsilon^{\frac{2n}{3}}}{\varepsilon^n}\Big(\int_M|\Phi(W_{\varepsilon,\xi})|^3\Big)^{\frac13}\Big(\frac{1}{\varepsilon^n}\int_M|\phi|^3_{\varepsilon,\xi}\Big)^{\frac13}\Big(\frac{1}{\varepsilon^n}\int_M\Big|\Big(\frac{\partial}{\partial \bar z_h}W_{\varepsilon,\xi(\bar z)}\Big)\Big|_{\bar z=0}\Big|^3\Big)^{\frac13} \\
&\le C\varepsilon^{-\frac{n}{3}}\|\Phi(W_{\varepsilon,\xi})\|_{H^1}.
\end{aligned}
$$

Now, if $n = 2$, by (5.1) we have $I_3 \le C\varepsilon^{\beta-n/3} = o(1)$, choosing β wisely. If $n = 3, 4$, by (5.2) we get

$$
I_3 \le C\varepsilon^{\frac{n+2}{2}-\frac{n}{3}} = C\varepsilon^{\frac{n}{6}+1}.
$$

This proves (4.2).

Proof of (4.3). Following the proof of [7, Lemma 5.1, step 2], we just have to prove that

$$
|G'_\varepsilon(W_{\varepsilon,\xi(\bar z)}+\phi_{\varepsilon,\xi(\bar z)})[Z^l_{\varepsilon,\xi(\bar z)}]| = o(1),
$$

Since $0 < \Phi(u) < \frac{1}{q}$,

$$
\begin{aligned}
|G'_\varepsilon(W_{\varepsilon,\xi(\bar z)}+\phi_{\varepsilon,\xi(\bar z)})[Z^l_{\varepsilon,\xi(\bar z)}]| &\le \Big|\frac{C}{\varepsilon^n}\int_M \Phi(W_{\varepsilon,\xi(\bar z)}+\phi_{\varepsilon,\xi(\bar z)})(W_{\varepsilon,\xi(\bar z)}+\phi_{\varepsilon,\xi(\bar z)})Z^l_{\varepsilon,\xi(\bar z)}\Big| \\
&\qquad + \Big|\frac{1}{\varepsilon^n}\int_M \Phi^2(W_{\varepsilon,\xi(\bar z)}+\phi_{\varepsilon,\xi(\bar z)})(W_{\varepsilon,\xi(\bar z)}+\phi_{\varepsilon,\xi(\bar z)})Z^l_{\varepsilon,\xi(\bar z)}\Big| \\
&\le \Big|\frac{C}{\varepsilon^n}\int_M \Phi(W_{\varepsilon,\xi(\bar z)}+\phi_{\varepsilon,\xi(\bar z)})(W_{\varepsilon,\xi(\bar z)}+\phi_{\varepsilon,\xi(\bar z)})Z^l_{\varepsilon,\xi(\bar z)}\Big| =: I_4.
\end{aligned}
$$

By (2.10) it can be proved easily that $\|Z^l_{\varepsilon,\xi(\bar z)}\|_\varepsilon = O(1)$. So we have

$$
\begin{aligned}
I_4 &\le \varepsilon^{-\frac{n}{3}}\Big(\int_M|\Phi(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi})|^3\Big)^{\frac13}\Big(\frac{1}{\varepsilon^n}\int_M|W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi}|^3\Big)^{\frac13}\Big(\frac{1}{\varepsilon^n}\int_M|Z^l_{\varepsilon,\xi(\bar z)}|^3\Big)^{\frac13} \\
&\le \varepsilon^{-\frac{n}{3}}\|\Phi(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi})\|_{H^1_g}(\|W_{\varepsilon,\xi}\|_{3,\varepsilon}+\|\phi_{\varepsilon,\xi}\|_\varepsilon)\|Z^l_{\varepsilon,\xi(\bar z)}\|_\varepsilon \\
&\le \varepsilon^{-\frac{n}{3}}\|\Phi(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi})\|_{H^1_g}.
\end{aligned}
$$

Now, if $n = 2$, by (5.2) we have that

$$I_3 \le c\varepsilon^{\beta - n/3} = o(1),$$

and if $n = 3, 4$ by (5.2) we have that

$$I_3 \le c\varepsilon^{\frac{n+2}{2} - \frac{n}{3}} = c\varepsilon^{\frac{n}{6}+1}. \qquad \square$$

