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Research Article

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An elliptic equation with an indefinite sublinear boundary condition

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Abstract: We investigate the problem

$$\begin{cases} -\Delta u = |u|^{p-2}u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda b(x)|u|^{q-2}u & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded and smooth domain of $\mathbb{R}^N$ ($N \geq 2$), $1 < q < 2 < p$, $\lambda > 0$, and $b \in C^{1+\alpha}(\partial\Omega)$ for some $\alpha \in (0,1)$. We show that $\int_{\partial\Omega} b < 0$ is a necessary and sufficient condition for the existence of nontrivial non-negative solutions of this problem. Under the additional condition $b^+ \not\equiv 0$ we show that for $\lambda > 0$ sufficiently small this problem has two nontrivial non-negative solutions which converge to zero in $C(\overline{\Omega})$ as $\lambda \to 0$. When $p < 2^*$ we also provide the asymptotic profiles of these solutions.

Keywords: Semilinear elliptic equation, indefinite type problem, nonlinear boundary condition, asymptotic profiles

MSC 2010: 35J25, 35J61, 35J20, 35B09, 35B32

1 Introduction and statements of the main results

Let Ω be a bounded domain of $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary $\partial\Omega$. This article is concerned with the problem

$$\begin{cases} -\Delta u = |u|^{p-2}u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda b(x)|u|^{q-2}u & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where

- $\Delta = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}$ is the usual Laplacian in $\mathbb{R}^N$,
- $\lambda > 0$,
- $1 < q < 2 < p$,
- $b \in C^{1+\alpha}(\partial\Omega)$ with $\alpha \in (0,1)$,
- $\mathbf{n}$ is the unit outer normal to the boundary $\partial\Omega$.

Our purpose is to investigate the existence, non-existence and multiplicity of non-negative solutions of (P_λ). A function $u \in X := H^1(\Omega)$ is said to be a *solution* of (P_λ) if it is a weak solution, i.e. it satisfies

$$\int_\Omega \nabla u \nabla w - \int_\Omega |u|^{p-2}uw - \lambda \int_{\partial\Omega} b(x)|u|^{q-2}uw = 0 \quad \text{for all } w \in X.$$

In this case, $u \in W^{2,r}_{\mathrm{loc}}(\Omega) \cap C^\theta(\overline{\Omega})$ for some $r > N$ and $0 < \theta < 1$ (see [10, Theorem 9.11], [13, Theorem 2.2]).

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A solution of (P_λ) is said to be *positive* if it satisfies $u > 0$ on $\overline{\Omega}$, whereas it is said to be *nontrivial and non-negative* if it satisfies $u \geq 0$ and $u \not\equiv 0$. By the weak maximum principle [10, Theorem 9.1], nontrivial non-negative solutions of (P_λ) satisfy $u > 0$ in Ω. In addition, if u is a positive solution of (P_λ), then $u \in C^{2+\theta}(\overline{\Omega})$ for some $\theta \in (0, 1)$.

If u is a nontrivial non-negative solution of (P_λ), then there holds

$$\int_\Omega u^{p-1} + \lambda \int_{\partial\Omega} b(x) u^{q-1} = 0.$$

It follows that $u \equiv 0$ is the only non-negative solution of (P_λ) if $b \geq 0$. We then shall assume $b^- \not\equiv 0$ throughout this article.

In view of the condition $1 < q < 2 < p$ and its weak formulation, when $b^+ \not\equiv 0$, then (P_λ) belongs to the class of concave-convex type problems, which has been widely investigated, mostly for Dirichlet boundary conditions, since the work of Ambrosetti, Brezis and Cerami [3]. To the best of our knowledge, very few works have been devoted to concave-convex problems under Neumann boundary conditions.

Tarfulea [17] considered the problem

$$\begin{cases} -\Delta u = \lambda |u|^{q-2} u + a(x)|u|^{p-2} u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $a \in C(\overline{\Omega})$. He proved that $\int_\Omega a < 0$ is a necessary and sufficient condition for the existence of a positive solution of (1.1). By making use of the sub-supersolutions method, the author proved the existence of $\Lambda > 0$ such that problem (P_λ) has at least one positive solution for $\lambda < \Lambda$ which converges to 0 in $L^\infty(\Omega)$ as $\lambda \to 0^+$, and no positive solution for $\lambda > \Lambda$.

Garcia-Azorero, Peral and Rossi [8] dealt with the problem

$$\begin{cases} -\Delta u + u = |u|^{p-2} u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda |u|^{q-2} u & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

By means of a variational approach, they proved that if $1 < q < 2$ and $p = \frac{2N}{N-2}$ when $N > 2$, then there exists $\Lambda_1 > 0$ such that (1.2) has at least two positive solutions for $\lambda < \Lambda_1$, at least one positive solution for $\lambda = \Lambda_1$, and no positive solution for $\lambda > \Lambda_1$.

In [2], Alama investigated the problem

$$\begin{cases} -\Delta u = \mu u + b(x) u^{q-1} + \gamma u^{p-1} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mu \in \mathbb{R}$ and $\gamma > 0$. A special difficulty in this problem is the possible existence of *dead core* solutions when b changes sign. Using variational, bifurcation and sub-supersolutions techniques, the author proved existence, non-existence and multiplicity results for non-negative solutions in accordance with γ and μ. Moreover, these solutions are shown to be positive in the set where $b > 0$.

First we shall prove that the condition

$$\int_{\partial\Omega} b < 0 \tag{1.3}$$

is necessary for the existence of nontrivial non-negative solutions of (P_λ). This type of condition goes back (at least) to Bandle, Pozio and Tesei [4]. Next, we show that (1.3) and $b^+ \not\equiv 0$ yield the existence of two nontrivial non-negative solutions of (P_λ). Whenever (1.3) holds, we set

$$c^* = \left(\frac{-\int_{\partial\Omega} b}{|\Omega|} \right)^{\frac{1}{p-q}}.$$

We now state our main result.

Theorem 1.1. *The following statements hold:*

(i) *Problem (P_λ) has a nontrivial non-negative solution if and only if $\int_{\partial\Omega} b < 0$.*

(ii) *If $\int_{\partial\Omega} b < 0$, then there exists $\lambda_0 > 0$ such that (P_λ) has a nontrivial non-negative solution $u_{2,\lambda}$ for $0 < \lambda < \lambda_0$. Moreover, $u_{2,\lambda} \to 0$ in $C^{2+\theta}(\overline{\Omega})$ for some $\theta \in (0,1)$ as $\lambda \to 0^+$. If in addition $b^+ \not\equiv 0$, then (P_λ) has another nontrivial non-negative solution $u_{1,\lambda}$ for $0 < \lambda < \lambda_0$, which also satisfies $u_{1,\lambda} \to 0$ in $H^1(\Omega) \cap C^\theta(\overline{\Omega})$ for some $\theta \in (0,1)$ as $\lambda \to 0^+$. More precisely:*

(a) *Assume $p < 2^*$. If $\lambda_n \to 0^+$, then, up to a subsequence, $\lambda_n^{-1/(2-q)} u_{1,\lambda_n} \to w_0$ in $H^1(\Omega) \cap C^\theta(\overline{\Omega})$ for some $\theta \in (0,1)$, where w_0 is a nontrivial non-negative ground state solution of*

$$\begin{cases} \Delta w = 0 & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \mathbf{n}} = b(x)|w|^{q-2}w & \text{on } \partial\Omega. \end{cases} \tag{P_w}$$

Furthermore, $w_0 > 0$ in Ω, the set $\{x \in \partial\Omega : w_0 = 0\}$ has no interior points in the relative topology of $\partial\Omega$, and it is contained in $\{x \in \partial\Omega : b(x) \le 0\}$.

