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Research Article

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A result of uniqueness of solutions of the Shigesada–Kawasaki–Teramoto equations

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Abstract: We derive the uniqueness of weak solutions to the Shigesada–Kawasaki–Teramoto (SKT) systems using the adjoint problem argument. Combining with [10], we then derive the well-posedness for the SKT systems in space dimension $d \leq 4$.

Keywords: Well-posedness, quasi-linear parabolic equations, global existence

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1 Introduction

Uniqueness is an important issue to address when one considers the global well-posedness for a system of differential equations. For systems of partial differential equations like cross diffusion systems, the uniqueness has remained a challenge for solutions with mild regularity since the comparison/maximum principle for the cross diffusion systems like SKT is not available. We also note here that in our recent work [10], we showed a weak maximum principle for non-negativeness of solutions that allowed us to prove the existence of positive weak solutions of SKT systems directly using finite difference approximations, a priori estimates, and passage to the limit, which avoid the change of variables (or an entropy function) being used in other works as in, e.g., [5–7]. Together with our existence result for weak solutions of SKT systems in [10], this article provides the well-posedness for these systems in space dimension $d \leq 4$.

The available uniqueness results for the SKT systems are rather scarce and require high regularity of the solutions. In [12, Theorem 3.5], Yagi proved a uniqueness result for solutions in

$$\mathcal{C}((0,T];H^2(\Omega)) \cap \mathcal{C}^1((0,T];L^2(\Omega))$$

using an abstract theory for parabolic equations in space dimension 2. In [1, 2], Amann proved global existence and uniqueness results for solutions of general systems of parabolic equations with high regularity in space in the semigroup settings, $W^{1,p}(\Omega)$ for $p > n$, which require Hölder a priori estimates when applied to SKT equations.

In this work, we use the argument of adjoint problems to build specific test functions to show the uniqueness for solutions in a more general space setting, in $L^\infty(0,T;H^1(\Omega)^2)$ with time derivatives in $L^{4/3}(\Omega_T)^2$ for space dimension $d \leq 4$; see Remark 2.1 below. This argument of using adjoint problems has been used to show uniqueness results for scalar partial differential equations describing flows of gas or fluid in porous media or the spread of a certain biological population; see, e.g., [3, 4]. It is also systematically used in the context of linear equations in [9].

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Throughout our work, we denote by Ω an open bounded domain in $\mathbb{R}^d$, $d \leq 4$, and we set $\Omega_T = \Omega \times (0, T)$ for any $T > 0$. We aim to show the uniqueness result and then combine this result with our previous global existence results of weak solutions in [10] to show the global well-posedness for the following SKT system of diffusion reaction equations (see [11]):

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{p}(\mathbf{u}) + \mathbf{q}(\mathbf{u}) = \boldsymbol{\ell}(u) \text{ in } \Omega_T, \\ \partial_\nu \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega \times (0, T) \text{ or } \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega \times (0, T), \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x) \geq \mathbf{0} \text{ in } \Omega, \end{cases} \tag{1.1}$$

where $\mathbf{u} = (u, v)$ and

$$\begin{aligned} \mathbf{p}(\mathbf{u}) &= \begin{pmatrix} p_1(u, v) \\ p_2(u, v) \end{pmatrix} = \begin{pmatrix} (d_1 + a_{11}u + a_{12}v)u \\ (d_2 + a_{21}u + a_{22}v)v \end{pmatrix}, \\ \mathbf{q}(\mathbf{u}) &= \begin{pmatrix} q_1(u, v) \\ q_2(u, v) \end{pmatrix} = \begin{pmatrix} (b_1 u + c_1 v)u \\ (b_2 u + c_2 v)v \end{pmatrix}, \\ \boldsymbol{\ell}(\mathbf{u}) &= \begin{pmatrix} \ell_1(u) \\ \ell_2(v) \end{pmatrix} = \begin{pmatrix} a_1 u \\ a_2 v \end{pmatrix}. \end{aligned} \tag{1.2}$$

Here $a_{ij} \geq 0$, $b_i \geq 0$, $c_i \geq 0$, $a_i \geq 0$, $d_i \geq 0$ are such that

$$0 < a_{12}a_{21} < 64 a_{11}a_{22}. \tag{1.3}$$

It can be shown [13] that condition (1.3) is equivalent to

$$0 < a_{12}^2 < 8a_{11}a_{21} \quad \text{and} \quad 0 < a_{21}^2 < 8a_{22}a_{12}, \tag{1.4}$$

as far as existence and uniqueness of solutions are concerned.

One of the difficulties with the SKT equations is that they are not parabolic equations. Whereas Amann [1, 2] has proven the existence and uniqueness of regular solutions for general parabolic equations, which can be applied to SKT equations using L^p estimates, we proved in [10] the existence of weak solutions (see also [7]) to the SKT equations. It is important to validate this concept of weak solutions, to show that the weak solutions are unique. This is precisely what we are doing in this article in dimension $d \leq 4$.

Throughout the article, we often use the following alternate form of (1.1):

$$\partial_t \mathbf{u} - \nabla \cdot (\mathbf{P}(\mathbf{u}) \nabla \mathbf{u}) + \mathbf{q}(\mathbf{u}) = \boldsymbol{\ell}(\mathbf{u}),$$

where

$$\mathbf{P}(\mathbf{u}) = \begin{pmatrix} p_{11}(u, v) & p_{12}(u, v) \\ p_{21}(u, v) & p_{22}(u, v) \end{pmatrix} = \begin{pmatrix} d_1 + 2a_{11}u + a_{12}v & a_{12}u \\ a_{21}v & d_2 + a_{21}u + a_{22}v \end{pmatrix}. \tag{1.5}$$

When condition (1.4) is satisfied and $u \geq 0$, $v \geq 0$, we can prove that the matrix $\mathbf{P}(\mathbf{u})$ is (pointwise) positive definite and that

$$(\mathbf{P}(\mathbf{u})\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geq \alpha(u + v)|\boldsymbol{\xi}|^2 + d_0|\boldsymbol{\xi}|^2 \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^2, \tag{1.6}$$

where $d_0 = \min(d_1, d_2)$ and

$$0 < \alpha < \min(a_{11}, a_{12}, a_{21}, a_{22}, \delta_0).$$

Here we refer the readers to a proof of (1.6) in our recent article [10].

We consider later on the mappings

$$\mathcal{P} : \mathbf{u} = (u, v) \mapsto \mathbf{p} = (p_1, p_2), \quad \mathcal{Q} : \mathbf{u} = (u, v) \mapsto \mathbf{q} = (q_1, q_2)$$

and we observe that

$$\mathbf{P}(\mathbf{u}) = \frac{D\mathcal{P}}{D\mathbf{u}}(\mathbf{u}), \quad \mathbf{Q}(\mathbf{u}) = \frac{D\mathcal{Q}}{D\mathbf{u}}(\mathbf{u}), \tag{1.7}$$

and

$$\nabla \mathbf{p}(\mathbf{u}) = \mathbf{P}(\mathbf{u}) \nabla \mathbf{u}.$$

We see that the explicit form of $\mathbf{P}(\mathbf{u})$ is given in (1.5) and the one of $\mathbf{Q}(\mathbf{u})$ is

$$\mathbf{Q}(\mathbf{u}) = \begin{pmatrix} 2b_1 u + c_1 v & c_1 u \\ b_2 v & b_2 u + 2c_2 v \end{pmatrix}.$$

Note that (1.6) implies that, for $u, v \geq 0$, $\mathbf{P}(\mathbf{u})$ is invertible (as a 2×2 matrix), and that, pointwise (i.e. for a.e. $x \in \Omega$),

$$|\mathbf{P}(\mathbf{u})^{-1}|_{\mathcal{L}(\mathbb{R}^2)} \leq \frac{1}{d_0 + \alpha(u + v)}.$$

Our work is organized as follows: We show our main result in Section 2, where the uniqueness for weak solutions to the SKT system is derived using solutions of adjoint problems. Since the proof of the uniqueness relies on the existence of solutions to the adjoint problem, we show the existence for these problems in Section 2.1, together with the a priori estimates in dimension $d \leq 4$. We finally show in Section 3 that the newly derived uniqueness result in Section 2 together with our existence result in [10] leads to the global well-posedness for the SKT systems in space dimension $d \leq 4$.

