

Research Article

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The elliptic sinh-Gordon equation in a semi-strip

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Abstract: We study the elliptic sinh-Gordon equation posed in a semi-strip by applying the so-called Fokas method, a generalization of the inverse scattering transform for boundary value problems. Based on the spectral analysis for the Lax pair formulation, we show that the spectral functions can be characterized from the boundary values. We express the solution of the equation in terms of the unique solution of the matrix Riemann–Hilbert problem whose jump matrices are defined by the spectral functions. Moreover, we derive the global algebraic relation that involves the boundary values. In addition, it can be verified that the solution of the elliptic sinh-Gordon equation posed in the semi-strip exists if the spectral functions defined by the boundary values satisfy this global relation.

Keywords: Boundary value problems, elliptic PDEs, sinh-Gordon equation, integrable equations

MSC 2010: 47K15, 35Q55

1 Introduction

We study the boundary problem for the elliptic sinh-Gordon equation posed in a semi-strip,

$$q_{xx} + q_{yy} = \sinh q, \quad (x, y) \in \Omega, \quad (1.1)$$

where $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < \infty, 0 < y < L\}$. It is well known that the elliptic sinh-Gordon equation is completely integrable [1, 3] and hence equation (1.1) can be analyzed by the inverse scattering transform. Indeed, the classical inverse scattering transform was applied to solve the elliptic sinh-Gordon equation posed in the entire plane $\{-\infty < x, y < \infty\}$ (see [3, 15]). For more complicated or general domains, the so-called Fokas method can be applied to solve boundary value problems [2, 4–7] (see also the monograph [9] and references therein). It should be noted that the method can be considered as a significant extension of the inverse scattering transform for boundary value problems. Regarding the boundary value problem for the elliptic sinh-Gordon equation, the Fokas method has been applied to solve problem (1.1) posed in the half plane $\{-\infty < x < \infty, 0 < y < \infty\}$ (see [13]) and the quarter plane $\{0 < x, y < \infty\}$ (see [14]). It has been shown in [13, 14] that the solution of the equation can be expressed in terms of the unique solution of the matrix Riemann–Hilbert problem defined by the spectral functions. It also has been shown that the solution of the equation exists if the spectral functions determined by the boundary values satisfy the so-called global relation.

In this paper, we extend the results presented in [13, 14] to the semi-strip (see also [10, 11, 16] for analogous results). The rigorous analysis of the Fokas method involves the following steps:

(i) Assuming that a smooth solution $q(x, y)$ exists, we express the solution $q(x, y)$ in terms of the unique solution of the matrix Riemann–Hilbert problem defined by the spectral functions $\{a_j(k), b_j(k)\}$, $j = 1, 2, 3$. These spectral functions are defined by boundary values $\{q(x, 0), q_y(x, 0)\}$, $\{q(0, y), q_x(0, y)\}$ and

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$\{q(x, L), q_y(x, L)\}$, respectively. It should be remarked that the spectral functions satisfy the global algebraic relation that involves all boundary values:

$$a_3(k) = a_1(k)a_2(-k) + b_1(k)b_2(-k), \quad k \in \mathbb{C}^-, \quad (1.2a)$$

$$b_3(k)e^{-2\omega_2(k)L} = a_2(k)b_1(k) - a_1(k)b_2(k), \quad k \in \mathbb{C}^-. \quad (1.2b)$$

(ii) Given boundary values $\{g_0(x), g_1(x)\}$, $\{f_0(y), f_1(y)\}$ and $\{h_0(x), h_1(x)\}$, where

$$q(x, 0) = g_0(x), \quad q_y(x, 0) = g_1(x), \quad (1.3a)$$

$$q(0, y) = f_0(y), \quad q_x(0, y) = f_1(y), \quad (1.3b)$$

$$q(x, L) = h_0(x), \quad q_y(x, L) = h_1(x), \quad (1.3c)$$

we define the spectral functions and $q(x, y)$ in terms of the solution for the Riemann–Hilbert problem formulated in step (i). Assuming that the spectral functions satisfy the global relation (1.2), we prove that the function $q(x, y)$ defined by the solution of the Riemann–Hilbert problem solves equation (1.1) and satisfy the boundary values (1.3).

The outline of this work is the following. In Section 2, we introduce the Lax pair for the elliptic sinh-Gordon equation and eigenfunctions that satisfy both parts of the Lax pair. In Section 3, we discuss the spectral functions defined by the boundary values and the global algebraic relation that involves all boundary values. In Section 4, we formulate the matrix Riemann–Hilbert problem whose jump matrices are uniquely defined by the spectral functions. Analyzing the Riemann–Hilbert problem, we show that the solution of the equation exists if the boundary values satisfy the global relation. We end with concluding remarks in Section 5.

2 Preliminaries

It is well known that the elliptic sinh-Gordon equation (1.1) admits the following compatibility condition of the Lax pair [3, 13, 15]:

$$\mu_x + \omega_1(k)[\sigma_3, \mu] = Q(x, y, k)\mu, \quad (2.1a)$$

$$\mu_y + \omega_2(k)[\sigma_3, \mu] = i\tilde{Q}(x, y, k)\mu, \quad (2.1b)$$

where $k \in \mathbb{C}$ is a spectral parameter, μ is a 2×2 matrix-valued eigenfunction and

$$\omega_1(k) = -\frac{1}{2i}\left(k - \frac{1}{4k}\right), \quad \omega_2(k) = -\frac{1}{2}\left(k + \frac{1}{4k}\right), \quad Q(x, y, k) = \frac{1}{4} \begin{pmatrix} \frac{i}{2k}(\cosh q - 1) & -(r + \frac{\sinh q}{2k}) \\ r - \frac{\sinh q}{2k} & -\frac{i}{2k}(\cosh q - 1) \end{pmatrix}$$

with

$$\tilde{Q}(x, y, k) = Q(x, y, -k), \quad r(x, y) = iq_x(x, y) + q_y(x, y), \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the matrix commutator defined by $[\sigma_3, A] = \sigma_3 A - A \sigma_3$. We write $\hat{\sigma}_3 A = [\sigma_3, A]$ for the matrix commutator. It is convenient to use the notation

$$e^{\hat{\sigma}_3 \xi} A = e^{\sigma_3 \xi} A e^{-\sigma_3 \xi} = \begin{pmatrix} a_{11} & e^{2\xi} a_{12} \\ e^{-2\xi} a_{21} & a_{22} \end{pmatrix}.$$

