Research Article

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Large solutions to non-divergence structure semilinear elliptic equations with inhomogeneous term

https://doi.org/10.1515/anona-2017-0065
Received March 17, 2017; revised April 5, 2017; accepted April 6, 2017

Abstract: Motivated by the work [9], in this paper we investigate the infinite boundary value problem associated with the semilinear PDE $Lu = f(u) + h(x)$ on bounded smooth domains $\Omega \subseteq \mathbb{R}^n$, where L is a non-divergence structure uniformly elliptic operator with singular lower-order terms. In the equation, f is a continuous non-decreasing function that satisfies the Keller–Osserman condition, while h is a continuous function in Ω that may change sign, and which may be unbounded on Ω. Our purpose is two-fold. First we study some sufficient conditions on f and h that would ensure existence of boundary blow-up solutions of the above equation, in which we allow the lower-order coefficients to be singular on the boundary. The second objective is to provide sufficient conditions on f and h for the uniqueness of boundary blow-up solutions. However, to obtain uniqueness, we need the lower-order coefficients of L to be bounded in Ω, but we still allow h to be unbounded on Ω.

Keywords: Large solutions, existence and uniqueness, semilinear elliptic equation

MSC 2010: 35J60, 35J70

1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with C^2 boundary. We consider a uniformly elliptic nondivergence structure second order differential operator, namely,

$$Lu = a_{ij}(x)u_{x_i x_j} + b_i(x)u_{x_i} + c(x)u. \tag{1.1}$$

In (1.1) and throughout this paper, the summation convention over repeated indices from 1 to n is in effect. We will assume that $[a_{ij}(x)]$ is an $n \times n$ symmetric matrix of continuous real-valued functions on Ω such that

$$|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \text{for all } (x, \xi) \in \Omega \times \mathbb{R}^N, \tag{1.2}$$

for some constant $\Lambda \geq 1$.

In this paper we wish to study the question of existence and uniqueness of solutions to the following problem:

$$\begin{cases} Lu = f(u) + h(x), & x \in \Omega, \\ u = \infty, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

Here $h\colon \Omega \to \mathbb{R}$ is a continuous function which may be unbounded on Ω, and $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function with appropriate conditions to be specified later.

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To study the existence of solutions to (1.3) we assume the following conditions. The coefficients $b_i(x)$ $(1 \le i \le n)$ and $c(x)$ in (1.1) are locally bounded Borel functions such that:
(DB) $d(x) \sum |b_i(x)| = o(1)$ as $d(x) \to 0$,
(DC) $d^2(x)|c(x)| \le \eta(d(x))$ for all $x \in \Omega$, with $d(x) \le \delta_0$ and $c(x) \le 0$ for all $x \in \Omega$.
In (DB) and (DC) we have used $d(x)$ to denote the distance of $x \in \Omega$ from the boundary $\partial\Omega$, a notation we will continue to use throughout the paper. In (DC), $\delta_0 > 0$ is a positive constant and $\eta\colon (0, \delta_0] \to (0, \eta(\delta_0)]$ is an increasing function that satisfies the Dini condition

$$\int_0^{\delta_0} \frac{\eta(s)}{s}\, ds < \infty. \tag{1.4}$$

Following the pioneering works of Keller [15] and Osserman [19], problem (1.3) has been studied extensively by numerous authors when L is the Laplacian and $h \equiv 0$. The reader is referred to the monograph [11] and the references therein for more discussion related to such problems. In [3], Bandle and Marcus investigated existence and asymptotic boundary behavior of solutions to (1.3) when $h \equiv 0 \equiv c$ and $b_i \in C^\alpha(\overline{\Omega})$, $i = 1, \dots, n$, for some $0 < \alpha < 1$. Concerning problem (1.3) with $h(x) \not\equiv 0$ on Ω, Véron in [24], and Díaz and Letelier in [6], established the existence and uniqueness of positive solutions to (1.3) for the nonlinearity $f(t) = |t|^{p-1}t$, $p > 1$, and for a non-positive unbounded inhomogeneous term $h \in C(\Omega)$ with appropriate growth condition on the boundary.

As far as we are aware, it was García-Melián who first studied problem (1.3) for a sign-changing and unbounded inhomogeneous term h in the recent paper [9]. He studied the existence of a solution to problem (1.3) when L is the Laplacian, $f(t) = |t|^{p-1}t$ for $p > 1$ and h belonged to a large class of unbounded and sign-changing functions on Ω. He also obtained uniqueness of positive solutions of (1.3) for $h \in C(\Omega)$ with appropriate growth condition on the boundary, but bounded on Ω from above.

In this paper we wish to continue the aforementioned investigations with the objective of extending the results in several fronts. In all cases the class of inhomogeneous terms h we consider will include sign-changing and unbounded functions in Ω having appropriate growth conditions near the boundary. The necessity of some restriction on the growth of h near the boundary has already been noted in [9, Theorem 3]. As our first main result we will show the existence of solutions to (1.3), where the lower-order coefficients are allowed to be unbounded in Ω, and f comes from a wide class of nonlinearities. In addition, to the usual Keller–Osserman, we will require some mild conditions on f. As it turns out, if the inhomogeneous term h grows no faster than a suitable multiple of $f(\phi(d(x)))$ near the boundary of Ω, then problem (1.3) admits a solution. Here ϕ is a decreasing function on $(0, a)$, for some $a > 0$, that is associated with the nonlinearity f. For instance, $\phi(t) = t^{-2/(p-1)}$ when $f(t) = t^p$, $p > 1$, for $t > 0$. Our second main result concerns the asymptotic boundary estimates for solutions of (1.3) when the coefficients of L are bounded in Ω and, as a consequence, the uniqueness of solutions is obtained. These results are shown to hold for a large class of inhomogeneous terms h, which may change sign and be unbounded on Ω. In this regard, the asymptotic estimate and uniqueness results of this paper are new even when L is the Laplacian and $f(t) = |t|^{p-1}t$ for $p > 1$, as we do not require h to be bounded from above.

We point out that problem (1.3) was also considered in [10, 25]. We direct the reader to [9] for a discussion of problem (1.3) in these papers. The reader is referred to the recent papers [2, 4, 5, 8, 21, 26], and references therein, on asymptotic behavior and uniqueness of singular solutions related to the content of this paper. In particular, we draw attention to the paper [5] in which the authors make a systematic use of Karamata variation theory to study uniqueness of boundary blow-up solutions. To the best of our knowledge, [8] is the first paper to investigate existence of boundary blow-up solutions of equations with nonmonotonic nonlinearity.

The paper is organized as follows. In Section 2, we state the main conditions used on the nonlinearity f to study problem (1.3). In particular, we recall a lemma that will be useful in establishing the existence of solutions in the case when L has singular lower-order term coefficients. Section 3 is devoted to the study of existence of a solution to (1.3). In Section 4, we establish boundary asymptotic estimates of solutions to problem (1.3). Existence of positive solutions and uniqueness of solutions is investigated in Section 5. Finally, we have included an appendix where we prove some technical results that are used in the paper.

2 Preliminaries

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with C^2 boundary. Throughout the paper, it will be convenient to use the following notations for a given $\delta > 0$:

$$\Omega_\delta := \{x \in \Omega : d(x) < \delta\} \quad \text{and} \quad \Omega^\delta := \{x \in \Omega : d(x) > \delta\}.$$

Since Ω is a bounded C^2 domain, we note that there exists $\mu > 0$ such that $d \in C^2(\overline{\Omega}_\mu)$ and $|\nabla d(x)| = 1$ on Ω_μ. See [12, Lemma 14.16] for a proof.

By modifying the distance function d appropriately we can suppose that d is a positive C^2 function on Ω. For instance, one can use $(1-\psi)d + \psi$ instead of d, where $\psi \in C_c^2(\Omega)$ is a cut-off function with $0 \le \psi \le 1$ on Ω, $\psi \equiv 0$ on Ω_{μ_0} for some $0 < \mu_0 < \mu$, and $\psi \equiv 1$ on Ω^μ. Therefore, hereafter, we will always suppose that d is this modified distance function and that d is in $C^2(\overline{\Omega})$ with $|\nabla d| \equiv 1$ on Ω_μ.