2 Uniqueness result for SKT systems

As mentioned earlier, our uniqueness result is proven using an argument of an adjoint problem; see, e.g., [4], see also [9] in the context of linear parabolic problems. The existence of solutions of our adjoint problem will be granted if the solution $\mathbf{u}$ of (1.1) enjoys the following regularity properties:

$$\mathbf{u} \in L^\infty(0, T; H^1(\Omega)^2) \quad \text{and} \quad \partial_t \mathbf{u} \in L^{\frac{4}{3}}(\Omega_T)^2.$$

Remark 2.1. Although this was not explicitly stated in [10], the solutions that we constructed in dimension $d \leq 4$ belong to $L^\infty(0, T; H^1(\Omega)^2)$ with $\partial_t \mathbf{u} \in L^2(0, T; L^2(\Omega)^2)$; see Appendix A.3.

Introducing a test function $\boldsymbol{\varphi}$ which satisfies (2.2) below and the same boundary condition as $\mathbf{u}$, we multiply (1.1) by $\boldsymbol{\varphi}$, integrate, integrate by parts, and obtain the variational weak form of (1.1):

$$\begin{cases} \langle \partial_t \mathbf{u}, \boldsymbol{\varphi} \rangle - \langle \mathbf{p}(\mathbf{u}), \Delta \boldsymbol{\varphi} \rangle + \langle \mathbf{q}(\mathbf{u}), \boldsymbol{\varphi} \rangle = \langle \boldsymbol{\ell}(\mathbf{u}), \boldsymbol{\varphi} \rangle, \\ \partial_\nu \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega \times (0, T) \text{ or } \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega \times (0, T), \\ \mathbf{u}(x, 0) = \mathbf{u}_0 \text{ in } \Omega \end{cases} \tag{2.1}$$

for all test functions $\boldsymbol{\varphi}$ such that

$$\begin{cases} \boldsymbol{\varphi} \in L^2(0, T; H^2(\Omega)^2) \cap L^\infty(0, T; H^1(\Omega)^2) \text{ and } \partial_t \boldsymbol{\varphi} \in L^{\frac{4}{3}}(\Omega_T)^2, \\ \partial_\nu \boldsymbol{\varphi} = \mathbf{0} \text{ or } \boldsymbol{\varphi} = \mathbf{0} \text{ on } \partial\Omega \text{ } (\boldsymbol{\varphi} \text{ satisfies the same boundary condition as } \mathbf{u}). \end{cases} \tag{2.2}$$

Note that the boundary terms disappear because $\mathbf{p}(\mathbf{u})$ satisfies the same boundary condition as $\mathbf{u}$. To show that the solutions of (1.1) are unique, we introduce the difference of two solutions $\mathbf{u}_1, \mathbf{u}_2$ of (2.1), $\bar{\mathbf{u}} = \mathbf{u}_1 - \mathbf{u}_2$, and we will eventually show that $\bar{\mathbf{u}} = \mathbf{0}$ for a.e. $\mathbf{x} \in \Omega$ and $t > 0$.

We first observe that $\bar{\mathbf{u}}$ satisfies

$$\begin{cases} \langle \partial_t \bar{\mathbf{u}}, \boldsymbol{\varphi} \rangle - \langle \mathbf{p}(\mathbf{u}_1) - \mathbf{p}(\mathbf{u}_2), \Delta \boldsymbol{\varphi} \rangle + \langle \mathbf{q}(\mathbf{u}_1) - \mathbf{q}(\mathbf{u}_2), \boldsymbol{\varphi} \rangle = \langle \boldsymbol{\ell}(\mathbf{u}_1) - \boldsymbol{\ell}(\mathbf{u}_2), \boldsymbol{\varphi} \rangle, \\ \partial_\nu \bar{\mathbf{u}} = \mathbf{0} \text{ on } \partial\Omega \times (0, T) \text{ or } \bar{\mathbf{u}} = \mathbf{0} \text{ on } \partial\Omega \times (0, T), \\ \bar{\mathbf{u}}(x, 0) = \mathbf{0} \text{ in } \Omega \end{cases} \tag{2.3}$$

for any test function $\boldsymbol{\varphi}$ that satisfies (2.2).

Using the notations $\mathbf{P}(\star)$, $\mathbf{Q}(\star)$ introduced earlier in (1.7) and the relations (A.1), (A.2) from Lemma A.1, we find

$$\langle \bar{\mathbf{u}}, \boldsymbol{\varphi} \rangle_t - \langle \bar{\mathbf{u}}, \boldsymbol{\varphi}_t \rangle - \langle \mathbf{P}(\tilde{\mathbf{u}})\bar{\mathbf{u}}, \Delta \boldsymbol{\varphi} \rangle + \langle \mathbf{Q}(\tilde{\mathbf{u}})\bar{\mathbf{u}}, \boldsymbol{\varphi} \rangle = \langle \boldsymbol{\ell}(\bar{\mathbf{u}}), \boldsymbol{\varphi} \rangle,$$

where $\tilde{\mathbf{u}} = (\mathbf{u}_1 + \mathbf{u}_2)/2$.

Thus

$$\langle \bar{\mathbf{u}}, \boldsymbol{\varphi}\rangle_t - \langle \bar{\mathbf{u}}, \boldsymbol{\varphi}_t\rangle - \langle \bar{\mathbf{u}}, \mathbf{P}(\tilde{\mathbf{u}})^T \Delta\boldsymbol{\varphi}\rangle + \langle \bar{\mathbf{u}}, \mathbf{Q}(\tilde{\mathbf{u}})^T \boldsymbol{\varphi}\rangle = \langle \boldsymbol{\ell}(\bar{\mathbf{u}}), \boldsymbol{\varphi}\rangle. \tag{2.4}$$

We notice that $\tilde{\mathbf{u}} \in L^\infty(0, T; H^1(\Omega)^2)$ because $\mathbf{u}_1, \mathbf{u}_2 \in L^\infty(0, T; H^1(\Omega)^2)$.

We now consider the test function $\boldsymbol{\varphi}$ to be a solution of the following backward adjoint problem:

$$\begin{cases} -\partial_t\boldsymbol{\varphi} - \mathbf{P}(\tilde{\mathbf{u}})^T \Delta\boldsymbol{\varphi} + \mathbf{Q}(\tilde{\mathbf{u}})^T \boldsymbol{\varphi} = \boldsymbol{\varphi} \text{ in } \Omega_T, \\ \partial_\nu \boldsymbol{\varphi} = \mathbf{0} \text{ or } \boldsymbol{\varphi} = \mathbf{0} \text{ on } \partial\Omega \times (0, T), \\ \boldsymbol{\varphi}(T) = \boldsymbol{\chi}(\mathbf{x}) \text{ in } \Omega, \end{cases} \tag{2.5}$$

where $\boldsymbol{\chi}(\mathbf{x}) = (\chi^u(\mathbf{x}), \chi^v(\mathbf{x})) \in H^1(\Omega)^2$.