Conclusion of the proof of Lemma 4.2. To finish the proof of this lemma we need to prove the $\mathcal{C}^1$-convergence. We do this for the first partial derivative. We set $\xi(\overline{z}) := \exp_\xi(\overline{z})$ for $\overline{z} \in B_R^{n-1}(0)$. Then we have

$$\begin{aligned}
\frac{\partial}{\partial \overline{z}_1} J_\varepsilon(W_{\varepsilon,\xi(\overline{z})})|_{\overline{z}=0} &= J'_\varepsilon(W_{\varepsilon,\xi(\overline{z})})\Big[\frac{\partial}{\partial \overline{z}_1} W_{\varepsilon,\xi(\overline{z})}\Big]\Big|_{\overline{z}=0} \\
&= \frac{\varepsilon^2}{\varepsilon^n} \int_M c(x)\nabla_g W_{\varepsilon,\xi(\overline{z})} \nabla_g \frac{\partial}{\partial \overline{z}_1} W_{\varepsilon,\xi(\overline{z})}\, d\mu_g|_{\overline{z}=0} \\
&\quad + \frac{1}{\varepsilon^n} \int_M d(x) W_{\varepsilon,\xi(\overline{z})} \frac{\partial}{\partial \overline{z}_1} W_{\varepsilon,\xi(\overline{z})}\, d\mu_g|_{\overline{z}=0} \\
&\quad + \frac{1}{\varepsilon^n} \int_M b(x) W^{p-1}_{\varepsilon,\xi(\overline{z})} \frac{\partial}{\partial \overline{z}_1} W_{\varepsilon,\xi(\overline{z})}\, d\mu_g|_{\overline{z}=0} \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Next, we estimate each term. Set $x := \psi_\xi^\partial(y)$ and $\tilde{c}(y) := c(\psi_\xi^\partial(y)) = c(x)$. By (5.8), we have

$$\begin{aligned}
I_1 &= \frac{\varepsilon^2}{\varepsilon^n} \int_M c(x)\nabla_g W_{\varepsilon,\xi(\overline{z})} \nabla_g\Big(\frac{\partial}{\partial \overline{z}_1} W_{\varepsilon,\xi(\overline{z})}\Big)\, d\mu_g|_{\overline{z}=0} \\
&= \frac{\varepsilon^2}{\varepsilon^n} \int_{\mathbb{R}^n_+} \tilde{c}(y)|g_\xi(y)|^{\frac12} g_\xi^{ij}(y) \frac{\partial}{\partial y_i}\Big[\tilde{\gamma}(0) U_\varepsilon\Big(\sqrt{\tilde{A}(0)}\tilde{\mathcal{E}}(0,y)\Big)\chi(\tilde{\mathcal{E}}(0,y))\Big] \\
&\qquad \times \frac{\partial}{\partial y_j}\frac{\partial}{\partial \overline{z}_1}\Big[\tilde{\gamma}(\overline{z}) U_\varepsilon\Big(\sqrt{\tilde{A}(\overline{z})}\tilde{\mathcal{E}}(\overline{z},y)\Big)\chi(\tilde{\mathcal{E}}(\overline{z},y))\Big]\Big|_{\overline{z}=0}\, dy \\
&= \int_{\mathbb{R}^n} \tilde{c}(\varepsilon\zeta)|g_\xi(\varepsilon\zeta)|^{\frac12} g_\xi^{ij}(\varepsilon\zeta) \frac{\partial}{\partial \zeta_i}\Big[\tilde{\gamma}(0) U_\varepsilon\Big(\sqrt{\tilde{A}(0)}\tilde{\mathcal{E}}(0,\varepsilon\zeta)\Big)\chi(\tilde{\mathcal{E}}(0,\varepsilon\zeta))\Big] \\
&\qquad \times \frac{\partial}{\partial \zeta_j}\frac{\partial}{\partial \overline{z}_1}\Big[\tilde{\gamma}(\overline{z}) U_\varepsilon\Big(\sqrt{\tilde{A}(\overline{z})}\tilde{\mathcal{E}}(\overline{z},\varepsilon\zeta)\Big)\chi(\tilde{\mathcal{E}}(\overline{z},\varepsilon\zeta))\Big]\Big|_{\overline{z}=0}\, d\zeta.
\end{aligned}$$

