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Research Article

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Periodic impulsive fractional differential equations

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Abstract: This paper deals with the existence of periodic solutions of fractional differential equations with periodic impulses. The first part of the paper is devoted to the uniqueness, existence and asymptotic stability results for periodic solutions of impulsive fractional differential equations with varying lower limits for standard nonlinear cases as well as for cases of weak nonlinearities, equidistant and periodically shifted impulses. We also apply our result to an impulsive fractional Lorenz system. The second part extends the study to periodic impulsive fractional differential equations with fixed lower limit. We show that in general, there are no solutions with long periodic boundary value conditions for the case of bounded nonlinearities.

Keywords: Fractional differential equations, impulses, periodic solutions, existence result

MSC 2010: 34A08, 34A37, 34C25

1 Introduction

It is well known that fractional differential equations (FDE) have no nonconstant periodic solutions [7]. Then asymptotically periodic solutions are shown [1, 2, 15]. This paper is devoted to this problem. We first study impulsive FDE (IFDE) with Caputo derivatives when the lower limits are periodically varying at each impulses. We show that such kind of IFDE do may possess periodic solutions. We present several particular cases of IFDE along with nonexistence results. For completeness of the paper, we also discuss IFDE when the lower limit of the Caputo derivative is fixed, which is introduced in [5, 14]. Then there is also a problem on the definition of solutions for such kind of IFDE; this is mentioned in [16]. Here we just study periodic boundary value conditions on finite intervals. For more development, methods, and the current research on impulsive and functional fractional differential equations, we refer to [12].

The paper is organized as follows: Section 2 is devoted to the study of IFDE with varying lower limits for Caputo derivatives. First, in Section 2.1, we deal with general IFDE to show uniqueness, existence and asymptotic stability results for periodic solutions. In Section 2.2, we study IFDE with weak nonlinearities leading to the averaging theory for IFDE like in [11]. Section 2.3 is devoted to IFDE with equidistant and periodically shifted impulses. This specific form allows us to obtain a nonexistence result for periodic solutions which is applied to an impulsive fractional Lorenz system. Then we obtain Landesman–Lazer-type existence results for periodic solutions. Section 2.4 continues the investigation of IFDE with equidistant and periodically shifted impulses, but all are small. The corresponding limiting ODE is derived and, by using known results from numerical dynamics, the relationship between dynamics of the limiting ODE and origi-

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nal IFDE is studied. For completeness of the paper, Section 3 deals with IFDE with fixed lower limit of Caputo derivative. We extend some results of Section 2 to that case. Several concrete examples are given in the paper for illustration of theoretical results. Section 4 briefly outlines further possible ways for studies.

2 FDE with Caputo derivatives with varying lower limits

2.1 General impulses

Let $\mathbb{N}_0 = \{0, 1, \dots, \infty\}$. We consider

$$\begin{cases} {}^cD^q_{t_k,t}x(t) = f(t, x(t)), & t \in (t_k, t_{k+1}),\ k \in \mathbb{N}_0, \\ x(t_k^+) = x(t_k^-) + \Delta_k(x(t_k^-)), & k \in \mathbb{N}, \\ x(0) = x_0, \end{cases} \tag{2.1}$$

where $q \in (0, 1)$, ${}^cD^q_{t_k,t}x(t)$ is the generalized Caputo fractional derivative with lower limit at t_k. We suppose the following conditions:

(i) $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous and T-periodic in t.

(ii) There are constants $K > 0$, $L_k > 0$ such that

$$\|f(t, x) - f(t, y)\| \le K\|x - y\| \quad \text{and} \quad \|x + \Delta_k(x) - y - \Delta_k(y)\| \le L_k\|x - y\|$$

for any $t \in \mathbb{R}$, $x, y \in \mathbb{R}^m$ and any $k \in \mathbb{N}$.

(iii) There is $N \in \mathbb{N}$ such that $T = t_{N+1}$, $t_{k+N+1} = t_k + T$ and $\Delta_{k+N+1} = \Delta_k$ for any $k \in \mathbb{N}$.

It is well known [8] that under assumptions (i) and (ii), (2.1) has a unique solution on $\mathbb{R}_+$. So we can consider the Poincaré mapping

$$P(x_0) = x(T^-) + \Delta_{N+1}(x(T^-)).$$

Clearly fixed points of P determine T-periodic solutions of (2.1) (see [4, Lemma 2.2]).

Lemma 2.1. *Under assumptions* (i) *and* (ii), *there holds*

$$\|P(x) - P(y)\| \le \Theta\|x - y\| \quad \textit{for all } x,\ \textit{for all } y \in \mathbb{R}^m$$

for $\Theta = \prod_{k=0}^{N} L_{k+1}E_q(K(t_{k+1} - t_k)^q)$, *where* E_q *is the Mittag-Leffler function (see* [8, p. 40]*).*

Proof. On all intervals (t_k, t_{k+1}), $k = 0, 1, \dots, N$, equation (2.1) is equivalent to

$$x(t) = x(t_k^+) + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} f(s, x(s))\, ds. \tag{2.2}$$

Consider two solutions x and y of (2.1) with $x(0) = x_0$ and $y(0) = y_0$, respectively. Then by (2.2), we get

$$\begin{aligned} \|x(t) - y(t)\| &\le \|x(t_k^+) - y(t_k^+)\| + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \|f(s, x(s)) - f(s, y(s))\|\, ds \\ &\le \|x(t_k^+) - y(t_k^+)\| + \frac{K}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \|x(s) - y(s)\|\, ds. \end{aligned} \tag{2.3}$$

Applying a Gronwall fractional inequality [17, Corollary 2] to (2.3), we obtain

$$\|x(t) - y(t)\| \le \|x(t_k^+) - y(t_k^+)\| E_q(K(t - t_k)^q), \quad t \in (t_k, t_{k+1}). \tag{2.4}$$

Then using (2.4) with (ii), we get

$$\|x(t_{k+1}^+) - y(t_{k+1}^+)\| \le L_{k+1}E_q(K(t_{k+1} - t_k)^q)\|x(t_k^+) - y(t_k^+)\|, \quad k = 0, 1, \dots, N,$$

which implies

$$\|P(x_0) - P(y_0)\| \le \prod_{k=0}^{N} L_{k+1} E_q(K(t_{k+1} - t_k)^q)\|x_0 - y_0\|, \tag{2.5}$$

as desired. □

Now we can prove the following result.