Before showing the uniqueness result of solutions of (1.1) using the test function $\boldsymbol{\varphi}$ as a solution of (2.5), we first show the existence of $\boldsymbol{\varphi} = (\phi^u, \phi^v) \in L^2(0, T; H^2(\Omega)^2)$ with $\partial_t\boldsymbol{\varphi} \in L^{4/3}(\Omega_T)$ in the following section.

2.1 Existence of solutions for the adjoint systems

In this section, we continue to assume that $d \le 4$, and we show the existence of a solution $\boldsymbol{\varphi}$ of (2.5) satisfying (2.2) by building approximate systems where the classical existence theory can be applied to show the existence of approximate solutions. We suppose throughout this section that the functions $\tilde{\mathbf{u}} \ge \mathbf{0}$ in the first equation in (2.5) satisfies

$$\tilde{\mathbf{u}} \in L^\infty(0, T; H^1(\Omega)^2).$$

We observe that the diffusive matrix $\mathbf{P}(\tilde{\mathbf{u}})$ in (2.5) may not be uniformly parabolic[1] unless $\tilde{\mathbf{u}} \in L^\infty(\Omega_T)^2$. Thus we can not directly apply the classical results for parabolic equations to show the existence of $\boldsymbol{\varphi}$; see, e.g., [8, Theorem 5.1]. We therefore use an approximation approach as in [4].

The existence of a solution $\boldsymbol{\varphi}$ of (2.5) is obtained in three steps:

- Define approximations $\boldsymbol{\varphi}_\varepsilon$ of $\boldsymbol{\varphi}$, which are solutions of the approximate systems (2.7) below.
- Derive a priori estimates for the functions $\boldsymbol{\varphi}_\varepsilon$.
- Pass to the limit as $\varepsilon \to 0$ to show the existence of $\boldsymbol{\varphi}$, a solution of (2.5).

We start now with the first step of building approximate solutions $\boldsymbol{\varphi}_\varepsilon$.

2.1.1 Approximate adjoint systems

We know that $\tilde{\mathbf{u}} = (\mathbf{u}_1 + \mathbf{u}_2)/2 \ge \mathbf{0}$ and $\tilde{\mathbf{u}} \in L^\infty(0, T; H^1(\Omega)^2)$ (see Theorem A.2). We build approximations $\tilde{\mathbf{u}}_\varepsilon$ of $\tilde{\mathbf{u}}$, as a sequence in $L^\infty(\Omega_T)^2$, that converges to $\tilde{\mathbf{u}}$ in $L^4(\Omega_T)^2$. We can define such $\tilde{\mathbf{u}}_\varepsilon$ as follows:

$$\tilde{\mathbf{u}}_\varepsilon = \boldsymbol{\theta}_\varepsilon(\tilde{\mathbf{u}}),$$

where $\boldsymbol{\theta}_\varepsilon$ is a smooth function with derivative bounded by a constant independent of ε, which we assume to be 1, such that

$$\boldsymbol{\theta}_\varepsilon(\tilde{\mathbf{u}}) = \begin{cases} \tilde{\mathbf{u}} & \text{for } \tilde{\mathbf{u}} \le \frac{1}{\varepsilon}, \\ \frac{1}{\varepsilon} & \text{for } \tilde{\mathbf{u}} \ge \frac{2}{\varepsilon}. \end{cases}$$

We easily see that $\mathbf{u}_\varepsilon \in L^\infty(\Omega_T)^2$. Furthermore, we have $\tilde{\mathbf{u}}_\varepsilon \to \tilde{\mathbf{u}}$ a.e. and $|\tilde{\mathbf{u}}_\varepsilon|_{L^4} \le |\tilde{\mathbf{u}}|_{L^4} < \infty$, and we obtain by the Lebesgue dominated convergence that $\tilde{\mathbf{u}}_\varepsilon$ converges to $\tilde{\mathbf{u}}$ in $L^4(\Omega_T)^2$. Finally, we easily see the following by straightforward calculations:

$$\|\tilde{\mathbf{u}}_\varepsilon\|_{L^\infty(0,T;H^1(\Omega)^2)} \le \kappa \|\tilde{\mathbf{u}}\|_{L^\infty(0,T;H^1(\Omega)^2)}, \tag{2.6}$$

where κ depends on the maximum value of θ_ε', which is independent of ε.

1 A matrix $\mathbf{P}(\star)$ is uniformly parabolic if $\kappa_1|\boldsymbol{\xi}|^2 \le (\mathbf{P}(\star)\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \le \kappa_2|\boldsymbol{\xi}|^2$ for all $\xi \in \mathbb{R}^{2d}$ and some $\kappa_1, \kappa_2 > 0$; see the definition in, e.g., [8].

We then let $\boldsymbol{\varphi}_\varepsilon = (\varphi_u^\varepsilon, \varphi_v^\varepsilon)$ satisfy the following approximate system:

$$\begin{cases} -\partial_t \boldsymbol{\varphi}_\varepsilon - \mathbf{P}(\tilde{\mathbf{u}}_\varepsilon)^T \Delta \boldsymbol{\varphi}_\varepsilon + \mathbf{Q}(\tilde{\mathbf{u}}_\varepsilon)^T \boldsymbol{\varphi}_\varepsilon = \boldsymbol{\varphi}_\varepsilon \text{ in } \Omega_T, \\ \partial_\nu \boldsymbol{\varphi}_\varepsilon = \mathbf{0} \text{ or } \boldsymbol{\varphi}_\varepsilon = \mathbf{0} \text{ on } \partial\Omega \times (0,T), \\ \boldsymbol{\varphi}_\varepsilon(T) = \boldsymbol{\chi}(\mathbf{x}) \text{ in } \Omega. \end{cases} \tag{2.7}$$

We know that $\tilde{\mathbf{u}}_\varepsilon \in L^\infty(\Omega_T)^2$, which yields $\mathbf{P}(\tilde{\mathbf{u}}_\varepsilon) \in L^\infty(\Omega_T)^4$. This in turn implies that

$$(\mathbf{P}(\tilde{\mathbf{u}}_\varepsilon)\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \le \kappa(\varepsilon)|\boldsymbol{\xi}|^2,$$

where $\kappa(\varepsilon)$ is a constant depending on ε. This bound from above of $\mathbf{P}(\tilde{\mathbf{u}}_\varepsilon)$ and its bound from below in (1.6) give the uniform parabolic condition for the approximate system (2.5). The existence of a smooth function $\boldsymbol{\varphi}_\varepsilon$ is hence given by the classical theory of the equations of parabolic type; see, e.g., [8, Theorem 5.1]. We now bound the approximate solutions $\boldsymbol{\varphi}_\varepsilon$ independently of ε.

Lemma 2.2. *Assume that $d \le 4$ and $\tilde{\mathbf{u}} \in L^\infty(0,T;H^1(\Omega)^2)$. We then have the following a priori bounds independent of ε for the solution $\boldsymbol{\varphi}_\varepsilon$ of* (2.7)*:*

$$\sup_{t\in[0,T]} \|\boldsymbol{\varphi}_\varepsilon\|_{H^1(\Omega)^2} \le \kappa \|\boldsymbol{\chi}\|_{H^1(\Omega)^2}, \tag{2.8a}$$

$$\int_0^T (1 + \tilde{u}_\varepsilon + \tilde{v}_\varepsilon)|\Delta \boldsymbol{\varphi}_\varepsilon|^2 \, dt \le \kappa \|\boldsymbol{\chi}\|_{H^1(\Omega)^2} \tag{2.8b}$$

and

$$\|\partial_t \boldsymbol{\varphi}_\varepsilon\|_{L^{\frac{4}{3}}(\Omega_T)} \le \kappa \|\boldsymbol{\chi}\|. \tag{2.8c}$$

Here, in this lemma, κ depends on $\|\mathbf{u}\|_{L^\infty(0,T;H^1(\Omega)^2)}$ and on the coefficients but is independent of ε.