By a solution u of $Lu = (\ge, \le)\, H(x,u)$ we mean a twice weakly differentiable function on Ω such that

$$Lu(x) = (\ge, \le)\, H(x, u(x)) \quad \text{for almost every } x \in \Omega.$$

We start with the following extension of the classical maximum principle. We assume that $t \mapsto H(x,t)$ is non-decreasing on $\mathbb{R}$ for each $x \in \Omega$.

Lemma 2.1 (Comparison Principle). *Let $u, w \in W^{2,n}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$, and assume that $Lu \ge H(x,u)$ in Ω and $Lw \le H(x,w)$ in Ω. If $u \le w$ on $\partial\Omega$, then $u \le w$ in Ω.*

Proof. Given $\varepsilon > 0$,

$$L(w+\varepsilon) = Lw + \varepsilon c \le Lw \le H(x,w) \le H(x, w+\varepsilon) \quad \text{in } \Omega \qquad \text{and} \qquad u - (w+\varepsilon) < 0 \quad \text{on } \partial\Omega.$$

Let $\delta > 0$ sufficiently small such that $u - (w+\varepsilon) \le 0$ on $\partial\Omega^\delta$. Suppose that the open set

$$\Omega_0^\delta := \{x \in \Omega^\delta : u(x) > w(x) + \varepsilon\}$$

is non-empty. Since $L(u - (w+\varepsilon)) \ge H(x,u) - H(x, w+\varepsilon) \ge 0$ in Ω_0^δ and the coefficients of L are bounded on Ω^δ, the maximum principle applies (see [12, Theorem 9.1]) and we conclude that $u - (w+\varepsilon) \le 0$ on Ω_0^δ. This is an obvious contradiction to the assumption that Ω_0^δ is non-empty. Therefore, we must have $u \le w + \varepsilon$ in Ω^δ. Since $\delta > 0$ is arbitrary, we conclude that $u \le w + \varepsilon$ in Ω. Since $\varepsilon > 0$ is arbitrary, we find that $u \le w$ on Ω, as desired. □

We consider the following conditions on the nonlinearity f in (1.3):

(f1) $f : \mathbb{R} \to \mathbb{R}$ is a non-decreasing continuous function such that $f(0) = 0$ with $f(t) > 0$ for $t > 0$.

(f2) f satisfies the Keller–Osserman condition, namely,

$$\int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty, \quad \text{where } F(t) = \int_0^t f(s)\,ds.$$

(f3) We have

$$\liminf_{t\to\infty} \frac{F(t)}{tf(t)} > 0.$$

We make a few remarks about the above conditions.

Remark 2.2. (i) Any regularly varying function at infinity with index $1 < q < \infty$ satisfies conditions (f2) and (f3). We recall that f is said to be regularly varying at infinity of index $q \in \mathbb{R}$ if f is a measurable function defined on (a, ∞) for some $a > 0$ and

$$\lim_{t\to\infty} \frac{f(\xi t)}{f(t)} = \xi^q \quad \text{for all } \xi > 0.$$

(ii) $f(t) = t$ satisfies (f3) but not (f2), while $f(t) = e^t - 1$ satisfies (f2) but not (f3).

(iii) If f satisfies (f1), then it is clear that $F(t) \le tf(t)$ for all $t \ge 0$. Moreover, if f satisfies both (f1) and (f2), then

$$\liminf_{t\to\infty} \frac{F(t)}{tf(t)} \le \frac{1}{2}.$$

We refer to Lemma A.1 in Appendix A for a proof of this assertion.

The reader may find more on regularly varying functions and some basic information on Karamata regular variation theory in [14, 23].

It is a well-known fact, see [13, 22], that if f satisfies (f1) and the Keller–Osserman condition (f2), then

$$\lim_{t\to\infty} \frac{\sqrt{F(t)}}{f(t)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{t}{f(t)} = 0. \tag{2.1}$$

In fact, the following result holds for any f that satisfies (f1), (f2) and (f3).

Lemma 2.3. *Suppose that f satisfies* (f1), (f2) *and* (f3). *Then the following hold:*

(i) $\limsup_{t\to\infty} \frac{\sqrt{F(t)}}{f(t)\int_t^\infty F(s)^{-1/2}\,ds} < \infty$,

(ii) $\limsup_{t\to\infty} \frac{t}{f(t)(\int_t^\infty F(s)^{-1/2}\,ds)^2} < \infty$.

We refer the reader to [18] for a proof of the above lemma. We let ϕ to be the non-increasing function such that

$$\int_{\phi(t)}^{\infty} \frac{ds}{\sqrt{2F(s)}} = t, \qquad t > 0.$$

It follows that

$$\phi'(t) = -\sqrt{2F(\phi(t))} \quad \text{and} \quad \phi''(t) = f(\phi(t)), \qquad t > 0.$$

For later use, let us compute Lv, where $v(x) := \phi(\varrho(x))$ for some $\varrho \in C^2(\Omega)$:

$$\begin{aligned} Lv &= \phi''(\varrho)a_{ij}\varrho_{x_i}\varrho_{x_j} + \phi'(\varrho)(a_{ij}\varrho_{x_ix_j} + b_i\varrho_{xi}) + c\phi(\varrho) \\ &= f(\phi(\varrho))a_{ij}\varrho_{x_i}\varrho_{x_j} - \sqrt{2F(\phi(\varrho))}(a_{ij}\varrho_{x_ix_j} + b_i\varrho_{xi}) + c\phi(\varrho) \\ &= f(\phi(\varrho))\Big(a_{ij}\varrho_{x_i}\varrho_{x_j} - \frac{\sqrt{2F(\phi(\varrho))}}{f(\phi(\varrho))}(a_{ij}\varrho_{x_ix_j} + b_i\varrho_{xi}) + \frac{\phi(\varrho)c}{f(\phi(\varrho))}\Big). \end{aligned} \tag{2.2}$$

3 On existence of solutions to problem (1.3)

Throughout this section, we assume that the lower order coefficients $b_i(x)$ $(1 \le i \le n)$ and $c(x)$ of L satisfy conditions (DB) and (DC), respectively.

The next result, a consequence of these conditions and Lemma 2.3, will prove useful in establishing the existence of solutions to problem (1.3).

Corollary 3.1. *Suppose that f satisfies conditions* (f1), (f2) *and* (f3). *Then*

$$\lim_{d(x)\to 0} |c(x)|\frac{\phi(d(x))}{f(\phi(d(x)))} = 0 \quad \textit{and} \quad \lim_{d(x)\to 0} \frac{\sqrt{F(\phi(d(x)))}}{f(\phi(d(x)))}\sum_i |b_i(x)| = 0.$$

Proof. We prove the first limit and omit the second as the proof is similar. We have

$$\begin{aligned} \frac{|c(x)|\phi(d(x))}{f(\phi(d(x)))} &= \frac{d^2(x)|c(x)|\phi(d(x))}{d^2(x)f(\phi(d(x)))} \\ &= \frac{|c(x)|d^2(x)\phi(d(x))}{f(\phi(d(x)))\big(\int_{\phi(d(x))}^\infty F(s)^{-1/2}\,ds\big)^2} \\ &\le \frac{\eta(d(x))\phi(d(x))}{f(\phi(d(x)))\big(\int_{\phi(d(x))}^\infty F(s)^{-1/2}\,ds\big)^2}. \end{aligned}$$

On noting that $\eta(0+) = 0$, the claim follows from Lemma 2.3 (ii). □

Remark 3.2. It is clear that (DB) and (DC) hold when the coefficients $\mathbf{b}$ and c of L are bounded on Ω. Therefore, Corollary 3.1 holds when the coefficients of L are bounded. In fact, in this case, condition (f3) is not needed. One only needs to recall (2.1).