Note that the Lax pair (2.1) can be written as the simple formulation

$$d[e^{(\omega_1(k)x + \omega_2(k)y)\hat{\sigma}_3} \mu(x, y, k)] = e^{(\omega_1(k)x + \omega_2(k)y)\hat{\sigma}_3} W(x, y, k),$$

where the differential 1-form W is given by

$$W(x, y, k) = Q(x, y, k)\mu(x, y, k)dx + i\tilde{Q}(x, y, k)\mu(x, y, k)dy. \quad (2.2)$$

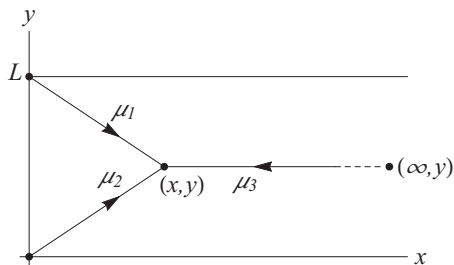


Figure 1: The three distinct points for the eigenfunctions μ_j , $j = 1, 2, 3$.

Hence, we define eigenfunctions that satisfy both parts of the Lax pair (2.1) as

$$\mu_j(x, y, k) = I + \int_{(x_j, y_j)}^{(x, y)} e^{-(\omega_1(k)(x-\xi) + \omega_2(k)(y-\eta))\hat{\sigma}_3} W_j(\xi, \eta, k), \quad (2.3)$$

where W_j is the differential form given in (2.2) with μ_j . Since the differential 1-form $W(x, y, k)$ is closed, the integration in (2.3) does not depend on paths [6, 14]. We choose three distinct points (x_j, y_j) , $j = 1, 2, 3$ (see Figure 1),

$$(x_1, y_1) = (0, L), \quad (x_2, y_2) = (0, 0), \quad (x_3, y_3) = (\infty, y).$$

More specifically, the eigenfunctions $\mu_j(x, y, k)$, $j = 1, 2, 3$, satisfy the following integral equations:

$$\begin{aligned} \mu_1(x, y, k) &= I + \int_0^x e^{-\omega_1(k)(x-\xi)\hat{\sigma}_3} (Q\mu_1)(\xi, y, k) d\xi - i \int_y^L e^{-(\omega_1(k)x + \omega_2(k)(y-\eta))\hat{\sigma}_3} (\tilde{Q}\mu_1)(0, \eta, k) d\eta, \\ \mu_2(x, y, k) &= I + \int_0^x e^{-\omega_1(k)(x-\xi)\hat{\sigma}_3} (Q\mu_2)(\xi, y, k) d\xi + i \int_0^y e^{-(\omega_1(k)x + \omega_2(k)(y-\eta))\hat{\sigma}_3} (\tilde{Q}\mu_2)(0, \eta, k) d\eta, \\ \mu_3(x, y, k) &= I - \int_x^\infty e^{-\omega_1(k)(x-\xi)\hat{\sigma}_3} (Q\mu_3)(\xi, y, k) d\xi. \end{aligned}$$

Note that the off-diagonal components of the matrix-valued eigenfunctions μ_j involve the explicit exponential terms and

$$\operatorname{Re} \omega_1(k) < 0 \quad \text{for } \operatorname{Im} k > 0, \quad \operatorname{Re} \omega_2(k) < 0 \quad \text{for } \operatorname{Re} k > 0.$$

Thus, let the domains D_j in the complex k -plane, $j = 1, \dots, 4$ (see Figure 2), be defined by

$$\begin{aligned} D_1 &= \{k \in \mathbb{C} : \operatorname{Re} \omega_1(k) < 0\} \cap \{k \in \mathbb{C} : \operatorname{Re} \omega_2(k) < 0\}, \\ D_2 &= \{k \in \mathbb{C} : \operatorname{Re} \omega_1(k) < 0\} \cap \{k \in \mathbb{C} : \operatorname{Re} \omega_2(k) > 0\}, \\ D_3 &= \{k \in \mathbb{C} : \operatorname{Re} \omega_1(k) > 0\} \cap \{k \in \mathbb{C} : \operatorname{Re} \omega_2(k) > 0\}, \\ D_4 &= \{k \in \mathbb{C} : \operatorname{Re} \omega_1(k) > 0\} \cap \{k \in \mathbb{C} : \operatorname{Re} \omega_2(k) < 0\}. \end{aligned}$$

As a result, the domains of analyticity and boundedness for the eigenfunctions can be determined:

- $\mu_1(x, y, k)$ is analytic and bounded for $k \in (D_2, D_4)$,
- $\mu_2(x, y, k)$ is analytic and bounded for $k \in (D_1, D_3)$,
- $\mu_3(x, y, k)$ is analytic and bounded for $k \in (D_3 \cup D_4, D_1 \cup D_2)$.