Proof of Lemma 2.2. Multiplying (2.7) by $\boldsymbol{\varphi}_\varepsilon$, we find

$$-\frac{1}{2}\frac{d}{dt}|\boldsymbol{\varphi}_\varepsilon|^2 - \langle \mathbf{P}(\tilde{\mathbf{u}}_\varepsilon)^T \Delta \boldsymbol{\varphi}_\varepsilon, \boldsymbol{\varphi}_\varepsilon \rangle + \langle \mathbf{Q}(\tilde{\mathbf{u}}_\varepsilon)^T \boldsymbol{\varphi}_\varepsilon, \boldsymbol{\varphi}_\varepsilon \rangle = |\boldsymbol{\varphi}_\varepsilon|^2. \tag{2.9}$$

Multiplying (2.7) by $-\Delta \boldsymbol{\varphi}_\varepsilon$, we also find, after integration by parts,

$$-\frac{1}{2}\frac{d}{dt}|\nabla \boldsymbol{\varphi}_\varepsilon|^2 + \langle \mathbf{P}(\tilde{\mathbf{u}}_\varepsilon)^T \Delta \boldsymbol{\varphi}_\varepsilon, \Delta \boldsymbol{\varphi}_\varepsilon \rangle - \langle \mathbf{Q}(\tilde{\mathbf{u}}_\varepsilon)^T \boldsymbol{\varphi}_\varepsilon, \Delta \boldsymbol{\varphi}_\varepsilon \rangle = |\nabla \boldsymbol{\varphi}_\varepsilon|^2. \tag{2.10}$$

Adding equations (2.9) and (2.10) and regrouping the terms, we find

$$\begin{aligned} &-\frac{1}{2}\frac{d}{dt}(|\boldsymbol{\varphi}_\varepsilon|^2 + |\nabla \boldsymbol{\varphi}_\varepsilon|^2) + \langle \mathbf{P}(\tilde{\mathbf{u}}_\varepsilon)^T \Delta \boldsymbol{\varphi}_\varepsilon, \Delta \boldsymbol{\varphi}_\varepsilon \rangle \\ &\qquad = \langle (\mathbf{P}(\tilde{\mathbf{u}}_\varepsilon) + \mathbf{Q}(\tilde{\mathbf{u}}_\varepsilon)^T) \boldsymbol{\varphi}_\varepsilon, \Delta \boldsymbol{\varphi}_\varepsilon \rangle - \langle \mathbf{Q}(\tilde{\mathbf{u}}_\varepsilon)^T \boldsymbol{\varphi}_\varepsilon, \boldsymbol{\varphi}_\varepsilon \rangle + |\boldsymbol{\varphi}_\varepsilon|^2 + |\nabla \boldsymbol{\varphi}_\varepsilon|^2. \end{aligned} \tag{2.11}$$

We bound the first two terms on the right-hand side of (2.11) as follows:

- We first bound the easier term $\langle \mathbf{Q}(\tilde{\mathbf{u}}_\varepsilon)^T \boldsymbol{\varphi}_\varepsilon, \boldsymbol{\varphi}_\varepsilon \rangle$ using the Hölder inequality for three functions with powers $(2,4,4)$, the Sobolev embedding from H^1 to L^4 in dimension $d \le 4$, and (2.6):
$$|\langle \mathbf{Q}(\tilde{\mathbf{u}}_\varepsilon)^T \boldsymbol{\varphi}_\varepsilon, \boldsymbol{\varphi}_\varepsilon \rangle| \le c_0 |\tilde{\mathbf{u}}_\varepsilon|_{L^2} |\boldsymbol{\varphi}_\varepsilon|_{L^4}^2 \le c_1 \|\tilde{\mathbf{u}}_\varepsilon\|_{H^1} \|\boldsymbol{\varphi}_\varepsilon\|_{H^1}^2 \le \kappa_1(\|\tilde{\mathbf{u}}\|_{L^\infty(0,T;H^1)}) \|\boldsymbol{\varphi}_\varepsilon\|_{H^1}^2.$$
Here $\kappa_1(\|\tilde{\mathbf{u}}\|_{L^\infty(0,T;H^1)})$ is a constant which depends on $\|\tilde{\mathbf{u}}\|_{L^\infty(0,T;H^1)}$ but not on ε.
- We now bound the term $\langle (\mathbf{P}(\tilde{\mathbf{u}}_\varepsilon) + \mathbf{Q}(\tilde{\mathbf{u}}_\varepsilon)^T) \boldsymbol{\varphi}_\varepsilon, \Delta \boldsymbol{\varphi}_\varepsilon \rangle$ using Hölder's inequality for three functions with powers $(4,4,2)$, the previously used Sobolev embedding from H^1 to L^4 which assumes $d \le 4$, the Young inequality, and (2.6):
$$\begin{aligned} |\langle (\mathbf{P}(\tilde{\mathbf{u}}_\varepsilon) + \mathbf{Q}(\tilde{\mathbf{u}}_\varepsilon)^T) \boldsymbol{\varphi}_\varepsilon, \Delta \boldsymbol{\varphi}_\varepsilon \rangle| &\le c_0 |\tilde{\mathbf{u}}_\varepsilon|_{L^4} |\boldsymbol{\varphi}_\varepsilon|_{L^4} |\Delta \boldsymbol{\varphi}_\varepsilon|_{L^2} \\ &\le c_1 \|\tilde{\mathbf{u}}_\varepsilon\|_{H^1} \|\boldsymbol{\varphi}_\varepsilon\|_{H^1} |\Delta \boldsymbol{\varphi}_\varepsilon|_{L^2} \\ &\le c_1 \|\tilde{\mathbf{u}}_\varepsilon\|_{L^\infty(0,T;H^1)} \|\boldsymbol{\varphi}_\varepsilon\|_{H^1} |\Delta \boldsymbol{\varphi}_\varepsilon|_{L^2} \\ &\le \kappa_2(\|\tilde{\mathbf{u}}\|_{L^\infty(0,T;H^1)}) \|\boldsymbol{\varphi}_\varepsilon\|_{H^1}^2 + \frac{d_0}{2} |\Delta \boldsymbol{\varphi}_\varepsilon|_{L^2}^2, \end{aligned}$$

where $d_0 = \min(d_1, d_2)$ and $\kappa_2(\|\tilde{\mathbf{u}}\|_{L^\infty(0,T;H^1)})$ is independent of ε.

Using these two bounds in (2.11), we find

$$-\frac{1}{2}\frac{d}{dt}(|\boldsymbol{\varphi}_\varepsilon|^2 + |\nabla\boldsymbol{\varphi}_\varepsilon|^2) + \langle \mathbf{P}(\tilde{\mathbf{u}}_\varepsilon)^T\Delta\boldsymbol{\varphi}_\varepsilon, \Delta\boldsymbol{\varphi}_\varepsilon\rangle \le \kappa(\|\tilde{\mathbf{u}}\|_{L^\infty(0,T;H^1)})(|\boldsymbol{\varphi}_\varepsilon|^2 + |\nabla\boldsymbol{\varphi}_\varepsilon|^2) + \frac{d_0}{2}|\Delta\boldsymbol{\varphi}_\varepsilon|_{L^2}^2,$$

where $\kappa(\|\tilde{\mathbf{u}}\|_{L^\infty(0,T;H^1)})$ is a constant which depends on $\|\tilde{\mathbf{u}}\|_{L^\infty(0,T;H^1)}$ but is independent of ε.