Let $\Xi_* := \inf_{\xi>0} g_*(\xi)$ and $\Xi^* := \sup_{\xi>0} g^*(\xi)$, where

$$g^*(\xi) := \xi - \limsup_{t\to\infty} \frac{f(\xi t)}{f(t)} \quad \text{and} \quad g_*(\xi) := \Lambda\xi - \liminf_{t\to\infty} \frac{f(\xi t)}{f(t)}.$$

Here, Λ is the ellipticity constant of L as noted in condition (1.2).

As an example, we observe that for any regularly varying function (at infinity) f of index $1 < q < \infty$, it can be easily seen that

$$\Xi_* = -\infty \quad \text{and} \quad \Xi^* = (q-1)\Big(\frac{1}{q}\Big)^{q/(q-1)}. \tag{3.1}$$

On the other hand, for $\varpi > 2$, we note that $f(t) = t\log^{\varpi}(|t|+1)$ is regularly varying at infinity of index $q = 1$ and satisfies (f1), (f2) and (f3). Computation shows that $\Xi_* = 0 = \Xi^*$.

Remark 3.3. On noting that $g^*(1) = 0$, we see that $\Xi^* \geq 0$. Moreover, we also have $\Xi_* \leq 0$ for any f that satisfies (f1) and (f3). We refer the reader to Lemma A.3 in Appendix A.

We need some conditions on f and h in order to prove the existence of a solution to (1.3). We require the following assumption on f:

(f4) $\lim_{t\to-\infty} f(t) = -\infty$.

We should note that if $f\colon [0,\infty) \to [0,\infty)$ satisfies (f1) and (f2), then the odd extension of f to $\mathbb{R}$ satisfies (f4).

In addition to condition (f4), we will also require some growth restrictions, near the boundary $\partial\Omega$, on the inhomogeneous term h in (1.3). We state one of these conditions on h as follows:

(h1) $\Theta^*(h) := \limsup_{d(x)\to 0} \frac{h(x)}{f(\phi(d(x)))} < \Xi^*$.

The main result of this section gives the existence of a solution to problem (1.3). We employ the sub-solution and super-solution technique to establish the result. In preparation for this, let us consider a function $H\colon \Omega\times\mathbb{R} \to \mathbb{R}$ which is non-decreasing in $\mathbb{R}$ in the second variable for each $x \in \Omega$, and $H(\cdot, t) \in C(\overline{\Omega})$ for each $t \in \mathbb{R}$.

From [18] we recall the following result on the solvability of a class of Dirichlet problems with continuous boundary data.

Lemma 3.4. *Given $g \in C(\partial\Omega)$, the following Dirichlet problem admits a solution $u \in W^{2,p}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ for each $1 \leq p < \infty$:*

$$\begin{cases} Lu = H(x,u) & \text{in } \Omega,\\ u = g & \text{on } \partial\Omega.\end{cases} \tag{3.2}$$

Remark 3.5. Suppose that all coefficients of L are bounded and belong to $C^\alpha(\Omega)$ for some $0 < \alpha < 1$. If, in addition to the hypotheses on H, we suppose $H \in C^\alpha(\Omega\times\mathbb{R})$, then problem (3.2) admits a solution $u \in C^2(\Omega)\cap C(\overline{\Omega})$. This is a consequence of the elliptic regularity theory, see [12, Theorem 9.19] with $k = 1$.

We will use Lemma 3.4 to study the following infinite boundary value problem.

$$\begin{cases} Lu = H(x,u) & \text{in } \Omega,\\ u = \infty & \text{on } \partial\Omega.\end{cases} \tag{3.3}$$

As a simple consequence of Lemma 3.4 we obtain the following.

Lemma 3.6. *Let $u_*, u^* \in W^{2,p}_{\mathrm{loc}}(\Omega)\cap C(\Omega)$ for $1 < p < \infty$ be such that*

$$Lu_* \geq H(x,u_*) \quad \text{in } \Omega, \qquad Lu^* \leq H(x,u^*) \quad \text{in } \Omega \qquad \text{and} \qquad u^* = \infty \quad \text{on } \partial\Omega.$$

Assume that $u_* \in L^\infty(\Omega)$ or $u_* = \infty$ on $\partial\Omega$. If $u_* \leq u^*$ in Ω, then there exists a solution $u \in W^{2,p}_{\mathrm{loc}}(\Omega)\cap C(\Omega)$ of (3.3) with $u_* \leq u \leq u^*$ a.e. in Ω.

Proof. Let us first consider the case when $u_* \in L^\infty(\Omega)$. For each positive integer $j \geq \sup_\Omega u_*$, by Lemma 3.4, we let $u_j \in W^{2,p}_{\mathrm{loc}} \cap C(\overline{\Omega})$ be a solution of

$$\begin{cases} Lu_j = H(x, u_j) & \text{in } \Omega, \\ u = j & \text{on } \partial\Omega. \end{cases}$$

By the Comparison Principle, Lemma 2.1, we have

$$u_* \leq u_j \leq u_{j+1} \leq u^* \quad \text{in } \Omega \text{ for all } j.$$

By proceeding as in the proof of [18, Lemma 3.2], we conclude that $\{u_j\}$ converges locally uniformly to a solution $u \in C^2(\Omega)$ of problem (3.3) on Ω with $u_* \leq u \leq u^*$.

Now assume that $u_* = \infty$ on $\partial\Omega$. Let $\mathcal{O}_j := \{x \in \Omega : d(x) > 1/j\}$ for $j = 1, 2, \dots$, and let $u_j \in W^{2,p}_{\mathrm{loc}}(\mathcal{O}_j) \cap C(\overline{\mathcal{O}}_j)$ be a solution of

$$\begin{cases} Lu_j = H(x, u_j) & \text{in } \mathcal{O}_j, \\ u_j = u_* & \text{on } \partial\mathcal{O}_j. \end{cases}$$

By the Comparison Principle, we see that $u_* \leq u_j \leq u^*$ on $\overline{\mathcal{O}}_j$ for each $j \geq 1$. Consequently,[1] we also note that $u_j \leq u_{j+1}$ in $\mathcal{O}_j$. Thus, we have

$$u_* \leq u_j \leq u_{j+1} \leq u^* \quad \text{on } \mathcal{O}_j.$$

Again, by proceeding as in the proof of [18, Lemma 3.2], we conclude that $\{u_j\}$ converges locally uniformly to a solution $u \in W^{2,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ of problem (3.3) with $u_* \leq u \leq u^*$ on Ω. □

For the rest of the paper, we will assume that f satisfies both conditions (f1) and (f2).

We are now ready to state the following theorem on the existence of a solution to (1.3).

Theorem 3.7. *Suppose that f satisfies* (f3) *and* (f4). *We also assume that $h \in C(\Omega)$ satisfies* (h1). *Then problem* (1.3) *has a solution $u \in W^{2,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ for $1 \leq p < \infty$. Moreover, u is the maximal solution.*

Proof. There is no loss of generality in supposing that $p \geq n$. Let $\{\mathcal{O}_j\}$ be the sequence of open subsets of Ω introduced in the proof of Lemma 3.6 above. Let us first show that the following has a solution:

$$\begin{cases} Lw = f(w) + h & \text{in } \mathcal{O}_j, \\ w = \infty & \text{on } \partial\mathcal{O}_j. \end{cases} \tag{3.4}$$