Theorem 2.2. *Suppose* (i)–(iii) *are satisfied. If* $\Theta < 1$, *then* (2.1) *has a unique T-periodic solution, which is in addition asymptotically stable.*

Proof. Inequality (2.5) implies that $P : \mathbb{R}^m \to \mathbb{R}^m$ is a contraction, so applying the Banach fixed point theorem yields that P has a unique fixed point x_0. Moreover, since

$$\|P^n(x_0) - P^n(y_0)\| \le \Theta^n \|x_0 - y_0\|$$

for any $y_0 \in \mathbb{R}^m$, we see that the corresponding periodic solution is asymptotically stable. □

2.2 Systems with weak nonlinearities

Now we consider (2.1) with small nonlinearities of the form

$$\begin{cases} {}^cD^q_{t_k,t}x(t) = \varepsilon f(t, x(t)), & t \in (t_k, t_{k+1}),\ k \in \mathbb{N}_0, \\ x(t_k^+) = x(t_k^-) + \varepsilon\Delta_k(x(t_k^-)), & k \in \mathbb{N}, \\ x(0) = x_0, \end{cases} \tag{2.6}$$

where ε is a small parameter. Then (2.6) has a unique solution $x(\varepsilon, t)$ and the Poincaré mapping is given by

$$P(\varepsilon, x_0) = x(\varepsilon, T^-) + \varepsilon\Delta_{N+1}(x(\varepsilon, T^-)).$$

We suppose that

(C1) f and Δ_k are C^2-smooth.

Then $P(\varepsilon, x_0)$ is C^2-smooth as well. Moreover, we have

$$x(\varepsilon, t) = x_0 + \varepsilon v(t) + O(\varepsilon^2),$$

and thus

$$\begin{cases} P(\varepsilon, x_0) = x_0 + \varepsilon M(x_0) + O(\varepsilon^2), \\ M(x_0) = v(T^-) + \Delta_{N+1}(x_0). \end{cases} \tag{2.7}$$

Note that $v(t)$ solves

$$\begin{cases} {}^cD^q_{t_k,t}v(t) = f(t, x_0), & t \in (t_k, t_{k+1}),\ k = 0, 1, \ldots, N, \\ v(t_k^+) = v(t_k^-) + \Delta_k(x_0), & k = 1, 2, \ldots, N+1, \\ v(0) = 0, \end{cases}$$

so we easily derive

$$v(T^-) = \sum_{k=1}^{N} \Delta_k(x_0) + \frac{1}{\Gamma(q)} \sum_{k=0}^{N} \int_{t_k}^{t_{k+1}} (t_{k+1} - s)^{q-1} f(s, x_0)\, ds.$$

Consequently, (2.7) gives

$$M(x_0) = \sum_{k=1}^{N+1} \Delta_k(x_0) + \frac{1}{\Gamma(q)} \sum_{k=0}^{N} \int_{t_k}^{t_{k+1}} (t_{k+1} - s)^{q-1} f(s, x_0)\, ds.$$

Now we are ready to prove the following result.

Theorem 2.3. *Suppose assumptions* (i) *and* (C1). *If there is a simple zero* $x_0 \in \mathbb{R}^m$ *of* M, *i.e.,* $M(x_0) = 0$ *and* $\det DM(x_0) \neq 0$, *then* (2.6) *has a unique* T*-periodic solution located near* x_0 *for any* $\varepsilon \neq 0$ *small. Moreover, if* $\Re\sigma(DM(x_0)) < 0$ *then it is asymptotically stable, and if* $\Re\sigma(DM(x_0)) \cap (0,\infty) \neq \emptyset$, *then it is unstable. If* $M(x_0) \neq 0$ *for any* $x_0 \in \mathbb{R}^m$, *then* (2.6) *has no* T*-periodic solution for* $\varepsilon \neq 0$ *small.*

Proof. To find a T-periodic solution of (2.6), we need to solve $P(\varepsilon, x_0) = x_0$, which by (2.7) is equivalent to

$$M(x_0) + O(\varepsilon) = 0. \tag{2.8}$$

If there is a simple zero x_0 of M, then (2.8) can be solved by the implicit function theorem to get its solution $x_0(\varepsilon)$ with $x_0(0) = x_0$. Moreover, $DP(\varepsilon, x_0(\varepsilon)) = 1 + \varepsilon DM(x_0) + O(\varepsilon^2)$. Thus stability and instability results follow directly by the known arguments (see [10]), so we omit details. □

2.3 Systems with equidistant and periodically shifted impulses

We consider

$$\begin{cases} {}^cD^q_{kh}x(t) = f(t, x(t)), & t \in (kh, (k+1)h),\ k \in \mathbb{N}_0, \\ x(kh^+) = x(kh^-) + \Delta_k, & k \in \mathbb{N}, \\ x(0) = x_0, \end{cases} \tag{2.9}$$

where $h > 0$ and $q \in (0, 1)$. We consider the norm $\|x\| = \max_{i=1,\dots,m}|x_i|$ for $x = (x_1, \dots, x_m) \in \mathbb{R}^m$. We suppose the following conditions:

(I) $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous, locally Lipschitz in x and T-periodic in t, where $T = (N+1)h$ for some $N \in \mathbb{N}$.

(II) There is a constant $M > 0$ such that $\|f(t, x)\| \leq M$ for any $t, x \in \mathbb{R}$.

(III) The $\Delta_k \in \mathbb{R}^m$ satisfy $\Delta_{k+N+1} = \Delta_k$ for any $k \in \mathbb{N}$.

We are looking for T-periodic solutions of (2.9) on $\mathbb{R}_+$. It is well known [8] that under above assumptions, (2.9) has a unique solution $x(x_0, t)$ on $\mathbb{R}_+$, which is continuous in $x_0 \in \mathbb{R}^m$, $t \in \mathbb{R}_+ \setminus \{kh \mid k \in \mathbb{N}\}$ and left continuous in t at impulsive points $\{kh \mid k \in \mathbb{N}\}$. So we can consider the Poincaré mapping

$$P(x_0) = x(x_0, T^+).$$

Clearly fixed points of P determine T-periodic solutions of (2.9).