For convenience, we write each column of $\mu_j(x, y, k)$ as follows:

$$\mu_1 = (\mu_1^{(2)}, \mu_1^{(4)}), \quad \mu_2 = (\mu_2^{(1)}, \mu_2^{(3)}), \quad \mu_3 = (\mu_3^{(34)}, \mu_3^{(12)}),$$

where the superscripts indicate the analytic and bounded domains D_j , $j = 1, \dots, 4$, for the columns of the matrix-valued eigenfunctions. Using integration by parts, in the appropriate domain, we know

$$\mu_j(x, y, k) = I + O\left(\frac{1}{k}\right) \quad \text{as } k \rightarrow \infty. \quad (2.4)$$

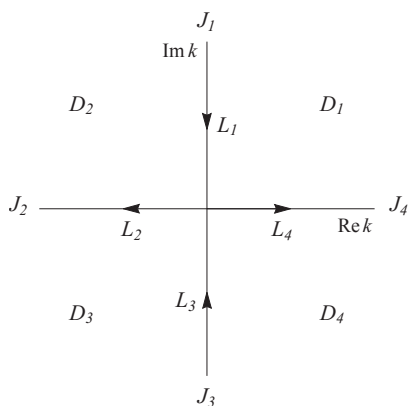


Figure 2: The domain D_j and the oriented contour L_j , $j = 1, \dots, 4$.

Since $\text{trace}(Q) = \text{trace}(\tilde{Q}) = 0$, equation (2.4) implies that $\det \mu_j = 1$, $j = 1, 2, 3$. The eigenfunctions enjoy the same symmetry as Q and \tilde{Q} :

$$\mu_{11}(x, y, k) = \mu_{22}(x, y, -k), \quad \mu_{21}(x, y, k) = -\mu_{12}(x, y, -k), \quad (2.5)$$

where the subscripts denote the (i, j) -component of the matrix.

Regarding the boundary values, we assume that $g_j, h_j \in H^1(\mathbb{R}^+)$ and $f_j \in H^1([0, L])$ for $j = 0, 1$.

3 Spectral analysis

3.1 Spectral functions

Note that the matrix eigenfunctions μ_1 , μ_2 and μ_3 are fundamental solutions of the Lax pair (2.1) and $\mu_1(0, L, k) = I$ and $\mu_2(0, 0, k) = I$. Hence, the eigenfunctions are related by the so-called spectral functions, also known as the scattering matrices, $S_1(k)$, $S_2(k)$ and $S_3(k)$:

$$\mu_3(x, y, k) = \mu_2(x, y, k) e^{-(\omega_1(k)x + \omega_2(k)y)\hat{\sigma}_3} S_1(k), \quad k \in (\mathbb{R}^+, \mathbb{R}^-), \quad (3.1a)$$

$$\mu_1(x, y, k) = \mu_2(x, y, k) e^{-(\omega_1(k)x + \omega_2(k)y)\hat{\sigma}_3} S_2(k), \quad k \in (i\mathbb{R}^+, i\mathbb{R}^-), \quad (3.1b)$$

$$\mu_3(x, y, k) = \mu_1(x, y, k) e^{-(\omega_1(k)x + \omega_2(k)(y-L))\hat{\sigma}_3} S_3(k), \quad k \in (\mathbb{R}^-, \mathbb{R}^+). \quad (3.1c)$$

Let $x = 0, y = 0$ in (3.1a) and (3.1b) and $x = 0, y = L$ in (3.1c). Then the spectral functions are given by

$$S_1(k) = \mu_3(0, 0, k), \quad S_2(k) = \mu_1(0, 0, k), \quad S_3(k) = \mu_3(0, L, k).$$

From the symmetry (2.5) of the eigenfunctions, we write the spectral functions $S_1(k)$, $S_2(k)$ and $S_3(k)$ as

$$S_j(k) = \begin{pmatrix} a_j(k) & -b_j(-k) \\ b_j(k) & a_j(-k) \end{pmatrix}, \quad j = 1, 2, 3.$$

Since $\det S_j(k) = 1$, $j = 1, 2, 3$, we find the following identities:

$$a_j(k)a_j(-k) + b_j(k)b_j(-k) = 1, \quad j = 1, 2, 3.$$

Motivated by the above arguments, we define

$$\Phi_1(x, k) = \mu_3(x, 0, k), \quad \Phi_2(y, k) = \mu_1(0, y, k), \quad \Phi_3(x, k) = \mu_3(x, L, k).$$

More specifically, the functions Φ_j , $j = 1, 2, 3$, satisfy the integral equations

$$\begin{aligned}\Phi_1(x, k) &= I - \int_x^\infty e^{-\omega_1(k)(x-\xi)\hat{\sigma}_3} (Q_0\Phi_1)(\xi, k) d\xi, \quad k \in (\mathbb{C}^-, \mathbb{C}^+), \quad 0 \leq x < \infty, \\ \Phi_2(y, k) &= I - i \int_y^L e^{-\omega_2(k)(y-\eta)\hat{\sigma}_3} (\tilde{Q}_0\Phi_2)(\eta, k) d\eta, \quad k \in \mathbb{C}, \quad 0 \leq y \leq L, \\ \Phi_3(x, k) &= I - \int_x^\infty e^{-\omega_1(k)(x-\xi)\hat{\sigma}_3} (Q_L\Phi_3)(\xi, k) d\xi, \quad k \in (\mathbb{C}^-, \mathbb{C}^+), \quad 0 \leq x < \infty,\end{aligned}$$

where $Q_0(x, k) = Q(x, 0, k)$, $\tilde{Q}_0(y, k) = Q(0, y, -k)$ and $Q_L(x, k) = Q(x, L, k)$. Note that

