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Research Article

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Hypercontractivity, supercontractivity, ultraboundedness and stability in semilinear problems

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Abstract: We study the Cauchy problem associated to a family of nonautonomous semilinear equations in the space of bounded and continuous functions over $\mathbb{R}^d$ and in L^p-spaces with respect to tight evolution systems of measures. Here, the linear part of the equation is a nonautonomous second-order elliptic operator with unbounded coefficients defined in $I \times \mathbb{R}^d$, (I being a right-halfline). To the above Cauchy problem we associate a nonlinear evolution operator, which we study in detail, proving some summability improving properties. We also study the stability of the null solution to the Cauchy problem.

Keywords: Nonautonomous second-order elliptic operators, semilinear parabolic equations, unbounded coefficients, hypercontractivity, supercontractivity, ultraboundedness, stability

MSC 2010: 35K58, 37L15

1 Introduction

This paper is devoted to continuing the analysis started in [5]. We consider a family of linear second-order differential operators $\mathcal{A}(t)$ acting on smooth functions ζ as

$$(\mathcal{A}(t)\zeta)(x) = \sum_{i,j=1}^{d} q_{ij}(t,x)D_{ij}\zeta(x) + \sum_{i=1}^{d} b_i(t,x)D_i\zeta(x), \quad t \in I,\ x \in \mathbb{R}^d, \tag{1.1}$$

where I is either an open right halfline or the whole $\mathbb{R}$. Then, given $T > s \in I$, we are interested in studying the nonlinear Cauchy problem

$$\begin{cases} D_t u(t,x) = (\mathcal{A}(t)u)(t,x) + \psi_u(t,x), & (t,x) \in (s,T] \times \mathbb{R}^d, \\ u(s,x) = f(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.2}$$

where $\psi_u(t,x) = \psi(t,x,u(t,x),\nabla_x u(t,x))$. We assume that the coefficients q_{ij} and b_i $(i,j = 1,\dots,d)$, possibly unbounded, are smooth enough, the diffusion matrix $Q = [q_{ij}]_{i,j=1,\dots,d}$ is uniformly elliptic and there exists a Lyapunov function φ for $\mathcal{A}(t)$ (see Hypothesis 2.1 (iii)). These assumptions yield that the linear part $\mathcal{A}(t)$ generates a linear evolution operator $\{G(t,s) : t \geq s \in I\}$ in $C_b(\mathbb{R}^d)$. More precisely, for every $f \in C_b(\mathbb{R}^d)$ and $s \in I$, the function $G(\cdot,s)f$ belongs to $C_b([s,+\infty) \times \mathbb{R}^d) \cap C^{1,2}((s,+\infty) \times \mathbb{R}^d)$, it is the unique bounded

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classical solution of the Cauchy problem (1.2), with $\psi_u \equiv 0$, and satisfies the estimate

$$\|G(t,s)f\|_\infty \le \|f\|_\infty, \quad t > s \in I,\ f \in C_b(\mathbb{R}^d). \tag{1.3}$$

We refer the reader to [9] for the construction of the evolution operator $G(t,s)$ and for further details.

Classical arguments can be adapted to our case to prove the existence of a unique local mild solution u_f of problem (1.2) for any $f \in C_b(\mathbb{R}^d)$, i.e., a function $u : [s,\tau] \times \mathbb{R}^d \to \mathbb{R}$ (for some $\tau > s$) such that

$$u(t,x) = (G(t,s)f)(x) + \int_s^t (G(t,r)\psi_u(r,\cdot))(x)\,dr, \quad t \in [s,\tau],\ x \in \mathbb{R}^d. \tag{1.4}$$

Under reasonable assumptions such a mild solution u_f is classical, defined in the whole $[s,+\infty)$ and satisfies the condition

$$\|u_f\|_\infty + \sup_{t\in(s,T)} \sqrt{t-s}\|\nabla_x u_f(t,\cdot)\|_\infty < +\infty$$

for any $T > s$. Hence, setting $\mathcal{N}(t,s)f = u_f(t,\cdot)$ for any $t > s$ we deduce that $\mathcal{N}(t,s)$ maps $C_b(\mathbb{R}^d)$ into $C_b^1(\mathbb{R}^d)$ and, from the uniqueness of the solution to (1.2), it follows that it satisfies the evolution law

$$\mathcal{N}(t,s)f = \mathcal{N}(t,r)\mathcal{N}(r,s)f$$

for any $r \in (s,t)$ and $f \in C_b(\mathbb{R}^d)$.

As in the linear case we are also interested to set problem (1.2) in an L^p-context. However, as it is already known from the linear case, the most natural L^p-setting where problems with unbounded coefficients can be studied is that related to the so-called *evolution systems of measures* [6], that is one-parameter families of Borel probability measures $\{\mu_t : t \in I\}$ such that

$$\int_{\mathbb{R}^d} G(t,s)f\,d\mu_t = \int_{\mathbb{R}^d} f\,d\mu_s, \quad f \in C_b(\mathbb{R}^d),\ t > s \in I. \tag{1.5}$$

When they exist, evolution families of measures are in general infinitely many, even the tight ones, where, roughly speaking, tight means that all the measures of the family are essentially concentrated on the same large ball (see Section 2 for a rigorous definition of tightness). Under additional assumptions on the coefficients of the operator $\mathcal{A}(t)$ (see Section 2), there exists a unique tight evolution system of measures $\{\mu_t : t \in I\}$, which has the peculiarity to be the unique system related to the asymptotic behavior of $G(t,s)$ as t tends to $+\infty$. We also mention that, typically, even if for $t \neq s$, the measures μ_t and μ_s are equivalent (being equivalent to the restriction of the Lebesgue measure to the Borel σ-algebra in $\mathbb{R}^d$), the corresponding L^p-spaces differ.

Formula (1.5) and the density of $C_b(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d,\mu_s)$ allow to extend $G(t,s)$ to a contraction from $L^p(\mathbb{R}^d,\mu_s)$ to $L^p(\mathbb{R}^d,\mu_t)$ for any $t > s$ and any $p \in [1,+\infty)$ and to prove very nice properties of $G(t,s)$ in these spaces.

In view of these facts, it is significant to extend $\mathcal{N}(t,s)$ to an operator from $L^p(\mathbb{R}^d,\mu_s)$ to $L^p(\mathbb{R}^d,\mu_t)$ for any $I \ni s < t$. This can be done if $p \ge p_0$ (see Hypothesis 2.1 (v)), $\psi(t,x,\cdot,\cdot)$ is Lipschitz continuous in $\mathbb{R}^{d+1}$ uniformly with respect to $(t,x) \in (s,T] \times \mathbb{R}^d$ and, in addition, $\sup_{t\in(s,T]} \sqrt{t-s}\|\psi(t,\cdot,0,0)\|_{L^p(\mathbb{R}^d,\mu_t)} < +\infty$. In particular, each operator $\mathcal{N}(t,s)$ is continuous from $L^p(\mathbb{R}^d,\mu_s)$ to $W^{1,p}(\mathbb{R}^d,\mu_t)$.

We stress that the first condition on ψ may seem too restrictive, but in fact it is not. Indeed, the Sobolev embedding theorems fail to hold, in general, when the Lebesgue measure is replaced by any of the measures μ_t. This can be easily seen in the particular case of the one-dimensional Ornstein–Uhlenbeck operator, where the evolution system of measures is replaced by a time-independent measure μ (the so-called invariant measure), which is the Gaussian centered at zero with covariance $1/2$. For any $\varepsilon > 0$, the function $x \mapsto \exp(2(2p+\varepsilon)^{-1}|x|^2)$ belongs to $W^{k,p}(\mathbb{R},\mu)$ for any $k \in \mathbb{N}$ but it does not belong to $L^{p+\varepsilon}(\mathbb{R}^d,\mu)$.

Under the previous assumptions, for any $f \in L^p(\mathbb{R}^d,\mu_s)$, $\mathcal{N}(\cdot,s)$ can be identified with the unique mild solution to problem (1.2) which belongs to $L^p((s,T)\times\mathbb{R}^d,\mu) \cap W_p^{0,1}(J\times\mathbb{R}^d,\mu)$, for any $J \Subset (s,T]$, such that $u_f(t,\cdot) \in W^{1,p}(\mathbb{R}^d,\mu_t)$ for almost every $t \in (s,T]$. Here, μ is the unique Borel measure on the σ-algebra of all the Borel subsets of $I \times \mathbb{R}^d$ which extends the map defined on the product of a Borel set $A \subset I$ and a Borel set $B \subset \mathbb{R}^d$ by

$$\mu(A \times B) := \int_A \mu_t(B)\,dt.$$

Since, as it has been stressed, in this context the Sobolev embedding theorems fail to hold in general, the summability improving properties of the nonlinear evolution operator $\mathcal{N}(t,s)$ are not immediate and true in all the cases. For this reason in Section 4 we investigate properties such as hypercontractivity, supercontractivity, ultraboundedness of the evolution operator $\mathcal{N}(t,s)$ and its spatial gradient. Differently from [5], where $\psi = \psi(t,u)$ and the hypercontractivity of $\mathcal{N}(t,s)$ is proved assuming $\psi(t,0) = 0$ for any $t > s$, here we consider a more general case. More precisely we assume that there exist $\xi_0 \geq 0$ and $\xi_1, \xi_2 \in \mathbb{R}$ such that $u\psi(t,x,u,v) \leq \xi_0|u| + \xi_1 u^2 + \xi_2|u||v|$ for any $t \geq s$, $x, v \in \mathbb{R}^d$, $u \in \mathbb{R}$. Under some other technical assumptions on the growth of the coefficients q_{ij} and b_i $(i,j = 1,\dots,d)$ as $|x| \to +\infty$, we show that as in the linear case, (see [3, 4]), the hypercontractivity and the supercontractivity of $\mathcal{N}(t,s)$ and $\nabla_x\mathcal{N}(t,s)$ are related to some logarithmic Sobolev inequalities with respect to the tight system $\{\mu_t : t \in I\}$. These estimates are the natural counterpart of the Sobolev embedding theorems in the context of invariant measures and evolution systems of measures.

For what concerns the ultraboundedness of $\mathcal{N}(t,s)$ and $\nabla_x\mathcal{N}(t,s)$ we first prove an Harnack-type estimate which establishes a pointwise estimate of $|\mathcal{N}(t,s)f|^p$ in terms of $G(t,s)|f|^p$ for any $f \in C_b(\mathbb{R}^d)$, $p > p_0$ and $t > s$. This estimate, together with the evolution law and the ultraboundedness of $G(t,s)$, allow us to conclude that, for any $f \in L^p(\mathbb{R}^d, \mu_s)$ and any $t > s$, the function $\mathcal{N}(t,s)f$ belongs to $W^{1,\infty}(\mathbb{R}^d, \mu_t)$ and to prove an estimate of $\|\mathcal{N}(t,s)f\|_{W^{1,\infty}(\mathbb{R}^d,\mu_t)}$ in terms of $\|f\|_{L^p(\mathbb{R}^d,\mu_s)}$.

Finally, assuming that $\psi(t,x,0,0) = 0$ for every $t \in (s,+\infty)$ and $x \in \mathbb{R}^d$, we prove that the trivial solution to the Cauchy problem (1.2) is exponentially stable both in $W^{1,p}(\mathbb{R}^d, \mu_t)$ and in $C^1_b(\mathbb{R}^d)$. This means that $\|u_f(t,\cdot)\|_X \leq C_X e^{-\omega_X t}$ as $t \to +\infty$ for some constants $C_X > 0$ and $\omega_X < 0$, both when $X = W^{1,p}(\mathbb{R}^d, \mu_t)$ and $X = C^1_b(\mathbb{R}^d)$. In the first case, the space X depends itself on t. We stress that, under sufficient conditions on the coefficients of the operators $\mathcal{A}(t)$, which include their convergence at infinity, in [2, 11] it has been proved that the measure μ_t weakly* converges to a measure μ, which turns out the invariant measure of the operator $\mathcal{A}_\infty$, whose coefficients are the limit as $t \to +\infty$ of the coefficients of the operator $\mathcal{A}(t)$. This gives more information on the convergence to zero of $\|u_f(t,\cdot)\|_{W^{1,p}(\mathbb{R}^d,\mu_t)}$ at infinity. We refer the reader also to [12] for the case of T-time periodic coefficients.

To get the exponential stability of the trivial solution in $C_b(\mathbb{R}^d)$, differently from [5] where a nonautonomous version of the principle of linearized stability is used and more restrictive assumptions on ψ are required, we let p tend to $+\infty$ in the decay estimate of $\|u_f(t,\cdot)\|_{W^{1,p}(\mathbb{R}^d,\mu_t)}$, since all the constants appearing in this estimate admit finite limit as p tends to $+\infty$. In particular, we stress that we do not need any additional assumptions on the differentiability of ψ but, on the other hand, we require that the mild solution u_f of (1.2) is actually classical.

Notations

For $k \geq 0$, by $C^k_b(\mathbb{R}^d)$ we mean the space of the functions in $C^k(\mathbb{R}^d)$ which are bounded together with all their derivatives up to the $[k]$-th order. $C^k_b(\mathbb{R}^d)$ is endowed with the norm

$$\|f\|_{C^k_b(\mathbb{R}^d)} = \sum_{|\alpha|\leq[k]} \|D^\alpha f\|_\infty + \sum_{|\alpha|=[k]} [D^\alpha f]_{C^{k-[k]}_b(\mathbb{R}^d)},$$

where $[k]$ denotes the integer part of k. When $k \notin \mathbb{N}$, we use the subscript "loc" to denote the space of all $f \in C^{[k]}(\mathbb{R}^d)$ such that the derivatives of order $[k]$ are $(k-[k])$-Hölder continuous in any compact subset of $\mathbb{R}^d$. Given an interval J, we denote by $B(J \times \mathbb{R}^d; \mathrm{Lip}(\mathbb{R}^{d+1}))$ and $C^{\alpha/2,\alpha}(J \times \mathbb{R}^d)$ $(\alpha \in (0,1))$, respectively, the set of all functions $f : J \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that $f(t,x,\cdot,\cdot)$ is Lipschitz continuous in $\mathbb{R}^{d+1}$, uniformly with respect to $(t,x) \in J \times \mathbb{R}^d$, and the usual parabolic Hölder space. The subscript "loc" has the same meaning as above.

We use the symbols $D_t f$, $D_i f$ and $D_{ij} f$ to denote respectively the time derivative $\frac{\partial f}{\partial t}$ and the spatial derivatives $\frac{\partial f}{\partial x_i}$ and $\frac{\partial^2 f}{\partial x_i \partial x_j}$ for any $i,j = 1,\dots,d$.

The open ball in $\mathbb{R}^d$ centered at 0 with radius $r > 0$ and its closure are denoted by B_r and $\overline{B}_r$, respectively. For any measurable set A, contained in $\mathbb{R}$ or in $\mathbb{R}^d$, we denote by $\mathbb{1}_A$ the characteristic function of A. Finally, we write $A \Subset B$ when A is compactly contained in B.

2 Assumptions and preliminary results

Let $\{\mathcal{A}(t) : t \in I\}$ be the family of linear second-order differential operators defined by (1.1).

Hypotheses 2.1. Our standing assumptions on the coefficients of the operators $\mathcal{A}(t)$ are as follows.

(i) The coefficients q_{ij}, b_i belong to $C^{\alpha/2,1+\alpha}_{\rm loc}(I \times \mathbb{R}^d)$ for any $i, j = 1, \ldots, d$ and some $\alpha \in (0, 1)$.

(ii) For every $(t, x) \in I \times \mathbb{R}^d$, the matrix $Q(t, x) = [q_{ij}(t, x)]_{ij}$ is symmetric and there exists a function
$$\kappa : I \times \mathbb{R}^d \to \mathbb{R},$$
with positive infimum κ_0, such that
$$\langle Q(t, x)\xi, \xi\rangle \geq \kappa(t, x)|\xi|^2$$
for any $(t, x) \in I \times \mathbb{R}^d$ and any $\xi \in \mathbb{R}^d$.

(iii) There exists a nonnegative function $\varphi \in C^2(\mathbb{R}^d)$, diverging to $+\infty$ as $|x| \to +\infty$, such that
$$(\mathcal{A}(t)\varphi)(x) \leq a - c\varphi(x)$$
for any $(t, x) \in I \times \mathbb{R}^d$ and some positive constants a and c.

(iv) There exists a locally bounded function $\rho : I \to \mathbb{R}^+$ such that
$$|\nabla_x q_{ij}(t, x)| \leq \rho(t)\kappa(t, x)$$
for any $(t, x) \in I \times \mathbb{R}^d$ and any $i, j = 1, \ldots, d$, where κ is defined in (ii).

(v) There exists a function $r : I \times \mathbb{R}^d \to \mathbb{R}$ such that
$$\langle \nabla_x b(t, x)\xi, \xi\rangle \leq r(t, x)|\xi|^2$$
for any $\xi \in \mathbb{R}^d$ and $(t, x) \in I \times \mathbb{R}^d$. Further, there exists $p_0 \in (1, 2]$ such that
$$+\infty > \sigma_{p_0} = \sup_{(t,x)\in I\times\mathbb{R}^d}\left(r(t, x) + \frac{d^3(\rho(t))^2\kappa(t, x)}{4\min\{p_0 - 1, 1\}}\right). \tag{2.1}$$

Under Hypotheses 2.1 (i)–(iii) (actually even under weaker assumptions) it is possible to associate an evolution operator $\{G(t, s) : t \geq s \in I\}$ to the operator $\mathcal{A}(t)$ in $C_b(\mathbb{R}^d)$, as described in the Introduction. The function $G(\cdot, \cdot)f$ is continuous in $\{(s, t, x) \in I \times I \times \mathbb{R}^d : s \leq t\}$ and
$$(G(t, s)f)(x) = \int_{\mathbb{R}^d} f(y)p(t, s, x, dy), \quad I \ni s < t,\ x \in \mathbb{R}^d, \tag{2.2}$$
where $p(t, s, x, dy)$ are probability measures for any $I \ni s < t$, $x, y \in \mathbb{R}^d$. This implies that
$$|G(t, s)f|^p \leq G(t, s)(|f|^p)$$
for any $I \ni s < t$, $f \in C_b(\mathbb{R}^d)$ and $p \geq 1$. Moreover, Hypotheses 2.1 (iv) and (v) yield the pointwise gradient estimates
$$|(\nabla_x G(t, s)f)(x)|^p \leq e^{p\sigma_p(t-s)}(G(t, s)|\nabla f|^p)(x), \quad f \in C^1_b(\mathbb{R}^d), \tag{2.3}$$
$$|(\nabla_x G(t, s)f)(x)|^p \leq \begin{cases} C^p_\varepsilon e^{p(\sigma_p+\varepsilon)(t-s)}(t - s)^{-\frac{p}{2}}(G(t, s)|f|^p)(x), & \text{if } \sigma_p < 0,\\ C^p_0(1 + (t - s)^{-\frac{p}{2}})(G(t, s)|f|^p)(x), & \text{otherwise},\end{cases} \tag{2.4}$$
for any $f \in C_b(\mathbb{R}^d)$, $t > s$, $x \in \mathbb{R}^d$, $p \in [p_0, +\infty)$, $\varepsilon > 0$ and some positive constants C_0 and C_ε, where σ_p is given by (2.1), with p instead of p_0. We stress that the pointwise estimates (2.3) and (2.4) have been proved with the constants C_0 and C_ε also depending of p. Actually, these constants may be taken independent of p. Indeed, consider for instance estimate (2.4). If $p \geq p_0$, then using the representation formula (2.2) we can

estimate

$$\begin{aligned}|\nabla_x G(t,s)f|^p=(|\nabla_x G(t,s)|^{p_0})^{\frac{p}{p_0}}&\le (C_{p_0}^{p_0}(1+(t-s)^{-\frac{p_0}{2}})G(t,s)|f|^{p_0})^{\frac{p}{p_0}}\\&\le 2^{\frac{p-p_0}{p_0}}C_{p_0}^p(1+(t-s)^{-\frac{p}{2}})(G(t,s)|f|^{p_0})^{\frac{p}{p_0}}\\&\le 2^{\frac{p-p_0}{p_0}}C_{p_0}^p(1+(t-s)^{-\frac{p}{2}})G(t,s)|f|^p\end{aligned}$$

for any $t > s \in I$ and $f \in C_b(\mathbb{R}^d)$, and, hence, estimate (2.4) holds true with a constant which can be taken independent of p.

Remark 2.2. The case $p = 1$ in estimate (2.3) is much more delicate and requires stronger assumptions. Indeed, as [1] shows, the algebraic condition $D_h q_{ij} + D_i q_{jh} + D_j q_{ih} = 0$ in $I \times \mathbb{R}^d$ for any $i, j, h \in \{1, \dots, d\}$ with $i \neq j \neq h$ is a necessary condition for (2.3) (with $p = 1$) to hold. For this reason, if the diffusion coefficients are bounded and independent of x, then the pointwise gradient estimate (2.3) holds true also with $p = 1$ and $\sigma_1 = r_0$, where r_0 is the supremum over $I \times \mathbb{R}^d$ of the function r in Hypothesis 2.1 (v).