Using the definition (5.6) of $\tilde{\mathcal{E}}$, we obtain

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^n_+} \tilde{c}(\varepsilon\zeta)\tilde{\gamma}(0)|g_\xi(\varepsilon\zeta)|^{\frac12} g_\xi^{ij}(\varepsilon\zeta)\Big[\Big(\frac{\partial}{\partial \zeta_i} U\Big(\sqrt{\tilde{A}(0)}\zeta\Big)\Big)\chi(\varepsilon\zeta) + U\Big(\sqrt{\tilde{A}(0)}\zeta\Big)\frac{\partial}{\partial \zeta_i}\chi(\varepsilon\zeta)\Big] \\
&\qquad \times \frac{\partial}{\partial \zeta_j}\frac{\partial}{\partial \overline{z}_1}\Big[\tilde{\gamma}(\overline{z}) U_\varepsilon\Big(\sqrt{\tilde{A}(\overline{z})}\tilde{\mathcal{E}}(\overline{z},\varepsilon\zeta)\Big)\chi(\tilde{\mathcal{E}}(\overline{z},\varepsilon\zeta))\Big]\Big|_{\overline{z}=0}\, d\zeta + O(\varepsilon) \\
&= \int_{\mathbb{R}^n_+} \tilde{c}(\varepsilon\zeta)\tilde{\gamma}(0)|g_\xi(\varepsilon\zeta)|^{\frac12} g_\xi^{ij}(\varepsilon\zeta)\Big[\Big(\frac{\partial}{\partial \zeta_i} U\Big(\sqrt{\tilde{A}(0)}\zeta\Big)\Big)\chi(\varepsilon\zeta) + U\Big(\sqrt{\tilde{A}(0)}\zeta\Big)\frac{\partial}{\partial \zeta_i}\chi(\varepsilon\zeta)\Big] \\
&\qquad \times \frac{\partial}{\partial \zeta_j}\Big[\frac{\partial}{\partial \overline{z}_1}\tilde{\gamma}(\overline{z})|_{\overline{z}=0} U\Big(\sqrt{\tilde{A}(0)}\zeta\Big)\chi(\varepsilon\zeta) + \tilde{\gamma}(0) U\Big(\sqrt{\tilde{A}(0)}\zeta\Big)\frac{\partial \chi}{\partial \zeta_k}(\varepsilon\zeta)\frac{\partial}{\partial \overline{z}_1}\mathcal{E}_k(\overline{z},\psi_\xi^\partial(\varepsilon\zeta))|_{\overline{z}=0} \\
&\qquad + \tilde{\gamma}(0)\chi(\varepsilon\zeta)\frac{\sqrt{\tilde{A}(0)}}{\varepsilon}\frac{\partial U}{\partial \zeta_k}\Big(\sqrt{\tilde{A}(0)}\zeta\Big)\frac{\partial}{\partial \overline{z}_1}\mathcal{E}_k(\overline{z},\psi_\xi^\partial(\varepsilon\zeta))|_{\overline{z}=0} \\
&\qquad + \tilde{\gamma}(0)\chi(\varepsilon\zeta)\frac{\partial U}{\partial \zeta_k}\Big(\sqrt{\tilde{A}(0)}\zeta\Big)\frac{\partial}{\partial \overline{z}_1}\sqrt{\tilde{A}(\overline{z})}\Big|_{\overline{z}=0}\zeta_k\Big]\, d\zeta + O(\varepsilon) \\
&=: D_1 + D_2 + D_3 + D_4 + O(\varepsilon),
\end{aligned}$$

where $\frac{\partial f}{\partial \zeta_k}(\cdot)$ denotes the derivative of the function f with respect to its k-th variable. Expanding $\tilde{c}(\varepsilon\zeta)$, by the exponential decay of U and its derivative, and by (2.5), (2.6) and (2.7), we get

$$
\begin{aligned}
D_1 &= \tilde{\gamma}(0)\int_{\mathbb{R}^n_+}(\tilde{c}(0)\delta_{ij}+O(\varepsilon|\zeta|))\Big[\frac{\partial}{\partial\zeta_i}\Big(U\Big(\sqrt{\tilde{A}(0)}\zeta\Big)\Big)\chi(\varepsilon\zeta)+O(\varepsilon|\zeta|)\Big]\\
&\quad\times\frac{\partial}{\partial\overline{z}_1}\tilde{\gamma}(\overline{z})|_{\overline{z}=0}\Big[\frac{\partial}{\partial\zeta_j}\Big(U(\sqrt{\tilde{A}(0)}\zeta)\Big)\chi(\varepsilon\zeta)+O(\varepsilon^2|\zeta|^2)\Big]\,d\zeta\\
&=\frac{1}{2}\frac{\partial}{\partial\overline{z}_1}(\tilde{\gamma}(\overline{z}))^2|_{\overline{z}=0}\tilde{c}(0)\int_{\mathbb{R}^n}\Big|\nabla_\zeta\Big(U\Big(\sqrt{\tilde{A}(0)}\zeta\Big)\Big)\Big|^2\,d\zeta+O(\varepsilon).
\end{aligned}
$$