$$S_j(k) = \Phi_j(0, k), \quad j = 1, 2, 3,$$

which immediately imply that the spectral functions $a_j(k)$ and $b_j(k)$, $j = 1, 3$, are analytic and bounded for $\operatorname{Im} k < 0$, while the spectral functions $a_2(k)$ and $b_2(k)$ are analytic and bounded for $k \in \mathbb{C}$ except for the essential singularities $k = 0$ and ∞ . Moreover, due to the symmetry (2.5), the functions Φ_j , $j = 1, 2, 3$, can also be written as

$$\Phi_j(x, k) = \begin{pmatrix} A_j(x, k) & -B_j(x, -k) \\ B_j(x, k) & A_j(x, -k) \end{pmatrix}, \quad j = 1, 3, \quad \Phi_2(y, k) = \begin{pmatrix} A_2(y, k) & -B_2(y, -k) \\ B_2(y, k) & A_2(y, -k) \end{pmatrix}.$$

In summary, we will define the integral representations for the spectral functions below.

Definition 3.1. Given $q(x, 0) = g_0(x)$ and $q_y(x, 0) = g_1(x)$, the map

$$\{g_0(x), g_1(x)\} \rightarrow \{a_1(k), b_1(k)\}$$

is defined by

$$\begin{aligned}a_1(k) &= 1 - \frac{1}{4} \int_0^\infty \left\{ \frac{i}{2k} (\cosh g_0(\xi) - 1) A_1(\xi, k) \right. \\ &\quad \left. - \left(i\dot{g}_0(\xi) + g_1(\xi) + \frac{1}{2k} \sinh g_0(\xi) \right) B_1(\xi, k) \right\} d\xi, \quad \operatorname{Im} k < 0, \quad (3.2a)\end{aligned}$$

$$\begin{aligned}b_1(k) &= -\frac{1}{4} \int_0^\infty e^{-2\omega_1(k)\xi} \left\{ \left(i\dot{g}_0(\xi) + g_1(\xi) - \frac{1}{2k} \sinh g_0(\xi) \right) A_1(\xi, k) \right. \\ &\quad \left. - \frac{i}{2k} (\cosh g_0(\xi) - 1) B_1(\xi, k) \right\} d\xi, \quad \operatorname{Im} k < 0, \quad (3.2b)\end{aligned}$$

where the functions A_1 and B_1 are the solutions of the x -part of the Lax pair (2.1a) with $y = 0$, i.e., A_1 and B_1 solve the following system of ordinary differential equations:

$$\begin{aligned}A_{1x} &= \frac{1}{4} \left[\frac{i}{2k} (\cosh g_0(x) - 1) A_1 - \left(i\dot{g}_0(x) + g_1(x) + \frac{1}{2k} \sinh g_0(x) \right) B_1 \right], \\ B_{1x} - 2\omega_1(k) B_1 &= \frac{1}{4} \left[\left(i\dot{g}_0(x) + g_1(x) - \frac{1}{2k} \sinh g_0(x) \right) A_1 - \frac{i}{2k} (\cosh g_0(x) - 1) B_1 \right]\end{aligned}$$

with $\lim_{x \rightarrow \infty} (A_1, B_1) = (1, 0)$.

Definition 3.2. Given $q(0, y) = f_0(y)$ and $q_x(0, y) = f_1(y)$, the map

$$\{f_0(y), f_1(y)\} \rightarrow \{a_2(k), b_2(k)\}$$

is defined by

$$a_2(k) = 1 + \frac{i}{4} \int_0^L \left\{ \frac{i}{2k} (\cosh f_0(\eta) - 1) A_2(\eta, k) + \left(i f_1(\eta) + \dot{f}_0(\eta) - \frac{1}{2k} \sinh f_0(\eta) \right) B_2(\eta, k) \right\} d\eta, \quad k \in \mathbb{C}, \quad (3.3a)$$

$$b_2(k) = -\frac{i}{4} \int_0^L e^{-2\omega_2(k)\eta} \left\{ \left(i f_1(\eta) + \dot{f}_0(\eta) + \frac{1}{2k} \sinh f_0(\eta) \right) A_2(\eta, k) + \frac{i}{2k} (\cosh f_0(\eta) - 1) B_2(\eta, k) \right\} d\eta, \quad k \in \mathbb{C}, \quad (3.3b)$$

where the functions A_2 and B_2 are the solutions of the y -part of the Lax pair (2.1b) with $x = 0$, i.e., A_2 and B_2 solve the following system of ordinary differential equations:

$$\begin{aligned} A_{2y} &= -\frac{i}{4} \left[\frac{i}{2k} (\cosh f_0(y) - 1) A_2 + \left(i f_0(y) + \dot{f}_1(y) - \frac{1}{2k} \sinh f_0(y) \right) B_2 \right], \\ B_{2y} - 2\omega_2(k) B_2 &= \frac{i}{4} \left[\left(i f_0(y) + \dot{f}_1(y) + \frac{1}{2k} \sinh f_0(y) \right) A_2 + \frac{i}{2k} (\cosh f_0(y) - 1) B_2 \right] \end{aligned}$$

with $\lim_{y \rightarrow \infty} (A_2, B_2) = (1, 0)$.