Under Hypotheses 2.1 we can also associate an evolution system of measures $\{\mu_t : t \in I\}$ with the operators $\mathcal{A}(t)$. Such a family of measures is tight, namely for every $\varepsilon > 0$ there exists $r > 0$ such that $\mu_s(\mathbb{R}^d \setminus B_r) < \varepsilon$ for any $s \in I$. The invariance property (1.5) and the density of $C_b(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d, \mu_s)$, $s \in I$, allows to extend $G(t, s)$ to a contraction from $L^p(\mathbb{R}^d, \mu_s)$ to $L^p(\mathbb{R}^d, \mu_t)$ for any $t > s$. As it has been stressed in the Introduction, in general evolution systems of measures are infinitely many, but, under suitable assumptions, there exists a unique tight evolution system of measures. This is, for instance, the case when Hypotheses 2.1 are satisfied as well as the following two conditions:

(i) q_{ij} and b_i belong to $C^{\alpha/2,1+\alpha}_{\rm loc}([a, +\infty) \times \mathbb{R}^d)$ for any $i, j = 1, \dots, d$ and some $a \in I$. Moreover, q_{ij} belongs to $C_b([a, +\infty) \times B_R)$ and $D_k q_{ij}$, b_j belong to $C_b([a, +\infty); L^p(B_R))$ for any $i, j, k \in \{1, \dots, d\}$, $R > 0$ and some $p > d + 2$.
(ii) There exists a constant $c > 0$ such that either $|Q(t, x)| \le c(1 + |x|)\varphi(x)$ and $\langle b(t, x), x\rangle \le c(1 + |x|^2)\varphi(x)$ for any $(t, x) \in [a, +\infty) \times \mathbb{R}^d$, or the diffusion coefficients are bounded in $[a, +\infty) \times \mathbb{R}^d$.

For more details and the proofs of the results that we have mentioned, we refer the reader to [9–11, 13].

3 The semilinear problem in a bounded time interval

Given $I \ni s < T$, we are interested in studying the Cauchy problem (1.2) both in the case when $f \in C_b(\mathbb{R}^d)$ and in the case when $f \in L^p(\mathbb{R}^d, \mu_s)$.

Hypotheses 3.1. Our standing assumptions on ψ are as follows.

(i) The function $\psi : [s, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is continuous. Moreover, there exists $\beta \in [0, 1)$ such that for any $R > 0$ and some constant $L_R > 0$

$$|\psi(t,x,u_1,(t-s)^{-\frac12}v_1)-\psi(t,x,u_2,(t-s)^{-\frac12}v_2)|\le L_R(t-s)^{-\beta}(|u_1-u_2|+|v_1-v_2|) \tag{3.1}$$

for any $t \in (s, T]$, $x \in \mathbb{R}^d$, $u_1, u_2 \in [-R, R]$, $v_1, v_2 \in \overline{B}_R$.
(ii) The function $\psi(\cdot, \cdot, 0, 0)$ belongs to $C_b([s, T] \times \mathbb{R}^d)$.

Theorem 3.2. *Under Hypotheses 2.1 and 3.1, for any $\overline{f} \in C_b(\mathbb{R}^d)$ there exist constants $r_0, \delta \in (0, T - s]$ such that, if $f \in C_b(\mathbb{R}^d)$ and $\|f - \overline{f}\|_\infty \le r_0$, then the nonlinear Cauchy problem* (1.2) *admits a unique mild solution $u_f \in C_b([s, s + \delta] \times \mathbb{R}^d) \cap C^{0,1}((s, s + \delta] \times \mathbb{R}^d)$ which satisfies the estimate*

$$\|u_f\|_\infty+\sup_{t\in(s,s+\delta]}\sqrt{t-s}\|\nabla_x u_f(t,\cdot)\|_\infty\le 2(1+2C_0+C_0\sqrt{\delta})(\|f\|_{C_b(\mathbb{R}^d)}+2\delta\|\psi(\cdot,\cdot,0,0)\|_{C_b([s,s+\delta]\times\mathbb{R}^d)}), \tag{3.2}$$

where C_0 is the constant in (2.4)*. Moreover, for any $R > 0$, $\theta \in (0, 1)$ and $t \in (s, s+\delta]$, $u_f(t, \cdot)$ belongs to $C^{1+\theta}(B_R)$ and there exists a positive constant $C_{R,T-s}$ such that*

$$\sup_{t\in(s,s+\delta]}(t-s)^{\frac{1+\theta}{2}}\|u_f(t,\cdot)\|_{C^{1+\theta}(B_R)}\le C_{R,T-s}\|f\|_\infty.$$

Finally, if $g \in C_b(\mathbb{R}^d)$ is such that $\|g - \overline{f}\|_\infty \le r_0$, then

$$\|u_f - u_g\|_\infty + \sup_{t\in(s,s+\delta]} \sqrt{t-s}\|\nabla_x u_f(t,\cdot) - \nabla_x u_g(t,\cdot)\|_\infty \le 2(1 + C_0 + C_0\sqrt{\delta})\|f - g\|_\infty. \tag{3.3}$$

Proof. Even if the proof is quite standard, for the reader's convenience we provide some details.

Fix $\overline{f} \in C_b(\mathbb{R}^d)$. Let $R_0 > 0$ be such that $R_0/(1+K_0) \ge 8\|\overline{f}\|_\infty$, where $K_0 = C_0(1+\sqrt{T-s})$ and C_0 is the constant in (2.4). Further, for any $\delta \in (0, T-s]$, let Y_δ be the set of all $u \in C_b([s, s+\delta]\times\mathbb{R}^d) \cap C^{0,1}((s, s+\delta)\times\mathbb{R}^d)$ such that $\|u\|_{Y_\delta} = \|u\|_{C_b((s,s+\delta]\times\mathbb{R}^d)} + \sup_{t\in(s,s+\delta]} \sqrt{t-s}\|\nabla_x u(t,\cdot)\|_\infty < +\infty$.

Step 1. We prove that there exists $\delta > 0$ such that, for any $f \in C_b(\mathbb{R}^d)$ satisfying the condition

$$\|f - \overline{f}\|_\infty \le r_0 := \frac{R_0}{4 + 4K_0},$$

there exists a mild solution to problem (1.2) defined in the time interval $[s, s+\delta]$. For this purpose, we consider the operator Γ, defined by the right-hand side of (1.4) for any $u \in B_{Y_\delta}(R_0)$ (the ball of Y_δ centered at zero with radius R_0). Clearly, the function ψ_u is continuous in $(s, s+\delta]\times\mathbb{R}^d$ and $\psi_u(t,\cdot)$ is bounded in $\mathbb{R}^d$ for any $t \in (s, s+\delta]$. Moreover, estimating $|\psi_u(t,x)| \le |\psi_u(t,x) - \psi(t,x,0,0)| + |\psi(t,x,0,0)|$ and taking (3.1) into account, we can easily show that the function $t \mapsto (t-s)^\beta\|\psi_u(t,\cdot)\|_\infty$ is bounded in $(s, s+\delta)$. Hence, Proposition A.1 and estimates (1.3) and (2.4) show that $\Gamma(u) \in Y_\delta$ for any $t \in (s, s+\delta]$ and $u \in B_{Y_\delta}(R_0)$. To show that, for a suitable $\delta \in (0,1]$, Γ is a 1/2-contraction in $B_{Y_\delta}(R_0)$, we observe that, using again (3.1), it follows that

$$\|\psi_u(t,\cdot) - \psi_v(t,\cdot)\|_\infty \le L_{R_0}(t-s)^{-\beta}\|u - v\|_{Y_\delta}, \quad t \in (s, s+\delta], \tag{3.4}$$

for any $u, v \in B_{Y_\delta}(R_0)$, where L_{R_0} is the constant in Hypothesis 3.1 (i). From this inequality and estimates (1.3) and (2.4) we conclude that $\|\Gamma(u) - \Gamma(v)\|_{Y_\delta} \le c_1\delta^{1-\beta}\|u - v\|_{Y_\delta}$ for any $u, v \in B_{Y_\delta}(R_0)$, where c_1, as the forthcoming constants, is independent of δ and u, if not otherwise specified. Hence, choosing δ properly, we can make Γ a 1/2-contraction in $B_{Y_\delta}(R_0)$.

It is also straightforward to see that Γ maps $B_{Y_\delta}(R_0)$ into itself, up to replacing δ with a smaller value if needed. It suffices to split $\Gamma(u) = (\Gamma(u) - \Gamma(0)) + \Gamma(0)$, use the previous result and estimate

$$\|\Gamma(0)\|_{Y_\delta} \le (1 + C_0 + C_0\sqrt{\delta})\|f\|_\infty + \delta(1 + 2C_0 + C_0\sqrt{\delta})\|\psi(\cdot,\cdot,0,0)\|_{C_b([s,T]\times\mathbb{R}^d)}.$$

As a consequence, Γ has a unique fixed point in $B_{Y_\delta}(R_0)$, which is a mild solution of (1.2) and satisfies (3.2).

Step 2. We prove the uniqueness of the mild solution u_f. For this purpose, let $u_1, u_2 \in Y_\delta$ be two mild solutions. By Lemma A.2, the function $r \mapsto h(r) := \|u_1(r,\cdot) - u_2(r,\cdot)\|_\infty + \sqrt{r-s}\|\nabla_x u_1(r,\cdot) - \nabla_x u_2(r,\cdot)\|_\infty$ is measurable in $(s, s+\delta)$. Moreover, using (3.4), we easily deduce that

$$\|D_x^j u_1(t,\cdot) - D_x^j u_2(t,\cdot)\|_\infty \le c_2(M)\int_s^t (t-r)^{-\frac{j}{2}}(r-s)^{-\beta}h(r)\,dr \tag{3.5}$$

for $j = 0, 1$, any $t \in [s, s+\delta]$, where $M = \max\{\|u_1\|_{Y_\delta}, \|u_2\|_{Y_\delta}\}$. Estimating $\sqrt{t-s}$ with $\sqrt{t-r} + \sqrt{r-s}$ for any $r \in (s,t)$, from (3.5), with $j = 1$, it follows that

$$\begin{aligned}\sqrt{t-s}\|\nabla_x u_1(t,\cdot) - \nabla_x u_2(t,\cdot)\|_\infty \le{}& c_2(M)\int_s^t (r-s)^{-\beta}h(r)\,dr + c_2(M)\int_s^t (t-r)^{-\frac{1}{2}}(r-s)^{\frac{1}{2}-\beta}\|u_1(r,\cdot) - u_2(r,\cdot)\|_\infty\,dr\\ &+ c_2(M)\int_s^t (t-r)^{-\frac{1}{2}}(r-s)^{1-\beta}\|\nabla_x u_1(r,\cdot) - \nabla_x u_2(r,\cdot)\|_\infty\,dr.\end{aligned} \tag{3.6}$$

Using (3.5), we estimate the last two integral terms in the right-hand side of (3.6), which we denote by $\mathcal{I}(t)$ and $\mathcal{J}(t)$. Replacing (3.5), with $j = 0$, in $\mathcal{I}(t)$, we get

$$\mathcal{I}(t) \le c_3(M)\delta^{1-\beta}\int_s^t (\sigma - s)^{-\beta}h(\sigma)\,d\sigma. \tag{3.7}$$

The same arguments show that $\mathcal{J}(t)$ can be estimated pointwise in $[s, s+\delta]$ by the right-hand side of (3.7), with $c_3(M)$ being possibly replaced by a larger constant $c_4(M)$. Summing up, we have proved that

$$\sqrt{t-s}\|\nabla_x u_1(t,\cdot) - \nabla_x u_2(t,\cdot)\|_\infty \le c_5(M)\delta^{1-\beta}\int_s^t (\sigma - s)^{-\beta} h(\sigma)\, d\sigma. \tag{3.8}$$

From (3.5) and (3.8) we conclude that

$$h(t) \le c_6(M,\delta)\int_s^t (r-s)^{-\beta} h(r)\, dr, \qquad t \in (s, s+\delta].$$

The generalized Gronwall lemma (see [7]) yields $h(t) \equiv 0$ for any $t \in (s, s+\delta)$, i.e., $u_1 \equiv u_2$ in $(s, s+\delta) \times \mathbb{R}^d$.

Step 3. We prove (3.2) and (3.3). Since $u_f = \Gamma(0) + (\Gamma(u_f) - \Gamma(0))$ and Γ is a 1/2-contraction in $B_{Y_\delta}(R_0)$, we conclude that $\|u_f\|_{Y_\delta} \le 2\|\Gamma(0)\|_{Y_\delta}$ and (3.2) follows from the estimate on $\|\Gamma(0)\|_{Y_\delta}$ proved above. Estimate (3.3) can be proved in the same way.

Step 4. We prove that $u_f(t,\cdot) \in C^{1+\theta}(B_R)$ for any $t \in (s, s+\delta]$, $R > 0$, $\theta \in (0,1)$, and

$$\sup_{t \in (s,s+\delta]} (t-s)^{\frac{1+\theta}{2}} \|u_f(t,\cdot)\|_{C^{1+\theta}(B_R)} \le c_7\|f\|_\infty$$

for some constant c_7, independent of f. For this purpose, we observe that the results in the previous steps show that the function ψ_u satisfies the estimate $(t-s)^\beta\|\psi_u(t,\cdot)\|_\infty \le c_8\|f\|_\infty$ for any $t \in (s, s+\delta]$, the constant c_8 being independent of f. Applying Proposition A.1 and estimate (A.5), we complete the proof. □

Corollary 3.3. *In addition to the assumption of Theorem 3.2 suppose that there exist $\beta \in [0,1)$ and $\gamma \in (0,1)$ such that $2\beta + \gamma < 2$ and*

$$|\psi(t, x, u, (t-s)^{-\frac{1}{2}}v) - \psi(t, y, u, (t-s)^{-\frac{1}{2}}v)| \le C_R(t-s)^{-\beta}|x-y|^\gamma \tag{3.9}$$

for any $t \in (s, T]$, $x, y, v \in B_R$, $u \in [-R, R]$, any $R > 0$ and some positive constant C_R. Then, for any $f \in C_b(\mathbb{R}^d)$, the mild solution u_f to problem (1.2) *belongs to $C^{1,2}((s, s+\delta] \times \mathbb{R}^d)$ and it is a classical solution to* (1.2).

Proof. Fix $R > 0$. Theorem 3.2 shows that $u_f(t,\cdot)$ belongs to $C^{1+\gamma}(B_R)$ and

$$\|\nabla_x u_f(t,\cdot)\|_{C^\gamma(B_R)} \le C_R(t-s)^{-\frac{1+\gamma}{2}}\|f\|_\infty$$

for any $t \in (s, s+\delta]$. Moreover, by interpolation from (3.2) it follows that $\|u_f(t,\cdot)\|_{C_b^\gamma(\mathbb{R}^d)} \le C(t-s)^{-\gamma/2}\|f\|_\infty$ for any $t \in (s, s+\delta]$. From these estimates, adding and subtracting $\psi(t, y, u(t,x), \nabla_x u(t,x))$, we deduce that $|\psi_u(t,x) - \psi_u(t,y)| \le C\|f\|_\infty(t-s)^{-\beta-\frac{\gamma}{2}}|x-y|^\gamma$ for any $t \in (s, s+\delta]$, $x, y \in \mathbb{R}^d$ such that $|x-y| \le R$ and some positive constant C, depending on R and u. As a byproduct, $\|\psi_u(t,\cdot)\|_{C^\gamma(B_R)} \le \widetilde{C}(t-s)^{-\beta-\frac{\gamma}{2}}\|f\|_\infty$ for any $t \in (s, s+\delta]$ and some positive constant $\widetilde{C}$, depending on R and u. Now, using Proposition A.1, we conclude that $u \in C^{1,2}((s, s+\delta] \times \mathbb{R}^d) \cap C^{0,2+\theta}_{\mathrm{loc}}((s, s+\delta] \times \mathbb{R}^d)$ for any $\theta < \gamma$ if $\gamma \le \alpha$, and for $\theta = \gamma$ otherwise. □

Remark 3.4. Suppose that (3.1) is replaced by the condition

$$|\psi(\cdot,\cdot,u_1,v_1) - \psi(\cdot,\cdot,u_2,v_2)| \le L_R(|u_1-u_2| + |v_1-v_2|)$$

in $[s, T] \times \mathbb{R}^d$, for any $R > 0$, $u_1, u_2 \in [-R, R]$, $v_1, v_2 \in B_R$ and some positive constant L_R. Then, the proof of the previous theorem can be repeated verbatim with $Y_\delta = C_b^{0,1}([s, s+\delta] \times \mathbb{R}^d)$, endowed with the natural norm, and we can show that the mild solution to problem (1.2) belongs to $C_b^{0,1}([s, s+\delta] \times \mathbb{R}^d)$ and $\|u_f\|_{C_b^{0,1}([s,s+\delta]\times\mathbb{R}^d)} \le \widetilde{C}_\delta\|f\|_{C_b^1(\mathbb{R}^d)}$ for some positive constant $\widetilde{C}_\delta$, independent of f.

We now provide some sufficient conditions for the mild solution to problem (1.2) to exist in the large. Such conditions will be crucial to define the nonlinear evolution operator associated with the Cauchy problem (1.2).

Hypotheses 3.5. We introduce the following assumptions.

(i) For any $R > 0$ there exists a positive constant L_R such that

$$|\psi(t,x,u_1,v_1) - \psi(t,x,u_2,v_2)| \le L_R(|u_2-u_1| + |v_2-v_1|)$$

for any $t \in [s,T]$, $x \in \mathbb{R}^d$, $u_1, u_2 \in [-R,R]$ and $v_1, v_2 \in \mathbb{R}^d$.

(ii) For any $\tau > s \in I$ there exist positive constants k_0, k_1 and a, and a function $\tilde{\varphi} \in C^2(\mathbb{R}^d)$ with nonnegative values and blowing up at infinity such that $u\psi(t,x,u,v) \le k_0(1+u^2) + k_1|u||v|$ and $\mathcal{A}\tilde{\varphi} + k_1|\nabla\tilde{\varphi}| \le a\tilde{\varphi}$ in $\mathbb{R}^d$ for any $t \in [s,\tau]$, $x, v \in \mathbb{R}^d$ and $u \in \mathbb{R}$.

In the rest of this section, for any $p \in [p_0,+\infty)$ and $T > s$ we denote by $[\psi]_{p,T}$ the supremum over (s,T) of the function $\sqrt{t-s}\,\|\psi(t,\cdot,0,0)\|_{L^p(\mathbb{R}^d,\mu_t)}$; $[\psi]_{\infty,T}$ is defined similarly, replacing $L^p(\mathbb{R}^d,\mu_t)$ by $C_b(\mathbb{R}^d)$.

Theorem 3.6. *Assume that Hypotheses 2.1, 3.1* (ii)*, 3.5 and condition* (3.9) *are satisfied. Then, for any $f \in C_b(\mathbb{R}^d)$, the classical solution u_f to problem* (1.2) *exists in* $[s,T]$. *If, further, the constant in Hypothesis 3.5* (i) *is independent of R, then for any $p \in [p_0,+\infty]$,*

$$\sup_{t\in(s,T)} (\|u_f(t,\cdot)\|_{L^p(\mathbb{R}^d,\mu_t)} + \sqrt{t-s}\,\|\nabla_x u_f(t,\cdot)\|_{L^p(\mathbb{R}^d,\mu_t)}) \le C_{T-s}(\|f\|_{L^p(\mathbb{R}^d,\mu_s)} + (\sqrt{T-s}+1)[\psi]_{p,T}), \tag{3.10}$$

$$\sup_{t\in(s,T)} (\|u_f(t,\cdot) - u_g(t,\cdot)\|_{L^p(\mathbb{R}^d,\mu_t)} + \sqrt{t-s}\,\|\nabla_x u_f(t,\cdot) - \nabla_x u_g(t,\cdot)\|_{L^p(\mathbb{R}^d,\mu_t)}) \le C_{T-s}\|f-g\|_{L^p(\mathbb{R}^d,\mu_s)} \tag{3.11}$$

for every $f, g \in C_b(\mathbb{R}^d)$, where $C_\tau = (\sqrt{\tau}+1)e^{d_1\tau^{3/2}+d_2}$ for some positive constants d_1 and d_2.

Proof. We split the proof into two steps.