(b) *$\lambda^{-1(p-q)} u_{2,\lambda} \to c^*$ in $C^{2+\theta}(\overline{\Omega})$ for some $\theta \in (0,1)$ as $\lambda \to 0^+$. In particular, $u_{2,\lambda}$ is a positive solution of (P_λ) for $\lambda > 0$ sufficiently small.*

Remark 1.2. From the assertion that $w_\lambda := \lambda^{-1/(p-q)} u_{2,\lambda} \to c^* > 0$ in $C^{2+\theta}(\overline{\Omega})$, we can deduce the fact that $u_{2,\lambda} = \lambda^{1/(p-q)} w_\lambda \to 0$ in $C^{2+\theta}(\overline{\Omega})$ as $\lambda \to 0^+$, and that $u_{2,\lambda} > 0$ in $\overline{\Omega}$ for sufficiently small $\lambda > 0$. Moreover, when $p < 2^*$, there holds $\frac{u_{1,\lambda}}{u_{2,\lambda}} = o(\lambda^\sigma)$ for any $\sigma < \frac{p-2}{(2-q)(p-q)}$ as $\lambda \to 0^+$. In particular, we have $u_{2,\lambda} > u_{1,\lambda} \ge 0$ for sufficiently small $\lambda > 0$, see Figure 1.

The rest of this article is devoted to the proof of Theorem 1.1. In the next subsection, we show Theorem 1.1 (ii). In Section 1.2, we assume $p < 2^*$ and follow a variational approach to obtain $u_{1,\lambda}$, $u_{2,\lambda}$ and their asymptotic profiles as $\lambda \to 0^+$. In Sections 1.3 and 1.4, we use these asymptotic profiles to remove the condition $p < 2^*$. More precisely, we employ the sub-supersolutions method to obtain a positive solution having some features of $u_{1,\lambda}$. In addition, we employ a Lyapunov–Schmidt-type reduction to obtain $u_{2,\lambda}$. Finally, in Section 1.5 we prove a positivity property for non-negative solutions of (P_λ) in the case $N = 1$.

Throughout this article we use the following notations and conventions:

- The infimum of an empty set is assumed to be ∞.
- Unless otherwise stated, for any $f \in L^1(\Omega)$ the integral $\int_\Omega f$ is considered with respect to the Lebesgue measure, whereas for any $g \in L^1(\partial\Omega)$ the integral $\int_{\partial\Omega} g$ is considered with respect to the surface measure.
- For $r \ge 1$ the Lebesgue norm in $L^r(\Omega)$ will be denoted by $\|\cdot\|_r$ and the usual norm of $H^1(\Omega)$ by $\|\cdot\|$.
- The strong and weak convergence are denoted by $\to$ and $\rightharpoonup$, respectively.
- The positive and negative parts of a function u are defined by $u^\pm := \max\{\pm u, 0\}$.
- If $U \subset \mathbb{R}^N$, then we denote the closure of U by $\overline{U}$ and the interior of U by $\operatorname{int} U$.
- The support of a measurable function f is denoted by $\operatorname{supp} f$.

1.1 A necessary condition

Proposition 1.3. *If (P_λ) has a nontrivial non-negative solution, then $\int_{\partial\Omega} b < 0$.*

Proof. Let u be a nontrivial non-negative solution of (P_λ). By the weak maximum principle we know that $u > 0$ in Ω. Moreover, by [12, Proposition 5.1] we have $u > 0$ in $\{x \in \partial\Omega : b(x) > 0\}$. Given $\varepsilon > 0$, we take $w = (u+\varepsilon)^{1-q}$ in the weak formulation of (P_λ) to get

$$(1-q)\int_\Omega |\nabla u|^2 (u+\varepsilon)^{-q} - \int_\Omega u^{p-1}(u+\varepsilon)^{1-q} - \lambda \int_{\partial\Omega} b\Big(\frac{u}{u+\varepsilon}\Big)^{q-1} = 0.$$

Since $q > 1$, we obtain

$$\lambda \int_{\Gamma_u} b\Big(\frac{u}{u+\varepsilon}\Big)^{q-1} < -\int_\Omega u^{p-1}(u+\varepsilon)^{1-q},$$

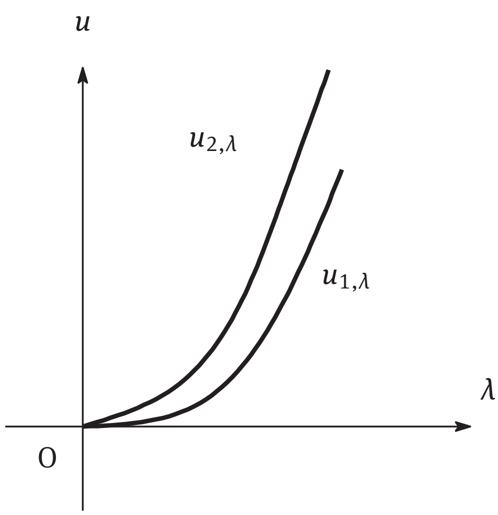

Figure 1: Ordering of $u_{1,\lambda}$ and $u_{2,\lambda}$.

where $\Gamma_u = \partial\Omega \cap \operatorname{supp} u$. Letting $\varepsilon \to 0$ and using the Lebesgue dominated convergence theorem, we get

$$\lambda \int_{\Gamma_u} b \leq -\int_{\Omega} u^{p-q},$$

so that $\int_{\Gamma_u} b < 0$. Now, since $b \leq 0$ in $\partial\Omega \setminus \Gamma_u$, we have

$$\int_{\partial\Omega} b = \int_{\Gamma_u} b + \int_{\partial\Omega\setminus\Gamma_u} b \leq \int_{\Gamma_u} b < 0,$$

as desired. □

1.2 The variational approach

Throughout this subsection, we assume that $p < 2^*$, so that weak solutions of (P_λ) are critical points of the C^1 functional I_λ, defined on X by

$$I_\lambda(u) := \frac{1}{2}E(u) - \frac{1}{p}A(u) - \frac{\lambda}{q}B(u),$$

where

$$E(u) = \int_{\Omega} |\nabla u|^2, \quad A(u) = \int_{\Omega} |u|^p, \quad B(u) = \int_{\partial\Omega} b(x)|u|^q.$$

Let us recall that $X = H^1(\Omega)$ is equipped with the usual norm

$$\|u\| = \left[\int_{\Omega} (|\nabla u|^2 + u^2)\right]^{\frac{1}{2}}.$$

We shall study the geometry of I_λ to obtain non-negative weak solutions of (P_λ) for a small $\lambda > 0$.

The next result will be used repeatedly in this section.

Lemma 1.4. *The following statements hold:*

(i) *If (u_n) is a sequence such that $u_n \rightharpoonup u_0$ in X and $\lim E(u_n) = 0$, then u_0 is a constant and $u_n \to u_0$ in X.*

(ii) *Assume $\int_{\partial\Omega} b < 0$. If $v \not\equiv 0$ and $B(v) \geq 0$, then v is not a constant.*

Proof. (i) Since $u_n \rightharpoonup u_0$ in X and E is weakly lower semicontinuous, we have $0 \le E(u_0) \le \lim E(u_n) = 0$. Hence, $E(u_0) = 0$, which implies that u_0 is a constant. In addition, if $u_n \not\to u_0$ in X, then $E(u_0) < \lim E(u_n) = 0$, which is a contradiction. Therefore, $u_n \to u_0$ in X.

(ii) If v is a non-zero constant and $B(v) \ge 0$, then $B(v) = |v|^p \int_{\partial\Omega} b < 0$, which yields a contradiction. □

Let us introduce some useful subsets of X:

$$E^+ = \{u \in X : E(u) > 0\},$$
$$A^+ = \{u \in X : A(u) > 0\},$$
$$B^\pm = \{u \in X : B(u) \gtrless 0\}, \quad B_0 = \{u \in X : B(u) = 0\}, \quad B_0^\pm = B^\pm \cup B_0.$$

The Nehari manifold associated to I_λ is given by

$$N_\lambda := \{u \in X \setminus \{0\} : \langle I'_\lambda(u), u\rangle = 0\} = \{u \in X \setminus \{0\} : E(u) = A(u) + \lambda B(u)\}.$$

We shall use the splitting

$$N_\lambda = N_\lambda^+ \cup N_\lambda^- \cup N_\lambda^0,$$

where

$$N_\lambda^\pm := \{u \in N_\lambda : \langle J'_\lambda(u), u\rangle \gtrless 0\} = \Big\{u \in N_\lambda : E(u) \lessgtr \lambda \frac{p-q}{p-2} B(u)\Big\}$$
$$= \Big\{u \in N_\lambda : E(u) \gtrless \frac{p-q}{2-q} A(u)\Big\}$$

and

$$N_\lambda^0 = \{u \in N_\lambda : \langle J'_\lambda(u), u\rangle = 0\}.$$

Here, $J_\lambda(u) = \langle I'_\lambda(u), u\rangle = E(u) - A(u) - \lambda B(u)$.