Lemma 2.4. *Under assumptions* (I)–(III), *there holds*

$$P^r(x_0) = x_0 + \sum_{k=1}^{r(N+1)} \Delta_k + \frac{1}{\Gamma(q)} \sum_{k=0}^{rN} \int_{kh}^{(k+1)h} ((k+1)h - s)^{q-1} f(s, x(x_0, s))\, ds \tag{2.10}$$

and

$$\|x(x_0, t) - x_0\| \leq \sum_{k=1}^{N} \|\Delta_k\| + \frac{M(N+1)h^q}{\Gamma(q+1)} \tag{2.11}$$

for $t \in I$ *and* $r \in \mathbb{N}$, *where* P^r *is the rth iteration of* P.

Proof. On all intervals $(kh, (k+1)h)$, $k \in \mathbb{N}_0$, equation (2.9) is equivalent to

$$x(x_0, t) = x(kh^+) + \frac{1}{\Gamma(q)} \int_{kh}^{t} (t-s)^{q-1} f(s, x(x_0, s))\, ds,$$

which implies

$$x(x_0, t) = x_0 + \sum_{k=1}^{n} \Delta_k + \frac{1}{\Gamma(q)} \sum_{k=0}^{n-1} \int_{kh}^{(k+1)h} ((k+1)h - s)^{q-1} f(s, x(x_0, s))\, ds + \frac{1}{\Gamma(q)} \int_{nh}^{t} (t-s)^{q-1} f(s, x(x_0, s))\, ds$$

for $t \in (nh, (n+1)h)$. This implies (2.10) since $P^r(x_0) = x(x_0, rT^+)$. Moreover, we have

$$\begin{aligned} |x(x_0, t) - x_0| &\le \sum_{k=1}^{N} \|\Delta_k\| + \frac{M}{\Gamma(q)} \sum_{k=0}^{N} \int_{kh}^{(k+1)h} ((k+1)h - s)^{q-1}\, ds \\ &= \sum_{k=1}^{N} \|\Delta_k\| + \frac{M(N+1)h^q}{\Gamma(q+1)} \end{aligned}$$

for $t \in I$. This implies (2.11). □

Now we can prove the following result.

Theorem 2.5. *Suppose assumptions* (I)–(III). *If*

$$\frac{\|\sum_{k=1}^{N+1} \Delta_k\|}{N+1} > \frac{Mh^q}{\Gamma(q+1)}, \tag{2.12}$$

then (2.9) *has no rT-periodic solution for any $r \in \mathbb{N}$.*

Proof. We need to solve $P^r(x_0) = x_0$, which is equivalent to

$$-\sum_{k=1}^{r(N+1)} \Delta_k = \frac{1}{\Gamma(q)} \sum_{k=0}^{rN} \int_{kh}^{(k+1)h} ((k+1)h - s)^{q-1} f(s, x(x_0, s))\, ds. \tag{2.13}$$

This gives

$$\begin{aligned} r\Big\|\sum_{k=1}^{N+1} \Delta_k\Big\| &\le \frac{1}{\Gamma(q)} \sum_{k=0}^{rN} \int_{kh}^{(k+1)h} ((k+1)h - s)^{q-1} |f(s, x(x_0, s))|\, ds \\ &\le \frac{Mr(N+1)h^q}{\Gamma(q+1)}. \end{aligned}$$

This contradicts (2.12). □

Example 2.6. We consider the Lorenz system

$$f(x_1, x_2, x_3) = (a(x_2 - x_1), x_1(b - x_3) - x_2, x_1x_2 - cx_3), \tag{2.14}$$

which for $q = 0.995$, $a = 10$, $b = 120$, $c = \frac{8}{3}$, and without impulses presents a chaotic attractor (see Fig. 1). The utilized numerical scheme to integrate Lorenz's system is the predictor-corrector Adams–Bashforth–Moulton method for FDEs [3].

Take $N = 1$ and $t_k = kh$, $k \in \mathbb{N}_0$. Let

$$\mathcal{C} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : -50 \le x_1 \le 50,\ -100 \le x_2 \le 100,\ 50 \le x_3 \le 200\},$$

which embeds Lorenz's attractor. To apply Theorem 2.5, we need to find

$$\begin{aligned} M &= \max_{x \in \mathcal{C}} \|f(x)\| \\ &= \max_{x \in \mathcal{C}} \max\{|a(x_2 - x_1)|, |x_1(b - x_3) - x_2|, |x_1x_2 - cx_3|\} \doteq 533.333. \end{aligned}$$

Let $\Delta_k = (\Delta_{k1}, \Delta_{k2}, \Delta_{k3})$. For $\Delta_{11} = \Delta_{21} = \bar{\Delta}_1$, $\Delta_{12} = 0$, $\Delta_{22} = \bar{\Delta}_2$, and $\Delta_{31} = \Delta_{32} = \bar{\Delta}_3$, condition (2.12) has the form

$$\max\Big\{|\bar{\Delta}_1|, \frac{|\bar{\Delta}_2|}{2}, |\bar{\Delta}_3|\Big\} > \frac{Mh^q}{\Gamma(q+1)} \doteq 2.19767$$

for $h = 0.004$. Note that $T = (N+1)h = 0.008$. Summarizing, we obtain the following proposition.