Thanks to the positivity of $\mathbf{P}(\star)$ in (1.5), we have, using (1.6),

$$-\frac{1}{2}\frac{d}{dt}(|\boldsymbol{\varphi}_\varepsilon|^2 + |\nabla\boldsymbol{\varphi}_\varepsilon|^2) + \Big(\frac{d_0}{2} + \alpha(\tilde{u}_\varepsilon + \tilde{v}_\varepsilon)\Big)|\Delta\boldsymbol{\varphi}_\varepsilon|^2 \le \kappa(\|\tilde{\mathbf{u}}\|_{L^\infty(0,T;H^1)})(|\boldsymbol{\varphi}_\varepsilon|^2 + |\nabla\boldsymbol{\varphi}_\varepsilon|^2).$$

This implies

$$-\frac{d}{dt}(|\boldsymbol{\varphi}_\varepsilon|^2 + |\nabla\boldsymbol{\varphi}_\varepsilon|^2) + \alpha(1 + \tilde{u}_\varepsilon + \tilde{v}_\varepsilon)|\Delta\boldsymbol{\varphi}_\varepsilon|^2 \le \kappa(|\boldsymbol{\varphi}_\varepsilon|^2 + |\nabla\boldsymbol{\varphi}_\varepsilon|^2), \tag{2.12}$$

where again $\kappa = \kappa(\|\tilde{\mathbf{u}}\|_{L^\infty(0,T;H^1)})$.

Recall that $\boldsymbol{\varphi}(T) = \boldsymbol{\chi} \in H^1(\Omega)^2$. Multiplying (2.12) by e^{2t} and integrating over $[t, T]$ for $t \in [0, T]$, we infer (2.8a) and (2.8b).

Now, to derive the bound independent of ε for $\partial_t\boldsymbol{\varphi}_\varepsilon$, we write using the first equation in (2.7):

$$\|\partial_t\boldsymbol{\varphi}_\varepsilon\|_{L^{\frac{4}{3}}} = \|\mathbf{P}(\tilde{\mathbf{u}}_\varepsilon)^T\Delta\boldsymbol{\varphi}_\varepsilon - \mathbf{Q}(\tilde{\mathbf{u}}_\varepsilon)^T\boldsymbol{\varphi}_\varepsilon + \boldsymbol{\varphi}_\varepsilon\|_{L^{\frac{4}{3}}}. \tag{2.13}$$

We bound the most challenging norm term $\|\mathbf{P}(\tilde{\mathbf{u}}_\varepsilon)^T\Delta\boldsymbol{\varphi}_\varepsilon\|_{L^{4/3}}$ on the right-hand side of (2.13). We consider a function $\mathbf{z} \in L^4(\Omega_T)^2$ and write

$$\int_{\Omega_T} \mathbf{P}(\tilde{\mathbf{u}}_\varepsilon)^T\Delta\boldsymbol{\varphi}_\varepsilon\mathbf{z}\,dx\,dt \le (\max_{i=1,2} d_i|\Omega|^{\frac{1}{4}} + 2\max_{i,j=1,2} a_{ij}(\|u_\varepsilon\|_{L^4} + \|v_\varepsilon\|_{L^4}))(\|\Delta\varphi_\varepsilon^u\|_{L^2} + \|\Delta\varphi_\varepsilon^v\|_{L^2})\|\mathbf{z}\|_{L^4}.$$

Observing that $\|u_\varepsilon\|_{L^4} \le \|u\|_{L^4} \le \|u\|_{L^\infty(0,T;H^1)}$, a similar bound for $\|v_\varepsilon\|_{L^4}$, and using (2.8b), we find the following bound for any $\mathbf{z} \in L^4(\Omega_T)$:

$$\int_{\Omega_T} \mathbf{P}(\tilde{\mathbf{u}}_\varepsilon)^T\Delta\boldsymbol{\varphi}_\varepsilon\mathbf{z}\,dx\,dt \le \kappa(\|\mathbf{u}\|_{L^4})|\Delta\boldsymbol{\varphi}_\varepsilon|\|\mathbf{z}\|_{L^4} \le \kappa(\|\mathbf{u}\|_{L^\infty(0,T;H^1)})\|\boldsymbol{\chi}\|\|\mathbf{z}\|_{L^4},$$

where κ depends on $\|\mathbf{u}\|_{L^\infty(0,T;H^1)}$ and on the coefficients d_i, a_{ij} but is independent of ε. This gives the a priori bound (2.8c). □

2.1.2 Passage to the limit for the solutions of the approximate systems

We now pass to the limit as $\varepsilon \to 0$ in the approximate adjoint system (2.7). From the a priori estimates (2.8a)–(2.8c) we have that there exists a subsequence of $\boldsymbol{\varphi}_\varepsilon$, still denoted by $\boldsymbol{\varphi}_\varepsilon$, such that as $\varepsilon \to 0$,

$$\begin{cases} \boldsymbol{\varphi}_\varepsilon \rightharpoonup \boldsymbol{\varphi} \text{ in } L^\infty(0, T; H^1(\Omega)^2) \text{ weak-star}, \\ \Delta\boldsymbol{\varphi}_\varepsilon \rightharpoonup \Delta\boldsymbol{\varphi} \text{ in } L^2(\Omega_T)^2 \text{ weakly}, \\ \partial_t\boldsymbol{\varphi}_\varepsilon \rightharpoonup \partial_t\boldsymbol{\varphi} \text{ in } L^{\frac{4}{3}}(\Omega_T)^2 \text{ weakly}. \end{cases} \tag{2.14}$$

We then pass to the limit term by term in (2.7), where the most challenging product term $\mathbf{P}(\tilde{\mathbf{u}}_\varepsilon)\Delta\boldsymbol{\varphi}_\varepsilon$ is treated as follows: for any $\mathbf{z} \in L^4(\Omega_T)^2$, we write

$$\begin{aligned} &\int_{\Omega_T} (\mathbf{P}(\tilde{\boldsymbol{\varphi}}_\varepsilon)^T\Delta\boldsymbol{\varphi}_\varepsilon - \mathbf{P}(\tilde{\boldsymbol{\varphi}})^T\Delta\boldsymbol{\varphi})\mathbf{z}\,dx\,dt \\ &\quad = \underbrace{\int_{\Omega_T} \Delta\boldsymbol{\varphi}_\varepsilon(\mathbf{P}(\tilde{\mathbf{u}}_\varepsilon) - \mathbf{P}(\tilde{\mathbf{u}}))\mathbf{z}\,dx\,dt}_{T_\varepsilon^1} + \underbrace{\int_{\Omega_T} (\Delta\boldsymbol{\varphi}_\varepsilon - \Delta\boldsymbol{\varphi})\mathbf{P}(\tilde{\mathbf{u}})\mathbf{z}\,dx\,dt}_{T_\varepsilon^2}. \end{aligned}$$

- We first deal with the easier term T_ε^2. We easily see that $T_\varepsilon^2 \to 0$ as $\varepsilon \to 0$ thanks to (2.14) and the facts that $\mathbf{P}(\tilde{\mathbf{u}}) \in L^4(\Omega_T)^4$ and $\mathbf{z} \in L^4(\Omega_T)^2$, which gives $\mathbf{P}(\tilde{\mathbf{u}})\mathbf{z} \in L^2(\Omega_T)^2$.
- For T_ε^1, as $\varepsilon \to 0$, we know that $\tilde{\mathbf{u}}_\varepsilon \to \tilde{\mathbf{u}}$ in $L^4(\Omega_T)^2$ strongly, which gives $\mathbf{P}(\tilde{\mathbf{u}}_\varepsilon) \to \mathbf{P}(\tilde{\mathbf{u}})$ in $L^4(\Omega_T)^4$ strongly. Thus $(\mathbf{P}(\tilde{\mathbf{u}}_\varepsilon) - \mathbf{P}(\tilde{\mathbf{u}}))\mathbf{z}$ converges to $\mathbf{0}$ in $L^2(\Omega_T)^2$ strongly. Since $\Delta\boldsymbol{\varphi}_\varepsilon$ is bounded in $L^2(\Omega_T)^2$ (thanks to (2.8b)), we conclude that $T_\varepsilon^1 \to 0$ as $\varepsilon \to 0$.