Similarly, $D_2 = O(\varepsilon)$. Also, we have

$$
\begin{aligned}
D_4 &= \tilde{\gamma}^2(0)\int_{\mathbb{R}^n_+}(\tilde{c}(0)\delta_{ij}+O(\varepsilon|\zeta|))\Big[\frac{\partial}{\partial\zeta_i}\Big(U\Big(\sqrt{\tilde{A}(0)}\zeta\Big)\Big)\chi(\varepsilon\zeta)+O(\varepsilon^2|\zeta|^2)\Big]\\
&\quad\times\frac{\partial}{\partial\zeta_j}\Big[\chi(\varepsilon\zeta)\frac{\partial U}{\partial\zeta_k}\Big(\sqrt{\tilde{A}(0)}\zeta\Big)\frac{\partial}{\partial\overline{z}_1}\sqrt{\tilde{A}(\overline{z})}|_{\overline{z}=0}\zeta_k\Big]\,d\zeta\\
&=\tilde{c}(0)\tilde{\gamma}^2(0)\frac{\partial}{\partial\overline{z}_1}\sqrt{\tilde{A}(\overline{z})}|_{\overline{z}=0}\int_{\mathbb{R}^n_+}\frac{\partial}{\partial\zeta_i}\Big(U\Big(\sqrt{\tilde{A}(0)}\zeta\Big)\Big)\frac{\partial}{\partial\zeta_i}\Big[\frac{\partial U}{\partial\zeta_k}\Big(\sqrt{\tilde{A}(0)}\zeta\Big)\zeta_k\Big]\,d\zeta+O(\varepsilon).
\end{aligned}
$$

Now, an elementary computation yields

$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial\overline{z}_1}\Big|\nabla_\zeta U\Big(\sqrt{\tilde{A}(\overline{z})}\zeta\Big)\Big|^2 &= \frac{\partial}{\partial\zeta_i}\Big(U\Big(\sqrt{\tilde{A}(\overline{z})}\zeta\Big)\Big)\frac{\partial}{\partial\zeta_i}\frac{\partial}{\partial\overline{z}_1}\Big(U\Big(\sqrt{\tilde{A}(\overline{z})}\zeta\Big)\Big)\\
&= \frac{\partial}{\partial\zeta_i}\Big(U\Big(\sqrt{\tilde{A}(\overline{z})}\zeta\Big)\Big)\frac{\partial}{\partial\zeta_i}\Big[\frac{\partial U}{\partial\xi_k}\Big(\sqrt{\tilde{A}(\overline{z})}\zeta\Big)\zeta_k\frac{\partial}{\partial\overline{z}_1}\sqrt{\tilde{A}(\overline{z})}\Big].
\end{aligned}
$$

Hence,

$$
D_4 = \frac{1}{2}\tilde{c}(0)\tilde{\gamma}^2(0)\int_{\mathbb{R}^n_+}\frac{\partial}{\partial\overline{z}_1}\Big(\Big|\nabla_\zeta U\Big(\sqrt{\tilde{A}(\overline{z})}\zeta\Big)\Big|^2\Big)\Big|_{\overline{z}=0}\,d\zeta+O(\varepsilon).
$$

We conclude that

$$
\begin{aligned}
D_1 + D_4 &= \frac{1}{2}\tilde{c}(0)\int_{\mathbb{R}^n_+}\frac{\partial}{\partial\overline{z}_1}\Big(\Big|\tilde{\gamma}(\overline{z})\nabla_\zeta U\Big(\sqrt{\tilde{A}(\overline{z})}\zeta\Big)\Big|^2\Big)\Big|_{\overline{z}=0}\,d\zeta+O(\varepsilon)\\
&= \frac{1}{2}c(\xi)\int_{\mathbb{R}^n_+}\frac{\partial}{\partial\overline{z}_1}\Big(\Big|\nabla_\zeta V^{\xi(\overline{z})}(\zeta)\Big|^2\Big)\Big|_{\overline{z}=0}\,d\zeta+O(\varepsilon).
\end{aligned}
$$

The term D_3 is more delicate since the factor $\frac{1}{\varepsilon}$ forces us to expand all factors up to the second order. In the light of (2.5), (2.6) and (2.7), with the convention that the matrix $(h_{ij})_{i,j=1,\dots,n}$ coincides with the second fundamental form when $i,j = 1,\dots,n-1$ and $h_{i,n} = h_{n,j} = 0$ for $i,j = 1,\dots,n$, we obtain