Step 1. We prove that, for any $f \in C_b(\mathbb{R}^d)$, u_f is defined in the whole $[s,T]$. To this end, we fix $f \in C_b(\mathbb{R}^d)$, denote by $[s,\tau_f)$ the maximal time domain where u_f is defined and assume, by contradiction, that $\tau_f < T$. We are going to prove that u_f is bounded in $[s,\tau_f)\times\mathbb{R}^d$. Once this is proved, we can use Hypotheses 3.5 (i) to deduce, adding and subtracting $\psi(t,x,0,0)$, that $|\psi(t,x,u_f(t,x),v)| \le C(1+|v|)$ for $t \in [s,T]$, $x, v \in \mathbb{R}^d$ and some constant $C > 0$, which depends on $\|u_f\|_{C_b([s,\tau_f)\times\mathbb{R}^d)}$ and $\|\psi(\cdot,\cdot,0,0)\|_{C_b([s,T]\times\mathbb{R}^d)}$. Applying the same arguments as in Step 2 of the proof of Theorem 3.2, we can show that also the function $t \mapsto \sqrt{t-s}\|\nabla_x u_f(t,\cdot)\|_\infty$ is bounded in $[s,\tau_f)\times\mathbb{R}^d$. This is enough to infer that u_f can be extended beyond τ_f, contradicting the maximality of the interval $[s,\tau_f)$.

To prove that u_f is bounded in $(s,\tau_f)\times\mathbb{R}^d$, we fix $b \in (0,\tau_f-s)$, $\lambda > a + k_0$ and we set

$$v_n(t,x) := e^{-\lambda(t-s)}u_f(t,x) - n^{-1}\tilde{\varphi}(x)$$

for any $(t,x) \in [s,s+b]\times\mathbb{R}^d$. A straightforward computation shows that

$$D_t v_n - \mathcal{A}v_n = e^{-\lambda(\cdot-s)}\psi(\cdot,\cdot,e^{\lambda(\cdot-s)}(v_n+n^{-1}\tilde{\varphi}), e^{\lambda(\cdot-s)}(\nabla_x v_n + n^{-1}\nabla\tilde{\varphi})) - \lambda(v_n+n^{-1}\tilde{\varphi}) + n^{-1}\mathcal{A}\tilde{\varphi} \tag{3.12}$$

in $(s,s+b]\times\mathbb{R}^d$. Since u_f is bounded in $[s,s+b]\times\mathbb{R}^d$ and $\tilde{\varphi}$ blows up at infinity, the function v_n admits a maximum point (t_n,x_n). If $v_n(t_n,x_n) \le 0$ for any n, then $u_f \le 0$ in $[s,s+b]\times\mathbb{R}^d$. Assume that $v_n(t_n,x_n) > 0$ for some n. If $t_n = s$, then $v_n(t_n,x_n) \le \sup_{\mathbb{R}^d} f$. If $t_n > s$, then $D_t v_n(t_n,x_n) - \mathcal{A}(t_n)v_n(t_n,x_n) \ge 0$, so that, multiplying both the sides of (3.12) by $v_n(t_n,x_n) + n^{-1}\tilde{\varphi}(x_n) > 0$ and using Hypotheses 3.5 (ii) we get $0 \le (-\lambda + k_0 + a)(v_n(t_n,x_n) + n^{-1}\tilde{\varphi}(x_n))^2 + k_0$, which clearly implies that, also in this case, u is bounded from above in $[s,s+b]$ by a constant, independent of b.

Repeating the same arguments with u_f being replaced by $-u_f$, we conclude that u_f is bounded also from below by a positive constant independent of b. Since b is arbitrary, it follows that $\|u_f\|_{C_b((s,\tau_f)\times\mathbb{R}^d)} < +\infty$ as claimed.

Step 2. Fix $f, g \in C_b(\mathbb{R}^d)$, $p \ge p_0$ and let $|||\cdot|||_p$ be the norm defined by

$$|||v|||_p = \sup_{t\in(s,T)} e^{-\omega(t-s)}(\|v(t,\cdot)\|_{L^p(\mathbb{R}^d,\mu_t)} + \sqrt{t-s}\|\nabla_x v(t,\cdot)\|_{L^p(\mathbb{R}^d,\mu_t)})$$

on smooth functions v, where ω is a positive constant to be chosen later on and to fix the ideas we assume that $p < +\infty$. From Hypothesis 3.5 (i), where L_R is replaced by a constant L, it follows that

$$\|\psi_{u_g}(r,\cdot) - \psi_{u_f}(r,\cdot)\|_{L^p(\mathbb{R}^d,\mu_r)} \le L\|u_g(r,\cdot) - u_f(r,\cdot)\|_{W^{1,p}(\mathbb{R}^d,\mu_r)}$$

for any $r \in (s, T]$. Hence, recalling that each operator $G(t, r)$ is a contraction from $L^p(\mathbb{R}^d, \mu_r)$ to $L^p(\mathbb{R}^d, \mu_t)$ and using the second pointwise gradient estimate in (2.4) and the invariance property of the family $\{\mu_t : t \in I\}$, we conclude that

$$\begin{aligned}
|||u_f - u_g|||_p \le{}& |||G(\cdot, s)(f-g)|||_p + L|||u_f - u_g|||_p \sup_{t\in(s,T)} \int_s^t e^{\omega(r-t)}\Big(1 + \frac{1}{\sqrt{r-s}}\Big)\,dr \\
&+ LC_0|||u_f - u_g|||_p \sup_{t\in(s,T)} \sqrt{t-s}\int_s^t e^{\omega(r-t)}\Big(1+\frac{1}{\sqrt{t-r}}\Big)\Big(1+\frac{1}{\sqrt{r-s}}\Big)\,dr \\
\le{}& [1 + C_0(1+\sqrt{T-s})]\|f-g\|_{L^p(\mathbb{R}^d,\mu_s)} \\
&+ L|||u_f - u_g|||_p\bigg[(1 + C_0\sqrt{T-s})\bigg(\frac{1}{\omega} + \int_s^t \frac{e^{\omega(r-t)}}{\sqrt{r-s}}\,dr\bigg) + \frac{\sqrt{\pi}}{\sqrt{\omega}}C_0\sqrt{T-s} \\
&+ C_0 \sup_{t\in(s,T)}\sqrt{t-s}\int_s^t \frac{e^{\omega(r-t)}\,dr}{\sqrt{t-r}\sqrt{r-s}}\bigg].
\end{aligned} \tag{3.13}$$

To estimate the integral terms in the last side of (3.13), we fix $\delta > 0$ and observe that

$$\int_s^t \frac{e^{\omega(r-t)}}{\sqrt{r-s}}\,dr = \int_s^{s+\delta}\frac{e^{\omega(r-t)}}{\sqrt{r-s}}\,dr + \bigg(\int_{s+\delta}^t \frac{e^{\omega(r-t)}}{\sqrt{r-s}}\,dr\bigg)^+ \le 2\sqrt{\delta} + \frac{1}{\sqrt{\delta}\omega}. \tag{3.14}$$

Hence, minimizing over $\delta > 0$, we conclude that the left-hand side of estimate (3.14) is bounded from above by $\sqrt{8}\omega^{-1/2}$. Splitting $\sqrt{t-s} \le \sqrt{t-r} + \sqrt{r-s}$ and arguing as above, also the last term in square brackets in the last side of (3.13) can be estimated by $(\sqrt{8} + \sqrt{\pi})\omega^{-1/2}$. It thus follows that

$$|||u_f - u_g|||_p \le [1 + C_0(1+\sqrt{T-s})]\|f-g\|_{L^p(\mathbb{R}^d,\mu_s)} + L(c_{T-s}\omega^{-\frac{1}{2}} + (1 + C_0\sqrt{T-s})\omega^{-1})|||u_f - u_g|||_p,$$

where $c_\tau = (\sqrt{8} + \sqrt{\pi})C_0(\sqrt{\tau}+1) + \sqrt{8}$. Choosing ω such that $c_{T-s}\omega^{-1/2} + (1 + C_0\sqrt{T-s})\omega^{-1} \le (2L)^{-1}$, we obtain

$$|||u_f - u_g|||_p \le 2[1 + C_0(1+\sqrt{T-s})]\|f-g\|_{L^p(\mathbb{R}^d,\mu_s)}$$

and estimate (3.11) follows at once.

Estimate (3.10) can be proved likewise. Hence, the details are omitted. □

As a consequence of Theorem 3.6 we prove the existence of a mild solution to problem (1.2) in the time domain (s, T) when $f \in L^p(\mathbb{R}^d, \mu_s)$, that is a function $u_f \in L^p((s, T)\times\mathbb{R}^d, \mu) \cap W^{0,1}_p(J\times\mathbb{R}^d, \mu)$, for any $J \Subset (s, T]$, such that $u_f(t, \cdot) \in W^{1,p}(\mathbb{R}^d, \mu_t)$ for almost every $t \in (s, T]$ and, for such values of t, the equality

$$u_f(t, x) = (G(t, s)f)(x) + \int_s^t (G(t, r)\psi_u(r, \cdot))(x)\,dr$$

holds true in $\mathbb{R}^d \setminus A_t$, where A_t is negligible with respect to the measure μ_t (or, equivalently, with respect to the restriction of the Lebesgue measure to the Borel σ-algebra in $\mathbb{R}^d$).

Corollary 3.7. *Under all the assumptions of Theorem 3.6, for any $f \in L^p(\mathbb{R}^d, \mu_s)$ ($p \ge p_0$) there exists a unique mild solution to the Cauchy problem* (1.2). *The function u_f satisfies estimates* (3.10) *and* (3.11) *with the supremum being replaced by the essential supremum and, as a byproduct, $u_f \in W^{0,1}_p((s, T)\times\mathbb{R}^d, \mu)$ if $p < 2$, and $u_f \in W^{0,1}_q((s, T)\times\mathbb{R}^d, \mu)$ for any $q < 2$ otherwise. Finally, if there exists $\gamma \in (0, 1)$ such that*

$$|\psi(t, x, \xi, \eta) - \psi(t, y, \xi, \eta)| \le C_{J,R}(1 + |\xi| + |\eta|)|x-y|^\gamma \tag{3.15}$$

for any $t \in J$, $x, y \in B_R$, $\eta \in \mathbb{R}^d$, $\xi \in \mathbb{R}$, $J \Subset (s, T]$, $R > 0$ and a constant $C_{J,R} > 0$, then, for any $f \in L^p(\mathbb{R}^d, \mu_s)$ and almost every $t \in (s, T)$, $u_f(t, \cdot)$ belongs to $W^{2,p}_{\rm loc}(\mathbb{R}^d)$. Moreover, $u_f \in W^{1,2}_{p,{\rm loc}}((s, T)\times\mathbb{R}^d)$ and satisfies the equation $D_t u_f = \mathcal{A}u_f + \psi_{u_f}$.

Proof. Fix $f \in L^p(\mathbb{R}^d, \mu_s)$ and let $(f_n) \subset C_b(\mathbb{R}^d)$ be a sequence converging to f in $L^p(\mathbb{R}^d, \mu_s)$. By (3.11), $(u_{f_n}(t,\cdot))$ is a Cauchy sequence in $W^{1,p}(\mathbb{R}^d, \mu_t)$ for any $t \in (s, T]$. Hence, there exists a function v such that $u_{f_n}(t,\cdot)$ converges to $v(t,\cdot)$ in $W^{1,p}(\mathbb{R}^d, \mu_t)$ for any $t \in (s, T]$. Moreover, writing (3.10), with f being replaced by f_n, and letting n tend to $+\infty$ we deduce that v satisfies (3.10) as well.

Next, using (3.11) we can estimate

$$\begin{aligned}\|u_{f_n} - u_{f_m}\|^p_{L^p((s,T)\times\mathbb{R}^d,\mu)} &= \int_s^T \|u_{f_n}(t,\cdot) - u_{f_m}(t,\cdot)\|^p_{L^p(\mathbb{R}^d,\mu_t)}\,dt\\ &\le C^p_{T-s}(T-s)\|f_n - f_m\|^p_{L^p(\mathbb{R}^d,\mu_s)}\end{aligned}$$

and

$$\begin{aligned}\|\nabla_x u_{f_n} - \nabla_x u_{f_m}\|^q_{L^q((s,T)\times\mathbb{R}^d,\mu)} &= \int_s^T \|\nabla_x u_{f_n}(t,\cdot) - \nabla_x u_{f_m}(t,\cdot)\|^q_{L^q(\mathbb{R}^d,\mu_t)}\,dt\\ &\le \frac{2C^q_{T-s}}{2-q}(T-s)^{1-\frac{q}{2}}\|f_n - f_m\|^q_{L^q(\mathbb{R}^d,\mu_s)}\end{aligned}$$

for any $q \in [1, 2)$ if $p \ge 2$ and for $p = q$ otherwise. Hence, recalling that $L^p(\mathbb{R}^d, \mu_t) \hookrightarrow L^q(\mathbb{R}^d, \mu_t)$ for any $t \in I$, we conclude that the sequence (u_{f_n}) converges in $L^p((s, T) \times \mathbb{R}^d, \mu) \cap W^{0,1}_q((s, T) \times \mathbb{R}^d, \mu)$ to a function, which we denote by u_f. Clearly, $v(t,\cdot) = u_f(t,\cdot)$ almost everywhere in $\mathbb{R}^d$ for almost every $t \in (s, T)$. Letting n tend to $+\infty$ in formula (1.4), with f_n replacing f, we deduce that u_f is a mild solution to problem (1.2). The uniqueness follows, arguing as in the proof of Theorem 3.2 with the obvious changes.

Let us now prove the last part of the statement. We again use an approximation argument. Fix $t > s \in I$ and $R > 0$. At a first step, we estimate the norm of the operator $G(t, r)$ in $\mathcal{L}(L^p(\mathbb{R}^d, \mu_r), L^p(B_{R+1}))$ and in $\mathcal{L}(L^p(\mathbb{R}^d, \mu_r), W^{2,p}(B_{R+1}))$, for any $r \in [s, t)$. In the rest of the proof, we denote by c a positive constant, possibly depending on R, but being independent of t, r and $f \in L^p(\mathbb{R}^d, \mu_r)$, which may vary from line to line. Since there exists a positive and continuous function $\rho : I \times \mathbb{R}^d \to \mathbb{R}$ such that $\mu_r = \rho(r,\cdot)dx$, the spaces $L^p(B_M)$ and $L^p(B_M, \mu_r)$ coincide and their norms are equivalent for any $M > 0$. From this remark, the interior L^p-estimates in Theorem A.3, with $u = G(\cdot, s)f$ and the contractiveness of $G(t, r)$ from $L^p(\mathbb{R}^d, \mu_r)$ to $L^p(\mathbb{R}^d, \mu_t)$, imply that

$$\|G(t,r)f\|_{W^{2,p}(B_{R+1})} \le c(t-r)^{-1}\|f\|_{L^p(\mathbb{R}^d,\mu_r)}, \quad s < r < t < T, \tag{3.16}$$

first for any $f \in C_b(\mathbb{R}^d)$, and then, by density, for any $f \in L^p(\mathbb{R}^d, \mu_r)$. Since, for $\theta \in (0, 1)$,

$$(L^p(\mathbb{R}^d,\mu_r), L^p(\mathbb{R}^d,\mu_r))_{\theta,p} = L^p(\mathbb{R}^d,\mu_r) \quad\text{and}\quad (W^{1,p}(B_{R+1}), W^{2,p}(B_{R+1}))_{\theta,p} = W^{1+\theta,p}(B_{R+1}),$$

with equivalence of the corresponding norms, by an interpolation argument and (3.16) we deduce that $\|G(t,r)\|_{\mathcal{L}(L^p(\mathbb{R}^d,\mu_r),W^{1+\theta,p}(B_{R+1}))} \le c(t-r)^{-\frac{1+\theta}{2}}$ for any $s < r < t < T$. Hence, if for any $n \in \mathbb{N}$ we consider the function z_n, which is the integral term in (1.4), with u being replaced by u_{f_n}, and use (3.11) and the fact that $\psi \in B([s, T] \times \mathbb{R}^d; \mathrm{Lip}(\mathbb{R}^{d+1}))$, then we get

$$\begin{aligned}\|\nabla_x z_n(t,\cdot) - \nabla_x z_m(t,\cdot)\|_{W^{\theta,p}(B_{R+1})} &\le c\int_s^t (t-r)^{-\frac{1+\theta}{2}}\|u_{f_n}(r,\cdot) - u_{f_m}(r,\cdot)\|_{L^p(\mathbb{R}^d,\mu_r)}\,dr\\ &\quad + c\int_s^t (t-r)^{-\frac{1+\theta}{2}}\|\nabla_x u_{f_n}(r,\cdot) - \nabla_x u_{f_m}(r,\cdot)\|_{L^p(\mathbb{R}^d,\mu_r)}\,dr\\ &\le c(t-s)^{-\frac{\theta}{2}}\|f_n - f_m\|_{L^p(\mathbb{R}^d,\mu_s)}\end{aligned}$$

for any $n \in \mathbb{N}$. We have so proved that, for any $\theta \in (0, 1)$ and almost every $t \in (s, T]$, the function $u_f(t,\cdot)$ belongs to $W^{1+\theta}(B_{R+1})$ and

$$\|u_{f_n}(t,\cdot) - u_{f_m}(t,\cdot)\|_{W^{1+\theta,p}(B_{R+1})} \le c(t-s)^{-\frac{1+\theta}{2}}\|f_n - f_m\|_{L^p(\mathbb{R}^d,\mu_s)}, \quad m, n \in \mathbb{N}.$$

Similarly,

$$\|u_{f_n}(t,\cdot)\|_{W^{1+\theta,p}(B_{R+1})} \le c(t-s)^{-\frac{1+\theta}{2}}(\|f\|_{L^p(\mathbb{R}^d,\mu_s)} + \|\psi(\cdot,\cdot,0,0)\|_\infty), \quad n \in \mathbb{N}.$$

Using these estimates, we can now show that $\psi_{u_f}(r,\cdot)\in W^{\theta,p}(B_{R+1})$, for any $\theta<\gamma$. For this purpose, we add and subtract $\psi(t,y,u_{f_n}(t,x),\nabla_x u_{f_n}(t,x))$, use condition (3.15) and the Lipschitz continuity of ψ with respect to the last two variables to infer that

$$\begin{aligned}|\psi_{u_{f_n}}(t,x)-\psi_{u_{f_n}}(t,y)|&\le c|u_{f_n}(t,x)-u_{f_n}(t,y)|+c|\nabla_x u_{f_n}(t,x)-\nabla_x u_{f_n}(t,y)|\\&\quad+c|x-y|^{\gamma}(1+|u_{f_n}(t,x)|+|\nabla_x u_{f_n}(t,x)|)\end{aligned}\tag{3.17}$$

and

$$|\psi_{u_{f_n}}(t,x)-\psi_{u_{f_m}}(t,x)|\le c(|u_{f_n}(t,x)-u_{f_m}(t,x)|+|\nabla_x u_{f_n}(t,x)-\nabla_x u_{f_m}(t,x)|)\tag{3.18}$$

for any $t\in(s,T)$, $x,y\in\mathbb{R}^d$ and $m,n\in\mathbb{N}$. Hence, using (3.17) we obtain

$$\begin{aligned}&|\psi_{u_{f_n}}(t,x)-\psi_{u_{f_n}}(t,y)-\psi_{u_{f_m}}(t,x)+\psi_{u_{f_m}}(t,y)|\\&\quad\le c\big[|u_{f_n}(t,x)-u_{f_n}(t,y)|+|\nabla_x u_{f_n}(t,x)-\nabla_x u_{f_n}(t,y)|+|u_{f_m}(t,x)-u_{f_m}(t,y)|+|\nabla_x u_{f_m}(t,x)-\nabla_x u_{f_m}(t,y)|\\&\qquad\quad+|x-y|^{\gamma}(1+|u_{f_n}(t,x)|+|u_{f_m}(t,x)|+|\nabla_x u_{f_n}(t,x)|+|\nabla_x u_{f_m}(t,x)|)\big]\\&\quad=:\mathcal{I}(t,x,y)\end{aligned}$$

and, using (3.18),

$$\begin{aligned}|\psi_{u_{f_n}}(t,x)-\psi_{u_{f_n}}(t,y)-\psi_{u_{f_m}}(t,x)+\psi_{u_{f_m}}(t,y)|&\le c\big[|u_{f_n}(t,x)-u_{f_m}(t,x)|+|\nabla_x u_{f_n}(t,x)-\nabla_x u_{f_m}(t,x)|\\&\qquad+|u_{f_n}(t,y)-u_{f_m}(t,y)|+|\nabla_x u_{f_n}(t,y)-\nabla_x u_{f_m}(t,y)|\big]\\&=:\mathcal{J}(t,x,y).\end{aligned}$$

From these two estimates we conclude that

$$|\psi_{u_{f_n}}(t,x)-\psi_{u_{f_n}}(t,y)-\psi_{u_{f_m}}(t,x)+\psi_{u_{f_m}}(t,y)|^p\le(\mathcal{I}(t,x,y))^{\beta p}(\mathcal{J}(t,x,y))^{(1-\beta)p}$$

for any $(t,x)\in(s,T)\times\mathbb{R}^d$, any $\beta\in(0,1)$ and any $m,n\in\mathbb{N}$. Hence, for any $\theta<\gamma$ and β, such that $(0,1)\ni\theta'=\theta/\beta+d(1-\beta)/(p\beta)$, a long but straightforward computation reveals that

$$\begin{aligned}&[\psi_{u_{f_n}}(t,\cdot)-\psi_{u_{f_m}}(t,\cdot)]_{W^{\theta,p}(B_{R+1})}\\&\qquad\le c\|u_{f_n}(t,\cdot)-u_{f_m}(t,\cdot)\|^{(1-\beta)}_{W^{1,p}(\mathbb{R}^d,\mu_t)}(\|u_{f_n}(t,\cdot)\|^{\beta}_{W^{1+\theta',p}(B_{R+1})}+\|u_{f_m}(t,\cdot)\|^{\beta}_{W^{1+\theta',p}(B_{R+1})}+1)\end{aligned}$$

and, consequently,

$$\|\psi_{u_{f_n}}(t,\cdot)-\psi_{u_{f_m}}(t,\cdot)\|_{W^{\theta,p}(B_{R+1})}\le c(t-s)^{\frac{\beta-\theta'}{2}-1}\|f_n-f_m\|^{1-\beta}_{L^p(\mathbb{R}^d,\mu_s)}$$

for any $t\in(s,T)$. We are almost done. Indeed, by interpolation from Proposition A.3 we deduce that $\|G(t,r)\|_{\mathcal{L}(W^{\theta,p}(B_{R+1}),W^{2,p}(B_R))}\le c(t-r)^{-1+\theta/2}$. From this and the previous estimate we conclude that

$$\|z_n(t,\cdot)-z_m(t,\cdot)\|_{W^{2,p}(B_R)}\le c(t-s)^{\frac{\beta+\theta-\theta'}{2}-1}\|f_n-f_m\|^{1-\beta}_{L^p(\mathbb{R}^d,\mu_s)},\quad m,n\in\mathbb{N},$$

for any $t\in(s,T]$ and $\beta>\theta'$, so that

$$\|u_{f_n}(t,\cdot)-u_{f_m}(t,\cdot)\|_{W^{2,p}(B_R)}\le c(t-s)^{-1}\|f_n-f_m\|^{1-\beta}_{L^p(\mathbb{R}^d,\mu_s)}$$

for any $m,n\in\mathbb{N}$, thanks to (3.16). From this estimate it is easy to deduce that (u_{f_n}) is a Cauchy sequence in $W^{0,2}_{p,\mathrm{loc}}((s,T)\times\mathbb{R}^d)$. Since u_{f_n} is a classical solution to problem (1.2), we conclude that $(D_t u_{f_n})$ is a Cauchy sequence in $L^p_{\mathrm{loc}}((s,T)\times\mathbb{R}^d)$. It thus follows that $u_f\in W^{1,2}_{p,\mathrm{loc}}((s,T)\times\mathbb{R}^d)$ and it solves the equation

$$D_t u_f=\mathcal{A}u_f+\psi_{u_f}$$

in $(s,T)\times\mathbb{R}^d$. □

The arguments in the proof of Theorem 3.6 and Corollary 3.7 allow us to prove the following result.