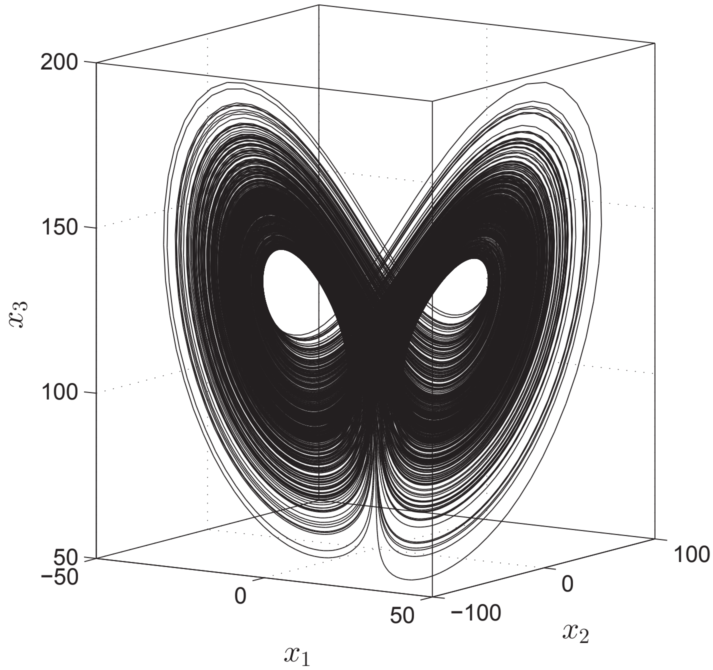

Figure 1: Chaotic attractor of (2.14).

Proposition 2.7. *Consider* (2.9) *for* $N = 1$, $h = 0.004$, $q = 0.995$ *and with* (2.14) *for* $a = 10$, $b = 120$ *and* $c = \frac{8}{3}$. *Let* $\Delta_{11} = \Delta_{21} = \bar{\Delta}_1$, $\Delta_{12} = 0$, $\Delta_{22} = \bar{\Delta}_2$, *and* $\Delta_{31} = \Delta_{32} = \bar{\Delta}_3$. *If either* $|\bar{\Delta}_1| > 2.19767$ *or* $|\bar{\Delta}_2| > 4.39535$ *or* $|\bar{\Delta}_3| > 2.19767$, *then our system has no* $0.008r$ *periodic solution in* $\mathbb{C}$ *for any* $r \in \mathbb{N}$.

Now we present existence results. Let $f(t,x) = (f_1(t,x), \dots, f_m(t,x))$, $\Delta_k = (\Delta_{k1}, \dots, \Delta_{km})$ and set

$$\hat{x}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m) \in \mathbb{R}^{m-1}$$

for any $i = 1, \dots, m$. Now we are ready to prove the following result.

Theorem 2.8. *Suppose* (I)–(III) *and, in addition, there are constants* $f^i_\pm \in \mathbb{R}$, $i = 1, \dots, m$, *such that either*

$$\liminf_{x_i \to \infty} f_i(t,x) \ge f^i_+ > f^i_- \ge \limsup_{x_i \to -\infty} f_i(t,x) \tag{C1}$$

uniformly with respect to $t \in I$ *and* $\hat{x}_i \in \mathbb{R}^{m-1}$ *for all* $i = 1, \dots, m$, *or*

$$\limsup_{x_i \to \infty} f_i(t,x) \le f^i_+ < f^i_- \le \liminf_{x_i \to -\infty} f_i(t,x) \tag{C2}$$

uniformly with respect to $t \in I$ *and* $\hat{x}_i \in \mathbb{R}^{m-1}$ *for all* $i = 1, \dots, m$. *If*

$$-\frac{\sum_{k=1}^{N+1} \Delta_{ki}}{N+1} \in \frac{h^q}{\Gamma(q+1)} (f^i_-, f^i_+) \tag{2.15}$$

for all $i = 1, \dots, m$, *then* (2.9) *has a* T*-periodic solution.*

Proof. We need to solve (2.13). For simplicity, we set

$$F(x_0) = \frac{1}{\Gamma(q)} \sum_{k=0}^{N} \int\limits_{kh}^{(k+1)h} ((k+1)h - s)^{q-1} f(s, x(x_0, s))\, ds.$$

So we need to solve

$$-\sum_{k=1}^{N+1} \Delta_k = F(x_0). \tag{2.16}$$

Let $F(x_0) = (F_1(x_0), \ldots, F_m(x_0))$. By (2.11), there holds $\lim_{x_{0i}\to\pm\infty} x_i(x_0, t) = \pm\infty$ uniformly with respect to $t \in I$ and $\hat{x}_{0i} \in \mathbb{R}^{m-1}$, where $x_0 = (x_{01}, \ldots, x_{0m})$ and $x(x_0, t) = (x_1(x_0, t), \ldots, x_m(x_0, t))$. So by the first possibility (C1), for any $\varepsilon > 0$, there is $m_\varepsilon > 0$ such that

$$
\begin{aligned}
f_i(s, x(x_0, s)) &> f_+^i - \varepsilon \quad \text{for all } x_{0i} > m_\varepsilon,\\
f_i(s, x(x_0, s)) &< f_-^i + \varepsilon \quad \text{for all } x_{0i} < -m_\varepsilon
\end{aligned}
$$

uniformly with respect to $s \in I$ and $\hat{x}_{0i} \in \mathbb{R}^{m-1}$ for all $i = 1, \ldots, m$. This implies

$$
F_i(x_0) \geq \frac{1}{\Gamma(q)} \sum_{k=0}^{N} \int_{kh}^{(k+1)h} ((k+1)h - s)^{q-1}(f_+^i - \varepsilon)\, ds = \frac{(f_+^i - \varepsilon)(N+1)h^q}{\Gamma(q+1)}
$$

for any $x_{0i} \geq m_\varepsilon$ and $\hat{x}_{0i} \in \mathbb{R}^{m-1}$, $\|\hat{x}_{0i}\| \leq m_\varepsilon$, while

$$
F_i(x_0) \leq \frac{(f_-^i + \varepsilon)(N+1)h^q}{\Gamma(q+1)}
$$

for any $x_{0i} \leq -m_\varepsilon$, $\hat{x}_{0i} \in \mathbb{R}^{m-1}$, $\|\hat{x}_{0i}\| \leq m_\varepsilon$, and for all $i = 1, \ldots, m$. Further, let us take $G : \mathbb{R}^m \to \mathbb{R}^m$ and $H : [0, 1] \times \mathbb{R}^m \to \mathbb{R}^m$ given by

$$
\begin{aligned}
G_i(x_0) &= \frac{(f_+^i - \varepsilon)(N+1)h^q}{2\Gamma(q+1)m_\varepsilon}(x_{0i} + m_\varepsilon) - \frac{(f_-^i + \varepsilon)(N+1)h^q}{2\Gamma(q+1)m_\varepsilon}(x_{0i} - m_\varepsilon),\\
H(\lambda, x_0) &= (1-\lambda)F(x_0) + \lambda G(x_0).
\end{aligned}
$$