Note that any nontrivial weak solution of (P_λ) belongs to N_λ. Furthermore, it follows from the implicit function theorem that $N_\lambda \setminus N_\lambda^0$ is a C^1 manifold and local minimizers of the restriction of I_λ to this manifold are critical points of I_λ (see for instance [7, Theorem 2.3]).

To analyze the structure of $N_\lambda^\pm$, we consider the fibering maps corresponding to I_λ, which are defined, for $u \neq 0$, as follows:

$$j_u(t) := I_\lambda(tu) = \frac{t^2}{2} E(u) - \frac{t^p}{p} A(u) - \lambda \frac{t^q}{q} B(u), \quad t > 0.$$

It is easy to see that

$$j'_u(1) = 0 \lessgtr j''_u(1) \iff u \in N_\lambda^\pm$$

and, more generally,

$$j'_u(t) = 0 \lessgtr j''_u(t) \iff tu \in N_\lambda^\pm.$$

Having this characterization in mind, we look for conditions under which j_u has a critical point. Set

$$i_u(t) := t^{-q} j_u(t) = \frac{t^{2-q}}{2} E(u) - \frac{t^{p-q}}{p} A(u) - \lambda B(u), \quad t > 0.$$

Let $u \in E^+ \cap A^+ \cap B^+$. Then i_u has a global maximum $i_u(t^*)$ at some $t^* > 0$ and, moreover, t^* is unique. If $i_u(t^*) > 0$, then j_u has a global maximum which is positive and a local minimum which is negative. Moreover, these are the only critical points of j_u.

We shall require a condition on λ that provides $i_u(t^*) > 0$. Note that

$$i'_u(t) = \frac{2-q}{2} t^{1-q} E(u) - \frac{p-q}{p} t^{p-q-1} A(u) = 0$$

if and only if

$$t = t^* := \Big(\frac{p(2-q)E(u)}{2(p-q)A(u)}\Big)^{\frac{1}{p-2}}.$$

Moreover,

$$i_u(t^*) = \frac{p-2}{2(p-q)}\Big(\frac{p(2-q)}{2(p-q)}\Big)^{\frac{2-q}{p-2}} \frac{E(u)^{\frac{p-q}{p-2}}}{A(u)^{\frac{2-q}{p-2}}} - \frac{\lambda}{q} B(u) > 0$$

if and only if

$$0 < \lambda < C_{pq} \frac{E(u)^{\frac{p-q}{p-2}}}{B(u)A(u)^{\frac{2-q}{p-2}}}, \tag{1.4}$$

where

$$C_{pq} = \Big(\frac{q(p-2)}{2(p-q)}\Big)\Big(\frac{p(2-q)}{2(p-q)}\Big)^{\frac{2-q}{p-2}}.$$

Note that

$$F(u) = \frac{E(u)^{\frac{p-q}{p-2}}}{B(u)A(u)^{\frac{2-q}{p-2}}}$$

satisfies $F(tu) = F(u)$ for $t > 0$, i.e. F is homogeneous of order 0.

We then introduce

$$\lambda_0 = \inf\{E(u)^{\frac{p-q}{p-2}} : u \in E^+ \cap A^+ \cap B^+,\ C_{pq}^{-1} B(u) A(u)^{\frac{2-q}{p-2}} = 1\}.$$

Note that if $E^+ \cap A^+ \cap B^+ = \emptyset$, then $\lambda_0 = \infty$.

We then deduce the following result, which provides sufficient conditions for the existence of critical points of j_u.

Proposition 1.5. *The following statements hold:*

(i) *If either $u \in A^+ \cap B^-$ or $u \in E^+ \cap A^+ \cap B_0$, then j_u has a positive global maximum at some $t_1 > 0$, i.e. $j_u'(t_1) = 0 > j_u''(t_1)$ and $j_u(t) < j_u(t_1)$ for $t \neq t_1$. Moreover, t_1 is the unique critical point of j_u.*

(ii) *Assume $\int_{\partial\Omega} b < 0$. Then $\lambda_0 > 0$ and for any $0 < \lambda < \lambda_0$ and $u \in E^+ \cap A^+ \cap B^+$ the map j_u has a negative local minimum at $t_1 > 0$ and a positive global maximum at $t_2 > t_1$, which are the only critical points of j_u.*

Proof. The first assertion is straightforward from the definition of j_u. Let us assume now $\int_{\partial\Omega} b < 0$. First, we show that $\lambda_0 > 0$. Assume $\lambda_0 = 0$, so that we can choose $u_n \in E^+ \cap A^+ \cap B^+$ satisfying

$$E(u_n) \to 0 \quad \text{and} \quad C_{pq}^{-1} B(u_n) A(u_n)^{\frac{2-q}{p-2}} = 1.$$

If (u_n) is bounded in X, then we may assume that $u_n \rightharpoonup u_0$ for some $u_0 \in X$ and $u_n \to u_0$ in $L^p(\Omega)$ and $L^q(\Omega)$. It follows from Lemma 1.4 (i) that u_0 is a constant and $u_n \to u_0$ in X. From $u_n \in A^+ \cap B^+$ we deduce that $u_0 \in A_0^+ \cap B_0^+$. In addition, there holds

$$C_{pq}^{-1} B(u_0) A(u_0)^{\frac{2-q}{p-2}} = 1,$$

so that $u_0 \not\equiv 0$. From Lemma 1.4 we get a contradiction.

Let us assume now that $\|u_n\| \to \infty$. Set $v_n = \frac{u_n}{\|u_n\|}$, so that $\|v_n\| = 1$. We may assume that $v_n \rightharpoonup v_0$ and $v_n \to v_0$ in $L^p(\Omega)$. Since $E(v_n) \to 0$ and $v_n \in A^+$, we can argue as for u_n to reach a contradiction. Finally, for any $u \in E^+ \cap A^+ \cap B^+$ we have

$$\lambda_0 \le C_{pq} \frac{E(u)^{\frac{p-q}{p-2}}}{B(u)A(u)^{\frac{2-q}{p-2}}}.$$

Thus, if $0 < \lambda < \lambda_0$, then $i_u(t^*) > 0$ from (1.4). □

Proposition 1.6. *We have the following results:*

(i) *N_λ^0 is empty.*

(ii) *If $b^+ \not\equiv 0$ and $\int_{\partial\Omega} b < 0$, then N_λ^+ is non-empty for $0 < \lambda < \lambda_0$.*

(iii) *If $b^- \not\equiv 0$, then N_λ^- is non-empty.*

Proof. (i) From Proposition 1.5 it follows that there is no $t > 0$ such that $j_u'(t) = j_u''(t) = 0$, i.e. N_λ^0 is empty.

(ii) Since $b^+ \not\equiv 0$, we can find $u \in B^+$. Moreover, since $\int_{\partial\Omega} b < 0$, by Lemma 1.4 we have $u \in E^+ \cap A^+$. By Proposition 1.5 we infer that for $0 < \lambda < \lambda_0$ there are $0 < t_1 < t_2$ such that $t_1 u \in N_\lambda^+$ and $t_2 u \in N_\lambda^-$.

(iii) Since $b^- \not\equiv 0$, we can find $u \in B^-$, so that $u \in A^+ \cap B^-$. By Proposition 1.5 we infer that there exists $t_1 > 0$ such that $t_1 u \in N_\lambda^-$. □

The following result provides some properties of N_λ^+.