To this end, let v_j be a solution of $Lv_j = f(v_j)$ in $\mathcal{O}_j$ and $v_j = \infty$ on $\partial\mathcal{O}_j$. If $h \geq 0$ in $\mathcal{O}_j$, then $w_j := v_j$ is a super-solution of (3.4). Otherwise, let z_j be a solution of $Lz_j = \min_{\overline{\mathcal{O}}_j} h$ in $\mathcal{O}_j$ with $z_j = 0$ on $\partial\mathcal{O}_j$. Note that $z_j > 0$. Then $w_j := v_j + z_j$ satisfies

$$Lw_j = L(v_j + z_j) = f(v_j) + \min_{\overline{\mathcal{O}}_j} h \leq f(v_j + z_j) + h(x) = f(w_j) + h(x), \quad x \in \mathcal{O}_j,$$

and

$$w_j = \infty \quad \text{on } \partial\mathcal{O}_j.$$

Thus, in any case w_j is a super-solution of (3.4). We now proceed to construct a sub-solution of (3.4). To this end, we claim that there are positive constants A and α such that

$$w = A\phi(d(x)) - \alpha$$

is a sub-solution of (1.3) in Ω. To see this, let $\varrho := d$ in (2.2), and we estimate (2.2) in Ω_μ as follows:

$$\begin{aligned} Lw &\geq f(\phi(d))\Big[A - A\frac{\sqrt{2F(\phi(d))}}{f(\phi(d))}\Big(|L_0 d| + \sum_i |b_i(x)|\Big) - \frac{A|c(x)|\phi(d)}{f(\phi(d))}\Big] - \alpha c \\ &\geq f(\phi(d))\Big[A - A\frac{\sqrt{2F(\phi(d))}}{f(\phi(d))}\Big(|L_0 d| + \sum_i |b_i(x)|\Big) - \frac{A|c(x)|\phi(d)}{f(\phi(d))}\Big]. \end{aligned} \tag{3.5}$$

1 Note that $u_{j+1} \geq u_*$ on $\mathcal{O}_{j+1}$ and hence $u_j = u_* \leq u_{j+1}$ on $\partial\mathcal{O}_j$. Therefore, the Comparison Principle applies.

In the last inequality, we have used the fact that $-\alpha c \geq 0$ for any $\alpha > 0$. By (h1), since $\Xi^* > \Theta^*$, we let A such that $g^*(A) > \Theta^*$. Let us choose $\varepsilon > 0$ sufficiently small so that

$$\Theta^* + 2\varepsilon < g^*(A) = A - \limsup_{d(x)\to 0} \frac{f(A\phi(d(x)))}{f(\phi(d(x)))}.$$

Therefore, there exists $0 < \delta < \mu$ such that for $x \in \Omega_\delta$, we have

$$A - \varepsilon > \Theta^* + \frac{\varepsilon}{2} + \frac{f(A\phi(d(x)))}{f(\phi(d(x)))}.$$

Recalling (2.1) and Corollary 3.1, we can take $\delta > 0$ sufficiently small so that for all $x \in \Omega_\delta$,

$$A\frac{\sqrt{2F(\phi(d))}}{f(\phi(d))}\Big(|L_0 d| + \sum_i |b_i(x)|\Big) + \frac{A|c(x)|\phi(d)}{f(\phi(d))} < \varepsilon. \tag{3.6}$$

Likewise, by the definition of $\Theta^* := \Theta^*(h)$, we have (by shrinking δ further, if necessary)

$$\frac{h(x)}{f(\phi(d(x)))} < \Theta^* + \frac{\varepsilon}{2} \quad \text{for all } x \in \Omega_\delta. \tag{3.7}$$

Putting (3.5), (3.6) and (3.7) together, we find that $w(x) = A\phi(d(x)) - \alpha$ satisfies the following for all $x \in \Omega_\delta$:

$$\begin{aligned} Lw &\geq f(\phi(d(x)))(A - \varepsilon) \\ &\geq f(\phi(d(x)))\Big[\frac{h(x)}{f(\phi(d(x)))} + \frac{f(A\phi(d(x)))}{f(\phi(d(x)))}\Big] \\ &= h(x) + f(A\phi(d(x))) \\ &\geq f(w) + h(x) \quad \text{for any } \alpha > 0. \end{aligned}$$

Next we choose α so that w is a sub-solution of $Lw = f(w) + h(x)$ on Ω^δ. For this, let

$$\eta := \min\{AL\phi(d) : x \in \Omega^\delta\}.$$

We should point out that η is independent of α. The hypothesis on f shows that

$$f(A\phi(\delta) - \alpha) + \max\{h(x) : x \in \Omega^\delta\} \to -\infty \quad \text{as } \alpha \to \infty.$$

Thus, we can choose $\alpha > 0$ sufficiently large so that

$$f(A\phi(\delta) - \alpha) + \max\{h(x) : x \in \Omega^\delta\} \leq \eta.$$

Having fixed such α, we see that in Ω^δ the following holds:

$$\begin{aligned} Lw &= AL\phi(d(x)) - \alpha c \\ &\geq \eta \\ &\geq f(A\phi(\delta) - \alpha) + \max\{h(x) : x \in \Omega^\delta\} \\ &\geq f(A\phi(d(x)) - \alpha) + h(x) \\ &= f(w) + h(x). \end{aligned}$$

Thus, we have shown that w is a sub-solution of (1.3) in Ω. Moreover, for each positive integer j, the Comparison Principle shows that $w \leq w_j$ in $\mathcal{O}_j$. Therefore, by Lemma 3.6, problem (3.4) has a solution u_j such that $w \leq u_j \leq w_j$ in $\mathcal{O}_j$. Since u_j is bounded on $\mathcal{O}_{j-1}$, the Comparison Principle shows that $u_j \leq u_{j-1}$ on $\mathcal{O}_{j-1}$, that is, $\{u_k\}_{k=j-1}^{\infty}$ is a decreasing sequence in $\mathcal{O}_{j-1}$. Let us set

$$u(x) := \lim_{j\to\infty} u_j(x), \quad x \in \Omega.$$

Since $\{u_j\}$ is locally uniformly bounded, by standard Schauder estimates, we see that $u \in C^2(\Omega)$ satisfies $Lu = f(u) + h$ in Ω. From this and the inequality $w \leq u$ in Ω, we see that u is a solution of (1.3), as desired.

If v is any solution of (1.3), then Comparison Principle shows that $v \leq u_j$ on $\mathcal{O}_j$ for all $j \geq 1$. Consequently, we have $v \leq u$ in Ω, and therefore u is a maximal solution of (1.3), as claimed. □

Remark 3.8. If the coefficients of L are bounded in Ω, we note that condition (f3) is not needed in the above theorem, see Remark 3.2.

4 On the boundary asymptotic estimates of solutions to problem (1.3)

In this section we will assume that the coefficients $b_i(x)$ $(1 \le i \le n)$ and $c(x)$ are bounded on Ω. It should be recalled that we always assume conditions (f1) and (f2) on f.

To discuss asymptotic boundary estimates of solutions to (1.3), we need an additional condition on h which we now make explicit:

(h2) $\Theta_*(h) := \liminf_{d(x)\to 0} \frac{h(x)}{f(\phi(d(x)))} > \Xi_*$.

Remark 4.1. If h is bounded from above on Ω, then $\Theta_* \le \Theta^* \le 0$.

We have the following lemma on asymptotic boundary estimates of solutions to (1.3).