Note that $H(0, x_0) = F(x_0)$ and $H(1, x_0) = G(x_0)$. It is easy to check that by fixing $\varepsilon > 0$ sufficiently small, (2.15) implies

$$
H_i(\lambda, x_0) \neq -\sum_{k=1}^{N+1} \Delta_{ki} \tag{2.17}
$$

for any $x_{0i} = \pm m_\varepsilon$, $\hat{x}_{0i} \in \mathbb{R}^{m-1}$, $\|\hat{x}_{0i}\| \leq m_\varepsilon$. Similarly, we verify that (2.17) holds also in the second possibility (C2). Consequently, we derive

$$
H_i(\lambda, x_0) \neq -\sum_{k=1}^{N+1} \Delta_{ki}
$$

for any $\lambda \in [0, 1]$ and $x_0 \in \partial\Omega$ for $\Omega = \{x_0 \in \mathbb{R}^m \mid \|x_0\| \leq m_\varepsilon\}$, where $\partial\Omega$ is the border of Ω. This gives

$$
\deg\Big(F, \Omega, -\sum_{k=1}^{N+1} \Delta_k\Big) = \deg\Big(G, \Omega, -\sum_{k=1}^{N+1} \Delta_k\Big),
$$

where deg is the Brouwer topological degree. On the other hand, a linear equation $G(x_0) = -\sum_{k=1}^{N+1} \Delta_k$ has the only solution $\bar{x}_0$, which by (2.15) is located in Ω. Moreover, the linearization $DG(\bar{x}_0)$ is a nonsingular matrix, so $\deg(G, \Omega, -\sum_{k=1}^{N+1} \Delta_k) = \pm 1$, and thus $\deg(F, \Omega, -\sum_{k=1}^{N+1} \Delta_k) = \pm 1 \neq 0$. Summarizing, we see that (2.16) has a solution in Ω, and so (2.13) is solvable. □

Following the proof of Theorem 2.8, we have a modified result.

Theorem 2.9. *Suppose* (I)–(III) *and, in addition, there are constants $f_\pm^i \in \mathbb{R}$, $i = 1, \ldots, m$ and a permutation $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$ such that*

$$
\begin{aligned}
&\text{either} \quad \liminf_{x_i\to\infty} f_{\sigma(i)}(t, x) > f_+^{\sigma(i)} > f_-^{\sigma(i)} > \limsup_{x_i\to\infty} f_{\sigma(i)}(t, x)\\
&\text{or} \quad \limsup_{x_i\to\infty} f_{\sigma(i)}(t, x) < f_+^{\sigma(i)} < f_-^{\sigma(i)} < \liminf_{x_i\to\infty} f_{\sigma(i)}(t, x)
\end{aligned} \tag{C3}
$$

uniformly with respect to $t \in I$ and $\hat{x}_i \in \mathbb{R}^{m-1}$ for all $i = 1, \ldots, m$. If (2.15) *holds for all $i = 1, \ldots, m$, then* (2.9) *has a T-periodic solution.*

Clearly, either (C1) or (C2) implies (C3). Of course, assumptions (2.12) and (2.15) are complementary since

$$(f_-^i, f_+^i) \subset (-M, M)$$

for all $i = 1, \dots, m$. Now we consider concrete examples.

Example 2.10. Consider a system

$$f_i(t, x) = A_i \tanh x_i + B_i \cos t \cos x_i \sin\Big(\sum_{k=1}^{m} x_k\Big)$$

for $A_i > B_i > 0$ and all $i = 1, \dots, m$. Then we take

$$f_+^i = A_i - B_i, \quad f_-^i = -A_i + B_i, \quad M = \max_{i=1,\dots,m}\{A_i + B_i\}, \quad T = 2\pi, \quad h = \frac{2\pi}{N+1}.$$

Hence, assumption (2.12) has the form

$$\frac{\|\sum_{k=1}^{N+1} \Delta_k\|}{N+1} > \frac{\max_{i=1,\dots,m}\{A_i + B_i\} h^q}{\Gamma(q+1)},$$

and assumption (2.15) has the form

$$-\frac{\sum_{k=1}^{N+1} \Delta_{ki}}{N+1} \in \frac{h^q}{\Gamma(q+1)}(-A_i + B_i, A_i - B_i)$$

for all $i = 1, \dots, m$, respectively. Consequently, Theorem 2.8 can be applied.

2.4 Systems with small equidistant and shifted impulses

We consider

$$\begin{cases} {}^cD_{kh}^q x(t) = f(x(t)), & t \in (kh, (k+1)h),\ k \in \mathbb{N}_0, \\ x(kh^+) = x(kh^-) + \bar{\Delta} h^q, & k \in \mathbb{N}, \\ x(0) = x_0, \end{cases} \tag{2.18}$$

where $h > 0$, $q \in (0, 1)$, $\bar{\Delta} \in \mathbb{R}^m$, and $f : \mathbb{R}^m \to \mathbb{R}^m$ is Lipschitz. Under above assumptions [8], equation (2.18) has a unique solution $x(x_0, t)$ on $\mathbb{R}_+$, which is continuous in $x_0 \in \mathbb{R}^m$, $t \in \mathbb{R}_+ \setminus \{kh \mid k \in \mathbb{N}\}$ and left continuous in t at impulsive points $\{kh \mid k \in \mathbb{N}\}$. So we can consider the Poincaré mapping

$$P_h(x_0) = x(x_0, h^+).$$

On all intervals $(kh, (k+1)h)$, $k \in \mathbb{N}_0$, equation (2.18) is equivalent to

$$\begin{aligned} x(x_0, t) &= x(kh^+) + \frac{1}{\Gamma(q)} \int_{kh}^{t} (t-s)^{q-1} f(x(x_0, s))\, ds \\ &= x(kh^+) + \frac{1}{\Gamma(q)} \int_{0}^{t-kh} (t - kh - s)^{q-1} f(x(x(kh^+), s))\, ds. \end{aligned} \tag{2.19}$$