Proposition 3.8. *Under Hypotheses 2.1, the following properties are satisfied.*

(i) *Let $\psi \in C((s,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d)$ with $[\psi]_{\infty,T}+\sup_{(t,x)\in(s,T]\times\mathbb{R}^d}[\psi(t,x,\cdot,\cdot)]_{\mathrm{Lip}(\mathbb{R}^{d+1})}<+\infty$. Then, for any $f\in C_b(\mathbb{R}^d)$, the Cauchy problem* (1.2) *admits a unique mild solution $u_f\in C([s,T]\times\mathbb{R}^d)\cap C^{0,1}((s,T]\times\mathbb{R}^d)$ which satisfies* (3.10) *and* (3.11) *for any $p\in[p_0,+\infty]$.*

(ii) *Let $\psi\in C((s,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d)$ and $[\psi]_{p,T}+\sup_{(t,x)\in(s,T]\times\mathbb{R}^d}[\psi(t,x,\cdot,\cdot)]_{\mathrm{Lip}(\mathbb{R}^{d+1})}<+\infty$ for some $p\ge p_0$. Then, for any $f\in L^p(\mathbb{R}^d,\mu_s)$, the Cauchy problem* (1.2) *admits a unique mild solution u_f which belongs to $W^{0,1}_p((s,T)\times\mathbb{R}^d)$, if $p_0\le p<2$, and to $W^{0,1}_p(J\times\mathbb{R}^d)$ for any $J\Subset(s,T]$, if $p\ge 2$. Further, u_f satisfies* (3.10) *and* (3.11)*, with the supremum being replaced by the essential supremum.*

Proof. To prove property (i), it suffices to apply the Banach fixed point theorem in the space of all the functions $v\in C_b([s,T]\times\mathbb{R}^d)\cap C^{0,1}((s,T]\times\mathbb{R}^d)$ such that $|||v|||_\infty<+\infty$, where $|||\cdot|||_\infty$ is defined in Step 2 of the proof of Theorem 3.6, with $p=+\infty$. The uniqueness of the so obtained solution follows from the condition $\sup_{(t,x)\in(s,T]\times\mathbb{R}^d}[\psi(t,x,\cdot,\cdot)]_{\mathrm{Lip}(\mathbb{R}^{d+1})}<+\infty$, in a standard way.

To prove property (ii), one can argue by approximation. We fix $f\in L^p(\mathbb{R}^d,\mu_s)$, approximate it by a sequence $(f_n)\subset C_b(\mathbb{R}^d)$, converging to f in $L^p(\mathbb{R}^d,\mu_s)$, and introducing a standard sequence (ϑ_n) of cut-off functions. If we set $\psi_n=\vartheta_n\psi$ for any $n\in\mathbb{N}$, then each function ψ_n satisfies the assumptions in property (i) and $[\psi_n]_{p,T}\le[\psi]_{p,T}$. Therefore, the Cauchy problem (1.2), with f_n and ψ_n replacing f and ψ admits a unique mild solution $u\in C_b([s,T]\times\mathbb{R}^d)\cap C^{0,1}((s,T]\times\mathbb{R}^d)$, which satisfies (3.10) and (3.11) with f_n replacing f. The arguments in the first part of the proof of Corollary 3.7 allow us to prove the existence of a mild solution u_f to the Cauchy problem (1.2) with the properties in the statement of the proposition. The uniqueness of the solution follows also in this case from the condition $\sup_{(t,x)\in(s,T]\times\mathbb{R}^d}[\psi(t,x,\cdot,\cdot)]_{\mathrm{Lip}(\mathbb{R}^{d+1})}<+\infty$. □

4 The evolution operator and its summability improving properties

Suppose that, besides Hypotheses 2.1, the assumptions on ψ in Theorem 3.6 hold true for any $I\ni s<T$ or $\psi\in C(I\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d)\cap B(J\times\mathbb{R}^d;\mathrm{Lip}(\mathbb{R}^{d+1}))$ for each $J\Subset I$ and $\psi(\cdot,\cdot,0,0)\in C_b(\mathbb{R}^{d+1})$. Then, for any $f\in C_b(\mathbb{R}^d)$ and $s\in I$ the mild solution to problem (1.2) exists in the whole of $[s,+\infty)$. Hence, we can set $\mathcal{N}(t,s)f=u_f(t,\cdot)$ for any $t>s$. Each operator $\mathcal{N}(t,s)$ maps $C_b(\mathbb{R}^d)$ into $C^1_b(\mathbb{R}^d)$. Moreover, the uniqueness of the solution to problem (1.2) yields the *evolution law* $\mathcal{N}(t,s)f=\mathcal{N}(t,r)\mathcal{N}(r,s)f$ for any $r\in(s,t)$ and $f\in C_b(\mathbb{R}^d)$. Hence $\{\mathcal{N}(t,s):I\ni s<t\}$ is a nonlinear evolution operator in $C_b(\mathbb{R}^d)$. It can be extended to the L^p-setting, for any $p\ge p_0$, using the same arguments as in the first part of the proof of Corollary 3.7. Clearly, if $\psi(t,x,\cdot,\cdot)$ is Lipschitz continuous in $\mathbb{R}^{d+1}$, uniformly with respect to $(t,x)\times J\times\mathbb{R}^d$, for any $J\Subset I$, then by density, we still deduce that $\mathcal{N}(t,s)$ satisfies the evolution law and, moreover, each operator $\mathcal{N}(t,s)$ is bounded from $L^p(\mathbb{R}^d,\mu_s)$ to $W^{1,p}(\mathbb{R}^d,\mu_t)$ and

$$\|\mathcal{N}(t,s)f\|_{L^p(\mathbb{R}^d,\mu_t)}+\sqrt{t-s}\,\|\nabla_x\mathcal{N}(t,s)f\|_{L^p(\mathbb{R}^d,\mu_t)}\le C_{T-s}(\|f\|_{L^p(\mathbb{R}^d,\mu_s)}+(T-s+1)\|\psi(\cdot,\cdot,0,0)\|_\infty). \tag{4.1}$$

4.1 Continuity properties of the nonlinear evolution operator

In the following theorem, assuming the above conditions on ψ, we prove an interesting continuity property of the operator $\mathcal{N}(t,s)$.

Theorem 4.1. *Let $(f_n)\subset C_b(\mathbb{R}^d)$ be a bounded sequence converging to some function $f\in C_b(\mathbb{R}^d)$ pointwise in $\mathbb{R}^d$. Then, for any $s\in I$, $\mathcal{N}(\cdot,s)f_n$ and $\nabla_x\mathcal{N}(\cdot,s)f_n$ converge to $\mathcal{N}(\cdot,s)f$ and $\nabla_x\mathcal{N}(\cdot,s)f$, respectively, locally uniformly in $(s,+\infty)\times\mathbb{R}^d$.*

Proof. Let (f_n) and f be as in the statement. To ease the notation, we write u_{f_n} and u_f for $\mathcal{N}(\cdot,s)f_n$ and $\mathcal{N}(\cdot,s)f$, respectively. Moreover, we set $h_n(r,\cdot)=G(t,r)(|u_{f_n}(r,\cdot)-u_f(r,\cdot)|^p+|\nabla_x(u_{f_n}(r,\cdot)-u_f(r,\cdot))|^p)$ for any $n\in\mathbb{N}$, $t>s$ and $r\in(s,t]$, and we denote by $L_{R,T}$ any constant such that

$$|\psi(t,x,u_2,v_2)-\psi(t,x,u_1,v_1)|\le L_{R,T}(|u_2-u_1|+|v_2-v_1|) \tag{4.2}$$

for any $t \in [s, s+T]$, $x, v_1, v_2 \in \mathbb{R}^d$, $u_1, u_2 \in [-R, R]$ and $T > 0$. As a first step, formula (3.2) shows that, for any $T > 0$, there exists a positive constant M_T such that $\|u_f\|_\infty + \|u_{f_n}\|_\infty \le M_T$. Fix $p \in (1, 2)$. Using formula (1.4), we can estimate

$$|D_x^j u_{f_n}(t,x) - D_x^j u_f(t,x)|^p \le 2^{p-1}|(D_x^j G(t,s)(f_n - f))(x)|^p + 2^{p-1}\bigg|\int_s^t (D_x^j G(t,r)(\psi_{u_{f_n}}(r,\cdot) - \psi_{u_f}(r,\cdot))(x)\,dr\bigg|^p$$

for any $(t,x) \in (s,+\infty) \times \mathbb{R}^d$ and $j = 0, 1$. By the representation formula (2.2), Hölder inequality, estimates (2.4) and (4.2), we deduce that

$$|u_{f_n}(t,\cdot) - u_f(t,\cdot)|^p \le 2^{p-1}G(t,s)|f_n - f|^p + (4T)^{p-1}L_{M_T}^p \int_s^t h(r,\cdot)\,dr$$

and

$$|\nabla_x u_{f_n}(t,\cdot) - \nabla_x u_f(t,\cdot)|^p \le 2^{p-1}(t-s)^{-\frac{p}{2}} c_T G(t,s)|f_n - f|^p + (4T)^{p-1} c_T L_{M_T,T}^p \int_s^t (t-r)^{-\frac{p}{2}} h(r,\cdot)\,dr$$

in $\mathbb{R}^d$, for any $t \in (s, s+T)$ and some positive constant c_T. Hence, the function $h_n(\cdot, x)$ satisfies the differential inequality

$$h_n(t,x) \le C_{p,T}(t-s)^{-\frac{p}{2}}(G(t,s)|f_n - f|^p)(x) + C_{p,T}\int_s^t (t-r)^{-\frac{p}{2}} h_n(r,x)\,dr$$

for any $t \in (s, s+T)$ and $x \in \mathbb{R}^d$. Since $h_n(\cdot, x)$ is continuous in $(s,t]$ and $h_n(r,x) \le \widetilde{C}_T(r-s)^{-p/2}$ for some positive constant $\widetilde{C}_T$, independent of n, and any $r \in (s,t)$, we can apply [8, Lemma 7.1] and conclude that

$$h_n(t,x) \le C_{p,T}(t-s)^{-\frac{p}{2}}(G(t,s)|f_n - f|^p)(x) + C_{p,T}\int_s^t (t-r)^{-\frac{p}{2}}(r-s)^{-\frac{p}{2}}(G(r,s)|f_n - f|^p)(x)\,dr$$

for any $t \in (s, s+T)$. Hence,

$$\|h_n(t,\cdot)\|_{C_b(B_R)} \le C_{p,T}(t-s)^{-\frac{p}{2}}\|G(t,s)|f_n - f|^p\|_{C_b(B_R)} + C_{p,T}\int_s^t (t-r)^{-\frac{p}{2}}(r-s)^{-p/2}\|G(r,s)|f_n - f|^p\|_{C_b(B_R)}\,dr$$

for any $R > 0$. By [9, Proposition 3.1 (i)], $\|G(r,s)|f_n - f|^p\|_{C_b(B_R)}$ vanishes as $n \to +\infty$ for any $r > s$. Hence, by dominated convergence, $\|h_n(t,\cdot)\|_{C_b(B_R)}$ vanishes as $n \to +\infty$ for any $t \in (s, s+T)$, which means that, for any $t \in (s, s+T)$, $u_{f_n}(t,\cdot)$ and $\nabla_x u_{f_n}(t,\cdot)$ converge uniformly in B_R to $u_f(t,\cdot)$ and $\nabla_x u_{f_n}(t,\cdot)$, respectively. The arbitrariness of R and T yields the assertion. □

4.2 Hypercontractivity

Throughout this and the forthcoming subsections we set

$$\mathcal{F}(\zeta) = |\sqrt{Q}\nabla_x \zeta|^2, \quad \mathcal{G}(\zeta) = \sum_{i=1}^d |\sqrt{Q}\nabla_x D_i \zeta|^2$$

for any smooth enough function ζ. To begin with, we recall the following crucial result.

Lemma 4.2 ([4, Lemma 3.1]). *Assume that Hypotheses 2.1 hold true and fix $[a,b] \subset I$. If $f \in C_b^{1,2}([a,b] \times \mathbb{R}^d)$ and $f(r,\cdot)$ is constant outside a compact set K for every $r \in [a,b]$, then the function $r \mapsto \int_{\mathbb{R}^d} f(r,\cdot)\,d\mu_r$ is continuously differentiable in $[a,b]$ and*

$$D_r \int_{\mathbb{R}^d} f(r,\cdot)\,d\mu_r = \int_{\mathbb{R}^d} D_r f(r,\cdot)\,d\mu_r - \int_{\mathbb{R}^d} \mathcal{A}(r) f(r,\cdot)\,d\mu_r, \quad r \in [a,b].$$

Hypotheses 4.3. We introduce the following assumptions.

(i) $\psi \in B(I \times \mathbb{R}^d; \mathrm{Lip}(\mathbb{R}^{d+1})) \cap C(I \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d)$, condition (3.9) is satisfied in $[s, T]$, for any $T > s \in I$ and some constant which may depend also on s and T, and there exist two constants $\xi_0 \geq 0$ and ξ_1 such that $u\psi(t, x, u, v) \leq \xi_0|u| + \xi_1 u^2 + \xi_2|u||v|$ for any $t \geq s$, $x, v \in \mathbb{R}^d$ and $u \in \mathbb{R}$.

(ii) There exists a nonnegative function $\widetilde{\varphi} : \mathbb{R}^d \to \mathbb{R}$, blowing up at infinity such that $\mathcal{A}\widetilde{\varphi} + k_1|\nabla\widetilde{\varphi}| \leq a\widetilde{\varphi}$ in $\mathbb{R}^d$ for some locally bounded functions a, k_1.

(iii) There exist locally bounded functions $C_0, C_1, C_2 : I \to \mathbb{R}^+$ such that

$$|Q(t,x)x| \leq C_0(t)|x|^3\tilde{\varphi}(x), \quad \mathrm{Tr}(Q(t,x)) \leq C_1(t)(1+|x|^2)\tilde{\varphi}(x), \quad \langle b(t,x), x\rangle \leq C_2(t)|x|^2\tilde{\varphi}(x)$$

for any $t \in I$ and any $x \in \mathbb{R}^d$.

(iv) There exists a positive constant K such that

$$\int_{\mathbb{R}^d} |f|^q \log(|f|)\, d\mu_t \leq \|f\|^q_{L^q(\mathbb{R}^d,\mu_t)} \log(\|f\|_{L^q(\mathbb{R}^d,\mu_t)}) + Kq \int_{\mathbb{R}^d} |f|^{q-2}|\nabla f|^2 \mathbb{1}_{\{f\neq 0\}}\, d\mu_t \tag{4.3}$$

for any $t > s$, $f \in C^1_b(\mathbb{R}^d)$ and $q \in (1, +\infty)$.

Remark 4.4. (i) Hypothesis 4.3 (i) implies that $\psi(\cdot, \cdot, 0, 0)$ is bounded in $[s, +\infty) \times \mathbb{R}^d$ and

$$\|\psi(\cdot,\cdot,0,0)\|_{C_b([s,+\infty)\times\mathbb{R}^d)} \leq \xi_0.$$

(ii) Sufficient conditions for (4.3) to hold are given in [4]. In particular, (4.3) holds true when (2.3) is satisfied with $p = 1$ (see Remark 2.2).

We can now prove the main result of this subsection.

Theorem 4.5. *Let Hypotheses 2.1 and 4.3 be satisfied. Then, for any $f \in L^p(\mathbb{R}^d, \mu_s)$ $(p \geq p_0)$ and $t > s$, the function $\mathcal{N}(t,s)f$ belongs to $W^{1,p_\gamma(t)}(\mathbb{R}^d, \mu_t)$ and satisfies the estimates*

$$\|\mathcal{N}(t,s)f\|_{L^{p_\gamma(t)}(\mathbb{R}^d,\mu_t)} \leq e^{\omega_{p,\gamma}(t-s)}[\|f\|_{L^p(\mathbb{R}^d,\mu_s)} + \xi_0(t-s)], \tag{4.4}$$

$$\|\nabla_x\mathcal{N}(t,s)f\|_{L^{p_\gamma(t)}(\mathbb{R}^d,\mu_t)} \leq c_0(t-s)e^{\omega_{p,\sqrt{\gamma}}(t-s)}[\|f\|_{L^p(\mathbb{R}^d,\mu_s)} + \xi_0(t-s)] + c_1(t-s)\xi_0, \tag{4.5}$$

where $p_\gamma(t) := \gamma^{-1}(p-1)(e^{\kappa_0 K^{-1}(t-s)} - 1) + p$ for any $\gamma > 1$, κ_0 being the ellipticity constant in Hypothesis 2.1 (ii) *and K being the constant in* (4.3), *$\omega_{p,\sigma} = \xi_1 + (\xi_2^+)^2\sigma[(\sigma-1)(p-1)\kappa_0]^{-1}$ and the functions $c_0, c_1 : (0, +\infty) \to \mathbb{R}^+$ are continuous and blow up at zero.*

Proof. To begin with, we observe that it suffices to prove (4.4) and (4.5) for functions $f \in C^1_b(\mathbb{R}^d)$. Indeed, in the general case, the assertion follows approximating f with a sequence $(f_n) \subset C^1_b(\mathbb{R}^d)$ which converges to f in $L^p(\mathbb{R}^d, \mu_s)$. By (3.10), $\mathcal{N}(t,s)f_n$ converges to $\mathcal{N}(t,s)f$ in $W^{1,p}(\mathbb{R}^d, \mu_t)$ for almost every $t > s$. Hence, writing (4.4) and (4.5) with f being replaced by f_n and letting n tend to $+\infty$, the assertion follows at once by applying Fatou lemma.

We split the rest of the proof into two steps. In the first one we prove (4.4) and in the latter one (4.5).