$$
\begin{aligned}
D_3 &= \tilde{\gamma}^2(0)\sqrt{\tilde{A}(0)}\int_{\mathbb{R}^n_+}\frac{1}{\varepsilon}\tilde{c}(\varepsilon\zeta)|g_\xi(\varepsilon\zeta)|^{\frac{1}{2}}g_\xi^{ij}(\varepsilon\zeta)\Big[\frac{\partial}{\partial\zeta_i}\Big(U\Big(\sqrt{\tilde{A}(0)}\zeta\Big)\Big)\chi(\varepsilon\zeta)+O(\varepsilon^2|\zeta|^2)\Big]\\
&\quad\times\frac{\partial}{\partial\zeta_j}\Big[\chi(\varepsilon\zeta)\frac{\partial U}{\partial\xi_k}\Big(\sqrt{\tilde{A}(0)}\zeta\Big)\frac{\partial}{\partial\overline{z}_1}\mathcal{E}_k(\overline{z},\psi_\xi^\partial(\varepsilon\zeta))|_{\overline{z}=0}\Big]\,d\zeta\\
&= \tilde{\gamma}^2(0)\sqrt{\tilde{A}(0)}\int_{\mathbb{R}^n_+}\Big[\frac{\delta_{ij}\tilde{c}(0)}{\varepsilon}+2\tilde{c}(0)h_{ij}\zeta_n-\tilde{c}(0)(n-1)\delta_{ij}H\zeta_n+\delta_{ij}\frac{\partial\tilde{c}}{\partial\zeta_l}(0)\zeta_l\Big]\\
&\quad\times\frac{\partial}{\partial\zeta_i}\Big(U\Big(\sqrt{\tilde{A}(0)}\zeta\Big)\Big)\frac{\partial}{\partial\zeta_j}\Big[\frac{\partial U}{\partial\xi_k}\Big(\sqrt{\tilde{A}(0)}\zeta\Big)\frac{\partial}{\partial\overline{z}_1}\mathcal{E}_k(\overline{z},\psi_\xi^\partial(\varepsilon\zeta))|_{\overline{z}=0}\Big]\,d\zeta+O(\varepsilon).
\end{aligned}
$$

By Lemma 5.3, we have

$$\frac{\partial}{\partial \bar z_1}\mathcal{E}_k(\bar z, \psi_\xi^\partial(\varepsilon\zeta))|_{\bar z=0} = -\delta_{1k} + O(\varepsilon^2|\zeta|^2),$$
$$\frac{\partial}{\partial \zeta_j}\Big(\frac{\partial}{\partial \bar z_1}\mathcal{E}_k(\bar z, \psi_\xi^\partial(\varepsilon\zeta))|_{\bar z=0}\Big) = O(\varepsilon^2|\zeta|^2).$$

Moreover, since U is radial,

$$\frac{\partial U}{\partial \zeta_i}\Big(\sqrt{\tilde A(0)}\zeta\Big) = U'(\sqrt{\tilde A(0)}\zeta)\frac{\zeta_i}{|\zeta|}, \tag{5.9}$$

$$\frac{\partial}{\partial \zeta_1}\Big(\frac{U'(\sqrt{\tilde A(0)}\zeta)}{|\zeta|}\Big) = \Big(\frac{\sqrt{\tilde A(0)}U''(\zeta)}{|\zeta|^2} - \frac{U'(\zeta)}{|\zeta|^3}\Big)\zeta_1, \tag{5.10}$$

where $U' = \frac{\partial U}{\partial r}$, $U'' = \frac{\partial^2 U}{\partial r^2}$ and $r = |\zeta|$. Thus, we get

$$\begin{aligned}
D_3 &= -\tilde\gamma^2(0)\tilde A(0)\int_{\mathbb{R}^n_+}\Big[\frac{\delta_{ij}\tilde c(0)}{\varepsilon} + 2\tilde c(0)h_{ij}\zeta_n - \tilde c(0)(n-1)\delta_{ij}H\zeta_n + \delta_{ij}\frac{\partial \tilde c}{\partial \xi_l}(0)\zeta_l\Big]\\
&\qquad\times \frac{U'(\sqrt{\tilde A(0)}\zeta)}{|\zeta|}\zeta_i\frac{\partial}{\partial \zeta_j}\Big(\frac{U'(\sqrt{\tilde A(0)}\zeta)}{|\zeta|}\zeta_k\Big)\delta_{1k}\,d\zeta + O(\varepsilon)\\
&= -\tilde\gamma^2(0)\tilde A(0)\int_{\mathbb{R}^n_+}\Big[\frac{\delta_{ij}\tilde c(0)}{\varepsilon} + 2\tilde c(0)h_{ij}\zeta_n - \tilde c(0)(n-1)\delta_{ij}H\zeta_n + \delta_{ij}\frac{\partial \tilde c}{\partial \xi_l}(0)\zeta_l\Big]\\
&\qquad\times\Big(\frac{U'(\sqrt{\tilde A(0)}\zeta)}{|\zeta|}\Big)^2\zeta_i\delta_{j1} + \frac{U'(\sqrt{\tilde A(0)}\zeta)}{|\zeta|}\Big(\frac{\sqrt{\tilde A(0)}U''(\zeta)}{|\zeta|^2} - \frac{U'(\zeta)}{|\zeta|^3}\Big)\zeta_j\zeta_i\zeta_1\,d\zeta + O(\varepsilon).
\end{aligned}$$