Lemma 1.7. *Assume $b^+ \not\equiv 0$ and $\int_{\partial\Omega} b < 0$. Then, for $0 < \lambda < \lambda_0$, we have the following:*
(i) $N_\lambda^+ \subset B^+$.
(ii) *N_λ^+ is bounded in X.*
(iii) *$I_\lambda(u) < 0$ for any $u \in N_\lambda^+$.*

Proof. (i) Let $u \in N_\lambda^+$. Then $0 \le E(u) < \lambda \frac{p-q}{p-2} B(u)$, i.e. $u \in B^+$.

(ii) Assume $(u_n) \subset N_\lambda^+$ and $\|u_n\| \to \infty$. Set $v_n = \frac{u_n}{\|u_n\|}$. It follows that $\|v_n\| = 1$, so we may assume that $v_n \rightharpoonup v_0$ in X, $B(v_n)$ is bounded and $v_n \to v_0$ in $L^p(\Omega)$ (implying $A(v) \to A(v_0)$). Since $u_n \in N_\lambda^+$, we see that

$$E(v_n) < \lambda \frac{p-q}{p-2} B(v_n) \|u_n\|^{q-2},$$

and thus $\limsup_n E(v_n) \le 0$. Lemma 1.4 (i) yields that v_0 is a constant and $v_n \to v_0$ in X. Consequently, $\|v_0\| = 1$ and v_0 is a non-zero constant. On the other hand, since $u_n \in N_\lambda^+$, we have $v_n \in N_\lambda^+$, so $v_n \in B^+$. It follows that $v_0 \in B_0^+$, a contradiction.

(iii) Let $u \in N_\lambda^+$, so that $u \in B^+$. Hence u is not a constant and $E(u) > 0$. Thus $u \in E^+ \cap A^+ \cap B^+$ and by Proposition 1.5 (ii) we infer that $I_\lambda(u) < 0$ and $t > 1$ if $j_u'(t) > 0$. □

Proposition 1.8. *Assume $b^+ \not\equiv 0$ and $\int_{\partial\Omega} b < 0$. Then, for any $0 < \lambda < \lambda_0$, there exists $u_{1,\lambda} \ge 0$ such that $I_\lambda(u_{1,\lambda}) = \min_{N_\lambda^+} I_\lambda < 0$. In particular, $u_{1,\lambda}$ is a nontrivial non-negative solution of (P_λ).*

Proof. Let $0 < \lambda < \lambda_0$. By Proposition 1.6 we know that N_λ^+ is non-empty. We consider a minimizing sequence $(u_n) \subset N_\lambda^+$, i.e.

$$I_\lambda(u_n) \to \inf_{N_\lambda^+} I_\lambda < 0.$$

Since (u_n) is bounded in X, we may assume that $u_n \rightharpoonup u_0$ in X, $u_n \to u_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. It follows that

$$I_\lambda(u_0) \le \liminf_n I_\lambda(u_n) = \inf_{N_\lambda^+} I_\lambda(u) < 0,$$

so that $u_0 \not\equiv 0$. Moreover, as $u_n \in B^+$, we have $u_0 \in B_0^+$ and u_0 is not a constant. So $u_0 \in E^+ \cap A^+ \cap B^+$. Since $0 < \lambda < \lambda_0$, Proposition 1.5 yields that $t_1 u_0 \in N_\lambda^+$ for some $t_1 > 0$. Assume $u_n \not\to u_0$. If $1 < t_1$, then we have

$$I_\lambda(t_1 u_0) = j_{u_0}(t_1) \le j_{u_0}(1) < \limsup j_{u_n}(1) = \limsup I_\lambda(u_n) = \inf_{N_\lambda^+} I_\lambda,$$

which is impossible. If $t_1 \le 1$, then $j_{u_n}'(t_1) \le 0$ for every n, so that

$$j_{u_0}'(t_1) < \limsup j_{u_n}'(t_1) \le 0,$$

which is a contradiction. Therefore, $u_n \to u_0$. Now, since $u_n \to u_0$, we have $j_{u_0}'(1) = 0 \le j_{u_0}''(1)$. But $j_{u_0}''(1) = 0$ is impossible by Proposition 1.6 (i). Thus $u_0 \in N_\lambda^+$ and $I_\lambda(u_0) = \inf_{N_\lambda^+} I_\lambda$. We set $u_{1,\lambda} = u_0$. □

Next, we obtain a nontrivial non-negative weak solution of (P_λ), which achieves $\inf_{N_\lambda^-} I_\lambda$ for $\lambda \in (0, \lambda_0)$. The following result provides some properties of N_λ^-.

Lemma 1.9. *Assume $\int_{\partial\Omega} b < 0$. Then $I_\lambda(u) > 0$ for $0 < \lambda < \lambda_0$ and any $u \in N_\lambda^-$.*

Proof. Let $u \in N_\lambda^-$. If $u \in B_0$, then u is not a constant, so $u \in E^+ \cap A^+$. Thus, by Proposition 1.5, j_u has a positive global maximum at $t = 1$. The same conclusion holds if $u \in B^-$. Finally, if $u \in B^+$, then u is not a constant. Hence $u \in E^+ \cap A^+$, and since $0 < \lambda < \lambda_0$, Proposition 1.5 yields again that j_u has a positive global maximum at $t = 1$. □

Proposition 1.10. *Let $\int_{\partial\Omega} b < 0$. Then for any $\lambda \in (0, \lambda_0)$ there exists $u_{2,\lambda} \ge 0$ such that $I_\lambda(u_{2,\lambda}) = \min_{N_\lambda^-} I_\lambda > 0$. In particular, $u_{2,\lambda}$ is a non-negative solution of (P_λ).*

Proof. First of all, since $\int_{\partial\Omega} b < 0$, we have $b^- \not\equiv 0$, so that by Proposition 1.6 we know that N_λ^- is non-empty. In addition, since $I_\lambda(u) > 0$ for $u \in N_\lambda^-$, we can choose $u_n \in N_\lambda^-$ such that

$$I_\lambda(u_n) \to \inf_{N_\lambda^-} I_\lambda \geq 0.$$

We claim that (u_n) is bounded in X. Indeed, there exists $C > 0$ such that $I_\lambda(u_n) \leq C$. Since $u_n \in N_\lambda$, we deduce

$$\Big(\frac{1}{2} - \frac{1}{p}\Big)E(u_n) - \lambda\Big(\frac{1}{q} - \frac{1}{p}\Big)B(u_n) = I_\lambda(u_n) \leq C.$$

Assume $\|u_n\| \to \infty$ and set $v_n = \frac{u_n}{\|u_n\|}$, so that $\|v_n\| = 1$. We may assume that $v_n \rightharpoonup v_0$ in X and $v_n \to v_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. Then, from

$$\Big(\frac{1}{2} - \frac{1}{p}\Big)E(v_n) \leq \lambda\Big(\frac{1}{q} - \frac{1}{p}\Big)B(v_n)\|u_n\|^{q-2} + C\|u_n\|^{-2},$$

we infer that $\limsup_n E(v_n) \leq 0$. Lemma 1.4 (i) yields that v_0 is a constant and $v_n \to v_0$ in X, which implies $\|v_0\| = 1$. On the other hand, since $u_n \in N_\lambda$, we have

$$E(u_n) = \lambda B(u_n) + A(u_n).$$

Dividing by $\|u_n\|^p$ and passing to the limit as $n \to \infty$, we get $A(v_0) = 0$, i.e. $v_0 = 0$, which is impossible. Hence (u_n) is bounded. We may then assume that $u_n \rightharpoonup u_0$ in X and $u_n \to u_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. If $u_0 \equiv 0$, then we set $v_n = \frac{u_n}{\|u_n\|}$. From

$$E(u_n) < \frac{p-q}{2-q}A(u_n)$$

we get

$$E(v_n) < \frac{p-q}{2-q}A(v_n)\|u_n\|^{p-2} \to 0.$$

So we can assume that $v_n \to v_0$ with v_0 constant. Moreover, from

$$E(u_n) = \lambda B(u_n) + A(u_n)$$

we deduce that $B(v_n) \to 0$, i.e. $B(v_0) = 0$, which contradicts $\int_{\partial\Omega} b < 0$. Thus, $u_0 \not\equiv 0$. By Proposition 1.5 we infer the existence of $t_2 > 0$ such that $t_2 u_0 \in N_\lambda^-$. Assume $u_n \not\to u_0$. Then, since $u_n \in N_\lambda^-$, we get

$$I_\lambda(t_2 u_0) < \limsup I_\lambda(t_2 u_n) \leq \limsup I_\lambda(u_n) = \inf_{N_\lambda^-} I_\lambda,$$

which is a contradiction. Therefore, $u_n \to u_0$. In particular, we get $j_{u_0}'(1) = 0$ and $j_{u_0}''(1) < 0$. Since N_λ^0 is empty for $\lambda \in (0, \lambda_0)$, we infer that $u_0 \in N_\lambda^-$ and $I_\lambda(u_0) = \inf_{N_\lambda^-} I_\lambda$. We set $u_{2,\lambda} = u_0$. □

We now discuss the asymptotic profiles of $u_{1,\lambda}$ and $u_{2,\lambda}$ as $\lambda \to 0^+$.