We hence have $\mathbf{P}(\tilde{\mathbf{u}}_\varepsilon)\Delta\boldsymbol{\varphi}_\varepsilon \rightharpoonup \mathbf{P}(\tilde{\mathbf{u}})\Delta\boldsymbol{\varphi}$ weakly in $L^{4/3}(\Omega_T)^2$ as $\varepsilon \to 0$. We thus conclude that $\boldsymbol{\varphi}$ is a weak solution of (2.5) as below.

Proposition 2.3. *Under the assumptions that $d \le 4$ and $\tilde{\mathbf{u}} \in L^\infty(0, T; H^1(\Omega)^2)$, the adjoint system* (2.5) *admits a solution $\boldsymbol{\varphi}$ in $L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)^2)$ such that $\partial_t\boldsymbol{\varphi} \in L^{4/3}(\Omega_T)^2$.*

We now resume the work of showing the uniqueness of the solution $\mathbf{u}$ of (1.1).

2.2 Uniqueness result for the SKT system

We recall that $\mathbf{u}_1, \mathbf{u}_2$ are two solutions of (2.1), and we have written $\bar{\mathbf{u}} = \mathbf{u}_1 - \mathbf{u}_2$; we will eventually show that $\bar{\mathbf{u}} = \mathbf{0}$ for a.e. $\mathbf{x} \in \Omega$ and $t > 0$. We also recall that $\bar{\mathbf{u}}$ satisfies (2.3).

Using the existence result of the adjoint problem in Theorem 2.3, we have a solution

$$\boldsymbol{\varphi} = (\phi^u, \phi^v) \in L^2(0, T; H^2(\Omega)^2)$$

of (2.5) with $\partial_t\boldsymbol{\varphi} \in L^{4/3}(\Omega_T)$.

We henceforth infer from (2.4) and (2.5) that

$$\langle \bar{\mathbf{u}}, \boldsymbol{\varphi}\rangle_t + \langle \bar{\mathbf{u}}, \boldsymbol{\varphi}\rangle = \langle \boldsymbol{\ell}(\bar{\mathbf{u}}), \boldsymbol{\varphi}\rangle. \tag{2.15}$$

We look at the first component in (2.15):

$$\langle u_1(t) - u_2(t), \varphi^u\rangle_t = (a_1 - 1)\langle u_1(t) - u_2(t)), \varphi^u\rangle.$$

Multiplying the equation by $e^{-(a_1-1)t}$ and integrating over the time interval $[0, T]$, we find

$$\langle u_1(T) - u_2(T), \chi^u\rangle = 0.$$

This is true for any $\chi^u \in H^1(\Omega)$, and we thus find $u_1(T) = u_2(T)$ for a.e. $\mathbf{x} \in \Omega$. The argument is also valid for any other time $t < T$ which gives $u_1(t) = u_2(t)$ a.e. Similarly, we have $v_1(t) = v_2(t)$ a.e.

We have thus shown the following result.

Theorem 2.4 (Uniqueness). *In space dimension $d \le 4$, the SKT system* (1.1) *admits at most one weak solution $\mathbf{u} \ge \mathbf{0}$ such that*

$$\mathbf{u} \in L^\infty(0, T; H^1(\Omega)^2) \quad \text{and} \quad \partial_t\mathbf{u} \in L^{\frac{4}{3}}(\Omega_T)^2.$$

3 Global well-posedness for the SKT system

In this section, we assume that $d \le 4$ and show that our uniqueness result in Section 2 yields the global well-posedness for solutions of the SKT system (1.1) with the following initial datum conditions:

$$\mathbf{u}_0 \in L^2(\Omega)^2 \quad \text{and} \quad \nabla\mathbf{p}(\mathbf{u}_0) \in L^2(\Omega)^4. \tag{3.1}$$

Remark 3.1. Thanks to the existence result in our prior work [10], whose main result is stated as Theorem A.2, we see that for all $T > 0$, under the assumptions that the space dimension $d \le 4$ and the initial data satisfies (3.1), the SKT system (1.1) possesses solutions $\mathbf{u} \in L^\infty(0, T; H^1(\Omega)^2)$ with $\partial_t\mathbf{u} \in L^{4/3}(\Omega_T)^{4/3}$ as consequences of (A.3c) and (A.4a). Theorem 2.4 thus applies and gives the uniqueness of such a solution $\mathbf{u}$ of (1.1). We then conclude that the solution $\mathbf{u}$ exists globally and uniquely.

Our main result in this section is as follows.

Theorem 3.2. *Suppose that $d \le 4$, that $\mathbf{u}_0$ satisfies* (3.1), *and that the coefficients satisfy* (1.4). *System* (1.1) *possesses a unique global solution $\mathbf{u} \in L^\infty(0,\infty; H^1(\Omega)^2)$ with $\partial_t \mathbf{u} \in L^2(\Omega \times (0,\infty))^2$. Furthermore, the mapping $\mathbf{u}_0 \mapsto \mathbf{u}$ is continuous from $L^q(\Omega)$ into $L^2(\Omega)$ endowed with the norm $|\star|_w$*

$$|\star|_w = \sup_{v \in H^1} \frac{\langle \star, v\rangle}{\|v\|}.$$

Here $q = \max(2d/(6-d), 4d/(d+2))$.

To show the continuous dependance on the initial data, we suppose that $\mathbf{u}_1$ and $\mathbf{u}_2$ are two solutions with initial data $\mathbf{u}_1(0), \mathbf{u}_2(0)$ satisfying (3.1). We proceed as in Section 2 by denoting $\bar{\mathbf{u}} = \mathbf{u}_1 - \mathbf{u}_2$, $\tilde{\mathbf{u}} = (\mathbf{u}_1 + \mathbf{u}_2)/2$, and recall from (2.4) that

$$\langle \bar{\mathbf{u}}, \boldsymbol{\varphi}\rangle_t - \langle \bar{\mathbf{u}}, \boldsymbol{\varphi}_t\rangle - \big\langle \bar{\mathbf{u}}, \mathbf{P}(\tilde{\mathbf{u}})^T \Delta\boldsymbol{\varphi}\big\rangle + \big\langle \bar{\mathbf{u}}, \mathbf{Q}(\tilde{\mathbf{u}})^T \boldsymbol{\varphi}\big\rangle = \langle \boldsymbol{\ell}(\bar{\mathbf{u}}), \boldsymbol{\varphi}\rangle, \tag{3.2}$$

where $\boldsymbol{\varphi}$ solves the following adjoint problem:

$$\begin{cases} -\partial_t \boldsymbol{\varphi} - \mathbf{P}(\tilde{\mathbf{u}})^T \Delta\boldsymbol{\varphi} + \mathbf{Q}(\tilde{\mathbf{u}})^T \boldsymbol{\varphi} = \boldsymbol{\ell}(\boldsymbol{\varphi}) \text{ in } \Omega_\tau = \Omega \times (0,\tau), \\ \partial_\nu \boldsymbol{\varphi} = \mathbf{0} \text{ or } \boldsymbol{\varphi} = \mathbf{0} \text{ on } \partial\Omega \times (0,\tau), \\ \boldsymbol{\varphi}(\tau) = \boldsymbol{\chi} \text{ in } \Omega, \end{cases} \tag{3.3}$$

for $\boldsymbol{\chi}(\mathbf{x}) = (\chi^u(\mathbf{x}), \chi^v(\mathbf{x})) \in H^1(\Omega)^2$ (arbitrary) with appropriate compatible boundary conditions.