Now, by symmetry considerations, for $i = 1, \dots, n-1$, any term containing ζ_i to an odd power vanishes, and, since $h_{in} = h_{nj} = 0$, we get that

$$\begin{aligned}
D_3 &= -\tilde\gamma^2(0)\tilde A(0)\frac{\partial \tilde c}{\partial \xi_l}(0)\int_{\mathbb{R}^n_+}\delta_{ij}\Big(\frac{U'(\sqrt{\tilde A(0)}\zeta)}{|\zeta|}\Big)^2\zeta_i\zeta_l\delta_{j1}\\
&\qquad + \delta_{ij}\frac{U'(\sqrt{\tilde A(0)}\zeta)}{|\zeta|}\Big(\frac{\sqrt{\tilde A(0)}U''(\zeta)}{|\zeta|^2} - \frac{U'(\zeta)}{|\zeta|^3}\Big)\zeta_j\zeta_i\zeta_1\zeta_l\,d\zeta + O(\varepsilon)\\
&= -\tilde\gamma^2(0)\tilde A(0)\frac{\partial \tilde c}{\partial \xi_1}(0)\int_{\mathbb{R}^n_+}\Big(\frac{U'(\sqrt{\tilde A(0)}\zeta)}{|\zeta|}\Big)^2\zeta_1^2\\
&\qquad + \frac{U'(\sqrt{\tilde A(0)}\zeta)}{|\zeta|}\Big(\frac{\sqrt{\tilde A(0)}U''(\zeta)}{|\zeta|^2} - \frac{U'(\zeta)}{|\zeta|^3}\Big)\zeta_i\zeta_i\zeta_1^2\,d\zeta + O(\varepsilon)
\end{aligned}$$

Notice that, by (5.9) and (5.10), we have

$$\frac{1}{2}\frac{\partial}{\partial \zeta_1}\Big|\nabla_\zeta U\Big(\sqrt{\tilde A(0)}\zeta\Big)\Big|^2 = \tilde A(0)\Big(\frac{U'(\sqrt{\tilde A(0)}\zeta)}{|\zeta|}\Big)^2\zeta_1 + \tilde A(0)\frac{U'(\sqrt{\tilde A(0)}\zeta)}{|\zeta|}\Big(\frac{\sqrt{\tilde A(0)}U''(\zeta)}{|\zeta|^2} - \frac{U'(\zeta)}{|\zeta|^3}\Big)\zeta_i\zeta_i\zeta_1,$$

so

$$\begin{aligned}
D_3 &= -\tilde\gamma^2(0)\frac{\partial \tilde c}{\partial \xi_1}(0)\frac{1}{2}\int_{\mathbb{R}^n_+}\frac{\partial}{\partial \zeta_1}\Big|\nabla_\zeta U\Big(\sqrt{\tilde A(0)}\zeta\Big)\Big|^2\zeta_1\,d\zeta\\
&= \tilde\gamma^2(0)\frac{\partial \tilde c}{\partial \xi_1}(0)\frac{1}{2}\int_{\mathbb{R}^n_+}\frac{\partial}{\partial \zeta_1}\Big|\nabla_\zeta U\Big(\sqrt{\tilde A(0)}\zeta\Big)\Big|^2\,d\zeta\\
&= \frac{\partial \tilde c}{\partial \xi_1}(0)\frac{1}{2}\int_{\mathbb{R}^n_+}|\nabla V^\xi(\zeta)|^2\,d\zeta = \frac{\partial}{\partial \bar z_1}c(\xi(\bar z))|_{\bar z=0}\frac{1}{2}\int_{\mathbb{R}^n_+}|\nabla V^\xi(\zeta)|^2\,d\zeta + O(\varepsilon).
\end{aligned}$$

Consequently, we obtain

$$I_1 = \frac{1}{2} \int\limits_{\mathbb{R}^n_+} \frac{\partial}{\partial \bar{z}_1} \big[c(\xi(\bar{z})) |\nabla V^{\xi(\bar{z})}(\zeta)|^2 \big] \big|_{\bar{z}=0} \, d\zeta + O(\varepsilon).$$