Lemma 4.2. *Let $h \in C(\Omega)$.*

(i) *If h satisfies* (h2), *then there exists a positive constant A^* such that for any solution $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$ of* (1.3), *we have*

$$\limsup_{d(x)\to 0} \frac{u(x)}{\phi(d(x))} \le A^*. \tag{4.1}$$

(ii) *If h satisfies* (h1) *and is bounded from above, then there exists a positive constant A_* such that for any solution $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$ of* (1.3), *we have*

$$A_* \le \liminf_{d(x)\to 0} \frac{u(x)}{\phi(d(x))}. \tag{4.2}$$

Proof. For any $0 < \rho < \mu$, let us consider the sets

$$\Omega_\rho^- := \{x \in \Omega : \rho < d(x) < \mu\} \quad \text{and} \quad \Omega_\rho^+ := \{x \in \Omega : 0 < d(x) < \mu - \rho\}.$$

Proof of (4.1). For an appropriate choice of a positive constant A^*, we will show that

$$w^*(x) := A^*\phi(d(x) - \rho), \quad x \in \Omega_\rho^-,$$

is a super-solution of (1.3) on Ω_ρ^- for all $0 < \rho < \mu$ and sufficiently small μ. To this end, we estimate (2.2) with $\varrho(x) := d(x) - \rho$ as follows. Recalling that $c \le 0$ in Ω, we estimate

$$Lw^* \le f(\phi(d(x) - \rho))\Big[A^*\Lambda + \frac{A^*\sqrt{2F(\phi(d-\rho))}}{f(\phi(d-\rho))}(|L_0 d| + \|\mathbf{b}\|_{L^\infty})\Big]. \tag{4.3}$$

Let $\varepsilon > 0$, to be specified later. By (2.1), we can take $\mu > 0$ sufficiently small so that for all $x \in \Omega_\rho^-$,

$$\frac{A^*\sqrt{2F(\phi(d-\rho))}}{f(\phi(d-\rho))}(|L_0 d| + \|\mathbf{b}\|_{L^\infty}) < \varepsilon. \tag{4.4}$$

Therefore, from (4.3) and (4.4), we conclude that for $x \in \Omega_\mu^-$,

$$Lw^* \le f(\phi(d-\rho)))(A^*\Lambda + \varepsilon).$$

Now, let ε be chosen so that $\Theta_* - 2\varepsilon > \Xi_*$. We pick a number $A^* := A^*(\Lambda, f, \Theta_*(h)) > 0$ such that

$$\Theta_* - 2\varepsilon > g^*(A^*) = A^*\Lambda - \liminf_{0<d(x)-\rho\to 0} \frac{f(A^*\phi(d(x)-\rho))}{f(\phi(d(x)-\rho))}.$$

That is,

$$A^*\Lambda + \varepsilon < \Theta_* - \varepsilon + \liminf_{0<d(x)-\rho\to 0} \frac{f(A^*\phi(d(x)-\rho))}{f(\phi(d(x)-\rho))}.$$

We also suppose that μ is sufficiently small, so that for $(x, \rho) \in \Omega_\rho^- \times (0, \mu)$,

$$A^*\Lambda + \varepsilon < \Theta_* - \frac{\varepsilon}{2} + \frac{f(A^*\phi(d(x)-\rho))}{f(\phi(d(x)-\rho))}.$$

Let us now suppose that $\Theta_*(h) \le 0$. In this case, for $(x, \rho) \in \Omega_\rho^- \times (0, \mu)$, we have

$$\begin{aligned} Lw^* &\le f(\phi(d(x)-\rho))(A^*\Lambda + \varepsilon) \\ &\le f(\phi(d(x)-\rho))\Big[\Theta_* - \frac{\varepsilon}{2} + \frac{f(A^*\phi(d(x)-\rho))}{f(\phi(d(x)-\rho))}\Big] \\ &= \Big(\Theta_* - \frac{\varepsilon}{2}\Big) f(\phi(d(x)-\rho)) + f(A^*\phi(d(x)-\rho)) \\ &\le \Big(\Theta_* - \frac{\varepsilon}{2}\Big) f(\phi(d(x))) + f(A^*\phi(d(x)-\rho)) \\ &\le h(x) + f(w^*), \end{aligned}$$

from the definition of Θ_*. Let $B^* := \max\{u(x) : d(x) \ge \mu\}$. Then $u \le w^* + B^*$ on $\partial\Omega_\rho^-$ and

$$Lu = f(u) + h, \quad L(w^* + B^*) \le Lw^* \le f(w^*) + h \le f(w^* + B^*) + h \quad \text{on } \Omega_\rho^-.$$

By the Comparison Principle, Lemma 2.1, we conclude that $u \le w^* + B^*$ in Ω_ρ^-. Therefore,

$$\frac{u(x)}{\phi(d(x)-\rho)} - \frac{B^*}{\phi(d(x)-\rho)} \le A^* \quad \text{for } x \in \Omega_\rho^-.$$

On letting $\rho \to 0^+$, we see that the following holds on Ω_μ:

$$\frac{u(x)}{\phi(d(x))} - \frac{B^*}{\phi(d(x))} \le A^*.$$

Now, let $d(x) \to 0$ to obtain (4.1) when $\Theta_*(h) \le 0$.

We now consider the case $\Theta_*(h) > 0$. Then h is non-negative near $\partial\Omega$. Let $\{\mathcal{O}_j\}$ be the sequence of open subsets of Ω defined as in the proof of Lemma 3.6 and $m := \min\{h(x) : x \in \overline{\Omega}\}$. We note that $m > -\infty$. Let v_j be a solution of (3.4) with $h \equiv m$. If u is any solution of (1.3), then $Lu = f(u) + h \ge f(u) + m$. By the Comparison Principle, $u \le v_j$ in $\mathcal{O}_j$ for all j. Proceeding as in the proof of Theorem 3.7, one can show that $v = \lim_{j\to\infty} v_j$ is a solution of (1.3) on Ω with $h \equiv m$ on Ω. Clearly, $u \le v$ in Ω, and since v is a large solution of $Lv = f(v) + m$ on Ω and $\Theta_*(m) = 0$, the case considered above applied to v shows that

$$\limsup_{d(x)\to 0} \frac{u(x)}{\phi(d(x))} \le \limsup_{d(x)\to 0} \frac{v(x)}{\phi(d(x))} \le A^*,$$

where $A^* = A^*(\Lambda, f, \Theta_*(h))$, and therefore (4.1) holds.

Proof of (4.2). We consider the function

$$w_*(x) := A_*\phi(d(x)+\rho), \quad x \in \Omega_\rho^+.$$

For an appropriate choice of A_*, we show that w_* is a sub-solution in Ω_ρ^+ for all $\rho < \mu$, assuming that μ is sufficiently small. We assume first that $\Xi^* > 0$. Let us fix $\varepsilon > 0$ such that $3\varepsilon < \Xi^*$. Then, by definition, there exists a positive real number A_* such that $\Xi^*/2 + 3\varepsilon/2 < g^*(A_*)$. That is,

$$\frac{1}{2}\Xi^* + \limsup_{d(x)+\rho\to 0} \frac{f(A_*\phi(d(x)+\rho))}{f(\phi(d(x)+\rho))} < A_* - \frac{3}{2}\varepsilon.$$

Therefore, assuming that μ is sufficiently small, the following holds in Ω_ρ^+ for all $0 < \rho < \mu$:

$$\frac{1}{2}\Xi^* + \frac{f(A_*\phi(d(x)+\rho))}{f(\phi(d(x)+\rho))} < A_*\lambda - \varepsilon.$$

By shrinking μ if necessary, we can invoke (2.1) to estimate

$$A_*\Big[\frac{\sqrt{2F(\phi(d(x)+\rho))}}{f(\phi(d(x)+\rho))}(|L_0 d| + \|\mathbf{b}\|_{L^\infty}) + \frac{\phi(d(x)+\rho)}{f(\phi(d(x)+\rho))}\|c\|_{L^\infty}\Big] < \varepsilon \quad \text{for all } x \in \Omega_\rho^+ \text{ and } 0 < \rho < \mu.$$

Thus, for any $0 < \rho < \mu$, the following chain of inequalities hold on Ω_ρ^+:

$$\begin{aligned} Lw_* &\geq f(\phi(d(x)+\rho))\Big[A_* - A_* \frac{\sqrt{2F(\phi(d(x)+\rho))}}{f(\phi(d(x)+\rho))}(|L_0 d| + \|\mathbf{b}\|_{L^\infty}) - \frac{A_* \|c\|_{L^\infty}\phi(d(x)+\rho)}{f(\phi(d(x)+\rho))}\Big] \\ &\geq f(\phi(d(x)+\rho))(A_* - \varepsilon) \\ &\geq f(\phi(d(x)+\rho))\Big(\frac{1}{2}\Xi^* + \frac{f(A_*\phi(d(x)+\rho))}{f(\phi(d(x)+\rho))}\Big) \\ &\geq \frac{1}{2}\Xi^* f(\phi(d(x)+\rho)) + f(A_*\phi(d(x)+\rho)). \end{aligned} \tag{4.5}$$