Step 1. Fix $f \in C^1_b(\mathbb{R}^d)$, $n \in \mathbb{N}$, $\varepsilon > 0$ and set

$$\beta_{n,\varepsilon}(t) := \|v_{n,\varepsilon}(t,\cdot)\|_{p_\gamma(t)}$$

for any $t > s$, where $v_{n,\varepsilon} = (\vartheta_n^2\mathcal{N}(\cdot,s)f + \varepsilon)^{1/2}$ and $\vartheta_n = \zeta(n^{-1}|x|)$ for any $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$. Here, ζ is a smooth function such that $\mathbb{1}_{B_1} \leq \zeta \leq \mathbb{1}_{B_2}$. Moreover, we set

$$\varphi_{1,n} = \left|\zeta'\left(\frac{|\cdot|}{n}\right)\right|\tilde{\varphi}, \quad \varphi_{2,n} = \left|\zeta''\left(\frac{|\cdot|}{n}\right)\right|\tilde{\varphi}$$

for any $n \in \mathbb{N}$. We recall that in [9, Theorem 5.4] it has been proved that $\sup_{t\in I}\|\tilde{\varphi}\|_{L^1(\mathbb{R}^d,\mu_t)} < +\infty$. Hence, the functions $t \mapsto \|\varphi_{j,n}\|_{L^p(\mathbb{R}^d,\mu_t)}$ $(j = 1, 2)$ are bounded in I and pointwise converge to zero as $n \to +\infty$.

By definition, the function $u = \mathcal{N}(\cdot, s)f$ belongs to $C^{0,1}_b([s,\tau] \times \mathbb{R}^d)$ for any $\tau > s$ and is a classical solution to problem (1.2). Moreover, Lemma 4.2 shows that $\beta_{n,\varepsilon}$ is differentiable in $(s, +\infty)$ and a straightforward computation reveals that

$$\beta'_{n,\varepsilon}(t) = -\frac{p'_\gamma(t)}{p_\gamma(t)}\beta_{n,\varepsilon}(t)\log(\beta_{n,\varepsilon}(t)) + \frac{1}{p_\gamma(t)}(\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)}\int_{\mathbb{R}^d}\{D_t[(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)}] - \mathcal{A}(t)[(v_{n,\varepsilon}(t,\cdot)^{p_\gamma(t)}]\}\, d\mu_t.$$

Taking into account that

$$
\begin{aligned}
D_t[(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)}] - \mathcal{A}[(v_{n,\varepsilon}(t,\cdot)^{p_\gamma(t)}] = {}& p_\gamma'(t)(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)}\log(v_{n,\varepsilon}(t,\cdot))\\
&- p_\gamma(t)(p_\gamma(t)-1)(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-2}(\mathcal{F}(v_{n,\varepsilon}))(t,\cdot)\\
&+ p_\gamma(t)(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-1}(D_t v_{n,\varepsilon}(t,\cdot) - \mathcal{A}(t)v_{n,\varepsilon}(t,\cdot))
\end{aligned}
$$

and

$$
\begin{aligned}
D_t v_{n,\varepsilon} - \mathcal{A}v_{n,\varepsilon} &= \vartheta_n^2 v_{n,\varepsilon}^{-1}u\psi_u - \varepsilon\vartheta_n^2\mathcal{F}(u)v_{n,\varepsilon}^{-3} - \operatorname{Tr}(QD^2\vartheta_n)v_{n,\varepsilon}^{-1}\vartheta_n u^2 - \varepsilon u^2 v_{n,\varepsilon}^{-3}\mathcal{F}(\vartheta_n)\\
&\quad - \langle b,\nabla\vartheta_n\rangle v_{n,\varepsilon}^{-1}\vartheta_n u^2 - 2(2\varepsilon u + \vartheta_n^2u^3)\langle Q\nabla\vartheta_n,\nabla_x u\rangle\vartheta_n v_{n,\varepsilon}^{-3}\\
&=: \vartheta_n^2 v_{n,\varepsilon}^{-1}u\psi_u - \varepsilon\vartheta_n^2\mathcal{F}(u)v_{n,\varepsilon}^{-3} + g_{n,\varepsilon}(t,\cdot),
\end{aligned}
$$

we deduce

$$
\begin{aligned}
\beta_{n,\varepsilon}'(t) = {}& (\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)}\int_{\mathbb{R}^d}(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-1}g_{n,\varepsilon}(t,\cdot)\,d\mu_t - \frac{p_\gamma'(t)}{p_\gamma(t)}\beta_{n,\varepsilon}(t)\log(\beta_{n,\varepsilon}(t))\\
&- (p_\gamma(t)-1)(\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)}\int_{\mathbb{R}^d}(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-2}(\mathcal{F}(v_{n,\varepsilon}))(t,\cdot)\,d\mu_t\\
&+ \frac{p_\gamma'(t)}{p_\gamma(t)}(\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)}\int_{\mathbb{R}^d}(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)}\log(v_{n,\varepsilon}(t,\cdot))\,d\mu_t\\
&+ (\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)}\int_{\mathbb{R}^d}(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-2}\vartheta_n^2 u(t,\cdot)\psi_u(t,\cdot)\,d\mu_t\\
&- \varepsilon(\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)}\int_{\mathbb{R}^d}\vartheta_n^2(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-4}(\mathcal{F}(u))(t,\cdot))\,d\mu_t.
\end{aligned}
$$

Using Hypotheses 4.3 (i), (iv), the expression of the function $t \mapsto p_\gamma(t)$ and Hypothesis 2.1 (ii), we can estimate

$$
\begin{aligned}
\beta_{n,\varepsilon}'(t) \le {}& (\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)}\int_{\mathbb{R}^d}(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-1}g_{n,\varepsilon}(t,\cdot)\,d\mu_t\\
&+ \xi_0(\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)}\int_{\mathbb{R}^d}\vartheta_n^2|u(t,\cdot)|(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-2}\,d\mu_t + \xi_1\beta_{n,\varepsilon}(t)\\
&- \varepsilon\xi_1(\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)}\int_{\mathbb{R}^d}(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-2}\,d\mu_t\\
&- (p-1)(1-\gamma^{-1})(\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)}\int_{\mathbb{R}^d}(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-2}(\mathcal{F}(v_{n,\varepsilon}))(t,\cdot)\,d\mu_t\\
&+ \xi_2^+(\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)}\int_{\mathbb{R}^d}\vartheta_n^2(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-2}|u(t,\cdot)||\nabla_x u(t,\cdot)|\,d\mu_t\\
&- \varepsilon(\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)}\int_{\mathbb{R}^d}\vartheta_n^2(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-4}\mathcal{F}(u(t,\cdot))\,d\mu_t.
\end{aligned}
\tag{4.6}
$$

Further, since $\mathcal{F}(v_{n,\varepsilon}) = \vartheta_n^2\mathcal{F}(\vartheta_n)u^4v_{n,\varepsilon}^{-2} + \vartheta_n^4u^2\mathcal{F}(u)v_{n,\varepsilon}^{-2} + 2\vartheta_n^2u^3v_{n,\varepsilon}^{-2}\langle Q\nabla_x u,\nabla\vartheta_n\rangle$ and

$$
\begin{aligned}
&\int_{\mathbb{R}^d}(v(t,\cdot))^{p_\gamma(t)-4}\vartheta_n^2(u(t,\cdot))^3\langle Q(t,\cdot)\nabla_x u(t,\cdot),\nabla\vartheta_n\rangle\,d\mu_t\\
&\quad\le \delta\int_{\mathbb{R}^d}(v(t,\cdot))^{p_\gamma(t)-4}\vartheta_n^4(u(t,\cdot))^2(\mathcal{F}(u))(t,\cdot)\,d\mu_t + \frac{1}{\delta}\int_{\mathbb{R}^d}(v(t,\cdot))^{p_\gamma(t)-4}(u(t,\cdot))^4(\mathcal{F}(\vartheta_n))(t,\cdot)\,d\mu_t,
\end{aligned}
$$

it follows that

$$\int_{\mathbb{R}^d} (v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-2}(\mathcal{F}(v_{n,\varepsilon}))(t,\cdot)\,d\mu_t \geq (1-\delta)\int_{\mathbb{R}^d} \vartheta_n^4(u(t,\cdot))^2(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-4}(\mathcal{F}(u))(t,\cdot)\,d\mu_t - C_{\varepsilon,\delta}(t)\int_{\mathbb{R}^d}\varphi_{1,n}\,d\mu_t$$

for any $\delta > 0$ and some continuous function $C_{\varepsilon,\delta} : [s,+\infty) \to \mathbb{R}^+$. Moreover, applying Hölder and Young inequalities and Hypothesis 2.1 (ii) we can infer that

$$\int_{\mathbb{R}^d} \vartheta_n^2(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-2}|u(t,\cdot)||\nabla_x u(t,\cdot)|d\mu_t \leq \frac{\delta_1}{\kappa_0}\int_{\mathbb{R}^d}\vartheta_n^4(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-4}|u(t,\cdot)|^2(\mathcal{F}(u))(t,\cdot)\,d\mu_t + \frac{1}{4\delta_1}(\beta_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)}$$

for any $\delta_1 > 0$ and

$$\int_{\mathbb{R}^d}\vartheta_n^2 u(t,\cdot)(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-2}\,d\mu_t \leq \int_{\mathbb{R}^d}(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-1}\,d\mu_t \leq (\beta_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-1}.$$

Hence,

$$\begin{aligned}
\beta'_{n,\varepsilon}(t) \leq{}& \xi_0 + \Big(\xi_1 + \frac{\xi_2^+}{4\delta_1}\Big)\beta_{n,\varepsilon}(t) + (\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)}\int_{\mathbb{R}^d}(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-1}g_{n,\varepsilon}(t,\cdot)\,d\mu_t\\
&- [(p-1)(1-\gamma^{-1})(1-\delta) - \kappa_0^{-1}\xi_2^+\delta_1](\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)}\int_{\mathbb{R}^d}\vartheta_n^4|u(t,\cdot)|^2(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-4}(\mathcal{F}(u))(t,\cdot)\,d\mu_t\\
&+ \widetilde{C}_{\varepsilon,\delta,p,\gamma}(t)(\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)}\int_{\mathbb{R}^d}\varphi_{1,n}\,d\mu_t\\
&- \varepsilon(\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)}\int_{\mathbb{R}^d}\vartheta_n^2(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-4}(\mathcal{F}(u))(t,\cdot)\,d\mu_t\\
&- \varepsilon\xi_1(\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)}\int_{\mathbb{R}^d}(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-2}\,d\mu_t
\end{aligned}$$

for some continuous function $\widetilde{C}_{\varepsilon,\delta,p,\gamma} : [s,+\infty) \to \mathbb{R}^+$. Now, we estimate the integral term containing g_n. We begin by observing that

$$\begin{aligned}
&-2\int_{\mathbb{R}^d}(2\varepsilon u(t,\cdot) + \vartheta_n^2(u(t,\cdot))^3)\langle Q(t,\cdot)\nabla\vartheta_n, \nabla_x u\rangle\vartheta_n(v_{n,\varepsilon}(t,\cdot))^{p(t)-4}\,d\mu_t\\
&\leq 4\varepsilon\delta_2\int_{\mathbb{R}^d}\vartheta_n^2(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-4}(\mathcal{F}(u))(t,\cdot)\,d\mu_t + \varepsilon\delta_2^{-1}\int_{\mathbb{R}^d}|u(t,\cdot)|^2(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-4}(\mathcal{F}(\vartheta_n))(t,\cdot)\,d\mu_t\\
&\quad + \delta_2\int_{\mathbb{R}^d}\vartheta_n^4|u(t,\cdot)|^2(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-4}(\mathcal{F}(u))(t,\cdot)\,d\mu_t\\
&\quad + \delta_2^{-1}\int_{\mathbb{R}^d}\vartheta_n^2|u(t,\cdot)|^4(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-4}(\mathcal{F}(\vartheta_n))(t,\cdot)\,d\mu_t\\
&\leq 4\varepsilon\delta_2\int_{\mathbb{R}^d}\vartheta_n^2(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-4}(\mathcal{F}(u))(t,\cdot)\,d\mu_t + \widetilde{C}_{\varepsilon,\delta_2}(t)\int_{\mathbb{R}^d}\varphi_{1,n}\,d\mu_t\\
&\quad + \delta_2\int_{\mathbb{R}^d}\vartheta_n^4|u(t,\cdot)|^2(v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-4}(\mathcal{F}(u))(t,\cdot)\,d\mu_t
\end{aligned}$$

for some continuous function $\widetilde{C}_{\varepsilon,\delta_2} : [s,+\infty) \to \mathbb{R}^+$. Moreover,

$$-\int_{\mathbb{R}^d} (v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-4} u(t,\cdot)[\vartheta_n \mathcal{A}(t)\vartheta_n - \varepsilon u(t,\cdot)(\mathcal{F}(\vartheta_n))(t,\cdot)]\, d\mu_t \le \overline{C}_\varepsilon(t) \int_{\mathbb{R}^d} (\varphi_{1,n} + \varphi_{2,n})\, d\mu_t$$

for some positive and continuous function $\overline{C}_\varepsilon : [s,+\infty) \to \mathbb{R}^+$. Hence, replacing these estimates in (4.6), we get

$$\begin{aligned}
\beta'_{n,\varepsilon}(t) \le{}& \xi_0 + \Big(\xi_1 + \frac{\xi_2^+}{4\delta_1}\Big)\beta_{n,\varepsilon}(t) + \widehat{C}_{\varepsilon,\delta,\delta_2,p}(t)(\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)} \int_{\mathbb{R}^d} (\varphi_{1,n} + \varphi_{2,n})\, d\mu_t \\
& - [(p-1)(1-\gamma^{-1})(1-\delta) - \kappa_0^{-1}\xi_2^+\delta_1 - \delta_2](\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)} \\
& \times \int_{\mathbb{R}^d} \vartheta_n^4 |u(t,\cdot)|^2 (v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-4} (\mathcal{F}(u))(t,\cdot)\, d\mu_t \\
& - \varepsilon(1-4\delta_2)(\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)} \int_{\mathbb{R}^d} \vartheta_n^2 (v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-4} (\mathcal{F}(u))(t,\cdot)\, d\mu_t \\
& - \varepsilon\xi_1(\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)} \int_{\mathbb{R}^d} (v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-2}\, d\mu_t,
\end{aligned} \tag{4.7}$$

where, again, $\widehat{C}_{\varepsilon,\delta,\delta_2,p} : (s,+\infty) \to \mathbb{R}^+$ is a continuous function. Choosing $\delta = \frac{1}{2}$, $\delta_1 = (p-1)(1-\gamma^{-1})\frac{\kappa_0}{4\xi_2}$ if $\xi_2 > 0$, $\delta_1 = 0$ otherwise, and then δ_2 small enough we obtain

$$\begin{aligned}
\beta'_{n,\varepsilon}(t) \le{}& \xi_0 + \omega_{p,\gamma}\beta_{n,\varepsilon}(t) + \widehat{C}_{\varepsilon,1/2,\delta_2,p}(t)(\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)} \|\varphi_{1,n} + \varphi_{2,n}\|_{L^1(\mathbb{R}^d,\mu_t)} \\
& - \varepsilon\xi_1(\beta_{n,\varepsilon}(t))^{1-p_\gamma(t)} \int_{\mathbb{R}^d} (v_{n,\varepsilon}(t,\cdot))^{p_\gamma(t)-2}\, d\mu_t.
\end{aligned} \tag{4.8}$$

Hence, integrating (4.8) between s and t and letting first $n \to +\infty$ and then $\varepsilon \to 0^+$, by dominated convergence we get

$$\|u(t,\cdot)\|_{L^{p_\gamma(t)}(\mathbb{R}^d,\mu_t)} \le \|f\|_{L^p(\mathbb{R}^d,\mu_s)} + \xi_0(t-s) + \omega_{p,\gamma} \int_s^t \|u(r,\cdot)\|_{L^{p(r)}(\mathbb{R}^d,\mu_r)}\, dr.$$

Applying the Gronwall lemma, we conclude the proof of (4.4).

Step 2. To check estimate (4.5), we arbitrarily fix $\gamma \in (1,+\infty)$, $t > s$ and we take

$$\varepsilon = \frac{K}{2\kappa_0} \log\Big(\frac{\gamma e^{\kappa_0 K^{-1}(t-s)}}{\gamma + e^{\kappa_0 K^{-1}(t-s)} - 1}\Big), \qquad \gamma' = \gamma \frac{e^{\kappa_0 K^{-1}(t-s-\varepsilon)} - 1}{e^{\kappa_0 K^{-1}(t-s)} - 1}. \tag{4.9}$$

With these choices of ε and γ', we have

$$p_{\gamma'}(t-\varepsilon) = p_\gamma(t).$$

From Step 1, we know that $\mathcal{N}(t-\varepsilon,s)f \in L^{p_{\gamma'}(t-\varepsilon)}(\mathbb{R}^d,\mu_{t-\varepsilon})$ and

$$\|\mathcal{N}(t-\varepsilon,s)f\|_{L^{p_\gamma(t)}(\mathbb{R}^d,\mu_{t-\varepsilon})} \le e^{\omega_{p,\gamma'}(t-s-\varepsilon)}(\|f\|_{L^p(\mathbb{R}^d,\mu_s)} + \xi_0(t-s)). \tag{4.10}$$

By the evolution law and estimates (4.10) and (4.1) we get

$$\begin{aligned}
&\sup_{\tau\in(t-\varepsilon,T)} \sqrt{\tau - t + \varepsilon}\, \|\nabla_x \mathcal{N}(\tau,s)f\|_{L^{p_\gamma(t)}(\mathbb{R}^d,\mu_\tau)} \\
&\quad \le C_{T-t+\varepsilon}\{e^{\omega_{p,\gamma'}(t-s-\varepsilon)}(\|f\|_{L^p(\mathbb{R}^d,\mu_s)} + \xi_0(t-s)) + (T-t+\varepsilon+1)\|\psi(\cdot,\cdot,0,0)\|_\infty\}
\end{aligned}$$

for any $T > t-\varepsilon$. In particular, taking $T = t$ and using Remark 4.4 (i) to estimate $\|\psi(\cdot,\cdot,0,0)\|_\infty \le \xi_0$, we get

$$\|\nabla_x \mathcal{N}(t,s)f\|_{L^{p_\gamma(t)}(\mathbb{R}^d,\mu_t)} \le \frac{C_\varepsilon}{\sqrt{\varepsilon}} e^{-\varepsilon\omega_{p,\gamma'}} e^{\omega_{p,\gamma'}(t-s)}[\|f\|_{L^p(\mathbb{R}^d,\mu_s)} + \xi_0(t-s)] + \Big(\sqrt{\varepsilon} + \frac{1}{\sqrt{\varepsilon}}\Big)C_\varepsilon\xi_0. \tag{4.11}$$

Replacing the value of ε in the expression of γ' (see (4.9)), we deduce that

$$\gamma' \geq \inf_{\delta \geq 1} \gamma(\delta-1)^{-1}\left[\left(\frac{\delta(\gamma+\delta-1)}{\gamma}\right)^{1/2}-1\right] = \sqrt{\gamma}$$

and, since the function $\sigma \mapsto \omega_{p,\sigma}$ is decreasing, $\omega_{p,\gamma'} \leq \omega_{p,\sqrt{\gamma}}$. Finally, observing that $e^{-\varepsilon\omega_{p,\gamma'}}$ is bounded in $(s,+\infty)$, $\varepsilon < (2\kappa_0)^{-1}K\log(\gamma)$ (which follows from (4.9) recalling that $\gamma' \geq \sqrt{\gamma}$) and $\varepsilon \sim (2\gamma)^{-1}(\gamma-1)(t-s)$ as $t-s \to 0^+$, formula (4.5) follows immediately replacing in (4.11) the value of ε given by (4.9). ☐

Remark 4.6. As the proof of Theorem 4.5 shows, if $\xi_2 \leq 0$, then we can take $\gamma = 1$ and $\omega_{p,1} = \xi_1$ in (4.4).

4.3 Supercontractivity

In the next theorem we prove a stronger result than Theorem 4.5, i.e., we prove that the nonlinear evolution operator $\mathcal{N}(t,s)$ satisfies a supercontractivity property. For this purpose, we introduce the following additional assumption.

Hypothesis 4.7. There exists a decreasing function $\nu : (0,+\infty) \to \mathbb{R}^+$ blowing up as σ tends to 0^+ such that

$$\int_{\mathbb{R}^d} |f|^p \log(|f|)\,d\mu_r - \|f\|^p_{L^p(\mathbb{R}^d,\mu_r)} \log(\|f\|_{L^p(\mathbb{R}^d,\mu_r)}) \leq \frac{\nu(\sigma)}{p}\|f\|^p_{L^p(\mathbb{R}^d,\mu_r)} + \sigma p \int_{\mathbb{R}^d} |f|^{p-2}|\nabla f|^2 \mathbb{1}_{\{f\neq 0\}}\,d\mu_r \tag{4.12}$$

for any $r \in I$, $\sigma > 0$ and $f \in C^1_b(\mathbb{R}^d)$.

Remark 4.8. Sufficient conditions for (4.12) to hold are given in [3]. In particular, it holds true when (2.3) is satisfied with $p = 1$ (see Remark 2.2) and there exist $K > 0$ and $R > 1$ such that $\langle b(t,x), x\rangle \leq -K|x|^2 \log|x|$ for any $t \in I$ and $|x| \geq R$.