Hence

$$x((k+1)h^+) = P_h(x(kh^+)) \tag{2.20}$$

and

$$P_h(x_0) = x(x_0, h^+) = x_0 + \bar{\Delta} h^q + \frac{1}{\Gamma(q)} \int_0^h (h-s)^{q-1} f(x(x_0, s))\, ds. \tag{2.21}$$

Inserting

$$x(x_0, t) = x_0 + h^q y(x_0, t), \quad t \in [0, h],$$

into (2.19), we get

$$\begin{aligned} y(x_0, t) &= \frac{1}{\Gamma(q)h^q} \int_0^t (t-s)^{q-1} f(x_0 + h^q y(x_0, s))\, ds \\ &= \frac{1}{\Gamma(q+1)} f(x_0) + \frac{1}{\Gamma(q)h^q} \int_0^t (t-s)^{q-1} \big(f(x_0 + h^q y(x_0, s)) - f(x_0)\big)\, ds \\ &= \frac{1}{\Gamma(q+1)} f(x_0) + O(h^q) \end{aligned}$$

since

$$\begin{aligned} \Big\| \int_0^t (t-s)^{q-1} \big(f(x_0 + h^q y(x_0, s)) - f(x_0)\big)\, ds \Big\| &\le \int_0^t (t-s)^{q-1} \| f(x_0 + h^q y(x_0, s)) - f(x_0) \|\, ds \\ &\le h^q \max_{t \in [0,h]} \| y(x_0, t) \| L_{\mathrm{loc}} \frac{t^q}{q} \\ &\le h^{2q} \max_{t \in [0,h]} \| y(x_0, t) \| \frac{L_{\mathrm{loc}}}{q}, \end{aligned}$$

where L_{loc} is a local Lipschitz constant of f. Then

$$x(x_0, t) = x_0 + \frac{h^q}{\Gamma(q+1)} f(x_0) + O(h^{2q}), \quad t \in [0, h],$$

and (2.21) gives

$$P_h(x_0) = x_0 + \bar{\Delta} h^q + \frac{h^q}{\Gamma(q+1)} f(x_0) + O(h^{2q}).$$

Hence (2.20) becomes

$$x((k+1)h^+) = x(kh^+) + h^q \Big(\bar{\Delta} + \frac{1}{\Gamma(q+1)} f(x(kh^+)) \Big) + O(h^{2q}). \tag{2.22}$$

We note that (2.19) implies

$$\| x(x_0, t) - x(kh^+) \| = O(h^q) \tag{2.23}$$

for $t \in [kh, (k+1)h]$. We see that (2.22) is leading to its approximation

$$z((k+1)h^+) = z(kh^+) + h^q \Big(\bar{\Delta} + \frac{1}{\Gamma(q+1)} f(z(kh^+)) \Big),$$

which is the Euler numerical approximation of

$$z'(t) = \bar{\Delta} + \frac{1}{\Gamma(q+1)} f(z(t)). \tag{2.24}$$

Using (2.23) and the known results about the Euler approximation method [6], we arrive at the following result.

Theorem 2.11. *Let $z(t)$ be a solution of (2.24) with $z(0) = x_0$ on $[0, T]$. Then the solution $x(x_0, t)$ of (2.18) exists on $[0, T]$ and satisfies*

$$x(x_0, t) = z(th^{q-1}) + O(h^q)$$

for $t \in [0, Th^{1-q}]$. If $z(t)$ is a stable (hyperbolic) periodic solution of (2.24), then there is a stable (hyperbolic) invariant curve of Poincaré mapping P_h of (2.18) in a $O(h^q)$ neighborhood of $z(t)$. Of course, h is sufficiently small.

Now we extend (2.18) for periodic impulses of the form

$$\begin{cases} {}^c D^q_{kh} x(t) = f(x(t)), & t \in (kh, (k+1)h),\ k \in \mathbb{N}_0, \\ x(kh^+) = x(kh^-) + \bar{\Delta}_k h^q, & k \in \mathbb{N}, \\ x(0) = x_0, \end{cases} \tag{2.25}$$

where the $\bar{\Delta}_k \in \mathbb{R}^m$ satisfy $\bar{\Delta}_{k+N+1} = \bar{\Delta}_k$ for any $k \in \mathbb{N}$ and some $N \in \mathbb{N}$. So we can consider the Poincaré mapping

$$P_h(x_0) = x(x_0, (N+1)h^+),$$

which has a form

$$P_h = P_{N+1,h} \circ \cdots \circ P_{1,h} \tag{2.26}$$

for

$$P_{k,h}(x_0) = \bar{\Delta}_k h^q + x(x_0, h).$$

We already know

$$P_{k,h}(x_0) = \bar{\Delta}_k h^q + x(x_0, h) = x_0 + \bar{\Delta}_k h^q + \frac{h^q}{\Gamma(q+1)} f(x_0) + O(h^{2q}).$$

Then (2.26) implies

$$P_h(x_0) = x_0 + h^q \sum_{k=1}^{N+1} \bar{\Delta}_k + \frac{(N+1)h^q}{\Gamma(q+1)} f(x_0) + O(h^{2q}).$$

On the other hand, the ODE

$$z'(t) = \frac{\sum_{k=1}^{N+1} \bar{\Delta}_k}{N+1} + \frac{1}{\Gamma(q+1)} f(z(t)) \tag{2.27}$$

has the Euler numerical approximation

$$x_0 + h^q \Big(\frac{\sum_{k=1}^{N+1} \bar{\Delta}_k}{N+1} + \frac{1}{\Gamma(q+1)} f(x_0) \Big)$$

with the step size h^q, and its $N+1$ iteration has a form

$$x_0 + h^q \sum_{k=1}^{N+1} \bar{\Delta}_k + \frac{(N+1)h^q}{\Gamma(q+1)} f(x_0) + O(h^{2q}),$$

the same one as of P_h. Hence instead of (2.24) we have (2.27), and Theorem 2.11 is directly extended to (2.25) with (2.27).