The existence of a solution that satisfies the following a priori estimates was proven in Lemma 2.2.

Lemma 3.3 (A priori estimates). *Assume that $d \le 4$ and $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v}) \in L^\infty(0,T; H^1(\Omega)^2)$. We then have the following a priori bounds independent of $\tau \in [0,T]$ for the solutions $\boldsymbol{\varphi}$ of* (3.3)*:*

$$\sup_{t \in [0,\tau]} \|\boldsymbol{\varphi}\|_{H^1(\Omega)^2} \le \kappa \|\boldsymbol{\chi}\|_{H^1(\Omega)^2}, \tag{3.4a}$$

$$\int_0^\tau (1 + \tilde{u} + \tilde{v})|\Delta\boldsymbol{\varphi}|^2 \, dt \le \kappa \|\boldsymbol{\chi}\|_{H^1(\Omega)^2}. \tag{3.4b}$$

Here, κ depends on $\|\tilde{\mathbf{u}}\|_{L^\infty(0,T;H^1(\Omega)^2)}$, T, and on the coefficients but is independent of τ.

We now continue to show the continuous dependance of $\mathbf{u}$ on the data. We find from equations (3.2) and (3.3) that

$$\langle \bar{\mathbf{u}}, \boldsymbol{\varphi}\rangle_t = 0.$$

Thus

$$\langle \bar{\mathbf{u}}(\tau), \boldsymbol{\chi}\rangle = \langle \bar{\mathbf{u}}(0), \boldsymbol{\varphi}(0)\rangle,$$

where we have used $\boldsymbol{\varphi}(\tau) = \boldsymbol{\chi}$.

We now use the first equation in (3.3) and find

$$\begin{aligned} \langle \bar{\mathbf{u}}(\tau), \boldsymbol{\chi}\rangle &= \Big\langle \bar{\mathbf{u}}(0), \boldsymbol{\chi} + \int_0^\tau [\mathbf{P}(\tilde{\mathbf{u}})^T \Delta\boldsymbol{\varphi} - \mathbf{Q}(\tilde{\mathbf{u}})\boldsymbol{\varphi} + \boldsymbol{\ell}(\boldsymbol{\varphi})] \, dt \Big\rangle \\ &= \langle \bar{\mathbf{u}}(0), \boldsymbol{\chi}\rangle + \Big\langle \bar{\mathbf{u}}(0), \int_0^\tau [\mathbf{P}(\tilde{\mathbf{u}})^T \Delta\boldsymbol{\varphi} - \mathbf{Q}(\tilde{\mathbf{u}})\boldsymbol{\varphi} + \boldsymbol{\ell}(\boldsymbol{\varphi})] \, dt \Big\rangle \end{aligned} \tag{3.5}$$

We next bound the typical terms on the right-hand side of (3.5):

- We first bound a typical term $\langle \bar{u}(0), \tilde{u}\Delta\phi^u\rangle$ in $\langle \bar{\mathbf{u}}(0), \mathbf{P}^T(\tilde{\mathbf{u}})\Delta\boldsymbol{\varphi}\rangle$ as follows:

$$\begin{aligned} \langle \bar{u}(0), \tilde{u}\Delta\phi^u\rangle &= \langle \underbrace{\bar{u}(0)}_{\in L^{\frac{4d}{d+2}}}, \underbrace{\tilde{u}^{\frac{1}{2}}}_{\in L^{\frac{4d}{d-2}}} \underbrace{\tilde{u}^{\frac{1}{2}}\Delta\phi^u}_{\in L^2}\rangle \\ &\le |\bar{u}(0)|_{L^{\frac{4d}{d+2}}} |\tilde{u}|^{\frac{1}{2}}_{L^{\frac{2d}{d-2}}} |\tilde{u}^{\frac{1}{2}}\Delta\phi^u|_{L^2} \\ &\le c|\bar{u}(0)|_{L^{\frac{4d}{d+2}}} |\tilde{u}|^{\frac{1}{2}}_{L^\infty(0,T;H^1)} |\tilde{u}^{\frac{1}{2}}\Delta\phi^u|_{L^2}. \end{aligned}$$

Thus, by (3.4b), we find

$$\left\langle \bar{\mathbf{u}}(0), \int_0^\tau \tilde{\mathbf{u}}\Delta\boldsymbol{\varphi}\,dt \right\rangle \leq c|\bar{\mathbf{u}}(0)|_{L^{\frac{4d}{d+2}}} |\tilde{\mathbf{u}}|^{\frac{1}{2}}_{L^\infty(0,T;H^1)} \int_0^\tau |\tilde{\mathbf{u}}^{\frac{1}{2}}\Delta\boldsymbol{\varphi}|_{L^2} \leq cT^{\frac{1}{2}}|\bar{\mathbf{u}}(0)|_{L^{\frac{4d}{d+2}}} |\tilde{\mathbf{u}}|^{\frac{1}{2}}_{L^\infty(0,T;H^1)} \|\boldsymbol{\chi}\|^{\frac{1}{2}}_{H^1}. \tag{3.6}$$

- We now bound the term $\langle \bar{\mathbf{u}}(0), \int_0^\tau \mathbf{Q}(\tilde{\mathbf{u}})\boldsymbol{\varphi}\,dt\rangle$ by bounding its typical term $\int_0^\tau \langle \bar{u}(0), \tilde{u}^2\phi^u\rangle\,dt$:

$$\langle \underbrace{\bar{u}(0)}_{\in L^{\frac{2d}{6-d}}}, \underbrace{\tilde{u}^2}_{\in L^{\frac{d}{d-2}}} \underbrace{\phi^u}_{\in L^{\frac{2d}{d-2}}} \rangle \leq |\bar{u}(0)|_{L^{\frac{2d}{6-d}}} |\tilde{u}|^2_{L^{\frac{2d}{d-2}}} |\phi^u|_{L^{\frac{2d}{d-2}}} \leq c|\bar{u}(0)|_{L^{\frac{2d}{6-d}}} \|\tilde{u}\|^2_{L^\infty(0,T;H^1)} \|\phi^u\|_{L^\infty(0,T;H^1)}.$$

Therefore, by (3.4a), we find

$$\left\langle \bar{\mathbf{u}}(0), \int_0^\tau \mathbf{Q}(\tilde{\mathbf{u}})\boldsymbol{\varphi}\,dt \right\rangle \leq cT|\bar{\mathbf{u}}(0)|_{L^{\frac{2d}{6-d}}} \|\tilde{\mathbf{u}}\|^2_{L^\infty(0,T;H^1)} \|\boldsymbol{\chi}\|_{H^1}. \tag{3.7}$$

- We finally bound

$$\left\langle \bar{\mathbf{u}}(0), \int_0^\tau \ell(\boldsymbol{\varphi})\,dt \right\rangle \leq c|\bar{\mathbf{u}}(0)| \int_0^\tau |\boldsymbol{\varphi}|\,dt \leq cT^{\frac{1}{2}}|\bar{\mathbf{u}}(0)|\|\boldsymbol{\chi}\|^{\frac{1}{2}}. \tag{3.8}$$

We therefore infer from (3.5)–(3.8) that

$$\langle \bar{\mathbf{u}}(\tau), \boldsymbol{\chi}\rangle \leq \langle \bar{\mathbf{u}}(0), \boldsymbol{\chi}\rangle + \kappa([|\tilde{\mathbf{u}}|^{\frac{1}{2}}_{L^\infty(0,T;H^1)} + 1]\|\boldsymbol{\chi}\|^{\frac{1}{2}}_{H^1} + \|\tilde{\mathbf{u}}\|^2_{L^\infty(0,T;H^1)}\|\boldsymbol{\chi}\|_{H^1})|\bar{\mathbf{u}}(0)|_{L^q(\Omega)^2},$$

where $\kappa = \kappa(T)$ is independent of τ and $q = \max(2d/(6-d), 4d/(d+2))$.