For the second term, setting $\tilde{d}(y) = d(\psi^{\partial}_{\xi}(y)) = d(x)$, we obtain in an analogous way

$$\begin{aligned}
I_2 &= \int\limits_{\mathbb{R}^n_+} \tilde{d}(\varepsilon\zeta) |g_\xi(\varepsilon\zeta)|^{\frac{1}{2}} \tilde{\gamma}(0) U\Big(\sqrt{\tilde{A}(0)}\zeta\Big) \chi(\varepsilon\zeta) \frac{\partial}{\partial \bar{z}_1} \Big[\tilde{\gamma}(\bar{z}) U_\varepsilon\Big(\sqrt{\tilde{A}(\bar{z})} \tilde{\mathcal{E}}(\bar{z}, \varepsilon\zeta)\Big) \chi(\tilde{\mathcal{E}}(\bar{z}, \varepsilon\zeta)) \Big] \Big|_{\bar{z}=0} \, d\zeta \\
&= \int\limits_{\mathbb{R}^n} \tilde{d}(\varepsilon\zeta) |g_\xi(\varepsilon\zeta)|^{\frac{1}{2}} \tilde{\gamma}(0) U\Big(\sqrt{\tilde{A}(0)}\zeta\Big) \chi(\varepsilon\zeta) \\
&\qquad \times \Bigg[\frac{\partial}{\partial \bar{z}_1} \tilde{\gamma}(\bar{z})|_{\bar{z}=0} U\Big(\sqrt{\tilde{A}(0)}\zeta\Big) \chi(\varepsilon\zeta) + \tilde{\gamma}(0) \chi(\varepsilon\zeta) \frac{\sqrt{\tilde{A}(0)}}{\varepsilon} \frac{\partial U}{\partial \xi_k}\Big(\sqrt{\tilde{A}(0)}\zeta\Big) \frac{\partial}{\partial \bar{z}_1} \mathcal{E}_k(\bar{z}, \psi^{\partial}_{\xi}(\varepsilon\zeta))|_{\bar{z}=0} \\
&\qquad\quad + \tilde{\gamma}(0) \chi(\varepsilon\zeta) \frac{\partial U}{\partial \xi_k}\Big(\sqrt{\tilde{A}(0)}\zeta\Big) \frac{\partial}{\partial \bar{z}_1} \sqrt{\tilde{A}(\bar{z})} \Big|_{\bar{z}=0} \zeta_k + \tilde{\gamma}(0) U\Big(\sqrt{\tilde{A}(0)}\zeta\Big) \frac{\partial}{\partial \bar{z}_1} [\chi(\tilde{\mathcal{E}}(\bar{z}, \varepsilon\zeta))] \Big|_{\bar{z}=0} \Bigg] \, d\zeta \\
&=: B_1 + B_2 + B_3 + B_4.
\end{aligned}$$

Expanding $\tilde{d}(\varepsilon\zeta)$, by the exponential decay of U and its derivative, and by (2.7) and the definition of $\tilde{\mathcal{E}}$, we get

$$\begin{aligned}
B_1 &= \int\limits_{\mathbb{R}^n_+} \tilde{d}(0) \tilde{\gamma}(0) \frac{\partial}{\partial \bar{z}_1} \tilde{\gamma}(\bar{z})|_{\bar{z}=0} U^2\Big(\sqrt{\tilde{A}(0)}\zeta\Big) \, d\zeta + O(\varepsilon) \\
&= \frac{1}{2} \tilde{d}(0) \frac{\partial}{\partial \bar{z}_1} \tilde{\gamma}^2(\bar{z})|_{\bar{z}=0} \int\limits_{\mathbb{R}^n} U^2\Big(\sqrt{\tilde{A}(0)}\zeta\Big) \, d\zeta.
\end{aligned}$$

As before, we obtain that $B_4 = O(\varepsilon)$ and

$$\begin{aligned}
B_3 &= \int\limits_{\mathbb{R}^n_+} \tilde{d}(0) \tilde{\gamma}^2(0) U\Big(\sqrt{\tilde{A}(0)}\zeta\Big) \frac{\partial U}{\partial \xi_k}\Big(\sqrt{\tilde{A}(0)}\zeta\Big) \frac{\partial}{\partial \bar{z}_1} \sqrt{\tilde{A}(\bar{z})} \Big|_{\bar{z}=0} \zeta_k \, d\zeta + O(\varepsilon) \\
&= \frac{1}{2} \int\limits_{\mathbb{R}^n_+} \tilde{d}(0) \tilde{\gamma}^2(0) \frac{\partial}{\partial \bar{z}_1} \Big(U\Big(\sqrt{\tilde{A}(\bar{z})}\zeta\Big) \Big)^2 \Big|_{\bar{z}=0} \, d\zeta + O(\varepsilon).
\end{aligned}$$