3 FDE with Caputo derivatives with fixed lower limits

Let $T = (N+1)h$ for a $N \in \mathbb{N}$ and $h > 0$. We consider

$$\begin{cases} {}^c D^q_0 x(t) = f(t, x(t)), & t \in [0, T], \\ x(kh^+) = x(kh^-) + \Delta_k, & k \in \{1, \dots, N\}, \\ x(0) = x_0, \end{cases} \tag{3.1}$$

where $q \in (0,1)$, ${}^c D^q_0 x(t)$ is the generalized Caputo fractional derivative with lower limit at 0. We suppose the following conditions

(ci) $f : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous and locally Lipschitz in x uniformly for $t \in [0, T]$.

(cii) There is a constant $M > 0$ such that $|f(t, x)| \le K$ for any $t \in [0, T]$ and $x \in \mathbb{R}^m$.

We are looking for solutions of (3.1) satisfying $x(0) = x(T)$. Following [16], by a solution of (3.1) we mean a function $x(t)$ which is continuous in $x_0 \in \mathbb{R}^m$, $t \in [0, T] \setminus \{kh \mid k = 1, \dots, N\}$ and left continuous in t at impulsive points kh, and satisfying

$$x(t) = x_0 + \sum_{i=1}^{k} \Delta_i + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds, \quad t \in (t_k, t_{k+1}]. \tag{3.2}$$

It is known that under above assumptions, (3.2) has a unique solution $x(x_0, t)$ on $[0, T]$

Now we can prove the following result.

Theorem 3.1. *Suppose assumptions* (ci)–(cii). *If*

$$\frac{\|\sum_{k=1}^{N} \Delta_k\|}{(N+1)^q} > \frac{Mh^q}{\Gamma(q+1)}, \tag{3.3}$$

then (3.1) *has no solution satisfying* $x(0) = x(T)$. *Note that* $T = (N+1)h$ *for some* $N \in \mathbb{N}$.

Proof. We need to solve

$$x(0) = x(T) = x_0 + \sum_{i=1}^{N} \Delta_i + \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} f(s, x(s))\, ds,$$

which is equivalent to

$$-\sum_{k=1}^{N} \Delta_k = \frac{1}{\Gamma(q)} \int_0^{(N+1)h} ((N+1)h - s)^{q-1} f(s, x(x_0, s))\, ds.$$

This gives

$$\begin{aligned} \Big\|\sum_{k=1}^{N} \Delta_k\Big\| &\le \frac{1}{\Gamma(q)} \int_0^{(N+1)h} ((N+1)h - s)^{q-1} |f(s, x(x_0, s))|\, ds \\ &\le \frac{M(N+1)^q h^q}{\Gamma(q+1)}. \end{aligned}$$

This contradicts (3.3). □

Remark 3.2. Note that (3.3) is equivalent to

$$\frac{\|\sum_{k=1}^{N} \Delta_k\|}{T^q} > \frac{M}{\Gamma(q+1)}. \tag{3.4}$$

Now we suppose that

(ciii) $\Delta_{k+p} = \Delta_k$ for all $k \in \mathbb{N}$ and some $p \in \mathbb{N}$.

Then we have the following corollary.

Corollary 3.3. *Suppose assumptions* (ci)–(ciii). *Then there hold the following assertions:*

(i) *If*

$$\frac{\|\sum_{k=1}^{p} \Delta_k\|}{(p+1)^q} > \frac{Mh^q}{\Gamma(q+1)}, \tag{3.5}$$

then (3.1) *has no solution satisfying* $x(0) = x(T)$ *with* $T = (rp+1)h$ *for any* $r \in \mathbb{N}$.

(ii) *If*

$$\sum_{k=1}^{p} \Delta_k \ne 0, \tag{3.6}$$

then there is no solution of (3.1) *satisfying* $x(0) = x(T)$ *with* $T = (rp+1)h$ *for any* $r \in \mathbb{N}$ *such that*

$$r > \sqrt[1-q]{\frac{M(p+1)^q h^q}{\|\sum_{k=1}^{p} \Delta_k\| \Gamma(q+1)}}. \tag{3.7}$$

Proof. We need to verify (3.3) for $N = rp$. First, we note that

$$\frac{\|\sum_{k=1}^{N}\Delta_k\|}{(N+1)^q} = \frac{\|\sum_{k=1}^{rp}\Delta_k\|}{(rp+1)^q} \geq r^{1-q}\frac{\|\sum_{k=1}^{p}\Delta_k\|}{(p+1)^q} \tag{3.8}$$

since $r \geq 1$. Assuming (3.5), we have that (3.8) implies (3.3). This proves (i). Next, assuming (3.6) and (3.7), we obtain that (3.8) implies (3.3). This proves (ii). □

Of course, (ii) implies (i). Next, supposing $h > 0$ as a small variable, we consider

$$\begin{cases} {}^cD_0^q x(t) = f(x(t)), & t \in (kh, (k+1)h),\ k \in \mathbb{N}_0, \\ x(kh^+) = x(kh^-) + \bar{\Delta}_k h^q, & k \in \mathbb{N}, \\ x(0) = x_0, \end{cases} \tag{3.9}$$

where $q \in (0,1)$, $f : \mathbb{R}^m \to \mathbb{R}^m$ satisfies (ci), (cii) and the $\bar{\Delta}_k \in \mathbb{R}^m$ satisfy
(civ) $\bar{\Delta}_{k+p} = \bar{\Delta}_k$ for all $k \in \mathbb{N}$ and some $p \in \mathbb{N}$.
Then condition (3.5) becomes

$$\frac{\|\sum_{k=1}^{p}\bar{\Delta}_k\|}{(p+1)^q} > \frac{M}{\Gamma(q+1)},$$

condition (3.6) becomes

$$\sum_{k=1}^{p}\bar{\Delta}_k \neq 0,$$

and (3.7) becomes

$$r > \sqrt[1-q]{\frac{M(p+1)^q}{\|\sum_{k=1}^{p}\bar{\Delta}_k\|\Gamma(q+1)}}. \tag{3.10}$$

Summarizing, we have the following result.

Corollary 3.4. *Under assumptions* (ci), (cii) *and* (civ), *there is no solution of* (3.9) *satisfying* $x(0) = x(T)$ *for any* $T = (rp+1)h$ *with* $r \in \mathbb{N}$ *satisfying* (3.10).

Corollary 3.4 states that under assumptions (ci), (cii) and (civ), (3.9) has no "periodic" solutions with large periods.