Lemma 1.11. *Assume $b^+ \not\equiv 0$ and $\int_{\partial\Omega} b < 0$. Then, for $0 < \lambda < \lambda_0$, there holds*

$$I_\lambda(u_{1,\lambda}) < -D_0 \lambda^{\frac{2}{2-q}}$$

for some $D_0 > 0$.

Proof. Let $u \in N_\lambda^+$. Thus, $u \in A^+ \cap E^+ \cap B^+$. Then

$$I_\lambda(u) \leq \tilde{I}_\lambda(u) := \frac{1}{2}E(u) - \frac{\lambda}{q}B(u).$$

Thus $I_\lambda(tu) \leq \tilde{I}_\lambda(tu)$ for every $t > 0$. Note that $\tilde{I}_\lambda(tu)$ has a global minimum point t_0 given by

$$t_0 = \Big(\frac{\lambda B(u)}{E(u)}\Big)^{\frac{1}{2-q}}$$

and

$$\tilde{I}_\lambda(t_0 u) = -\frac{2-q}{2q}\lambda t_0^q B(u) = -\frac{2-q}{2q}\frac{(\lambda B(u))^{\frac{2}{2-q}}}{E(u)^{\frac{q}{2-q}}} = -D_0\lambda^{\frac{2}{2-q}},$$

where

$$D_0 = \frac{2-q}{2q}\frac{B(u)^{\frac{2}{2-q}}}{E(u)^{\frac{q}{2-q}}}.$$

It follows that if $I_\lambda(tu)$ has a local minimum at t_1, then

$$I_\lambda(t_1 u) < -D_0\lambda^{\frac{2}{2-q}}$$

with $D_0 > 0$. Therefore, $I_\lambda(u) < -D_0\lambda^{\frac{2}{2-q}}$ for every $u \in N_\lambda^+$. In particular,

$$I_\lambda(u_{1,\lambda}) < -D_0\lambda^{\frac{2}{2-q}}. \qquad \square$$

We now determine the asymptotic profile of $u_{1,\lambda}$ as $\lambda \to 0^+$.

Proposition 1.12. *Assume $b^+ \not\equiv 0$ and $\int_{\partial\Omega} b < 0$. Then $u_{1,\lambda} \to 0$ in X as $\lambda \to 0^+$. Moreover, if $\lambda_n \to 0^+$, then, up to a subsequence, $\lambda_n^{-1/(2-q)} u_{1,\lambda_n} \to w_0$ in X, where w_0 is a non-negative ground state solution, i.e. a least energy solution of*

$$\begin{cases} \Delta w = 0 & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \mathbf{n}} = b(x)|w|^{q-2}w & \text{on } \partial\Omega. \end{cases} \tag{P_w}$$

Proof. First we show that $u_{1,\lambda}$ remains bounded in X as $\lambda \to 0^+$. Indeed, assume that $\|u_{1,\lambda}\| \to \infty$ and set $v_\lambda = \frac{u_{1,\lambda}}{\|u_{1,\lambda}\|}$. We may then assume that for some $v_0 \in X$ we have $v_\lambda \rightharpoonup v_0$ in X and $v_\lambda \to v_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. Since $u_{1,\lambda} \in N_\lambda$, we have

$$E(v_\lambda)\|u_{1,\lambda}\|^{2-p} = A(v_\lambda) + \lambda B(v_\lambda)\|u_{1,\lambda}\|^{q-p}.$$

Passing to the limit as $\lambda \to 0^+$, we obtain $A(v_0) = 0$, i.e. $v_0 \equiv 0$. From $u_{1,\lambda} \in N_\lambda^+$ we have

$$E(v_\lambda) < \lambda\frac{p-q}{p-2}B(v_\lambda)\|u_{1,\lambda}\|^{q-2},$$

so that $\limsup_\lambda E(v_\lambda) \leq 0$. By Lemma 1.4 (i) we infer that v_0 is a constant and $v_\lambda \to 0$ in X, which contradicts $\|v_\lambda\| = 1$ for every λ. Thus $u_{1,\lambda}$ stays bounded in X as $\lambda \to 0^+$.

Hence we may assume that $u_{1,\lambda} \rightharpoonup u_0$ in X and $u_{1,\lambda} \to u_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$ as $\lambda \to 0^+$. Since $u_{1,\lambda} \in N_\lambda^+$, we observe that

$$E(u_{1,\lambda}) < \lambda\frac{p-q}{p-2}B(u_{1,\lambda}). \tag{1.5}$$

Passing to the limit as $\lambda \to 0^+$, we get $\limsup_\lambda E(u_{1,\lambda}) \leq 0$. Lemma 1.4 (ii) provides that u_0 is a constant and $u_{1,\lambda} \to u_0$ in X. Since $u_{1,\lambda} \in B^+$, we have $u_0 \in B_0^+$, and $\int_{\partial\Omega} b < 0$ implies that $u_0 = 0$.

Let $w_\lambda = \lambda^{-1/(2-q)}u_{1,\lambda}$. We claim that w_λ remains bounded in X as $\lambda \to 0^+$. Indeed, from (1.5) we have

$$E(w_\lambda) < \frac{p-q}{p-2}B(w_\lambda).$$

Let us assume that $\|w_\lambda\| \to \infty$ and set $\psi_\lambda = \frac{w_\lambda}{\|w_\lambda\|}$. We may assume that $\psi_\lambda \rightharpoonup \psi_0$ and $\psi_\lambda \to \psi_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. It follows that

$$E(\psi_\lambda) < \frac{p-q}{p-2}B(\psi_\lambda)\|w_\lambda\|^{q-2},$$

so that $\limsup_\lambda E(\psi_\lambda) \leq 0$. By Lemma 1.4 (i) we infer that ψ_0 is a constant and $\psi_\lambda \to \psi_0$ in X. On the other hand, from $u_{1,\lambda} \in B^+$ we have $\psi_\lambda \in B^+$, and consequently $\psi_0 \in B_0^+$. From $\int_{\partial\Omega} b < 0$ we infer that $\psi_0 \equiv 0$, which contradicts $\|\psi_0\| = 1$. Hence w_λ stays bounded in X as $\lambda \to 0^+$, and we may assume that $w_\lambda \rightharpoonup w_0$ in X and $w_\lambda \to w_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. Note that w_λ satisfies

$$\int_\Omega \nabla w_\lambda \nabla w - \lambda^{\frac{p-2}{2-q}}\int_\Omega w_\lambda^{p-1}w - \int_{\partial\Omega} b(x)w_\lambda^{q-1}w = 0 \quad \text{for all } w \in X.$$

Taking $w = w_\lambda - w_0$ and letting $\lambda \to 0$, we deduce that $w_\lambda \to w_0$ in X. Moreover, w_0 is a weak solution of (P_w). We claim that $w_0 \not\equiv 0$. Indeed, by Lemma 1.11 we have

$$I_\lambda(u_{1,\lambda}) < -D_0 \lambda^{\frac{2}{2-q}},$$

with $D_0 > 0$. Hence

$$\frac{\lambda^{\frac{2}{2-q}}}{2} E(w_\lambda) - \frac{\lambda^{\frac{p}{2-q}}}{p} A(w_\lambda) - \frac{\lambda^{\frac{2}{2-q}}}{q} B(w_n) < -D_0 \lambda^{\frac{2}{2-q}},$$

so that

$$\frac{1}{2} E(w_\lambda) - \frac{\lambda^{\frac{p-2}{2-q}}}{p} A(w_\lambda) - B(w_\lambda) < -D_0.$$

Letting $\lambda \to 0$, we obtain

$$\frac{1}{2} E(w_0) - B(w_0) \le -D_0,$$

and consequently $w_0 \not\equiv 0$.