Taking the supremum over $\boldsymbol{\chi}$ with $\|\boldsymbol{\chi}\| = \|\chi\|_{H^1(\Omega)^2} \leq 1$, we conclude that

$$\sup_{\chi\in H^1:\|\chi\|\leq 1} \langle \mathbf{u}_1(\tau) - \mathbf{u}_2(\tau), \boldsymbol{\chi}\rangle \leq |\mathbf{u}_1(0) - \mathbf{u}_2(0)| + \kappa|\mathbf{u}_1(0) - \mathbf{u}_2(0)|_{L^q(\Omega)^2},$$

where $\kappa = \kappa(T, \|\mathbf{u}_1 + \mathbf{u}_2\|_{L^\infty(0,T;H^1)})$.

A Appendices

A.1 A technical lemma

Lemma A.1. *Suppose that* $\mathbf{p}$, $\mathbf{q}$ *are as in* (1.2) *and* $\mathbf{P}$, $\mathbf{Q}$ *are as in* (1.7). *We then have*

$$\mathbf{p}(\mathbf{u}_1) - \mathbf{p}(\mathbf{u}_2) = \mathbf{P}(\tilde{\mathbf{u}})\bar{\mathbf{u}} \tag{A.1}$$

and

$$\mathbf{q}(\mathbf{u}_1) - \mathbf{q}(\mathbf{u}_2) = \mathbf{Q}(\tilde{\mathbf{u}})\bar{\mathbf{u}}, \tag{A.2}$$

where $\tilde{\mathbf{u}} = (\mathbf{u}_1 + \mathbf{u}_2)/2$ *and* $\bar{\mathbf{u}} = \mathbf{u}_1 - \mathbf{u}_2$.

Proof. We write

$$\begin{aligned}
\mathbf{p}(\mathbf{u}_1) - \mathbf{p}(\mathbf{u}_2) &= \mathbf{p}(\mathbf{u}_2 + \bar{\mathbf{u}}) - \mathbf{p}(\mathbf{u}_2) = \int_0^1 \frac{d}{dt}\mathbf{p}(\mathbf{u}_2 + t\bar{\mathbf{u}})\,dt = \int_0^1 \frac{D\mathcal{P}}{D\mathbf{u}}(\mathbf{u}_2 + t\bar{\mathbf{u}}) \cdot \bar{\mathbf{u}}\,dt \\
&= \int_0^1 \begin{pmatrix} d_1 + 2a_{11}(u_2 + t\bar{u}) + a_{12}(v_2 + t\bar{v}) & a_{12}(u_2 + t\bar{u}) \\ a_{21}(v_2 + t\bar{v}) & d_2 + a_{21}(u_2 + t\bar{u}) + 2a_{22}(v_2 + t\bar{v}) \end{pmatrix} \cdot \bar{\mathbf{u}}\,dt \\
&= \begin{pmatrix} d_1 + a_{11}(u_1 + u_2) + a_{12}(v_1 + v_2)/2 & a_{12}(u_1 + u_2)/2 \\ a_{21}(v_1 + v_2)/2 & d_2 + a_{21}(u_1 + u_2)/2 + a_{22}(v_1 + v_2) \end{pmatrix} \cdot \bar{\mathbf{u}} \\
&= \mathbf{P}(\tilde{\mathbf{u}}) \cdot \bar{\mathbf{u}}.
\end{aligned}$$

We thus proved (A.1) and we can derive (A.2) in the same fashion. □

A.2 Existence result for SKT systems

In [10, Theorem 3.1], we proved the following existence result for the SKT system (1.1).

Theorem A.2 (Existence of solutions for the SKT).
(i) *We assume that* $d \leq 4$, *that conditions* (1.4) *hold, and that* $\mathbf{u}_0$ *is given,* $\mathbf{u}_0 \in L^2(\Omega)^2$, $\mathbf{u}_0 \geq 0$. *Then equation* (1.1) *possesses a solution* $\mathbf{u} \geq \mathbf{0}$ *such that, for every* $T > 0$,

$$\mathbf{u} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)^2), \tag{A.3a}$$

$$(\sqrt{u} + \sqrt{v})(|\nabla u| + |\nabla v|) \in L^2(0, T; L^2(\Omega)), \tag{A.3b}$$

$$\mathbf{u} \in L^4(0, T; L^4(\Omega)), \tag{A.3c}$$

with the norms in these spaces bounded by a constant depending on T, *on the coefficients, and on the norms in* $L^2(\Omega)$ *of* u_0 *and* v_0.
(ii) *If, in addition,* $\nabla \mathbf{p}(\mathbf{u}_0) \in L^2(\Omega)^4$, *then the solution* $\mathbf{u}$ *also satisfies*

$$\nabla \mathbf{p}(\mathbf{u}) \in L^\infty(0, T; L^2(\Omega)^4), \quad (1 + |u| + |v|)^{\frac{1}{2}}(|\partial_t u| + |\partial_t v|) \in L^2(0, T; L^2(\Omega)), \tag{A.4a}$$

$$\Delta \mathbf{p}(\mathbf{u}) \in L^2(0, T; L^2(\Omega)^2), \tag{A.4b}$$

with the norms in these spaces bounded by a constant depending on the norms of $\mathbf{u}_0$ *and* $\nabla \mathbf{p}(\mathbf{u}_0)$ *in* L^2 *(and on* T *and the coefficients).*

A.3 Additional regularity of weak solutions

Although this was not explicitly stated in [10], the solutions that we constructed in dimension $d \leq 4$ belong to $L^\infty_t(H^1)$ with $\partial_t \mathbf{u}$ in $L^2_t(L^2)$:

- From (A.3a) and (A.4a) we have $\mathbf{u}, \nabla \mathbf{p}(\mathbf{u}) \in L^\infty(0, T; L^2(\Omega)^2)$. To show that $\nabla \mathbf{u} \in L^\infty(0, T; L^2(\Omega)^2)$, we note that (1.6) implies, for $u, v \geq 0$, that $\mathbf{P}(\mathbf{u})$ is invertible (as a 2×2 matrix), and that, pointwise (i.e. for a.e. $x \in \Omega$)

$$|\mathbf{P}(\mathbf{u})^{-1}|_{\mathcal{L}(\mathbb{R}^2)} \leq \frac{1}{d_0 + \alpha(u + v)}.$$

  We thus find $\nabla \mathbf{u} \in L^\infty(0, T; L^2(\Omega)^2)$ which says that $\mathbf{u} \in L^\infty(0, T; H^1(\Omega)^2)$.
- From (A.4a) we have $\partial_t \mathbf{u} \in L^2(\Omega_T)^2$.

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