Definition 3.3. Given $q(x, L) = h_0(x)$ and $q_y(x, L) = h_1(x)$, the map

$$\{h_0(x), h_1(x)\} \rightarrow \{a_3(k), b_3(k)\}$$

is defined by

$$a_3(k) = 1 - \frac{1}{4} \int_0^\infty \left\{ \frac{i}{2k} (\cosh h_0(\xi) - 1) A_3(\xi, k) - \left(i \dot{h}_0(\xi) + h_1(\xi) + \frac{1}{2k} \sinh h_0(\xi) \right) B_3(\xi, k) \right\} d\xi, \quad \text{Im } k < 0, \quad (3.4a)$$

$$b_3(k) = -\frac{1}{4} \int_0^\infty e^{-2\omega_1(k)\xi} \left\{ \left(i \dot{h}_0(\xi) + h_1(\xi) - \frac{1}{2k} \sinh h_0(\xi) \right) A_3(\xi, k) - \frac{i}{2k} (\cosh h_0(\xi) - 1) B_3(\xi, k) \right\} d\xi, \quad \text{Im } k < 0, \quad (3.4b)$$

where the functions A_3 and B_3 are the solutions of the x -part of the Lax pair (2.1a) with $y = L$, i.e., A_3 and B_3 solve the following system of ordinary differential equations:

$$\begin{aligned} A_{3x} &= \frac{1}{4} \left[\frac{i}{2k} (\cosh h_0(x) - 1) A_3 - \left(i \dot{h}_0(x) + h_1(x) + \frac{1}{2k} \sinh h_0(x) \right) B_3 \right], \\ B_{3x} - 2\omega_1(k) B_3 &= \frac{1}{4} \left[\left(i \dot{h}_0(x) + h_1(x) - \frac{1}{2k} \sinh h_0(x) \right) A_3 - \frac{i}{2k} (\cosh h_0(x) - 1) B_3 \right] \end{aligned}$$

with $\lim_{x \rightarrow \infty} (A_3, B_3) = (1, 0)$.

In what follows, we derive the global relation that involves all boundary values. It should be noted that the global relation plays a crucial role in the implementation of the Fokas method for boundary value problems. Evaluating equations (3.1a) and (3.1b) at $x = 0$ and $y = L$, we find

$$\begin{aligned} S_3(k) &= \mu_2(0, L, k) e^{-\omega_2(k)L\hat{\sigma}_3} S_1(k), \\ I &= \mu_2(0, L, k) e^{-\omega_2(k)L\hat{\sigma}_3} S_2(k). \end{aligned}$$

We then obtain the following global relation in terms of the scattering matrices:

$$e^{\omega_2(k)L\hat{\sigma}_3} S_3(k) = S_2^{-1}(k) S_1(k). \quad (3.5)$$

In particular, the global relation (3.5) can be written as

$$a_3(k) = a_1(k)a_2(-k) + b_1(k)b_2(-k), \quad k \in \mathbb{C}^-, \quad (3.6a)$$

$$b_3(k)e^{-2\omega_2(k)L} = a_2(k)b_1(k) - a_1(k)b_2(k), \quad k \in \mathbb{C}^-. \quad (3.6b)$$

3.2 Spectral analysis at boundary values

In this section we characterize the boundary values in terms of the spectral functions. This can be done by the spectral analysis for the Lax pair at $x = 0$, $y = 0$ and $y = L$, respectively.

Proposition 3.4. *The inverse map*

$$\{a_1(k), b_1(k)\} \rightarrow \{q(x, 0), q_y(x, 0)\}$$

to the map defined in Definition 3.1 is given by

$$\begin{aligned} \cosh q(x, 0) &= 1 - 8i \lim_{k \rightarrow \infty} k M_{11x}^{(x)} - 8 \lim_{k \rightarrow \infty} (k M_{21}^{(x)})^2, \\ i q_x(x, 0) + q_y(x, 0) &= -4i \lim_{k \rightarrow \infty} k M_{21}^{(x)}, \end{aligned}$$

where $M^{(x)}$ is the solution of the matrix Riemann–Hilbert problem

$$M_-^{(x)}(x, k) = M_+^{(x)}(x, k) J^{(x)}(x, k), \quad k \in \mathbb{R} \quad (3.7)$$

with the jump matrix $J^{(x)}$ given by

$$J^{(x)}(x, k) = \begin{pmatrix} 1 & \frac{b_1(-k)}{a_1(k)} e^{-2\omega_1(k)x} \\ \frac{b_1(k)}{a_1(-k)} e^{2\omega_1(k)x} & \frac{1}{a_1(k)a_1(-k)} \end{pmatrix}, \quad k \in \mathbb{R}. \quad (3.8)$$

Proof. Evaluating equation (3.1a) at $y = 0$, we find

$$\mu_3(x, 0, k) = \mu_2(x, 0, k) e^{-\omega_1(k)x \hat{\sigma}_3} S_1(k), \quad k \in (\mathbb{R}^+, \mathbb{R}^-), \quad 0 \leq x < \infty.$$

Note that the eigenfunction $\mu_2^{(1)}(x, 0, k)$ is analytic and bounded for $k \in D_1 \cup D_2$ and $\mu_2^{(3)}(x, 0, k)$ is analytic and bounded for $k \in D_3 \cup D_4$. Thus, we formulate the matrix Riemann–Hilbert problem (3.7) with the jump matrix $J^{(x)}(x, k)$ given by (3.8), where the sectionally meromorphic functions $M_{\pm}^{(x)}$ are defined by

$$\begin{aligned} M_+^{(x)}(x, k) &= \left(\frac{\mu_2^{(1)}(x, 0, k)}{a_1(-k)}, \mu_3^{(12)}(x, 0, k) \right), \quad \operatorname{Im} k > 0, \\ M_-^{(x)}(x, k) &= \left(\mu_3^{(34)}(x, 0, k), \frac{\mu_2^{(3)}(x, 0, k)}{a_1(k)} \right), \quad \operatorname{Im} k < 0 \end{aligned}$$

with $\det M_{\pm}^{(x)} = 1$ and $M_{\pm}^{(x)} = I + O(\frac{1}{k})$ as $k \rightarrow \infty$. The remaining proof follows arguments similar to arguments in [14]. \square