Recall that $\Xi^* > 0$. Since h is bounded from above on Ω, we can assume μ is small enough so that

$$\frac{1}{2}\Xi^* f(\phi(d(x)+\rho)) \geq h(x), \quad x \in \Omega_\rho^+.$$

Thus, for $\Xi^* > 0$ and sufficiently small $\mu > 0$, we have shown that

$$Lw_* \geq f(w^*) + h(x), \quad x \in \Omega_\rho^+, \text{ for all } 0 < \rho < \mu.$$

Now let us suppose that $\Xi^* = 0$. Then there exists $A_* > 0$ such that for $\varepsilon > 0$ and $d + \rho$ small, we have

$$A_* - \frac{f(A_*\phi(d+\rho))}{f(\phi(d+\rho))} > -\varepsilon,$$

which can be rewritten as

$$\frac{f(A_*\phi(d+\rho))}{f(\phi(d+\rho))} - 2\varepsilon < A_* - \varepsilon.$$

Using this in estimate (4.5), we find

$$Lw_* \geq f(A_*\phi(d+\rho)) - 2\varepsilon f(\phi(d+\rho)). \tag{4.6}$$

By (h1), we recall that $-\infty \leq \Theta^* < \Xi^* = 0$. There is no loss in generality if we assume that $\Theta^* > -\infty$. At this point we use condition (h1), that is,

$$\limsup_{d(x)\to 0} \frac{h(x)}{f(\phi(d))} = \Theta^* < 0. \tag{4.7}$$

On noting that $f(\phi(d+\rho)) \leq f(\phi(d))$, and using (4.7), we find

$$\frac{h(x)}{f(\phi(d+\rho))} \leq \frac{1}{2}\Theta^*.$$

Choosing $\varepsilon = -\Theta^*/4$ in the above inequality, we find

$$h(x) \leq -2\varepsilon f(\phi(d+\rho)).$$

Inserting the latter estimate into (4.6), for sufficiently small $\rho > 0$, yields

$$Lw_* \geq f(w_*) + h(x), \quad x \in \Omega_\rho^+.$$

In conclusion, we have shown that in either of the cases, the following holds:

$$Lw_* \geq f(w_*) + h(x), \quad x \in \Omega_\rho^+.$$

Let $B_* := A_*\phi(\mu)$. Note that $w_* - B_* \leq u$ on $\partial\Omega_\rho^+$ and

$$L(w_* - B_*) \geq Lw_* \geq f(w_*) + h \geq f(w_* - B_*) + h \quad \text{and} \quad L(u) = f(u) + h \quad \text{in } \Omega_\rho^+.$$

Therefore, $w_* - B_* \leq u$ in Ω_ρ^+ and

$$A_* \leq \frac{u(x)}{\phi(d(x)+\rho)} + \frac{B_*}{\phi(d(x)+\rho)} \quad \text{for } x \in \Omega_\rho^+.$$

On letting $\rho \to 0^+$, we see that

$$A_* \le \frac{u(x)}{\phi(d(x))} + \frac{B_*}{\phi(d(x))} \quad \text{on } \Omega_\mu.$$

On recalling that $\phi(d(x)) \to \infty$ as $d(x) \to 0$, we get

$$A_* \le \liminf_{d(x)\to 0} \frac{u(x)}{\phi(d(x))},$$

and the proof is complete. □

In Lemma 4.2, we required h in (1.3) to be bounded from above in order to get the asymptotic estimate (4.2). Next, we wish to remove this restriction to allow h to be unbounded on Ω. However, the growth of h near the boundary of Ω needs to be constrained in such a way that a Dirichlet problem with zero boundary data is solvable. We use the notation $h^+(x) := \max\{h(x), 0\}$.

(h3) We assume that $d^2(x)h^+(x)$ is bounded in Ω and that the following Dirichlet problem admits a solution $w \in W^{2,n}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$:

$$\begin{cases} Lw = h^+(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.8}$$

Suppose that $h \in C(\Omega)$ satisfies (h3). Then we note that there exists a positive constant C such that $h^+(x) \le Cd^{-2}(x)$ in Ω. Therefore, we have

$$0 \le \frac{h^+(x)}{f(\phi(d(x)))} \le \frac{C}{d^2(x)f(\phi(d(x)))}, \quad x \in \Omega. \tag{4.9}$$

By Lemma 2.3 (ii), we note that

$$\lim_{t\to\infty} \frac{1}{\left(\int_t^\infty F(s)^{-1/2}\,ds\right)^2 f(t)} = 0. \tag{4.10}$$

Thus, from (4.9) and (4.10), we conclude that

$$\Theta_*(h^+) = \Theta^*(h^+) = \lim_{d(x)\to 0} \frac{h^+(x)}{f(\phi(d(x)))} = 0.$$

Consequently, we see that

$$\Theta^*(h) = \Theta^*(-h^-) \quad \text{and} \quad \Theta_*(h) = \Theta_*(-h^-). \tag{4.11}$$

The following result complements Lemma 4.2.

Theorem 4.3. *Suppose that $h \in C(\Omega)$ satisfies* (h1), (h2) *and* (h3). *Then there exist constants $0 < A_* \le A^*$ such that both estimates* (4.1) *and* (4.2) *hold for any solution $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$ of* (1.3).

Proof. Estimate (4.1) is proved in Lemma 4.2. Thus, it only remains to show (4.2). For $j = 1, 2, \dots,$ let v_j be the solution of

$$Lv_j = f(v_j) - \min\{j, h^-\} \quad \text{in } \Omega, \qquad v_j = j \quad \text{on } \partial\Omega.$$

The Comparison Principle shows that $\{v_j\}$ is an increasing sequence. Let w be the unique solution of (4.8). Since $w < 0$ on Ω, we find that

$$L(v_j + w) = f(v_j) - \min\{j, h^-\} + h^+ \ge f(v_j + w) + h \quad \text{in } \Omega, \qquad v_j + w = j \quad \text{on } \partial\Omega.$$

Now, if u is any solution of (1.3) in Ω, then again by the Comparison Principle, we find

$$v_j + w(x) \le u(x) \quad \text{for all } j.$$

Letting $j \to \infty$, we obtain

$$v + w \le u \quad \text{in } \Omega,$$

where

$$v(x) := \lim_{j\to\infty} v_j(x), \quad x \in \Omega.$$

Then v is a large solution of $Lv = f(v) - h^-$ on Ω.

Since h satisfies (h1) and (h3), we see that $\Theta^*(-h^-) = \Theta^*(h) < \Xi^*$. Since, in addition, $-h^-$ is bounded from above we invoke Lemma 4.2 (ii) to infer that there exists a constant $A_* > 0$ such that v satisfies the asymptotic estimate (4.2). Since w is bounded on Ω, we have

$$A_* \le \liminf_{d(x)\to 0} \frac{v(x)}{\phi(d(x))} \le \limsup_{d(x)\to 0} \frac{u(x)}{\phi(d(x))},$$

and this concludes the proof of (4.2). □

The following corollary is noteworthy.

Corollary 4.4. *Let $h \in C(\Omega)$ and suppose that $|h|$ satisfies* (h3). *If $\Xi_* < 0 < \Xi^*$, then there exist constants $0 < A_* \le A^*$ such that for any solution u of* (1.3),

$$A_* \le \liminf_{d(x)\to 0} \frac{u(x)}{\phi(d(x))} \le \limsup_{d(x)\to 0} \frac{u(x)}{\phi(d(x))} \le A^*. \tag{4.12}$$

Proof. Since $|h|$ satisfies (h3), we recall from (4.11) that $\Theta^*(h) = \Theta_*(h) = 0$. Therefore, by hypothesis, h satisfies (h1) and (h2). Thus, the conclusion of the corollary follows from Theorem 4.3. □

Remark 4.5. Suppose that $h \in C(\Omega)$ satisfies $d^2(x)|h(x)| \le \eta(d(x))$ for all $x \in \Omega$, with $d(x) \le \delta_0$, for some δ_0, where $\eta\colon (0, \delta_0] \to (0, \eta(\delta_0)]$ is an increasing function that satisfies (1.4). According to [1, Theorem 4] or [16, Theorem 5.2], we note that $|h|$ satisfies condition (h3). Therefore, if f satisfies (f1) and is regularly varying at infinity of order $q > 1$, then (3.1), together with Corollary 4.4, shows that any solution u of (1.3) satisfies (4.12) for some constants $0 < A_* \le A^*$.