Theorem 4.9. *Let Hypotheses 2.1, 4.3* (i)–(iii) *and 4.7 be satisfied. Then, for any $t > s \in I$, $p_0 \leq p < q < +\infty$ and any $f \in L^p(\mathbb{R}^d,\mu_s)$, $\mathcal{N}(t,s)f$ belongs to $W^{1,q}(\mathbb{R}^d,\mu_t)$ and*

$$\|\mathcal{N}(t,s)f\|_{L^q(\mathbb{R}^d,\mu_t)} \leq c_2(t-s)(\|f\|_{L^p(\mathbb{R}^d,\mu_s)} + \xi_0(t-s)), \tag{4.13}$$

$$\|\nabla_x\mathcal{N}(t,s)f\|_{L^q(\mathbb{R}^d,\mu_t)} \leq c_3(t-s)\|f\|_{L^p(\mathbb{R}^d,\mu_s)} + c_4(t-s)\xi_0. \tag{4.14}$$

Here, $c_2, c_3, c_4 : (0,+\infty) \to \mathbb{R}^+$ are continuous functions such that $\lim_{r\to 0^+} c_k(r) = +\infty$ $(k = 2,3,4)$.

Proof. The proof of this result follows the same lines of the proof of Theorem 4.5. For this reason we use the notation therein introduced and we limit ourselves to sketching it in the case when $f \in C^1_b(\mathbb{R}^d)$.

Step 1. Here, we prove (4.13). For any $\sigma > 0$ and any $t \geq s$, we set

$$p(t) = e^{\kappa_0(2\sigma)^{-1}(t-s)}(p-1)+1, \quad m(t) = \nu(\sigma)(p^{-1}-(p(t))^{-1}), \quad \zeta_{n,\varepsilon}(t) = e^{-m(t)}\beta_{n,\varepsilon}(t).$$

The function $\zeta_{n,\varepsilon}$ is differentiable in $(s,+\infty)$ and arguing as in the proof of the quoted theorem, using (4.12) instead of (4.3) and the definition of $m(t)$ and $p(t)$, we deduce that

$$\begin{aligned}\zeta'_{n,\varepsilon}(t) = \Bigg[&(\beta_{n,\varepsilon}(t))^{1-p(t)}\int_{\mathbb{R}^d}(v_{n,\varepsilon}(t,\cdot))^{p(t)-1}g_{n,\varepsilon}(t,\cdot)\,d\mu_t\\ &-\frac{p-1}{2}(\beta_{n,\varepsilon}(t))^{1-p(t)}\int_{\mathbb{R}^d}(v_{n,\varepsilon}(t,\cdot))^{p(t)-2}(\mathcal{F}(v_{n,\varepsilon}))(t,\cdot)\,d\mu_t\\ &+(\beta_{n,\varepsilon}(t))^{1-p(t)}\int_{\mathbb{R}^d}(v_{n,\varepsilon}(t,\cdot))^{p(t)-2}\vartheta_n^2 u(t,\cdot)\psi_u(t,\cdot)\,d\mu_t\\ &-\varepsilon(\beta_{n,\varepsilon}(t))^{1-p(t)}\int_{\mathbb{R}^d}\vartheta_n^2(v_{n,\varepsilon}(t,\cdot))^{p(t)-4}(\mathcal{F}(u))(t,\cdot)\,d\mu_t\Bigg]e^{-m(t)}\end{aligned}$$

and the same arguments used to prove (4.7) show that, if $\delta_2 < \frac{1}{4}$, then

$$\zeta'_{n,\varepsilon}(t) \le \xi_0 e^{-m(t)} + \Big(\xi_1 + \frac{\xi_2^+}{4\delta_1}\Big)\zeta_{n,\varepsilon}(t) + \Big(\widehat{C}_{\varepsilon,\delta,\delta_2,p}(t)(\beta_{n,\varepsilon}(t))^{1-p(t)} \int_{\mathbb{R}^d} (\varphi_{1,n} + \varphi_{2,n})\, d\mu_t$$
$$- [2^{-1}(p-1)(1-\delta) - \kappa_0^{-1}\xi_2^+\delta_1 - \delta_2](\beta_{n,\varepsilon}(t))^{1-p(t)} \int_{\mathbb{R}^d} \vartheta_n^4 |u(t,\cdot)|^2 (v_{n,\varepsilon}(t,\cdot))^{p(t)-4}(\mathcal{F}(u))(t,\cdot)\, d\mu_t$$
$$- \varepsilon\xi_1(\beta_{n,\varepsilon}(t))^{1-p(t)} \int_{\mathbb{R}^d} (v_{n,\varepsilon}(t,\cdot))^{p(t)-2}\, d\mu_t\Big) e^{-m(t)}.$$

Choosing $\delta = \frac{1}{2}$, $\delta_1 = (p-1)\kappa_0(8\xi_2)^{-1}$ if $\xi_2 > 0$, $\delta_1 = 0$ otherwise, and $\delta_2 = [(p-1)\wedge 2]/8$ we get

$$\zeta'_{n,\varepsilon}(t) \le \widetilde{\omega}_p \zeta_{n,\varepsilon}(t) + e^{-m(t)}\Big[\xi_0 + \widehat{C}_{\varepsilon,\delta_2,p}(t)(\beta_{n,\varepsilon}(t))^{1-p(t)} \|\varphi_{1,n} + \varphi_{2,n}\|_{L^1(\mathbb{R}^d,\mu_t)}$$
$$- \varepsilon\xi_1(\beta_{n,\varepsilon}(t))^{1-p(t)} \int_{\mathbb{R}^d} (v_{n,\varepsilon}(t,\cdot))^{p(t)-2}\, d\mu_t\Big], \tag{4.15}$$

where $\widetilde{\omega}_p = \xi_1 + 2(\xi_2^+)^2(\kappa_0(p-1))^{-1}$. Hence, integrating (4.15) between s and t and letting first $n \to +\infty$ and then $\varepsilon \to 0^+$, by dominated convergence we get

$$e^{-m(t)}\|u_f(t,\cdot)\|_{L^{p(t)}(\mathbb{R}^d,\mu_t)} \le \xi_0(t-s) + \|f\|_{L^p(\mathbb{R}^d,\mu_s)} + \widetilde{\omega}_p \int_s^t e^{-m(r)}\|u(r,\cdot)\|_{L^{p(r)}(\mathbb{R}^d,\mu_r)}\, dr,$$

which yields

$$\|u_f(t,\cdot)\|_{L^{p(t)}(\mathbb{R}^d,\mu_t)} \le e^{\widetilde{\omega}_p(t-s)+m(t)}(\xi_0(t-s) + \|f\|_{L^p(\mathbb{R}^d,\mu_s)}).$$

Now, for any $q > p$ and $t > s$, we fix $\sigma = \kappa_0(t-s)(2\log(q-1) - 2\log(p-1))^{-1}$. We get $p(t) = q$ and from the previous inequality the claim follows with

$$c_2(r) = \exp(\widetilde{\omega}_p r + (p^{-1} - q^{-1})\nu(\kappa_0 r(2\log(q-1) - 2\log(p-1))^{-1})).$$

Step 2. Fix $q > p$. By Step 1, $\mathcal{N}((t+s)/2, s)f$ belongs to $L^q(\mathbb{R}^d, \mu_{(t+s)/2})$ and

$$\Big\|\mathcal{N}\Big(\frac{t+s}{2}, s\Big)f\Big\|_{L^q(\mathbb{R}^d,\mu_{(t+s)/s})} \le c_2\Big(\frac{t-s}{2}\Big)\Big(\|f\|_{L^p(\mathbb{R}^d,\mu_s)} + \xi_0\frac{t-s}{2}\Big).$$

The same arguments used in Step 2 of the proof of Theorem 4.5 show that $\mathcal{N}(t,s)f \in W^{1,q}(\mathbb{R}^d,\mu_\tau)$ for any $\tau > \frac{t+s}{2}$ and

$$\sqrt{\frac{t-s}{2}}\|\nabla_x \mathcal{N}(t,s)f\|_{L^q(\mathbb{R}^d,\mu_\tau)} \le C_{(t-s)/2}\Big[c_2\Big(\frac{t-s}{2}\Big)\Big(\|f\|_{L^p(\mathbb{R}^d,\mu_s)} + \xi_0\frac{t-s}{2}\Big) + \Big(\frac{t-s}{2} + 1\Big)\xi_0\Big].$$

Estimate (4.14) follows with $c_3(r) = \sqrt{\frac{2}{r}}C_{r/2}c_2(\frac{2}{r})$, $c_4(r) = C_{r/2}[c_2(\frac{2}{r})\sqrt{\frac{2}{r}} + \sqrt{\frac{2}{r}} + \sqrt{\frac{2}{r}}]$. □

4.4 Ultraboundedness

To begin with, we prove a sort of Harnack inequality, which besides the interest in its own will be crucial to prove the ultraboundedness of the nonlinear evolution operator $\mathcal{N}(t,s)$.

Proposition 4.10. *Let Hypotheses 2.1* (i)–(iii) *and 4.3* (i)–(iii) *be satisfied. Further, suppose that estimate* (2.3) *holds, with $p = 1$ and some constant $\sigma_1 \in \mathbb{R}$. Then, for any $f \in C_b(\mathbb{R}^d)$, $p > 1$, $t > s$ and $x, y \in \mathbb{R}^d$, the following estimate holds true:*

$$|(\mathcal{N}(t,s)f)(x)|^p \le \exp\Big(p(1+\xi_1^+)(t-s) + p\Theta(t-s)\frac{(|x-y| + \xi_2^+(t-s))^2}{4\kappa_0(t-s)^2(p-1)}\Big)[(G(t,s)|f|^p)(y) + \xi_0^p], \tag{4.16}$$

where $\Theta(r) = (e^{2\sigma_1 r} - 1)/(2\sigma_1)$ if $\sigma_1 > 0$ and $\Theta(r) = r$ otherwise.

Proof. To begin with, we observe that it suffices to prove (4.16) for functions in $C^1_b(\mathbb{R}^d)$. Indeed, if $f \in C_b(\mathbb{R}^d)$, we can determine a sequence $(f_n) \subset C^1_b(\mathbb{R}^d)$, bounded with respect to the sup-norm and converging to f locally uniformly in $\mathbb{R}^d$. Writing (4.16) with f replaced by f_n and using Theorem 4.1 and [9, Proposition 3.1 (i)], we can let n tend to $+\infty$ and complete the proof.

So, let us fix $f \in C^1_b(\mathbb{R}^d)$ and set $\Phi_n(r) := [G(t,r)(\vartheta_n^2 v_\varepsilon(r,\cdot))](\phi(r)) + \xi_0^p$ for any $n \in \mathbb{N}$ and $r \in (s,t)$, where $v_\varepsilon = (u_f^2+\varepsilon)^{p/2}$, $u_f = \mathcal{N}(\cdot, s)f$ (see Theorem 3.6), $\phi(r) = (r-s)(t-s)^{-1}x + (t-r)(t-s)^{-1}y$ and (ϑ_n) is a standard sequence of cut-off functions. We note that $\Phi_n(r) \ge C_\Phi > 0$ for any $r \in [s,t]$ and any $n \ge n_0$. This is clear if $\xi_0 > 0$. Suppose that $\xi_0 = 0$. If $r < t$, then $\Phi_n(r)$ is positive since $v_\varepsilon > 0$. If $r = t$, then $\Phi_n(t) = (\vartheta_n(x))^2 v_\varepsilon(t,x)$ which is positive if we choose $n \in \mathbb{N}$ large enough such that $x \in \operatorname{supp}(\vartheta_n)$. Moreover, $\Phi_n \in C^1((s,t))$. Hence $\log(\Phi_n) \in C^1((s,t))$ and we have

$$\begin{aligned}\frac{d}{dr}\log(\Phi_n(r)) = \frac{1}{\Phi_n(r)}\{&[G(t,r)(\vartheta_n^2 D_t v_\varepsilon(r,\cdot)) - \mathcal{A}(\vartheta_n^2 v_\varepsilon(r,\cdot)))](\phi(r))\\ &+ (t-s)^{-1}\langle [\nabla_x G(t,r)(\vartheta_n^2 v_\varepsilon(r,\cdot))](\phi(r)), x-y\rangle\}.\end{aligned}$$

We observe that

$$\begin{aligned}D_t(\vartheta_n^2 v_\varepsilon) - \mathcal{A}(\vartheta_n^2 v_\varepsilon) = {}& p\vartheta_n^2(u_f^2+\varepsilon)^{\frac{p}{2}-1}u_f\psi_{u_f} - p\vartheta_n^2 v_\varepsilon^{1-\frac{4}{p}}((p-1)u_f^2+\varepsilon)\mathcal{F}(u_f)\\ &- 4pv_\varepsilon^{1-\frac{2}{p}}\vartheta_n u_f\langle Q\nabla\vartheta_n, \nabla_x u_f\rangle - 2\vartheta_n v_\varepsilon\mathcal{A}\vartheta_n - 2v_\varepsilon\mathcal{F}(\vartheta_n),\end{aligned}$$

and

$$\begin{aligned}|\nabla_x G(t,r)(\vartheta_n^2 v_\varepsilon(r,\cdot))| &\le e^{\sigma_1(t-r)}G(t,r)|\nabla_x(\vartheta_n^2 v_\varepsilon(r,\cdot))|\\ &\le pe^{\sigma_1(t-r)}G(t,r)(\vartheta_n^2(v_\varepsilon(r,\cdot))^{1-\frac{2}{p}}|u_f(r,\cdot)|\kappa_0^{-\frac{1}{2}}((\mathcal{F}(u_f))(r,\cdot))^{\frac{1}{2}})\\ &\quad + e^{\sigma_1(t-r)}G(t,r)(2\vartheta_n|\nabla\vartheta_n|v_\varepsilon(r,\cdot)).\end{aligned}$$

Hence, we get

$$\begin{aligned}\frac{d}{dr}\log\Phi_n(r) \le \frac{1}{\Phi_n(r)}\Big\{&p\frac{|x-y|}{t-s}e^{\sigma_1(t-r)}G(t,r)[\vartheta_n^2(v_\varepsilon(r,\cdot))^{1-\frac{2}{p}}|u_f(r,\cdot)|\kappa_0^{-\frac{1}{2}}((\mathcal{F}(u_f))(r,\cdot))^{\frac{1}{2}}]\\ &- G(t,r)\zeta_{n,\varepsilon}(r,\cdot) + \frac{|x-y|}{t-s}e^{\sigma_1(t-r)}G(t,r)(2\vartheta_n|\nabla\vartheta_n|v_\varepsilon(r,\cdot))\Big\}(\phi(r)),\end{aligned}$$

where

$$\zeta_{n,\varepsilon} = 2\vartheta_n(\mathcal{A}\vartheta_n)v_\varepsilon + 2\mathcal{F}(\vartheta_n)v_\varepsilon + 4p\vartheta_n v_\varepsilon^{1-\frac{2}{p}}u_f\langle Q\nabla\vartheta_n, \nabla_x u_f\rangle - p\vartheta_n^2 v_{n,\varepsilon}^{1-\frac{2}{p}}u_f\psi_{u_f} + p\vartheta_n^2 v_\varepsilon^{1-\frac{4}{p}}((p-1)u_f+\varepsilon)\mathcal{F}(u_f).$$

From Hypothesis 4.3 (i) it follows that

$$\begin{aligned}\frac{d}{dr}\log\Phi_n(r) \le \frac{1}{\Phi_n(r)}G(t,r)\Big\{&-2\vartheta_n(\mathcal{A}(r)\vartheta_n)v_\varepsilon(r,\cdot) + p\xi_0\vartheta_n^2 v_\varepsilon^{1-\frac{1}{p}} + \xi_1^+\vartheta_n^2 p v_\varepsilon(r,\cdot)\\ &+ 4p(v_\varepsilon(r,\cdot))^{1-\frac{2}{p}}\vartheta_n|u_f(r,\cdot)||\langle Q(r,\cdot)\nabla\vartheta_n, \nabla_x u_f(r,\cdot)\rangle|\\ &- p\vartheta_n^2 v_\varepsilon(r,\cdot)\Big[((p-1)(u_f(r,\cdot))^2+\varepsilon)(h_\varepsilon(r,\cdot))^2\\ &- |u_f(r,\cdot)|h_\varepsilon(r,\cdot)\frac{e^{\sigma_1(t-r)}|x-y| + \xi_2^+(t-s)}{\sqrt{\kappa_0}(t-s)}\Big]\Big\}(\phi(r))\\ &+ \frac{|x-y|}{t-s}e^{\sigma_1(t-r)}\{G(t,r)[|\nabla\vartheta_n|v_\varepsilon(r,\cdot)]\}(\phi(r)),\end{aligned}\tag{4.17}$$

where $h_\varepsilon = (u_f^2+\varepsilon)^{-1}\sqrt{\mathcal{F}(u_f)}$. Using the Cauchy–Schwarz inequality, we can estimate

$$\begin{aligned}v_\varepsilon^{1-\frac{2}{p}}\vartheta_n|u_f||\langle Q\nabla\vartheta_n, \nabla_x u_f\rangle| &\le v_\varepsilon^{1-\frac{2}{p}}\vartheta_n|u_f|\sqrt{\mathcal{F}(\vartheta_n)}\sqrt{\mathcal{F}(u_f)}\\ &\le \delta\vartheta_n^2 v_\varepsilon h_\varepsilon^2 u_f^2 + \frac{1}{4\delta}v_\varepsilon\mathcal{F}(\vartheta_n).\end{aligned}$$

Moreover, using formula (2.2), we can estimate

$$
\begin{aligned}
(G(t,r)(\vartheta_n^2 v_\varepsilon^{1-1/p}))(\phi(r)) &\le ((G(t,r)(\vartheta_n^2 v_\varepsilon))(\phi(r)))^{1-\frac{1}{p}}((G(t,r)\vartheta_n)(\phi(r))^p\\
&\le ((G(t,r)(\vartheta_n^2 v_\varepsilon))(\phi(r)))^{1-\frac{1}{p}} \le (\Phi_n(r))^{1-\frac{1}{p}}.
\end{aligned}
$$

These two estimates replaced in (4.17) give

$$
\begin{aligned}
\frac{d}{dr}\log\Phi_n(r) \le \frac{1}{\Phi_n(r)}\bigg(G(t,r)\Big\{&[p\delta^{-1}(\mathcal{F}(\vartheta_n))(r,\cdot)-2\vartheta_n\mathcal{A}(r)\vartheta_n]v_\varepsilon(r,\cdot)+\xi_1^+\vartheta_n^2 p v_\varepsilon(r,\cdot)\\
&-p\vartheta_n^2 v_\varepsilon(r,\cdot)\Big[((p-1-\delta)(u_f(r,\cdot))^2+\varepsilon)(h_\varepsilon(r,\cdot))^2\\
&-|u_f(r,\cdot)|h_\varepsilon(r,\cdot)\frac{e^{\sigma_1(t-r)}|x-y|+\xi_2^+(t-s)}{\sqrt{\kappa_0}(t-s)}\Big]\Big\}\bigg)(\phi(r))\\
&+\frac{|x-y|}{t-s}e^{\sigma_1(t-r)}\{G(t,r)[|\nabla\vartheta_n|v_\varepsilon(r,\cdot)]\}(\phi(r))+p.
\end{aligned}
\tag{4.18}
$$

Straightforward computations show that $\mathcal{A}(r)\vartheta_n$ and $(\mathcal{F}(\vartheta_n))(r,\cdot)$ vanish pointwise in $\mathbb{R}^d$ as $n\to+\infty$, for any $r\in(s,t)$ and there exists a positive constant C such that $|\mathcal{A}(r)\vartheta_n|+(\mathcal{F}(\vartheta_n))(r,\cdot)\le C\tilde{\varphi}$ in $\mathbb{R}^d$ for any $n\in\mathbb{N}$, thanks to Hypothesis 4.3 (iii). By [9, Lemma 3.4] the function $G(t,\cdot)\tilde{\varphi}$ is bounded in $(s,t)\times B_R$ for any $R>0$. Hence, by dominated convergence we conclude that $G(t,r)(\vartheta_n(\mathcal{A}(r)\vartheta_n)v_\varepsilon(r,\cdot))$ vanishes as $n\to+\infty$, pointwise in $\mathbb{R}^d$, for any $r\in(s,t)$ and

$$
\|G(t,r)[p\delta^{-1}(\mathcal{F}(\vartheta_n))(r,\cdot)-2\vartheta_n\mathcal{A}(r)\vartheta_n]\|_{C_b(B_R)}\le C_{\delta,p,\|u_f\|_\infty}\sup_{r\in(s,t)}\|G(t,r)\tilde{\varphi}\|_{C_b(B_R)},\tag{4.19}
$$

where $R>\max\{|x|,|y|\}$. Similarly, the last but one term in (4.18) vanishes pointwise in $\mathbb{R}^d$ as $n\to+\infty$, for any $r\in(s,t)$ and

$$
|(G(t,r)[|\nabla\vartheta_n|v_\varepsilon(r,\cdot)])(\phi(r))|\le C_{p,\|u_f\|_\infty}\sup_{r\in(s,t)}\|G(t,r)\tilde{\varphi}\|_{C_b(B_R)}.\tag{4.20}
$$

Moreover, using the inequality $\alpha\beta^2-\gamma\beta\ge-\frac{\gamma^2}{4\alpha}$ for any $\alpha>0$ and $\beta,\gamma\in\mathbb{R}$, and that $G(t,s)g_1\le G(t,s)g_2$ for any $t>s$ and any $g_1\le g_2$, we deduce

$$
\begin{aligned}
\frac{d}{dr}\log(\Phi_n(r))\le{}&\frac{1}{\Phi_n(r)}(G(t,r)[p\delta^{-1}(\mathcal{F}(\vartheta_n))(r,\cdot)-2\vartheta_n\mathcal{A}(r)\vartheta_n]v_\varepsilon(r,\cdot))(\phi(r))+p(1+\xi_1^++e^{2\sigma_1^+(t-r)}\chi_\delta)\\
&+\frac{e^{\sigma_1(t-r)}|x-y|}{(t-s)\Phi_n(r)}(G(t,r)[|\nabla\vartheta_n|v_\varepsilon(r,\cdot)])(\phi(r)),
\end{aligned}
$$

where $\chi_\delta=(|x-y|+\xi_2^+(t-s))^2(4\kappa_0(t-s)^2(p-1-\delta))^{-1}$. Integrating both sides of the previous inequality in (s,t) and taking (4.19) and (4.20) into account to let $n\to+\infty$, we get

$$
\log\bigg(\frac{((u_f(t,x))^2+\varepsilon)^{\frac{p}{2}}+\xi_0^p}{(G(t,s)((f^2+\varepsilon)^{\frac{p}{2}}))(y)+\xi_0^p}\bigg)\le p[(1+\xi_1^+)(t-s)+\Theta(t-s)\chi_\delta],
$$

or even

$$
((u_f(t,x))^2+\varepsilon)^{p/2}\le\exp[p(1+\xi_1^+)(t-s)+p\Theta(t-s)\chi_\delta][(G(t,s)((f^2+\varepsilon)^{\frac{p}{2}}))(y)+\xi_0^p].
$$

By formula (2.2) we can let ε and δ tend to zero on both sides of the previous inequality and this yields the assertion. □

We can now prove the main result of this subsection. For this purpose, we set $\varphi_\lambda(x)=e^{\lambda|x|^2}$ for any $x\in\mathbb{R}^d$ and $\lambda>0$, and introduce the following additional assumption.