Proposition 3.5. *The inverse map*

$$\{a_2(k), b_2(k)\} \rightarrow \{q(0, y), q_x(0, y)\}$$

to the map defined in Definition 3.2 is given by

$$\begin{aligned} \cosh q(0, y) &= 1 + 8 \lim_{k \rightarrow \infty} k M_{11y}^{(y)} + 8 \lim_{k \rightarrow \infty} (k M_{21}^{(y)})^2, \\ i q_x(0, y) + q_y(0, y) &= -4i \lim_{k \rightarrow \infty} k M_{21}^{(y)}, \end{aligned}$$

where $M^{(y)}$ is the solution of the matrix Riemann–Hilbert problem

$$M_-^{(y)}(y, k) = M_+^{(y)}(y, k)J^{(y)}(y, k), \quad k \in i\mathbb{R}$$

with the jump matrix $J^{(y)}$ given by

$$J^{(y)}(y, k) = \begin{pmatrix} 1 & \frac{b_2(-k)}{a_2(k)} e^{-2\omega_2(k)y} \\ \frac{b_2(k)}{a_2(-k)} e^{2\omega_2(k)y} & \frac{1}{a_2(k)a_2(-k)} \end{pmatrix}, \quad k \in i\mathbb{R}. \quad (3.10)$$

Proof. The proof is similar to the arguments discussed in Proposition 3.4. From the spectral relation (3.1b) with $x = 0$, we find the jump matrix $J^{(y)}(y, k)$ given in (3.10) and the sectionally meromorphic functions $M_{\pm}^{(y)}$ given by

$$\begin{aligned} M_+^{(y)}(y, k) &= \left(\frac{\mu_2^{(1)}(0, y, k)}{a_2(-k)}, \mu_1^{(4)}(0, y, k) \right), \quad \operatorname{Re} k > 0, \\ M_-^{(y)}(y, k) &= \left(\mu_1^{(2)}(0, y, k), \frac{\mu_2^{(3)}(0, y, k)}{a_2(k)} \right), \quad \operatorname{Re} k < 0 \end{aligned}$$

with $\det M_{\pm}^{(y)} = 1$ and $M_{\pm}^{(y)} = I + O(\frac{1}{k})$ as $k \rightarrow \infty$. Note that the eigenfunction $\mu_2^{(1)}(0, y, k)$ is analytic and bounded for $k \in D_1 \cup D_4$ and the function $\mu_2^{(3)}(0, y, k)$ is analytic and bounded for $k \in D_2 \cup D_3$. \square

The spectral analysis at the boundary $y = L$ is similar to Proposition 3.4.

Proposition 3.6. *The inverse map*

$$\{a_3(k), b_3(k)\} \rightarrow \{q(x, L), q_y(x, L)\}$$

to the map defined in Definition 3.3 is given by

$$\begin{aligned} \cosh q(x, L) &= 1 - 8i \lim_{k \rightarrow \infty} k M_{11x}^{(L)} - 8 \lim_{k \rightarrow \infty} (k M_{21}^{(L)})^2, \\ iq_x(x, L) + q_y(x, L) &= -4i \lim_{k \rightarrow \infty} k M_{21}^{(L)}, \end{aligned}$$

where $M^{(L)}$ is the solution of the matrix Riemann–Hilbert problem

$$M_-^{(L)}(x, k) = M_+^{(L)}(x, k)J^{(L)}(x, k), \quad k \in \mathbb{R} \quad (3.12)$$

with the jump matrix $J^{(L)}$ given by

$$J^{(L)}(x, k) = \begin{pmatrix} 1 & \frac{b_3(-k)}{a_3(k)} e^{-2\omega_1(k)x} \\ \frac{b_3(k)}{a_3(-k)} e^{2\omega_1(k)x} & \frac{1}{a_3(k)a_3(-k)} \end{pmatrix}, \quad k \in \mathbb{R}. \quad (3.13)$$

4 Riemann–Hilbert problem

In this section, we first formulate the matrix Riemann–Hilbert problem under the assumption that the solution $q(x, y)$ of (1.1) exists and then we discuss the existence of the solution $q(x, y)$ satisfying the boundary conditions. Using equations (3.1) and the global relation (3.6), after tedious but straightforward calculations, we formulate the following matrix Riemann–Hilbert problem:

$$M_-(x, y, k) = M_+(x, y, k)J(x, y, k), \quad k \in \mathcal{L}, \quad (4.1)$$

where the oriented contours $\mathcal{L} = L_1 \cup L_2 \cup L_3 \cup L_4$ are given by (cf. Figure 2)

$$L_1 = D_1 \cap D_2, \quad L_2 = D_2 \cap D_3, \quad (4.2a)$$

$$L_3 = D_3 \cap D_4, \quad L_4 = D_4 \cap D_1, \quad (4.2b)$$

and the jump matrices are defined by

$$J_1(x, y, k) = \begin{pmatrix} 1 & 0 \\ \frac{b_2(k)}{a_1(-k)a_3(-k)} e^{2\theta(x, y, k)} & 1 \end{pmatrix}, \quad k \in L_1, \quad (4.3a)$$

$$J_2(x, y, k) = \begin{pmatrix} \frac{a_2(k)}{a_1(k)a_3(-k)} & -\frac{b_1(-k)}{a_1(k)} e^{-2\theta(x, y, k)} \\ -\frac{b_3(k)}{a_3(-k)} e^{2(\theta(x, y, k) - \omega_2(k)L)} & 1 \end{pmatrix}, \quad k \in L_2, \quad (4.3b)$$