Remark 4.6. Suppose that $h \in C(\Omega)$ satisfies (h1), (h2) and (h3). If $\overline{u}$ is a maximal solution of (1.3), then there exist constants $A \ge 1$ and $\delta > 0$ such that for any solution of u of (1.3),

$$\overline{u} \le Au \quad \text{in } \Omega_\delta.$$

This follows from Theorem 4.3 with $A := A^*/A_* \ge 1$, and it was obtained in [9] for $\Delta u = |u|^{q-1}u + h$, $q > 1$, under the assumption that h is bounded from above in Ω.

5 On uniqueness and existence of positive solutions to problem (1.3)

In this section, we continue to assume that all the coefficients of L are bounded in Ω.

Our next result shows that problem (1.3) admits at most one non-negative solution even when h is allowed to be unbounded in Ω.

Theorem 5.1. *Assume that f satisfies* (f4), *and $h \in C(\Omega)$ satisfies* (h1), (h2) *and* (h3). *If f is convex on* $[0, \infty)$, *then problem* (1.3) *has at most one non-negative solution in $W^{2,n}_{\mathrm{loc}}(\Omega)$.*

The proof is omitted since it is similar to that of [18, Theorem 4.4], see also [7–9, 17]. In particular, we mention [8] in which the authors prove uniqueness to the boundary blow-up problem associated with $\Delta u = f(u)$ in balls, where f is non-decreasing in $(0, \infty)$ and convex at infinity.

For our next result, we suppose that $f \in C^1(\mathbb{R})$. If $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$, then $f(u) \in W^{1,n}(\Omega)$. This follows from [12, Theorem 7.8]. Furthermore, we suppose that the coefficients of L are Hölder continuous in Ω. If $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$ is a solution of (1.3) with $h \equiv 0$, then, since $f(u) \in W^{1,n}_{\mathrm{loc}}(\Omega)$, from the elliptic regularity theory, we conclude that $u \in C^2(\Omega)$, see [12, Theorem 9.19].

In Theorem 3.7 we showed that problem (1.3) has a solution. This solution may change sign in Ω. In the next theorem, we consider the existence of a positive solution to (1.3). We use an adaptation of the argument in [9].

Theorem 5.2. *Let $0 < \beta < 1$. There exists a constant $C > 0$, depending on β and Ω, such that for every $h \in C(\Omega)$ satisfying*

$$\sup_{\Omega} d^{2-\beta}(x)h^{+}(x) \le C, \tag{5.1}$$

problem (1.3) *admits a positive solution in $W^{2,n}_{\mathrm{loc}}(\Omega)$.*

Proof. Let $U \in C^2(\Omega)$ be the maximal solution of (1.3) with $h \equiv 0$ on Ω (see the remark right after Theorem 5.1). By the tangency principle (see [20, Theorem 2.1.3]), we note that $\min_\Omega U > 0$. Let $w \in W^{2,n}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ be the unique solution of $Lw = h^+$ in Ω with $w = 0$ on $\partial\Omega$. The existence of such a solution is justified by [1, Theorem 4]. Moreover, $w < 0$ in Ω and there exists a positive constant $C_\beta > 0$ such that (see [1, Remark 6.2])

$$\sup_{\Omega} |d^{-\beta}(x)w(x)| \le C_\beta \sup_{\Omega} (d^{2-\beta}(x)h^{+}(x)).$$

Therefore, we have

$$|w(x)| = d^{\beta}(x)|d^{-\beta}(x)w(x)| \le C_\beta(\operatorname{diam}\Omega)^\beta \sup_{\Omega} d^{2-\beta}(x)h^{+}(x) \le CC_\beta(\operatorname{diam}\Omega)^\beta \quad \text{for all } x \in \Omega.$$

We choose $C > 0$ in (5.1) so that

$$CC_\beta(\operatorname{diam}\Omega)^\beta < \min_{\Omega} U(x).$$

Thus, $U + w > 0$ in Ω, and

$$L(U + w) = f(U) + h^{+} \ge f(U + w) + h \quad \text{in } \Omega.$$

Let $\{u_j\}$ be the sequence of solutions of (3.4) constructed in the proof of Theorem 3.7. Then, by the Comparison Principle, we note that $U + w \le u_j$ in $\mathcal{O}_j$ for all j. Therefore, $U + w \le u$ in Ω, showing that u is a positive solution of (1.3). □

A Appendix

We use this appendix to prove two lemmas and make some remarks, which may be of independent interest.

Lemma A.1. *Suppose that f satisfies* (f1). *If f satisfies the Keller–Osserman condition* (f2), *then*

$$\liminf_{t\to\infty} \frac{F(t)}{tf(t)} \le \frac{1}{2}. \tag{A.1}$$

Proof. Let

$$\alpha := \liminf_{t\to\infty} \frac{F(t)}{tf(t)}.$$

Given $\tau_0 > 0$, we see that

$$\ln\Big(\frac{F(t)}{F(\tau_0)}\Big) = \int_{\tau_0}^{t} \frac{f(s)}{F(s)}\,ds, \quad t > \tau_0.$$

That is,

$$F(t) = F(\tau_0)\exp\Big(\int_{\tau_0}^{t} \frac{f(s)}{F(s)}\,ds\Big), \quad t > \tau_0. \tag{A.2}$$

Now assume, contrary to (A.1), that $\alpha > 1/2$. We choose $\tau_0 > 0$ so that

$$\frac{F(t)}{tf(t)} \ge \frac{1}{2} \quad \text{for all } t \ge \tau_0. \tag{A.3}$$

Using (A.3) in (A.2), we obtain

$$F(t) \le F(\tau_0)\exp\Big(2\int_{\tau_0}^{t} \frac{ds}{s}\Big) = F(\tau_0)\tau_0^{-2}t^2 \quad \text{for all } t \ge \tau_0.$$

This shows that (f2) fails. □

Remark A.2. (i) We remark that if $t \mapsto t^{-1}f(t)$ is non-decreasing at infinity, then

$$\limsup_{t\to\infty} \frac{F(t)}{tf(t)} \le \frac{1}{2}.$$

To see this, suppose $t \mapsto t^{-1}f(t)$ is non-decreasing on (τ_0, ∞) for some $\tau_0 > 0$. Then

$$F(t) = F(\tau_0) + \int_{\tau_0}^{t} f(s)\,ds \le F(\tau_0) + \frac{f(t)}{t}\int_0^t s\,ds = F(\tau_0) + \frac{1}{2}tf(t), \quad t > \tau_0.$$

Therefore, we have

$$\frac{F(t)}{tf(t)} \le \frac{F(\tau_0)}{tf(t)} + \frac{1}{2}, \quad t > \tau_0,$$

from which the claim follows.

(ii) The converse of Lemma A.1 is false, as the example $f(t) = t\log^2(t+1)$ shows.