It remains to prove that w_0 is a ground state solution of (P_w), i.e.

$$I_b(w_0) = \min_{N_b} I_b,$$

where

$$I_b(u) = \frac{1}{2} E(u) - \frac{1}{q} B(u)$$

for $u \in X$ and

$$N_b = \{u \in X \setminus \{0\} : \langle I_b'(u), u \rangle = 0\} = \{u \in X \setminus \{0\} : E(u) = B(u)\}$$

is the Nehari manifold associated to I_b. Since $\int_{\partial\Omega} b < 0$, it is easily seen that there exists $w_b \neq 0$ such that $I_b(w_b) = \min_{N_b} I_b$. Note that $w_0 \in N_b$, and consequently $I_b(w_b) \le I_b(w_0)$. We now prove the reverse inequality. Since w_b is non-constant, we have $w_b \in B^+ \cap E^+$. We set $u_b = \lambda^{1/(2-q)} w_b$. Let $\lambda_n \to 0^+$. Since $u_b \in B^+ \cap E^+$ for every n, there exists $t_n > 0$ such that $t_n u_b \in N_{\lambda_n}^+$. Hence

$$t_n^2 E(u_b) < \lambda_n \frac{p-q}{p-2} t_n^q B(u_b),$$

i.e.

$$t_n^{2-q} < \frac{p-q}{p-2} \frac{B(w_b)}{E(w_b)} = \frac{p-q}{p-2}.$$

We may then assume that $t_n \to t_0$. We claim that $t_0 = 1$. Indeed, note that from $t_n u_b \in N_{\lambda_n}^+$ we infer that

$$t_n^2 E(u_b) = \lambda_n t_n^q B(u_b) + t_n^p A(u_b),$$

so

$$t_n^{2-q} E(w_b) = B(w_b) + t_n^{p-q} \lambda_n^{\frac{p-2}{2-q}} A(w_b).$$

From $E(w_b) = B(w_b)$ we infer that $t_0 = 1$, as claimed. Now, since $t_n u_b \in N_{\lambda_n}^+$, we have

$$I_{\lambda_n}(u_{1,\lambda_n}) \le I_{\lambda_n}(t_n u_b).$$

It follows that

$$I_{\lambda_n}(u_{1,\lambda_n}) \le \Big(\frac{1}{2} - \frac{1}{q}\Big) t_n^2 E(u_b) - \Big(\frac{1}{p} - \frac{1}{q}\Big) t_n^p A(u_b).$$

Hence

$$\frac{\lambda_n^{\frac{2}{2-q}}}{2} E(w_n) - \frac{\lambda_n^{\frac{p}{2-q}}}{p} A(w_n) - \frac{\lambda_n^{\frac{2}{2-q}}}{q} B(w_n) \le \frac{q-2}{2q} t_n^2 \lambda_n^{\frac{2}{2-q}} E(w_b) - \frac{q-p}{pq} \lambda_n^{\frac{p}{2-q}} t_n^p A(w_b),$$

i.e.

$$\frac{1}{2} E(w_n) - \frac{\lambda_n^{\frac{p-2}{2-q}}}{p} A(w_n) - \frac{1}{q} B(w_n) \le \frac{q-2}{2q} t_n^2 E(w_b) - \frac{q-p}{pq} \lambda_n^{\frac{p-2}{2-q}} t_n^p A(w_b).$$

Since $w_n \to w_0$ in X, we obtain

$$I_b(w_0) \le \Big(\frac{1}{2} - \frac{1}{q}\Big)E(w_b) = I_b(w_b).$$

Therefore, $I_b(w_0) = I_b(w_b)$, as claimed. □

We now consider the asymptotic behavior of $u_{2,\lambda}$ as $\lambda \to 0^+$. We shall prove that $u_{2,\lambda} \to 0$ in X as $\lambda \to 0^+$.

Lemma 1.13. *Assume $\int_{\partial\Omega} b < 0$. Then there exists a constant $C > 0$ such that $\|u_{2,\lambda}\| \le C$ as $\lambda \to 0^+$.*

Proof. First we show that there exists a constant $C_1 > 0$ such that $I_\lambda(u_{2,\lambda}) \le C_1$ for every $\lambda \in (0, \lambda_0)$. To this end, we consider the eigenvalue problem

$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Let λ_1 be the first eigenvalue of this problem and $\varphi_1 > 0$ be an eigenfunction associated to λ_1. Note that $\varphi_1 \in E^+ \cap A^+ \cap B_0$ and

$$j_{\varphi_1}(t) = \frac{t^2}{2}E(\varphi_1) - \frac{t^p}{p}A(\varphi_1),$$

so that j_{φ_1} has a global maximum at some $t_2 > 0$, which implies $t_2\varphi_1 \in N_\lambda^-$. Moreover, neither j_{φ_1} nor $t_2\varphi_1$ depend on $\lambda \in (0, \lambda_0)$. Let $C_1 = j_{\varphi_1}(t_2) = I_\lambda(t_2\varphi_1) > 0$. Since $I_\lambda(u_{2,\lambda}) = \min_{N_\lambda^-} I_\lambda$, we have

$$\Big(\frac{1}{2} - \frac{1}{p}\Big)E(u_{2,\lambda}) - \Big(\frac{1}{q} - \frac{1}{p}\Big)\lambda B(u_{2,\lambda}) = I_\lambda(u_{2,\lambda}) \le C_1.$$

Assume by contradiction that $\lambda_n \to 0$ and $\|u_{2,\lambda_n}\| \to \infty$. We set $v_n = \frac{u_{2,\lambda_n}}{\|u_{2,\lambda_n}\|}$ and assume that $v_n \rightharpoonup v_0$ in X. Then

$$\Big(\frac{1}{2} - \frac{1}{p}\Big)E(v_n) \le \Big(\frac{1}{q} - \frac{1}{p}\Big)\lambda B(v_n)\|u_{2,\lambda_n}\|^{q-2} + C_1\|u_{2,\lambda_n}\|^{-2}.$$

We obtain $\limsup E(v_n) \le 0$, and by Lemma 1.4 we infer that v_0 is a constant and $v_n \to v_0$ in X. In particular, $\|v_0\| = 1$. Moreover, from

$$E(u_{2,\lambda_n}) = \lambda_n B(u_{2,\lambda_n}) + A(u_{2,\lambda_n})$$

we get $A(v_n) \to 0$, i.e. $A(v_0) = 0$, which provides $v_0 = 0$, and we get a contradiction. Therefore, $(u_{2,\lambda})$ stays bounded in X as $\lambda \to 0$. □

Proposition 1.14. *Assume $\int_{\partial\Omega} b < 0$. Then $u_{2,\lambda} \to 0$ and $\lambda^{-1/(p-q)}u_{2,\lambda} \to c^*$ in X as $\lambda \to 0^+$.*

Proof. By Lemma 1.13, up to a subsequence, we have $u_{2,\lambda} \rightharpoonup u_0$ in X and $u_{2,\lambda} \to u_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$ as $\lambda \to 0$. Since $u_{2,\lambda}$ is a weak solution of (P_λ), it follows that $u_{2,\lambda} \to u_0$ in X and u_0 is a non-negative solution of

$$\begin{cases} -\Delta u = u^{p-1} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

But the only non-negative solution of this problem is $u \equiv 0$. Hence $u_0 \equiv 0$ and $u_{2,\lambda} \to 0$ in X as $\lambda \to 0$. We now set $w_\lambda = \lambda^{-1/(p-q)}u_{2,\lambda}$. Then w_λ is a non-negative solution of

$$\begin{cases} -\Delta w = \lambda^{\frac{p-2}{p-q}} w^{p-1} & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \mathbf{n}} = \lambda^{\frac{p-2}{p-q}} b(x)w^{q-1} & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