Thus,

$$\begin{aligned}
B_1 + B_3 &= \frac{1}{2} \tilde{d}(0) \int\limits_{\mathbb{R}^n_+} \frac{\partial}{\partial \bar{z}_1} \Big(\tilde{\gamma}^2(\bar{z}) U\Big(\sqrt{\tilde{A}(\bar{z})}\zeta\Big) \Big)^2 \Big|_{\bar{z}=0} \, d\zeta + O(\varepsilon) \\
&= \frac{1}{2} d(\xi) \int\limits_{\mathbb{R}^n_+} \frac{\partial}{\partial \bar{z}_1} (V^{\xi(\bar{z})}(\zeta))^2|_{\bar{z}=0} \, d\zeta + O(\varepsilon).
\end{aligned}$$

Again, we have to pay particular attention to the term containing $\frac{1}{\varepsilon}$ as a factor. From (2.7) and Lemma 5.3 we get

$$\begin{aligned}
B_2 &= \tilde{\gamma}^2(0) \sqrt{\tilde{A}(0)} \int\limits_{\mathbb{R}^n_+} \frac{\tilde{d}(\varepsilon\zeta)}{\varepsilon} |g_\xi(\varepsilon\zeta)|^{\frac{1}{2}} U\Big(\sqrt{\tilde{A}(0)}\zeta\Big) \frac{\partial U}{\partial \xi_k}\Big(\sqrt{\tilde{A}(0)}\zeta\Big) (-\delta_{1k} + O(\varepsilon^2|\zeta|^2)) \, d\zeta + O(\varepsilon) \\
&= -\tilde{\gamma}^2(0) \sqrt{\tilde{A}(0)} \int\limits_{\mathbb{R}^n_+} \Big(\frac{\tilde{d}(0)}{\varepsilon} + \frac{\partial \tilde{d}}{\partial \xi_l}(0) \zeta_l \Big) U\Big(\sqrt{\tilde{A}(0)}\zeta\Big) \frac{\partial U}{\partial \xi_1}\Big(\sqrt{\tilde{A}(0)}\zeta\Big) \, d\zeta + O(\varepsilon)
\end{aligned}$$

and, by (5.9),

$$\begin{aligned}
B_2 &= -\tilde{\gamma}^2(0) \sqrt{\tilde{A}(0)} \int\limits_{\mathbb{R}^n_+} \Big(\frac{\tilde{d}(0)}{\varepsilon} + \frac{\partial \tilde{d}}{\partial \xi_l}(0) \zeta_l \Big) U\Big(\sqrt{\tilde{A}(0)}\zeta\Big) \frac{U'(\sqrt{\tilde{A}(0)}\zeta)}{|\zeta|} \zeta_1 \, d\zeta + O(\varepsilon) \\
&= -\tilde{\gamma}^2(0) \sqrt{\tilde{A}(0)} \int\limits_{\mathbb{R}^n_+} \frac{\partial \tilde{d}}{\partial \xi_1}(0) U\Big(\sqrt{\tilde{A}(0)}\zeta\Big) \frac{U'(\sqrt{\tilde{A}(0)}\zeta)}{|\zeta|} \zeta_1^2 \, d\zeta + O(\varepsilon)
\end{aligned}$$

due to the symmetry. So,

$$
\begin{aligned}
B_2 &= -\tilde{\gamma}^2(0)\frac{\partial \tilde{d}}{\partial \xi_1}(0)\int_{\mathbb{R}^n_+} \frac{1}{2}\frac{\partial}{\partial \zeta_1}\Big[U\Big(\sqrt{\tilde{A}(0)}\zeta\Big)\Big]^2 \zeta_1 \, d\zeta + O(\varepsilon) \\
&= \tilde{\gamma}^2(0)\frac{\partial \tilde{d}}{\partial \xi_1}(0)\int_{\mathbb{R}^n_+} U^2\Big(\sqrt{\tilde{A}(0)}\zeta\Big)\, d\zeta + O(\varepsilon) \\
&= \frac{1}{2}\frac{\partial}{\partial \bar{z}_1} d(\xi)|_{\bar{z}=0}\int_{\mathbb{R}^n_+} (V^{\xi(\bar{z})}(\zeta))^2 \, d\zeta
\end{aligned}
$$

and

$$
I_2 = \frac{1}{2}\int_{\mathbb{R}^n_+} \frac{\partial}{\partial \bar{z}_1}\{d(\xi(\bar{z}))(V^{\xi(\bar{z})}(\zeta))^2\}\,\big|_{\bar{z}=0}\; d\zeta + O(\varepsilon).
$$

In a similar way we proceed for I_3, completing the proof. □

Funding: The first author is supported by CONACYT grant 237661 (México) and PAPIIT-DGAPA-UNAM grant IN104315 (Mexico). The second and third authors are partially supported by the GNAMPA project by INDAM. The second author is partially supported by the PRA project of the university of Pisa.

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