$$J_3(x, y, k) = \begin{pmatrix} 1 & -\frac{b_2(-k)}{a_1(k)a_3(k)} e^{-2\theta(x, y, k)} \\ 0 & 1 \end{pmatrix}, \quad k \in L_3, \quad (4.3c)$$

$$J_4(x, y, k) = J_1 J_2^{-1} J_3 = \begin{pmatrix} 1 & \frac{b_3(-k)}{a_3(k)} e^{-2(\theta(x, y, k) - \omega_2(k)L)} \\ \frac{b_1(k)}{a_1(-k)} e^{2\theta(x, y, k)} & \frac{a_2(-k)}{a_1(-k)a_3(k)} \end{pmatrix}, \quad k \in L_4 \quad (4.3d)$$

with $\theta(x, y, k) = \omega_1(k)x + \omega_2(k)y$. The matrix-valued functions M_{\pm} are sectionally meromorphic and defined as follows:

$$M_+(x, y, k) = \begin{cases} \left(\frac{\mu_2^{(1)}}{a_1(-k)}, \mu_3^{(12)} \right) & \text{for } k \in D_1, \\ \left(\mu_3^{(34)}, \frac{\mu_2^{(3)}}{a_1(k)} \right) & \text{for } k \in D_3, \end{cases}$$

$$M_-(x, y, k) = \begin{cases} \left(\frac{\mu_1^{(2)}}{a_3(-k)}, \mu_3^{(12)} \right) & \text{for } k \in D_2, \\ \left(\mu_3^{(34)}, \frac{\mu_1^{(4)}}{a_3(k)} \right) & \text{for } k \in D_4. \end{cases}$$

Note that $\det M_{\pm} = 1$ and $M_{\pm} = I + O(\frac{1}{k})$ as $k \rightarrow \infty$. Indeed, using the global relation (3.6), we find

$$a_2(k) - b_1(-k)b_3(k)e^{-2\omega_2(k)L} = a_1(k)a_3(-k),$$

which implies that $\det(J_2) = \det(J_4) = 1$.

The Riemann–Hilbert problem (4.1) can be solved by a Cauchy-type integral equation. Let $\tilde{J} = I - J$. Then equation (4.1) becomes

$$M_+(x, y, k) - M_-(x, y, k) = M_+(x, y, k)\tilde{J}(x, y, k).$$

Applying the Plemelj formula [9], the solution M of the Riemann–Hilbert problem (4.1) can be expressed as

$$M(x, y, k) = I + \frac{1}{2i\pi} \int_{\mathcal{L}} M_+(x, y, k') \tilde{J}(x, y, k') \frac{dk'}{k' - k}.$$

Expanding the integrand for k , we find

$$M(x, y, k) = I - \frac{1}{2i\pi k} \int_{\mathcal{L}} M_+(x, y, k') \tilde{J}(x, y, k') dk' + O\left(\frac{1}{k^2}\right), \quad k \rightarrow \infty.$$

Thus, we expand the solution M of the Riemann–Hilbert problem (4.1) as

$$M(x, y, k) = I + \frac{M^{(1)}(x, y)}{k} + \frac{M^{(2)}(x, y)}{k^2} + O\left(\frac{1}{k^3}\right), \quad k \rightarrow \infty. \quad (4.4)$$

If we substitute the expansion (4.4) into the x -part of the Lax pair (2.1a), the $(2, 1)$ -component at $O(1)$ yields

$$iq_x(x, y) + q_y(x, y) = -4iM_{21}^{(1)}(x, y) \quad (4.5)$$

and the $(1, 1)$ -component at $O(\frac{1}{k})$ implies

$$M_{11x}^{(1)}(x, y) = -\frac{1}{8i}(\cosh q(x, y) - 1) - \frac{1}{4}(iq_x(x, y) + q_y(x, y))M_{21}^{(1)}(x, y).$$

Simplifying the above equation with (4.5), we obtain the reconstruction formula for $q(x, y)$ in terms of the solution of the Riemann–Hilbert problem

$$\cosh q(x, y) = 1 - 8iM_{11x}^{(1)}(x, y) - 8(M_{21}^{(1)})^2.$$

Similarly, if we substitute the expansion (4.4) into the y -part of the Lax pair, the solution $q(x, y)$ can be written equivalently as

$$\cosh q(x, y) = 1 + 8M_{11y}^{(1)}(x, y) + 8(M_{21}^{(1)})^2.$$

Let us now state the existence theorem for the elliptic sinh-Gordon equation in the semi-strip.

Theorem 4.1. *Let the functions $g_j(x), h_j(x) \in H^1(\mathbb{R}^+)$ and $f_j(y) \in H^1([0, L])$, $j = 0, 1$, be given with the sufficiently small H^1 norms. Let the functions $a_j(k), b_j(k)$, $j = 1, 2, 3$, be given by (3.2), (3.3) and (3.4) in Definitions 3.1, 3.2 and 3.3, respectively. Suppose that the spectral functions a_j and b_j , $j = 1, 2, 3$, satisfy the global relation (3.6). Let $M(x, y, k)$ be the solution of the following matrix Riemann–Hilbert (RH) problem*

$$M_-(x, y, k) = M_+(x, y, k)J(x, y, k), \quad k \in \mathcal{L}, \quad (4.6)$$

where $\det(M_\pm) = 1$, $M_\pm = I + O(\frac{1}{k})$ as $k \rightarrow \infty$, the oriented contours \mathcal{L} are defined in (4.2) and the jump matrices J are given in (4.3).