We claim that w_λ stays bounded in X as $\lambda \to 0$. Indeed, assume that $\|w_\lambda\| \to \infty$ and $\psi_\lambda = \frac{w_\lambda}{\|w_\lambda\|} \rightharpoonup \psi_0$ in X with $\psi_\lambda \to \psi_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$ as $\lambda \to 0$. Let

$$c_\lambda = \Big(\frac{-\lambda\int_{\partial\Omega} b}{|\Omega|}\Big)^{\frac{1}{p-q}}.$$

We now use the fact that $c_\lambda \in N_\lambda^-$ for any $\lambda > 0$. Hence

$$I_\lambda(u_{2,\lambda}) \le I_\lambda(c_\lambda) = D\lambda^{\frac{p}{p-q}},$$

where

$$D = \frac{p-q}{pq} \frac{(-\int_{\partial\Omega} b)^{\frac{p}{p-q}}}{|\Omega|^{\frac{q}{p-q}}}.$$

Thus

$$\frac{p-2}{2p} \lambda^{\frac{2}{p-q}} E(w_\lambda) - \frac{p-q}{pq} \lambda^{\frac{p}{p-q}} B(w_\lambda) \leq D\lambda^{\frac{p}{p-q}},$$

so that

$$\frac{p-2}{2p} E(w_\lambda) - \frac{p-q}{pq} \lambda^{\frac{p-2}{p-q}} B(w_\lambda) \leq D\lambda^{\frac{p-2}{p-q}}.$$

Dividing the latter inequality by $\|w_\lambda\|^2$, we get $E(\psi_\lambda) \to 0$, and consequently $\psi_\lambda \to \psi_0$ in X as $\lambda \to 0$ and ψ_0 is a constant. Furthermore, integrating (1.6), we obtain

$$\int_\Omega w_\lambda^{p-1} + \int_{\partial\Omega} b w_\lambda^{q-1} = 0, \tag{1.7}$$

so that $\int_\Omega \psi_\lambda^{p-1} \to 0$, i.e. $\psi_0 = 0$, which is impossible since $\|\psi_0\| = 1$. Therefore, w_λ stays bounded in X as $\lambda \to 0$. We may then assume that $w_\lambda \rightharpoonup w_0$ in X and $w_\lambda \to w_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$ as $\lambda \to 0$. It follows that

$$\int_\Omega \nabla w_0 \nabla\phi = 0 \quad \text{for all } \phi \in X.$$

Hence w_0 is a constant and $w_\lambda \to w_0$ in X. It remains to show that $w_0 \neq 0$. If $w_0 = 0$, then we set again $\psi_\lambda = \frac{w_\lambda}{\|w_\lambda\|}$. From

$$E(w_\lambda) < \frac{p-2}{p-q} \lambda^{\frac{p-2}{p-q}} A(w_\lambda),$$

we infer that $E(\psi_\lambda) \to 0$, so that $\psi_\lambda \to \psi_0$ in X and ψ_0 is a constant. Moreover, from

$$0 \leq A(w_\lambda) + B(w_\lambda)$$

we have

$$-\|w_\lambda\|^{p-q} A(\psi_\lambda) \leq B(\psi_\lambda),$$

so that $B(\psi_0) \geq 0$. From $\int_{\partial\Omega} b < 0$ we deduce that $\psi_0 = 0$, which contradicts $\|\psi_0\| = 1$. Therefore, we have proved that w_0 is a non-zero constant. Finally, letting $\lambda \to 0$ in (1.7), we obtain $w_0^{p-1}|\Omega| = -w_0^{q-1} \int_{\partial\Omega} b$, i.e. $w_0 = c^*$. □

Remark 1.15. By a bootstrap argument based on elliptic regularity just as in the proof of [13, Theorem 2.2], we deduce that as $\lambda \to 0^+$, we have $w_\lambda \to c^* > 0$ in $W^{1,r}(\Omega)$ for $r > N$, and therefore in $C^\theta(\overline{\Omega})$ for some $\theta \in (0, 1)$. It follows that $w_\lambda > \frac{c^*}{2}$ on $\overline{\Omega}$ for sufficiently small $\lambda > 0$. Hence, an elliptic regularity argument yields that $w_\lambda \to c^*$ in $C^{2+\theta}(\overline{\Omega})$ for some $\theta \in (0, 1)$ as $\lambda \to 0^+$.

1.3 A result via sub-supersolutions

We now use the asymptotic profile of $u_{1,\lambda}$ as $\lambda \to 0$ to obtain U_λ by the sub-supersolutions method. Therefore, the condition $p < 2^*$ can be dropped. We shall consider an auxiliary problem first.

Lemma 1.16. *Assume that*

$$\int_{\partial\Omega} b < 0 \quad \textit{and} \quad 0 < \delta < -\frac{\int_{\partial\Omega} b}{|\Omega|}.$$

Then the problem

$$\begin{cases} -\Delta w = \delta |w|^{q-2} w & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \mathbf{n}} = b(x)|w|^{q-2} w & \text{on } \partial\Omega. \end{cases} \tag{$P_{b,\delta}$}$$

has a nontrivial non-negative solution w_δ.

Proof. First we claim that there exists $C > 0$ such that

$$\int_\Omega |\nabla w|^2 \geq C\|w\|^2 \quad \text{for all } w \text{ such that } \delta\|w\|_q^q + B(w) \geq 0.$$

Indeed, assume by contradiction that (w_n) is a sequence such that

$$\delta\|w_n\|_q^q + B(w_n) \geq 0 \quad \text{and} \quad \int_\Omega |\nabla w_n|^2 < \frac{1}{n}\|w_n\|^2.$$

Setting $v_n = \frac{w_n}{\|w_n\|}$, we may assume that $v_n \rightharpoonup v_0$ in X and $v_n \to v_0$ in $L^q(\Omega)$ and $L^q(\partial\Omega)$. Then $\limsup E(v_n) \leq 0$, so that, by Lemma 1.4, $v_n \to v_0$ in X and v_0 is a constant. On the other hand, we have $\delta\|v_0\|_q^q + B(v_0) \geq 0$, so that $\delta|\Omega| + \int_{\partial\Omega} b \geq 0$, which contradicts our assumption. The claim is thus proved. We now consider the functional

$$J_\delta(w) = \frac{1}{2}E(w) - \frac{1}{q}\delta\|w\|_q^q - \frac{1}{q}B(w), \qquad w \in X.$$

We claim that J_δ is bounded from below. Indeed, assume by contradiction that $J_\delta(w_n) \to -\infty$ for some sequence (w_n). Then $\delta\|w_n\|_q^q + B(w_n) \to \infty$, and consequently $\|w_n\| \to \infty$. From the claim above we deduce that $E(w_n) \geq C\|w_n\|^2$, and consequently $J_\delta(w_n) \to \infty$, a contradiction. Therefore, J_λ is bounded from below, and since it is weakly lower semicontinuous, it achieves its infimum. Consequently, choosing w_0 such that $\delta\|w_0\|_q^q + B(w_0) > 0$, we see that $J_\lambda(tw_0) < 0$ if $t > 0$ is small enough. It follows that the infimum of J_λ is negative, and consequently J_λ has a nontrivial critical point w_δ, which is a solution of $(P_{b,\delta})$. Since J_λ is even, we may choose w_δ non-negative. □

Proposition 1.17. *Assume $b^+ \not\equiv 0$ and $\int_{\partial\Omega} b < 0$. Then there exists $\Lambda_0 > 0$ such that (P_λ) has a nontrivial non-negative solution U_λ for $0 < \lambda < \Lambda_0$. Moreover, $U_\lambda \to 0$ in X as $\lambda \to 0^+$.*