Lemma A.3. *Suppose that f satisfies* (f1) *and* (f3). *Then $\Xi_* \le 0$.*

Proof. Let α be as in the proof of Lemma A.1. By (f3), we have $\alpha > 0$. Let us fix ρ so that $0 < \rho < \alpha$. Then there exists $\tau_0 := \tau(\rho) > 0$ such that

$$\frac{F(t)}{tf(t)} \ge \rho \quad \text{for all } t \ge \tau_0. \tag{A.4}$$

If $0 < \kappa < 1$, then, using (A.4) in (A.2), we find

$$\frac{F(\kappa t)}{F(t)} = \exp\biggl(-\int_{\kappa t}^{t} \frac{f(s)}{F(s)}\,ds\biggr) \ge \kappa^{1/\rho} \quad \text{for all } t \ge \frac{\tau_0}{\kappa}. \tag{A.5}$$

As a consequence of (A.5), for $0 < \kappa < 1$, we find

$$\kappa^{1/\rho} \le \frac{F(\kappa t)}{F(t)} \le \frac{2\kappa f(\kappa t)}{f(t/2)}, \quad t > \frac{\tau_0}{\kappa}. \tag{A.6}$$

Therefore, for $0 < \kappa < 1$, we have shown that

$$\frac{f(2\kappa t)}{f(t)} \ge \frac{1}{2}\kappa^{(1-\rho)/\rho}, \quad t \ge \frac{\tau_0}{2\kappa}. \tag{A.7}$$

For an arbitrary, but fixed $0 < \xi < 2$, we take $\kappa := \xi/2$ in (A.7) to find

$$\frac{f(\xi t)}{f(t)} \ge \frac{1}{2}\Bigl(\frac{\xi}{2}\Bigr)^{(1-\rho)/\rho}, \quad t \ge \frac{\tau_0}{\xi}.$$

Therefore,

$$\liminf_{t\to\infty} \frac{f(\xi t)}{f(t)} \ge \frac{1}{2}\Bigl(\frac{\xi}{2}\Bigr)^{(1-\rho)/\rho}, \quad 0 < \xi < 2.$$

Consequently, we get

$$g_*(\xi) \le \Lambda\xi - \frac{1}{2}\Bigl(\frac{\xi}{2}\Bigr)^{(1-\rho)/\rho}, \quad 0 < \xi < 2.$$

Therefore, since $0 < \rho < 1$, we note that the right-hand side tends to 0 as $\xi \to 0$. This shows that

$$\Xi_* = \inf_{\xi>0} g_*(\xi) \le 0,$$

and the proof is complete. □

We complement the above lemma with the following remark.

Remark A.4. Suppose that f satisfies (f1) and

$$\limsup_{t\to\infty} \frac{F(t)}{tf(t)} < \frac{1}{2}.$$

Then $\Xi_* = -\infty$. To see this let

$$\limsup_{t\to\infty} \frac{F(t)}{tf(t)} < \sigma < \frac{1}{2}. \tag{A.8}$$

Let $\kappa > 1$. Again, using (A.8) in (A.2), we can find some $\tau_0 > 0$ such that

$$\frac{F(\kappa t)}{F(t)} \geq \kappa^{1/\sigma}, \quad t > \tau_0.$$

Proceeding as in (A.6) we find

$$\frac{f(2\kappa t)}{f(t)} \geq \frac{1}{2}\kappa^{(1-\sigma)/\sigma}, \quad t \geq \tau_0.$$

That is,

$$\frac{f(\xi t)}{f(t)} \geq 2^{-1/\sigma}\xi^{(1-\sigma)/\sigma}, \quad t > \tau_0.$$

Consequently, we have

$$g_*(\xi) \leq \Lambda\xi - 2^{-1/\sigma}\xi^{(1-\sigma)/\sigma}.$$

Recalling that $1/\sigma > 2$, the assertion follows on letting $\xi \to \infty$.

References

[1] A. Ancona, On strong barriers and an inequality of Hardy for domains in $\mathbb{R}^n$, *J. Lond. Math. Soc. (2)* **34** (1986), no. 2, 274–290.

[2] C. Bandle and M. Marcus, "Large" solutions of semilinear elliptic equations: Existence, uniqueness and asymptotic behavior, *J. Anal. Math.* **58** (1992), 9–24.

[3] C. Bandle and M. Marcus, Asymptotic behaviour of solutions and their derivatives, for semilinear elliptic problems with blowup on the boundary, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **12** (1995), no. 2, 155–171.

[4] F. Cîrstea and V. Rădulescu, Uniqueness of the blow-up boundary solution of logistic equations with absorption, *C. R. Math. Acad. Sci. Paris* **335** (2002), no. 5, 447–452.

[5] F. Cîrstea and V. Rădulescu, Boundary blow-up in nonlinear elliptic equations of Bieberbach–Rademacher type, *Trans. Amer. Math. Soc.* **359** (2007), no. 7, 3275–3286.

[6] G. Díaz and R. Letelier, Explosive solutions of quasilinear elliptic equations: Existence and uniqueness, *Nonlinear Anal.* **20** (1993), no. 2, 97–125.

[7] M. Dindoš, Large solutions for Yamabe and similar problems on domains in Riemannian manifolds, *Trans. Amer. Math. Soc.* **363** (2011), no. 10, 5131–5178.

[8] S. Dumont, L. Dupaigne, O. Goubet and V. Rădulescu, Back to the Keller–Osserman condition for boundary blow-up solutions, *Adv. Nonlinear Stud.* **7** (2007), no. 2, 271–298.

[9] J. García-Melián, Large solutions for an elliptic equation with a nonhomogeneous term, *J. Math. Anal. Appl.* **434** (2016), 872–881.

[10] J. García-Melián, J. D. Rossi and J. C. Sabina de Lis, Elliptic systems with boundary blow-up: Existence, uniqueness and applications to removability of singularities, *Commun. Pure Appl. Anal.* **15** (2016), no. 2, 549–562.

[11] M. Ghergu and V. Rădulescu, *Singular Elliptic Problems: Bifurcation and Asymptotic Analysis*, Oxford Lecture Ser. Math. Appl. 37, Oxford University Press, Oxford, 2008.

[12] G. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, New York, 1977.

[13] F. Gladiali and G. Porru, Estimates for explosive solutions to p-Laplace equations, in: *Progress in Partial Differential Equations. Vol. 1* (Pont-à-Mousson 1997), Pitman Res. Notes Math. Ser. 383, Longman, Harlow (1998), 117–127.

[14] J. Karamata, Sur un mode de croissance régulière, Théorèmes fondamentaux, *Bull. Soc. Math. France* **61** (1933), 55–62.

[15] J. B. Keller, On solutions of $\Delta u = f(u)$, *Comm. Pure Appl. Math.* **10** (1957), 503–510.

[16] J. H. Michael, A general theory for linear elliptic partial differential equations, *J. Differential Equations* **23** (1977), no. 1, 1–29.

[17] M. Marcus and L. Véron, The boundary trace and generalized boundary value problems for semilinear elliptic equations with coercive absorption, *Comm. Pure Appl. Math.* **56** (2003), 1–43.

[18] A. Mohammed and G. Porru, Large solutions for non-divergence equations with singular lower order terms, *Nonlinear Anal. Real World Appl.* **35** (2017), 470–482.

[19] R. Osserman, On the inequality $\Delta u \geq f(u)$, *Pacific J. Math.* **7** (1957), 1641–1647.

[20] P. Pucci and J. Serrin, *The Maximum Principle*, Progr. Nonlinear Differential Equations Appl. 73, Birkhäuser, Basel, 2007.

[21] D. Repovš, Asymptotics for singular solutions of quasilinear elliptic equations with an absorption term, *J. Math. Anal. Appl.* **395** (2012), no. 1, 78–85.

[22] V. Rădulescu, Singular phenomena in nonlinear elliptic problems: from blow-up boundary solutions to equations with singular nonlinearities, in: *Handbook of Differential Equations: Stationary Partial Differential Equations. Vol. IV*, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam (2007), 485–593.

[23] E. Seneta, *Regularly Varying Functions*, Lecture Notes in Math. 508, Springer, Berlin, 1976.

[24] L. Véron, Semilinear elliptic equations with uniform blowup on the boundary, *J. Anal. Math.* **59** (1992), 231–250.

[25] Z. Zhang, A boundary blow-up elliptic problem with an inhomogeneous term, *Nonlinear Anal.* **68** (2008), 3428–3438.

[26] Z. Zhang, The existence and boundary behavior of large solutions to semilinear elliptic equations with nonlinear gradient terms, *Adv. Nonlinear Anal.* **3** (2014), no. 3, 165–185.