Hypothesis 4.11. For any $I\ni s<t$ and $\lambda>0$, the function $G(t,s)\varphi_\lambda$ belongs to $L^\infty(\mathbb{R}^d)$ and, for any $\delta>0$, $+\infty>M_{\delta,\lambda}:=\sup_{t-s\ge\delta}\|G(t,s)\varphi_\lambda\|_\infty$.

Remark 4.12. A sufficient condition for Hypothesis 4.11 to hold is given in [3, Theorem 4.3]. More precisely, it holds when (2.3) holds with $p=1$ and there exists $K>0$, $\beta,R>1$ such that $\langle b(t,x),x\rangle\le-K|x|^2(\log(|x|))^\beta$ for any $t\in I$ and $x\in\mathbb{R}^d\setminus B_R$.

Theorem 4.13. *Assume that Hypothesis 4.11 and the conditions in Proposition 4.10 are satisfied. Then, for any* $I \ni s < t$, $f \in L^p(\mathbb{R}^d, \mu_s)$ $(p \in [p_0, +\infty))$, *the function* $\mathcal{N}(t,s)f$ *belongs to* $W^{1,\infty}(\mathbb{R}^d)$ *and*

$$\|\mathcal{N}(t,s)f\|_\infty \le c_5(t-s)\|f\|_{L^p(\mathbb{R}^d,\mu_s)} + c_6(t-s)\xi_0, \tag{4.21}$$

$$\|\nabla_x\mathcal{N}(t,s)f\|_\infty \le c_7(t-s)\|f\|_{L^p(\mathbb{R}^d,\mu_s)} + c_8(t-s)\xi_0 \tag{4.22}$$

for some continuous functions $c_k : (0,+\infty) \to \mathbb{R}^+$ $(k = 5,6,7,8)$ *which blow up at zero.*

Proof. As usually, we prove the assertion for functions in $C_b^1(\mathbb{R}^d)$.

Step 1. We prove (4.21). So, let us fix $f \in C_b^1(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. By the invariance property of the family $\{\mu_t : t \in I\}$ and inequality (4.16), we can estimate

$$\begin{aligned}
\|f\|^p_{L^p(\mathbb{R}^d,\mu_s)} &= \int_{\mathbb{R}^d} (G(t,s)|f|^p)(y)\mu_t(dy)\\
&\ge \int_{B_R} [(G(t,s)|f|^p)(y) + \xi_0^p]\mu_t(dy) - \xi_0^p\\
&\ge |(\mathcal{N}(t,s)f)(x)|^p e^{-p\phi(t-s)} \int_{B_R} \exp\Big(-p\Theta(t-s)\frac{(|x-y| + \xi_2^+(t-s))^2}{4\kappa_0(t-s)^2(p-1)}\Big)\mu_t(dy) - \xi_0^p\\
&\ge |(\mathcal{N}(t,s)f)(x)|^p e^{-p\phi(t-s)} \exp\Big(-p\Theta(t-s)\frac{(|x| + R + \xi_2^+(t-s))^2}{4\kappa_0(t-s)^2(p-1)}\Big)\mu_t(B_R) - \xi_0^p,
\end{aligned}$$

where $\phi = 1 + \xi_1^+$. By the tightness of the family $\{\mu_t : t \in I\}$ we can fix $R > 0$ such that $\mu_t(B_R) \ge 2^{-p}$ for any $t \ge s$ and, from the previous chain of inequalities, we conclude that

$$|(\mathcal{N}(t,s)f)(x)|^p \le 2^p(\widetilde{C}(t-s))^p\varphi_{p\Lambda(t-s)}(x)(\|f\|^p_{L^p(\mathbb{R}^d,\mu_s)} + \xi_0^p), \tag{4.23}$$

where

$$\Lambda(r) = \exp\Big(\frac{\Theta(r)}{2\kappa_0 r^2(p-1)}\Big), \quad \widetilde{C}(r) = \exp\Big(\phi r + \Theta(r)\frac{(\xi_2^+ r + R)^2}{2\kappa_0 r^2(p-1)}\Big).$$

Now, using the evolution law and again (4.16), we can write

$$\begin{aligned}
|(\mathcal{N}(t,s)f)(x)|^p &= \Big|\Big(\mathcal{N}\Big(t,\frac{t+s}{2}\Big)\mathcal{N}\Big(\frac{t+s}{2},s\Big)\Big)(x)\Big|^p\\
&\le \Big[\Big(G\Big(t,\frac{t+s}{2}\Big)\Big|\mathcal{N}\Big(\frac{t+s}{2},s\Big)f\Big|^p\Big)(y) + \xi_0^p\Big]\\
&\quad \times \exp\Big(p\phi\frac{t-s}{2} + p\Theta\Big(\frac{t-s}{2}\Big)\frac{(2|x-y| + \xi_2^+(t-s))^2}{4\kappa_0(t-s)^2(p-1)}\Big)
\end{aligned} \tag{4.24}$$

for any $y \in \mathbb{R}^d$. From (2.2) and (4.23) we obtain

$$\begin{aligned}
\Big(G\Big(t,\frac{t+s}{2}\Big)\Big|\mathcal{N}\Big(\frac{t+s}{2},s\Big)f\Big|^p\Big)(y) &\le 2^p\Big(\widetilde{C}\Big(\frac{t-s}{2}\Big)\Big)^p(\|f\|^p_{L^p(\mathbb{R}^d,\mu_s)} + \xi_0^p)\Big(G\Big(t,\frac{t+s}{2}\Big)\varphi_{p\Lambda((t-s)/2)}\Big)(y)\\
&\le 2^p\Big(\widetilde{C}\Big(\frac{t-s}{2}\Big)\Big)^p(\|f\|^p_{L^p(\mathbb{R}^d,\mu_s)} + \xi_0^p)M_{(t-s)/2,p\Lambda(t-s)/2}.
\end{aligned} \tag{4.25}$$

From (4.24), (4.25), choosing $y = x$ in the exponential term, we get

$$|(\mathcal{N}(t,s)f)(x)| \le \Big[2\widetilde{C}\Big(\frac{t-s}{2}\Big)(\|f\|_{L^p(\mathbb{R}^d,\mu_s)} + \xi_0)M^{1/p}_{(t-s)/2,p\Lambda(t-s)/2} + \xi_0\Big]\exp\Big(\phi\frac{t-s}{2} + \Theta\Big(\frac{t-s}{2}\Big)\frac{(\xi_2^+)^2}{4\kappa_0(p-1)}\Big)$$

and (4.21) follows with

$$\begin{aligned}
c_5(r) &= 2\widetilde{C}\Big(\frac{r}{2}\Big)M^{1/p}_{r/2,p\Lambda r/2}\exp\Big[(1+\xi_1^+)\frac{r}{2} + \Theta\Big(\frac{r}{2}\Big)(\xi_2^+)^2(4\kappa_0(p-1))^{-1}\Big],\\
c_6(r) &= \Big(2\widetilde{C}\Big(\frac{r}{2}\Big)M^{1/p}_{r/2,p\Lambda r/2} + 1\Big)\exp\Big[(1+\xi_1^+)\frac{r}{2} + \Theta\Big(\frac{r}{2}\Big)(\xi_2^+)^2(4\kappa_0(p-1))^{-1}\Big].
\end{aligned}$$

Step 2. We fix $t > s$, $f \in C_b^1(\mathbb{R}^d)$. By Theorem 3.2, $\mathcal{N}(t,s)f \in C_b^1(\mathbb{R}^d)$ and, by Step 1,

$$\Big\|\mathcal{N}\Big(\frac{t+s}{2}, s\Big)f\Big\|_\infty \le c_5\Big(\frac{t-s}{2}\Big)\|f\|_{L^p(\mathbb{R}^d,\mu_s)} + c_6\Big(\frac{t-s}{2}\Big)\xi_0.$$

Hence, from (4.1) we get

$$\sqrt{\frac{t-s}{2}}\|\nabla_x\mathcal{N}(t,s)f\|_\infty \le \widetilde{C}_{(t-s)/2}\Big[c_5\Big(\frac{t-s}{2}\Big)\|f\|_{L^p(\mathbb{R}^d,\mu_s)} + c_6\Big(\frac{t-s}{2}\Big)\xi_0 + \frac{t-s}{2}\xi_0 + \xi_0\Big].$$

Taking $T = t$, estimate (4.22) follows with $c_7(r) = \sqrt{2}r^{-1/2}\widetilde{C}_{r/2}c_5(\frac{r}{2})$ and $c_8(r) = \widetilde{C}_{r/2}[c_6(\frac{r}{2})\sqrt{\frac{2}{r}} + \sqrt{\frac{r}{2}} + \sqrt{\frac{2}{r}}]$. □

5 Stability of the null solution

In this section we study the stability of the null solution to problem (1.2) both in the C_b- and L^p-settings. For this reason, we assume that $\psi(\cdot,\cdot,0,0) = 0$.

Theorem 5.1. *The following properties are satisfied.*

(i) *Let Hypotheses 2.1, 4.3* (i)–(iii) *hold true. Further, suppose that the constant $\omega_p = \xi_1 + (\xi_2^+)^2(4\kappa_0(p-1))^{-1}$ is negative, where ξ_1 and ξ_2 are defined in Hypothesis 4.3* (ii). *Then, for any $p \ge p_0$, there exists a positive constant K_p such that, for any $s \in I$, $f \in L^p(\mathbb{R}^d,\mu_s)$ and $j = 0, 1$,*

$$\|D_x^j\mathcal{N}(t,s)f\|_{L^p(\mathbb{R}^d,\mu_t)} \le K_p^j e^{\omega_p(t-s)}\|f\|_{L^p(\mathbb{R}^d,\mu_s)}, \quad t > s + j. \tag{5.1}$$

(ii) *Suppose that the assumptions of Theorem 3.6 are satisfied. Further, assume that Hypotheses 4.3* (i)–(iii) *hold with $\xi_1 < 0$. Then* (5.1) *holds true for any $f \in C_b(\mathbb{R}^d)$ with $p = +\infty$ and ω_p and K_p being replaced, respectively, by ξ_1 and $C_1e^{-\xi_1}$.*

Proof. (i) Estimate (5.1) can be obtained arguing as in the proof of Theorem 4.5, where now $p(t) = p$ for any $t \ge s$. As far as the gradient of $\mathcal{N}(t,s)f$ is concerned, we fix $t > s+1$ and observe that

$$\mathcal{N}(t,s)f = \mathcal{N}(t,t-1)\mathcal{N}(t-1,s)f.$$

Hence, from (4.1) we obtain

$$\|\nabla_x\mathcal{N}(t,s)f\|_{L^p(\mathbb{R}^d,\mu_t)} \le C_1\|\mathcal{N}(t-1,s)f\|_{L^p(\mathbb{R}^d,\mu_{t-1})} \le K_p e^{\omega_p(t-s)}\|f\|_{L^p(\mathbb{R}^d,\mu_s)},$$

where $K_p = C_1e^{-\omega_p}$.

(ii) The assertion follows easily letting p tend to $+\infty$ in (5.1). □

6 Examples

Here, we exhibit some classes of nonautonomous elliptic operators and some classes of nonlinear functions ψ which satisfy the assumptions of this paper.

Throughout this section, we fix a right-halfline I and assume that $\{\mathcal{A}(t) : t \in I\}$ is defined by (1.1) with

$$q_{ij}(t,x) = (1+|x|^2)^m q_{ij}^0(t), \quad b_i(t,x) = -x_i b(t)(1+|x|^2)^r, \quad (t,x) \in I \times \mathbb{R}^d,$$

for any $i,j = 1,\dots,d$, where $r > m$, the function $b \in C_{\rm loc}^{\alpha/2}(I)$ is bounded with positive infimum b_0 and $q_{ij}^0(t)$ are the entries of a symmetric and positive definite matrix $Q^0(t)$, for any $t \in I$, which satisfies the estimate $\langle Q^0(t)\xi,\xi\rangle \ge \lambda_0$ for any $t \in I$, $\xi \in \partial B(0,1) \subset \mathbb{R}^d$ and some positive constant λ_0. Then Hypotheses 2.1 are satisfied with $\varphi(x) = 1 + |x|^2$ for any $x \in \mathbb{R}^d$.

Example 6.1 (Local existence and regularity). Fix $T > s \in I$ and let us consider as nonlinear term the function $\psi : [s,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$\psi(t,x,u,v) = -\phi(t)g(x)u(1+|v|^2)^\alpha, \quad t \in [s,T],\ x,v \in \mathbb{R}^d,\ u \in \mathbb{R}.$$

Here, $\phi : [s, T] \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ are bounded and continuous functions with positive infimum,

$$|\phi(t)| \le M_0(t-s)^\eta$$

for any $t \in [s, T]$ and some positive constants M_0 and η, and $\alpha \in (0, \eta + 1)$. We claim that ψ satisfies Hypotheses 3.1 so that Theorem 3.2 can be applied to deduce that for any $f \in C_b(\mathbb{R}^d)$ there exists a unique local mild solution u_f to the Cauchy problem (1.2) with these choices of ψ and $\mathcal{A}$. Hypothesis 3.1 (ii) is trivially satisfied. To prove Hypothesis 3.1 (i), we fix $R > 1$, $u_1, u_2 \in [-R, R]$ and $v_1, v_2 \in B(0, R) \subset \mathbb{R}^d$. Then, for any $t \in (s, (s+1) \wedge T]$ and $x \in \mathbb{R}^d$, we can estimate

$$\begin{aligned}
&|\psi(t, x, u_1, (t-s)^{-\frac{1}{2}}v_1) - \psi(t, x, u_2, (t-s)^{-\frac{1}{2}}v_2)| \\
&\quad = \phi(t)g(x)|u_1(1 + (t-s)^{-1}|v_1|^2)^\alpha - u_2(1 + (t-s)^{-1}|v_2|^2)^\alpha| \\
&\quad \le \|g\|_\infty \phi(t)\{(1 + (t-s)^{-1}R^2)^\alpha|u_1 - u_2| + R|(1 + (t-s)^{-1}|v_1|^2)^\alpha - (1 + (t-s)^{-1}|v_2|^2)^\alpha|\} \\
&\quad \le C\|g\|_\infty (t-s)^{\eta-\alpha}\{(1 + R^2)^\alpha|u_1 - u_2| + R^2(1 \vee (1 + R^2)^{\alpha-1})|v_1 - v_2|\}
\end{aligned}$$

for some positive constant C, independent of t, x, u_j and v_j ($j = 1, 2$). Hence, Hypothesis 3.1 (i) is satisfied with $\beta = \alpha - \eta$ and

$$L_R = C\|g\|_\infty \max\{(1 + R^2)^\alpha, R^2(1 \vee (1 + R^2)^{\alpha-1})\}.$$

If, in addition, g is γ- Hölder continuous for some $\gamma \in (0, 1)$ and $2\alpha - 2\eta + \gamma < 2$, then, by Corollary 3.3, u_f is actually a classical solution to the Cauchy problem (1.2).

Example 6.2 (Global existence and stability). Fix $s \in I$ and let $\psi : [s, +\infty) \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be the function defined by

$$\psi(t, x, u, v) = \phi(t)g(x)(-h(u) + \chi(v)), \qquad t \ge s,\ x, v \in \mathbb{R}^d,\ u \in \mathbb{R},$$

where $\phi \in C_b((s, +\infty))$ has positive infimum ϕ_0 and $g \in C_b^\gamma(\mathbb{R}^d)$, for some $\gamma \in (0, 1)$, has positive infimum g_0. The function $h : \mathbb{R} \to \mathbb{R}$ is nonnegative, locally Lipschitz continuous in $\mathbb{R}$ and satisfies the conditions $h(0) = 0$ and $uh(u) \ge \gamma_1 u^2 - \gamma_0|u|$ for some positive constants γ_0 and γ_1. Finally, the function $\chi : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous and vanishes at 0. As it is easily seen, ψ satisfies Hypotheses 3.5, condition (3.9) and, clearly, $\psi(\cdot, \cdot, 0, 0)$ is a bounded function. Moreover, if we take $\tilde{\varphi}(x) = 1 + |x|^2$ for any $x \in \mathbb{R}^d$, then also the condition $\mathcal{A}\tilde{\varphi} + k_1|\nabla\tilde{\varphi}| \le a\tilde{\varphi}$ in $I \times \mathbb{R}^d$ is satisfied with some suitable positive constants a and k_1. Hence, by Theorem 3.6, problem (1.2) admits a unique global classical solution defined in the whole $[s, T] \times \mathbb{R}^d$.

If h is globally Lipschitz continuous, then ψ satisfies Hypotheses 4.3 (i) with

$$\xi_0 = \|\phi\|_\infty\|g\|_\infty\gamma_0, \quad \xi_1 = -\phi_0 g_0\gamma_1, \quad \xi_2 = [\chi]_{\mathrm{Lip}(\mathbb{R}^d)}\|\phi\|_\infty\|g\|_\infty.$$

For any $\sigma > \max\{m - 1, 0\}$, the function $\tilde{\varphi} : \mathbb{R}^d \to \mathbb{R}$, defined by $\tilde{\varphi}(x) = (1 + |x|^2)^\sigma$ for any $x \in \mathbb{R}^d$, satisfies Hypotheses 4.3 (ii)–(iii) for some suitable locally bounded functions a, k_1 and C_j ($j = 0, 1, 2$). Hence, Theorem 5.1 can be applied to deduce that, if

$$-\phi_0 g_0\gamma_1 + [s]^2_{\mathrm{Lip}(\mathbb{R}^d)}\|\phi\|^2_\infty\|g\|^2_\infty(4\lambda_0(p-1))^{-1} < 0,$$

then the functions $t \mapsto \|u_f(t, \cdot)\|_{L^p(\mathbb{R}^d, \mu_t)}$ and $t \mapsto \|\nabla_x u_f(t, \cdot)\|_{L^p(\mathbb{R}^d, \mu_t)}$ exponentially decay to zero as $t \to +\infty$ for any $f \in L^p(\mathbb{R}^d, \mu_s)$ and $p \ge p_0$, where λ_0 is the constant introduced at the beginning of this section. The same result holds for any $f \in C_b(\mathbb{R}^d)$ if we replace the $L^p(\mathbb{R}^d, \mu_t)$-norm by the L^∞-norm.

Example 6.3 (Summability improving properties). We take the same function ψ as in Example 6.2. In view of Remark 4.4, we assume that the diffusion coefficients are independent of x. Since $\langle\nabla_x b(t, x)\xi, \xi\rangle \le -b_0|\xi|^2$ for any $t \in I$ and $x, \xi \in \mathbb{R}^d$, and b_0 is positive, by [4, Theorem 3.3] Hypothesis 4.3 (iv) is satisfied. Then, by Theorem 4.5, estimates (4.4) and (4.5) are satisfied.