Example 3.5. We consider system (2.14) for $a = 10$, $b = 120$ and $c = \frac{8}{3}$. Motivated by [9, Theorem 2], for the sphere

$$\mathcal{S} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + (x_3 - a - b)^2 \leq R^2\}$$

with

$$R = \frac{(a+b)c}{\sqrt{4(c-1)}} = 134.263,$$

we find

$$M = \max_{x\in\mathcal{S}}\|f(x)\| = \max_{x\in\mathcal{S}_2}\max\{|a(x_2 - x_1)|, |x_1(b - x_3) - x_2|, |x_1x_2 - cx_3|\} \doteq 8077.26.$$

By Corollary 3.3, for x_1, $\bar{\Delta}_1$ applied at $3kh$, for x_2, $\bar{\Delta}_2$ applied at kh, and for x_3, $\bar{\Delta}_3$ applied at $2kh$, now condition (3.5) has the form

$$\max\{|\bar{\Delta}_1|, 3|\bar{\Delta}_2|, |\bar{\Delta}_3|\} > (p+1)^q\frac{Mh^q}{\Gamma(q+1)} \doteq 132.214$$

for $h = 0.004$ and $q = 0.995$.

For the cuboid from Example 2.6,

$$\mathcal{C} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : -50 \leq x_1 \leq 50,\ -100 \leq x_2 \leq 100,\ 50 \leq x_3 \leq 200\},$$

we have

$$M = \max_{x\in\mathcal{C}} \|f(x)\| \doteq 533.333.$$

Now condition (3.5) has the form

$$\max\{|\bar{\Delta}_1|, 3|\bar{\Delta}_2|, |\bar{\Delta}_3|\} > (p+1)^q \frac{Mh^q}{\Gamma(q+1)} \doteq 8.72997$$

again for $h = 0.004$ and $q = 0.995$.

Finally, for the completeness of the paper, we present the following existence result, whose proof is similar to Theorem 2.9, so we omit it.

Theorem 3.6. *Suppose* (ci), (cii) *and* (C3). *If*

$$-\frac{\sum_{k=1}^{N} \Delta_{ki}}{T^q} \in \frac{1}{\Gamma(q+1)}(f_-^i, f_+^i) \tag{3.11}$$

holds for all $i = 1, \ldots, m$, *then* (3.9) *has a solution satisfying* $x(0) = x(T)$.

Again, assumptions (3.4) and (3.11) are complementary. More related achievements are given in [16] and the references therein.

4 Conclusions

In this paper, fractional differential equations with periodic impulses are investigated. We establish new criteria for the uniqueness, existence and asymptotic stability of periodic solutions for impulsive fractional differential equations with either varying or fixed lower limits.

To conclude this paper, we roughly outline a possible further progress on this topic. Firstly, one can consider a nonhomogeneous linear case

$$\begin{cases} {}^cD_{t_k,t}^q x(t) = Ax(t) + f(t), & t \in (t_k, t_{k+1}),\ k = 0, 1, \ldots, N, \\ x(t_k^+) = (\mathbb{I} + B_k)x(t_k^-) + \Delta_k, & k = 1, 2, \ldots, N+1, \\ x(0) = x_0, \end{cases} \tag{4.1}$$

where $A, B_k : \mathbb{R}^m \to \mathbb{R}^m$ are matrices, $\Delta_k \in \mathbb{R}$, $\mathbb{I} : \mathbb{R}^m \to \mathbb{R}^m$ is the unit matrix, and then consider a semilinear case

$$\begin{cases} {}^cD_{t_k,t}^q x(t) = Ax(t) + f(t, x(t)), & t \in (t_k, t_{k+1}),\ k = 0, 1, \ldots, N, \\ x(t_k^+) = (\mathbb{I} + B_k)x(t_k^-) + \Delta_k(x(t_k^-)), & k = 1, 2, \ldots, N+1, \\ x(0) = x_0 \end{cases} \tag{4.2}$$

for achieving more specific results similar to the ones for impulsive ODEs [11] in the finite-dimensional case. Of course, we can extend (4.1) and (4.2) to an infinite-dimensional space, that is, we consider the associated impulsive fractional evolution equation by setting A as a generator of a C_0-semigroup on an infinite-dimensional Banach space, and then using the theory of semigroups with nonlinear functional analysis. In addition, the investigation of the existence of almost periodic solutions for (4.1) and (4.2) may be more interesting.

Secondly, one can also extend our recent results in [13] to consider linear and semilinear differential equations with periodic noninstantaneous impulses as follows:

$$\begin{cases} x'(t) = Ax(t), & t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \ldots, \\ x(t_i^+) = (\mathbb{I} + B_k)x(t_k^-) + B_k x(t_k^-), & k = 1, 2, \ldots, \\ x(t) = (\mathbb{I} + B_k)x(t_k^-) + B_k x(t_k^-), & t \in (t_k, s_k],\ k = 1, 2, \ldots, \\ x(s_k^+) = x(s_k^-), & k = 1, 2, \ldots, \end{cases} \tag{4.3}$$

and

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \dots, \\ x(t_i^+) = (\mathbb{I} + B_k)x(t_i^-) + B_k x(t_k^-), & k = 1, 2, \dots, \\ x(t) = (\mathbb{I} + B_k)x(t_k^-) + B_k x(t_k^-), & t \in (t_k, s_k],\ k = 1, 2, \dots, \\ x(s_k^+) = x(s_k^-), & k = 1, 2, \dots, \end{cases} \tag{4.4}$$

respectively, where t_k acts as an impulsive point and s_k acts as a junction point satisfying the conditions $t_0 = s_0 < t_1 < s_1 < t_2 < \cdots < t_k < s_k < t_{k+1} \cdots$, $t_k \to \infty$ and periodicity conditions $t_{k+N+1} = t_k + T$ and $s_{k+N+1} = s_k + T$. In addition, one can study the associated fractional order and infinite-dimensional cases for (4.3) and (4.4). The issues on the existence and stability of almost periodic solutions would be another interesting branch.

Finally, one can consider controllability and iterative learning control for the above equations with impulsive periodic controls arising from some real problems in engineering.

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