Then the Riemann–Hilbert problem is uniquely solvable and the function $q(x, y)$ defined by

$$iq_x + q_y = -4i \lim_{k \rightarrow \infty} kM_{21}, \quad \cosh q(x, y) = 1 - 8i \lim_{k \rightarrow \infty} kM_{11x} - 8 \lim_{k \rightarrow \infty} (kM_{21})^2 \quad (4.7)$$

solves the elliptic sinh-Gordon equation (1.1) satisfying the boundary conditions

$$q(x, 0) = g_0(x), \quad q_y(x, 0) = g_1(x), \quad (4.8a)$$

$$q(0, y) = f_0(y), \quad q_x(0, y) = f_1(y), \quad (4.8b)$$

$$q(x, L) = h_0(x), \quad q_y(x, L) = h_1(x). \quad (4.8c)$$

Proof. Using the dressing method and the vanishing lemma presented in [7, 9], we can prove the unique solvability of the Riemann–Hilbert problem (4.6) and then we can also verify that the $q(x, y)$ defined in (4.7) solves the elliptic sinh-Gordon equation (1.1).

The proof that $q(x, y)$ given in (4.7) satisfies the boundary values is similar to the argument presented in [14]. Here, we only state the proof of (4.8c). We first define

$$M^{(L)}(x, k) = \begin{cases} M(x, L, k)J_1(x, L, k) & \text{for } k \in D_1, \\ M(x, L, k) & \text{for } k \in D_2, \\ M(x, L, k)F(x, k) & \text{for } k \in D_3, \\ M(x, L, k)J_3^{-1}(x, L, k)F(x, k) & \text{for } k \in D_4, \end{cases} \quad (4.9)$$

where

$$F(x, k) = \begin{pmatrix} 1 & -\frac{b_2(-k)}{a_1(k)a_3(k)}e^{-2(\omega_1(k)x + \omega_2(k)L)} \\ 0 & 1 \end{pmatrix}.$$

It is convenient to denote $M(x, L, k)$ and $M^{(L)}(x, k)$ for $k \in D_j$, $j = 1, \dots, 4$, by $M_j(x, L, k)$ and $M_j^{(L)}(x, k)$, respectively. Then, equations (4.6) and (4.9) can be written as

$$M_2(x, L, k) = M_1J_1(x, L, k), \quad M_2(x, L, k) = M_3J_2(x, L, k), \quad (4.10a)$$

$$M_4(x, L, k) = M_3J_3(x, L, k), \quad M_4(x, L, k) = M_1J_4(x, L, k), \quad (4.10b)$$

and

$$M_1^{(L)}(x, k) = M_1(x, L, k)J_1(x, L, k), \quad M_2^{(L)}(x, k) = M_2(x, L, k), \quad (4.11a)$$

$$M_3^{(L)}(x, k) = M_3(x, L, k)F(x, k), \quad M_4^{(L)}(x, k) = M_4(x, L, k)J_3^{-1}(x, L, k)F(x, k). \quad (4.11b)$$

Combining (4.11) and (4.10), we find the jump conditions as

$$\begin{aligned} M_2^{(L)}(x, k) &= M_1^{(L)}(x, k), & M_3^{(L)}(x, k) &= M_2^{(L)}(x, k)J_2^{-1}(x, L, k)F(x, k), \\ M_4^{(L)}(x, k) &= M_3^{(L)}(x, k), & M_4^{(L)}(x, k) &= M_1^{(L)}(x, k)J_2^{-1}(x, L, k)F(x, k). \end{aligned}$$

Note that no jumps occur along the contours L_1 and L_3 and that $J_2^{-1}(x, L, k)F(x, k) = J^{(L)}(x, k)$, where $J^{(L)}(x, k)$ is given in (3.13). Thus, we define

$$\begin{aligned} M^{(L)}(x, k) &= M_+^{(L)}(x, k), & k \in D_1 \cup D_2, \\ M^{(L)}(x, k) &= M_-^{(L)}(x, k), & k \in D_3 \cup D_4 \end{aligned}$$

and then equations (4.10) are equivalent to the Riemann–Hilbert problem (3.12) with the jump matrix $J^{(L)}(x, k)$. Thus, evaluating (4.7) at $y = L$, we prove that the function $q(x, y)$ defined in (4.7) satisfies the boundary values (4.8c).

In a similar way, equations (4.8a) and (4.8b) can be proved. \square

5 Concluding remarks

In conclusion, we have studied the boundary value problem for the elliptic sinh-Gordon equation formulated in the semi-strip by the Fokas method, a generalization of the inverse scattering transform for boundary value problems. In particular, we have characterized the spectral functions in terms of the boundary values and we have derived the global algebraic relation that involves the boundary values. Furthermore, we have shown that the solution of the elliptic sinh-Gordon equation posed in the semi-strip exists provided that the spectral function defined by the boundary values satisfy the global relation. This solution can be expressed in terms of the unique solution of the matrix Riemann–Hilbert problem whose jump matrices are uniquely defined by the spectral functions. These spectral functions denoted by $\{a_j(k), b_j(k)\}$, $j = 1, 2, 3$, can be determined by the boundary values $\{q(x, 0), q_y(x, 0)\}$, $\{q(0, y), q_x(0, y)\}$ and $\{q(x, L), q_y(x, L)\}$. However, for a well-posed boundary value problem, it is required that a subset of the boundary values should be prescribed. In this respect, it is necessary to characterize unknown boundary values, called the Dirichlet-to-Neumann map [8]. This characterization can be done by analyzing the global relation as discussed in [10–12] and we will address this issue in the near future.

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