If, in addition, the power r in the drift coefficients is positive, then (4.12) holds true, by Remark 4.12. Indeed, in such a case we can estimate $\langle b(t, x), x\rangle = -b(t)|x|^2(1 + |x|^2)^r \le -b_0|x|^2(1 + |x|^2)^r \le -b_0|x|^{2+2r}$ for any $t \in I$ and $x \in \mathbb{R}^d$, so that, by Theorem 4.13, the nonlinear evolution operator $\mathcal{N}(t, s)$ satisfies estimates (4.21) and (4.22).

A Technical results

Proposition A.1. *Let Hypotheses 2.1 hold and let $g \in C((a,b] \times \mathbb{R}^d)$ satisfy*

$$[g]_{\gamma,\infty} := \sup_{r \in (a,b)} (r-a)^\gamma \|g(r,\cdot)\|_\infty < +\infty$$

for some $\gamma \in [0,1)$ and some $I \ni a < b$. Then the function $z : [a,b] \times \mathbb{R}^d \to \mathbb{R}$, defined by

$$z(t,x) := \int_a^t (G(t,r)g(r,\cdot))(x)\,dr, \quad t \in [a,b],\ x \in \mathbb{R}^d,$$

belongs to $C_b([a,b] \times \mathbb{R}^d) \cap C^{0,1+\theta}((a,b] \times \mathbb{R}^d)$ for any $\theta \in (0,1)$,

$$\|z\|_\infty \le \frac{(b-a)^{1-\gamma}}{1-\gamma}[g]_{\gamma,\infty}, \quad \|\nabla_x z(t,\cdot)\|_\infty \le c_{\gamma,a,b}(t-a)^{\frac{1}{2}-\gamma}[g]_{\gamma,\infty} \tag{A.1}$$

and

$$\|\nabla_x z(t,\cdot)\|_{C^\theta(B_R)} \le C_R[g]_{\gamma,\infty}(t-a)^{\frac{1-2\gamma-\theta}{2}} \tag{A.2}$$

for any $t \in (a,b]$, $R > 0$ and some positive constants $c_{\gamma,a,b}$ and C_R. In particular, if $\gamma \le \frac{1}{2}$, then $\nabla_x z$ is bounded in $(a,b] \times \mathbb{R}^d$.

Finally, if $[g]_{\gamma,\theta,R} := \sup_{t \in (a,b]}(t-a)^\gamma \|g(t,\cdot)\|_{C_b^\theta(B_R)} < +\infty$ for some $\theta \in (0,1)$ and any $R > 0$, then one has $z \in C^{0,2+\theta}_{\mathrm{loc}}((a,b] \times \mathbb{R}^d) \cap C^{1,2}((a,b] \times \mathbb{R}^d)$. Moreover,

$$\|z(t,\cdot)\|_{C_b^2(B_R)} \le c(t-a)^{\frac{\theta}{2}-\gamma}[g]_{\gamma,\theta,R+1}, \quad t \in (a,b], \tag{A.3}$$

and

$$\|z(t,\cdot)\|_{C_b^{2+\rho}(B_R)} \le c(t-a)^{\frac{\theta-\rho}{2}-\gamma}[g]_{\gamma,\theta,R+1}, \quad t \in (a,b], \tag{A.4}$$

where $\rho = \alpha$ if $\theta > \alpha$, whereas ρ can be arbitrarily fixed in $(0,\theta)$ otherwise

Proof. Throughout the proof, we will make use of [5, Proposition 2.7], where it has been shown that, for any $I \ni a < b$, $R > 0$, $\eta \in (0,1]$ and $\beta \in [\eta, 2+\alpha]$, there exist positive constants $C_\beta = C_\beta(a,b,R)$ and $C_{\eta,\beta} = C_{\eta,\beta}(a,b,R)$ such that for any $f \in C_b(\mathbb{R}^d) \cap C^\eta_{\mathrm{loc}}(\mathbb{R}^d)$,

$$\|G(t,s)f\|_{C^\beta(\overline{B}_R)} \le \begin{cases} C_\beta(t-s)^{-\frac{\beta}{2}}\|f\|_\infty, \\ C_{\eta,\beta}(t-s)^{-\frac{\beta-\eta}{2}}\|f\|_{C^\eta(\overline{B}_{R+1})}, \end{cases} \quad a \le s < t \le b. \tag{A.5}$$

To begin with, we observe that, for any $t \in (a,b]$ and $x \in \mathbb{R}^d$, the function $r \mapsto (G(t,r)g(r,\cdot))(x)$ is measurable in $(a,t]$. If g is bounded and uniformly continuous in $\mathbb{R}^{d+1}$, this is clear. Indeed, as it has been recalled in Section 2, the function $(t,s,x) \mapsto (G(t,s)f)(x)$ is continuous in $\{(t,s,x) \in I \times I \times \mathbb{R}^d : t \ge s\}$ for any $f \in C_b(\mathbb{R}^d)$. Hence, taking (1.3) into account and adding and subtracting $(G(t,r)g(r_0,\cdot))(x)$, we can estimate

$$|(G(t,r)g(r,\cdot))(x) - (G(t,r_0)g(r_0,\cdot))(x_0)| \le \|g(r,\cdot) - g(r_0,\cdot)\|_\infty + |(G(t,r)g(r_0,\cdot))(x) - (G(t,r_0)g(r_0,\cdot))(x_0)|$$

for any $(r,x), (r_0,x_0) \in [a,t] \times \mathbb{R}^d$, and the last side of the previous chain of inequalities vanishes as (r,x) tends to (r_0,x_0).

If the function g is as in the statement of the proposition, then we can approximate it by a sequence (g_n) of bounded and uniformly continuous functions in $\mathbb{R}^{d+1}$ which converge to g pointwise in $(a,b) \times \mathbb{R}^d$ and satisfy $\|g_n(r,\cdot)\| \le \|g(r,\cdot)\|_\infty$ for any $r \in (a,b)$.[1] Since the sequence (g_n) is bounded and pointwise converges to g in $(a,t] \times \mathbb{R}^d$, by [9, Proposition 3.1 (i)] $(G(t,\cdot)g_n(r,\cdot))(x)$ converges to $(G(t,\cdot)g(r,\cdot))(x)$ as $n \to +\infty$ pointwise in $(a,t]$. Hence, the function $r \mapsto (G(t,r)g(r,\cdot))(x)$ is measurable in $(a,t]$.

1 This can be done, for instance, setting $g_n(t,x) = \vartheta_n(t)(\overline{g}(t,\cdot) \star \rho_n)(x)$ for any $(t,x) \in \mathbb{R}^{d+1}$, $n \in \mathbb{N}$, where $\overline{g} : (a,+\infty) \times \mathbb{R}^d \to \mathbb{R}$ equals g in $(a,b) \times \mathbb{R}^d$ and $\overline{g}(t,\cdot) = g(b,\cdot)$ for any $t > b$, $(\vartheta_n) \subset C^\infty(\mathbb{R})$ is a sequence of smooth functions such that $\mathbb{1}_{[a+2/n,+\infty)} \le \vartheta_n \le \mathbb{1}_{[a+1/n,+\infty)}$ for any $n \in \mathbb{N}$ and "$\star$" denotes convolution with respect to the spatial variables.

Using again (1.3), we obtain $\|G(t,r)g(r,\cdot)\|_\infty \le \|g(r,\cdot)\|_\infty \le (r-a)^{-\gamma}[g]_{\gamma,\infty}$ for any $r \in (a,t]$. It thus follows that z is bounded and the first estimate in (A.1) follows.

Proving that z is continuous in $[a,b]\times\mathbb{R}^d$ is an easy task, based on estimate (1.3) and the dominated convergence theorem. Hence, the details are omitted.

Fix $\theta \in (0,1)$. The first estimate in (A.5) with $\beta = 1+\theta$ and the assumptions on g allow to differentiate z with respect to x_j ($j = 1,\dots,d$), under the integral sign, and obtain that $D_j z(t,\cdot)$ is locally θ-Hölder continuous in $\mathbb{R}^d$, uniformly with respect to $t \in (a,b)$, and

$$\|D_j z(t,\cdot)\|_{C^\theta(B_R)} \le C_R[g]_{\gamma,\infty}(t-a)^{\frac{1-2\gamma-\theta}{2}}, \quad t \in (a,b]. \tag{A.6}$$

To conclude that $D_j z$ is continuous in $(a,b]\times\mathbb{R}^d$, it suffices to prove that, for any $x \in \mathbb{R}^d$, the function $D_j z(\cdot,x)$ is continuous in $(a,b]$. For this purpose, we apply an interpolation argument. We fix $R > 0$ such that $x \in \overline{B}_R$. Applying the well-known interpolation estimate

$$\|f\|_{C^1(\overline{B}_R)} \le K\|f\|^{\theta/(1+\theta)}_{C(\overline{B}_R)}\|f\|^{1/(1+\theta)}_{C^{1+\theta}(B_R)}$$

with $f = z(t,\cdot) - z(t_0,\cdot)$ and $t, t_0 \in (a,b]$, from the continuity of z in $[a,b]\times\mathbb{R}^d$ and the local boundedness in $(a,b]$ of the function $t \mapsto \|f(t,\cdot)\|_{C^{1+\theta}(B_R)}$, we conclude that the function $D_j z(\cdot,x)$ is continuous in $(a,b]$. Hence, $z \in C^{0,1+\theta}_{\rm loc}((a,b]\times\mathbb{R}^d)$. Estimate (A.2) follows from (A.6). Further, estimate (2.4) and the assumption on g imply that

$$|D_j z(t,x)| \le C_0[g]_{\gamma,\infty}\int_a^t (r-a)^{-\gamma}(1+(t-r)^{-\frac12})\,dr = C'_{\gamma,a,b}(t-a)^{\frac12-\gamma}[g]_{\gamma,\infty}$$

for any $(t,x) \in (a,b]\times\mathbb{R}^d$, whence the second estimate in (A.1) follows at once.

Let us now assume that $\sup_{t\in(a,b)}(t-a)^\gamma\|g(t,\cdot)\|_{C^\theta_b(B_R)} < +\infty$ for any $R > 0$. Arguing as above and taking the second estimate in (A.5) with $\beta = 2$ (resp. $\beta = 2+\alpha$) into account, we can show that $z(t,\cdot) \in C^2_{\rm loc}(\mathbb{R}^d)$ (resp. $z(t,\cdot) \in C^{2+\alpha}_{\rm loc}(\mathbb{R}^d)$) for any $t \in (a,b]$ and (A.3) (resp. (A.4)) holds true. Applying the interpolation inequality

$$\|\varphi\|_{C^2(\overline{B}_R)} \le C\|\varphi\|^{\theta/(2+\theta)}_\infty\|\varphi\|^{2/(2+\theta)}_{C^{2+\theta}(\overline{B}_R)}$$

with $\varphi = z(t,\cdot) - z(t_0,\cdot)$ we deduce that the second-order spatial derivatives of z are continuous in $(a,b]\times B_R$ and, hence, in $(a,b]\times\mathbb{R}^d$ due to the arbitrariness of $R > 0$.

Finally, to prove the differentiability of z, we introduce the sequence (z_n), where

$$z_n(t,x) = \int_a^{t-\frac1n}(G(t,r)g(r,\cdot))(x)\,dr, \quad t \in [a+1/n,b],\ x \in \mathbb{R}^d,\ n \in \mathbb{N}.$$

As it is immediately seen, z_n converges to z, locally uniformly in $(a,b]\times\mathbb{R}^d$ and each function z_n is differentiable in $[a+1/n,b]\times\mathbb{R}^d$ with respect to t and

$$D_t z_n(t,x) = \int_a^{t-\frac1n}(\mathcal{A}(t)G(t,r)g(r,\cdot))(x)\,dr + \Big(G\Big(t,t-\frac1n\Big)g\Big(t-\frac1n,\cdot\Big)\Big)(x)$$

for such values of (t,x). Since $\|\mathcal{A}(t)G(t,r)g(r,\cdot)\|_{C_b(B_R)} \le C_R[g]_{\gamma,\infty}(t-r)^{\theta/2-\gamma}(r-a)^{-\gamma}$ for any $r \in (a,t)$, and $g(t-\frac1n,\cdot)$ converges to $g(t,\cdot)$ locally uniformly in $\mathbb{R}^d$, by [9, Proposition 3.6] and the dominated convergence theorem, we conclude that $D_t z_n$ converges locally uniformly in $(a,b]\times\mathbb{R}^d$ to $\mathcal{A}z + g$. Thus, we conclude that z is continuously differentiable in $(a,b]\times\mathbb{R}^d$ and, therein, $D_t z = \mathcal{A}z + g$. □

Lemma A.2. *Let J be an interval and let $g \in C(J\times\mathbb{R}^d)$ be such that $g(t,\cdot)$ is bounded in $\mathbb{R}^d$ for any $t \in J$. Then the function $t \mapsto \|g(t,\cdot)\|_\infty$ is measurable in J.*

Proof. To begin with, we observe that for any $n \in \mathbb{N}$ the function $t \mapsto \|g(t,\cdot)\|_{C(\overline{B}_n)}$ is continuous in J. This is a straightforward consequence of the uniform continuity of g in $J_0 \times B_n$ for any bounded interval J_0 compactly embedded into J. To complete the proof, it suffices to show that $z_n(t) := \|g(t,\cdot)\|_{C(\overline{B}_n)}$ converges to

$\|g(t,\cdot)\|_\infty$ for any $t \in J$. Clearly, for any fixed $t \in J$, the sequence $(z_n(t))$ is increasing and is bounded from above by $\|g(t,\cdot)\|_\infty$. To prove that $(z_n(t))$ converges to $\|g(t,\cdot)\|_\infty$, we fix a sequence $(x_n) \subset \mathbb{R}^d$ such that $|g(t,x_n)|$ tends to $\|g(t,\cdot)\|_\infty$ as $n \to +\infty$. For any $n \in \mathbb{N}$, let $k_n \in \mathbb{N}$ be such that $x_n \in B_{k_n}$. Without loss of generality, we can assume that the sequence (k_n) is increasing. Then $z_{k_n}(t) = \|g(t,\cdot)\|_{C(\overline{B}_{k_n})} \geq |g(t,x_n)|$ for any $n \in \mathbb{N}$. Hence, the sequence $(z_{k_n}(t))$ converges to $\|g(t,\cdot)\|_\infty$ and this is enough to conclude that the whole sequence $(z_n(t))$ converges to $\|g(t,\cdot)\|_\infty$ as $n \to +\infty$. □

Finally, we prove some interior L^p-estimates.

Proposition A.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and let $u \in C^{1,2}((s,T)\times\Omega)$ solve the equation $D_t u = \mathcal{A}u$ in $(s,T)\times\Omega$. Then, for any $x_0 \in \Omega$ and any radius $R_1 > 0$ such that $B_{R_1}(x_0) \Subset \Omega$, there exists a positive constant $\tilde{c} = \tilde{c}(R_1, x_0, s, T)$ such that*

$$(t-s)\|u(t,\cdot)\|_{W^{2,p}(B_{R_1}(x_0))} + \sqrt{t-s}\|u(t,\cdot)\|_{W^{1,p}(B_{R_1}(x_0))} \leq \tilde{c} \sup_{r\in(s,T)} \|u(r,\cdot)\|_{L^p(\Omega)}.$$

Proof. Throughout the proof, we denote by c a positive constant, independent of n and u, which may vary from line to line.

Let us fix $0 < R_1 < R_2$ such that $\overline{B_{R_2}(x_0)} \subset \Omega$ and a sequence of cut-off functions $(\vartheta_n) \subset C_c^\infty(\Omega)$ such that $\mathbb{1}_{B_{r_n}(x_0)} \leq \vartheta_n \leq \mathbb{1}_{B_{r_{n+1}}(x_0)}$ and $\|\vartheta_n\|_{C_b^k(\Omega)} \leq 2^{kn}c$ for any $n \in \mathbb{N}\cup\{0\}$ and $k = 0,1,2,3$, where

$$r_n := 2R_1 - R_2 + (2-2^{-n})(R_2 - R_1).$$

Since the function $u_n := \vartheta_n u$ solves the equation $D_t u_n = \mathcal{A}u_n + g_n$ in $(s,T)\times B_{r_{n+1}}(x_0)$, where

$$g_n = -u\mathcal{A}\vartheta_n - \langle Q\nabla_x u, \nabla\vartheta_n\rangle,$$

we can write

$$u_n(t,x) = (G_{n+1}^{\mathcal{D}}(t,s)\vartheta_n u(s,\cdot))(x) + \int_s^t (G_{n+1}^{\mathcal{D}}(t,\sigma)g_n(\sigma,\cdot))(x)\,d\sigma, \tag{A.7}$$

where $G_{n+1}^{\mathcal{D}}(t,s)$ is the evolution operator associated to the realization of the operator $\mathcal{A}$ in $L^p(B_{r_{n+1}}(x_0))$ with homogeneous Dirichlet boundary conditions. It is well known that

$$\|G^{\mathcal{D}}(t,r)\psi\|_{W^{2,p}(B_{r_{n+1}}(x_0))} \leq c(t-r)^{-1+\frac{\alpha}{2}}\|\psi\|_{W^{\alpha,p}(B_{r_{n+1}}(x_0))}$$

for any $\alpha \in (0,1)$, $\psi \in W^{\alpha,p}(B_{r_{n+1}}(x_0))$ and $s \leq r < t \leq T$. Since $g_n(\sigma,\cdot) \in W^{\alpha,p}(B_{r_{n+1}}(x_0))$ for any $\sigma \in (s,t)$, from (A.7) we obtain

$$(t-s)\|u(t,\cdot)\|_{W^{2,p}(B_{r_n}(x_0))} \leq c\|u(s,\cdot)\|_{L^p(B_{r_{n+1}}(x_0))} + c\int_s^t (t-\sigma)^{-1+\frac{\alpha}{2}}\|g_n(\sigma,\cdot)\|_{W^{\alpha,p}(B_{r_{n+1}}(x_0))}\,d\sigma.$$

Now, for any $n \in \mathbb{N}$ we set $\zeta_n := \sup_{t\in(s,T)}(t-s)\|u(t,\cdot)\|_{W^{2,p}(B_{r_n}(x_0))}$ and estimate the function under the integral sign. At first, we note that

$$\|g_n(\sigma,\cdot)\|_{W^{\alpha,p}(B_{r_{n+1}}(x_0))} \leq c\|\vartheta_n\|_{C_b^{2+\alpha}(B_{r_{n+1}}(x_0))}\|u(\sigma,\cdot)\|_{W^{1+\alpha,p}(B_{r_{n+1}}(x_0))}.$$

By interpolation and using Young's inequalities we obtain, for any $\sigma \in (s,t)$,

$$\begin{aligned}\|u(\sigma,\cdot)\|_{W^{1,p}(B_{r_{n+1}}(x_0))} &\leq c(\sigma-s)^{-\frac{1}{2}}\|u(\sigma,\cdot)\|_{L^p(B_{r_{n+1}}(x_0))}^{\frac{1}{2}}\sqrt{\zeta_{n+1}}\\ &\leq (\sigma-s)^{-\frac{1}{2}}(c\varepsilon^{-1}\|u(\sigma,\cdot)\|_{L^p(\Omega)} + \varepsilon\zeta_{n+1})\end{aligned}$$

and

$$\|\nabla_x u(\sigma,\cdot)\|_{W^{\alpha,p}(B_{r_{n+1}}(x_0))} \leq (\sigma-s)^{-\frac{1+\alpha}{2}}(c\varepsilon^{-\frac{1+\alpha}{1-\alpha}}\|u(\sigma,\cdot)\|_{L^p(\Omega)} + \varepsilon\zeta_{n+1}).$$

Collecting the above estimates together, we get

$$\zeta_n \leq 8^n c\varepsilon\zeta_{n+1} + c\sup_{r\in(s,T)}\|u(r,\cdot)\|_{L^p(\Omega)}(1 + 8^n\varepsilon^{-(1+\alpha)/(1-\alpha)}).$$

Now we fix $0 < \eta < 64^{-1/(1+\alpha)}$ and $\varepsilon = 8^{-n}c^{-1}\eta$. Multiplying both the sides of the previous inequality by η^n and summing up from 0 to N yields

$$\zeta_0 - \eta^{N+1}\zeta_{N+1} \leq c \sup_{r\in(s,T)} \|u(r,\cdot)\|_{L^p(\Omega)}. \tag{A.8}$$

Since $\{\zeta_n\}_{n\in\mathbb{N}}$ is bounded, taking the limit as $N \to +\infty$ on the left-hand side of (A.8) we conclude that

$$(t-s)\|u(t,\cdot)\|_{W^{2,p}(B_{R_1}(x_0))} \leq c \sup_{r\in(s,T)} \|u(r,\cdot)\|_{L^p(\Omega)}$$

for any $t \in (s,T)$. An interpolation argument gives $\|u(t,\cdot)\|_{W^{1,p}(B_{R_1}(x_0))} \leq c(t-s)^{-1/2}\sup_{r\in(s,T)} \|u(r,\cdot)\|_{L^p(\Omega)}$ for any $t \in (s,T)$, and this completes the proof. □

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