Proof. First we obtain a supersolution of (P_λ). To this end, we consider a nontrivial non-negative solution w_δ of $(P_{b,\delta})$. We set $\overline{u} = \lambda^{1/(2-q)}w_\delta$. Then $\overline{u}$ is a weak supersolution of (P_λ) if

$$\lambda^{\frac{1}{2-q}}\delta\int_\Omega w_\delta(x)^{q-1}v + \lambda^{\frac{1}{2-q}}\int_{\partial\Omega} b(x)w_\delta(x)^{q-1}v \geq \lambda^{\frac{p-1}{2-q}}\int_\Omega w_\delta(x)^{p-1}v + \lambda^{\frac{1}{2-q}}\int_{\partial\Omega} b(x)w_\delta(x)^{q-1}v$$

for every non-negative $v \in X$. It then suffices to have

$$\delta \geq \lambda^{\frac{p-2}{2-q}}w_\delta(x)^{p-q}$$

for a.e. $x \in \Omega$ such that $w_\delta(x) > 0$. This inequality is satisfied if

$$\lambda \leq \Lambda_0 := (\delta\|w_\delta\|_\infty^{q-p})^{\frac{2-q}{p-2}}.$$

On the other hand, since $b^+ \not\equiv 0$ there exist a non-empty, open and smooth $(N-1)$-dimensional surface $\Gamma_0 \subset \partial\Omega$ and $\eta_0 > 0$ such that $b \geq \eta_0$ in Γ_0. Let ϕ_1 be a positive eigenfunction associated to $\sigma_1(\lambda)$, the first eigenvalue of

$$\begin{cases} -\Delta\phi = \sigma\phi & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial\mathbf{n}} = \lambda\phi & \text{on } \Gamma_0, \\ \phi = 0 & \text{on } \Gamma_1, \end{cases}$$

where $\Gamma_1 = \partial\Omega \setminus \overline{\Gamma_0}$. Note that ϕ_1 is a weak solution of this problem (see Garcia-Melian, Rossi and Sabina de Lis [9]), i.e. $\phi_1 \in H^1_{\Gamma_1}(\Omega)$ and

$$\int_\Omega \nabla\phi\nabla v - \sigma_1\int_\Omega \phi v - \lambda\int_{\Gamma_0}\phi v = 0 \quad \text{for all } v \in H^1_{\Gamma_1}(\Omega),$$

where $H^1_{\Gamma_1}(\Omega) = \{v \in X : u|_{\Gamma_1} = 0\}$. From Agmon, Douglis and Nirenberg [1] and Stampacchia [16] we know that $\phi_1 \in C^{2+\theta}(\Omega \cup \Gamma_0 \cup \Gamma_1) \cap C^\theta(\overline{\Omega})$ for some $\theta \in (0,1)$, and thus, by the strong maximum principle and the

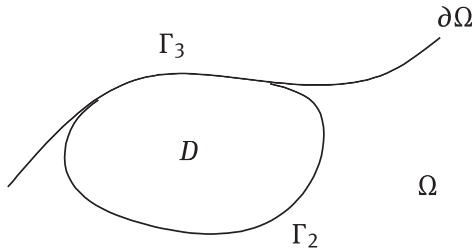

Figure 2: Subdomain D.

boundary point lemma, we have $\frac{\partial \phi_1}{\partial \mathbf{n}} < 0$ on Γ_1 and $\phi_1 > 0$ on $\Omega \cup \Gamma_0$. As for the $W^{2,p}$-regularity of ϕ_1, we know (cf. Beirão da Veiga [5, Theorem B]) that $\phi_1 \in W^{2,r}(\Omega)$ for some $r \in (1, \frac{4}{3})$. Note that $\sigma_1(\lambda) < 0$ for $\lambda > 0$. We set $\underline{u} = \varepsilon \phi_1$, where $\varepsilon > 0$. Then $\underline{u}$ is a weak subsolution of (P_λ) if

$$\varepsilon\Big(\lambda \int_{\Gamma_0} \phi_1 v + \sigma(\lambda) \int_{\Omega} \phi_1 v\Big) \leq \varepsilon^{p-1} \int_{\Omega} \phi_1^{p-1} v + \lambda \varepsilon^{q-1} \int_{\Gamma_0} b \phi_1^{q-1} v$$

for every non-negative $v \in X$. Since $\sigma_1(\lambda) < 0$, it then suffices to have $(\varepsilon \phi_1)^{2-q} \leq b$, which holds for $\varepsilon > 0$ sufficiently small.

Now, to apply the method of super and subsolutions we need to verify that $w_\delta > 0$ in a neighborhood of Γ_0. Let D be a smooth subdomain of Ω such that $\Gamma_2 := \Omega \cap \partial D$ and $\Gamma_3 := \partial D \setminus \overline{\Gamma_2}$ are non-empty, open and smooth $(N-1)$-dimensional surfaces. In addition, we assume that $\partial D = \Gamma_2 \cup \gamma \cup \Gamma_3$ with $\gamma = \overline{\Gamma_2} \cap \overline{\Gamma_3}$, see Figure 2. By assumption there exists a constant $d > 0$ such that $b > 0$ in $\Gamma_3 = \{x \in \partial\Omega : \operatorname{dist}(x, \Gamma_0) < d\}$. We then see that w_δ is a weak supersolution of the concave problem

$$\begin{cases} -\Delta u = \delta u^{q-1} & \text{in } D, \\ \dfrac{\partial u}{\partial \mathbf{n}} = b(x) u^{q-1} & \text{on } \Gamma_3, \\ u = 0 & \text{on } \Gamma_2. \end{cases} \tag{Q_b}$$

To construct a subsolution of (Q_b), we consider the problem

$$\begin{cases} -\Delta \phi = \lambda \phi & \text{in } D, \\ \dfrac{\partial \phi}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_3, \\ \phi = 0 & \text{on } \Gamma_2. \end{cases}$$

This eigenvalue problem possesses a smallest eigenvalue, which is positive. We denote by Φ_1 a positive eigenfunction associated to this eigenvalue. We see that Φ_1 is a weak subsolution of (Q_b) if $\|\Phi_1\|_{C(\overline{D})}$ is sufficiently small. Hence, the comparison principle [12, Proposition A.1] shows that $\Phi_1 \leq w_\delta$ on $\overline{D}$. In particular, $0 < \Phi_1 \leq w_\delta$ on Γ_3, as desired.

Finally, taking $\varepsilon > 0$ smaller if necessary, we have $\varepsilon \phi_1 \leq \overline{u}$ in Ω. By [11, Theorem 2] we deduce that (P_λ) has a solution U_λ which satisfies

$$\varepsilon \phi_1 \leq U_\lambda \leq \lambda^{\frac{1}{2-q}} w_\delta$$

in Ω for $\lambda < \Lambda_0$. In particular, we have $U_\lambda \to 0$ in $C(\overline{\Omega})$, and consequently in X, as $\lambda \to 0^+$. □

1.4 A bifurcation result

We now use a bifurcation technique to obtain V_λ for $\lambda > 0$ sufficiently close to 0 if $\int_{\partial\Omega} b < 0$. Saut and Schereur [14] have originally carried out this kind of bifurcation analysis by using the Lyapunov–Schmidt method. To the best of our knowledge, this approach has been first applied to the case of nonlinear boundary conditions in [18].

We consider the following problem, which corresponds to (P_λ) after the change of variable $w = \lambda^{-1/(p-q)}u$:

$$\begin{cases} -\Delta w = \lambda^{\frac{p-2}{p-q}} w^{p-1} & \text{in } \Omega, \\ \dfrac{\partial w}{\partial n} = \lambda^{\frac{p-2}{p-q}} b w^{q-1} & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

Let us recall that

$$c^* = \left(\frac{-\int_{\partial\Omega} b}{|\Omega|} \right)^{\frac{1}{p-q}}.$$

Proposition 1.18. *Assume $\int_{\partial\Omega} b < 0$. Then we have the following:*

(i) *If (1.8) has a sequence of non-negative solutions (λ_n, w_n) such that $\lambda_n \to 0^+$, $w_n \to c$ in $C(\overline{\Omega})$ and c is a positive constant, then $c = c^*$.*

(ii) *Conversely, (1.8) has, for $|\lambda|$ sufficiently small, a bifurcation branch $(\lambda, w(\lambda))$ of positive solutions (parameterized by λ) emanating from the trivial line $\{(0, c) : c \text{ is a positive constant}\}$ at $(0, c^*)$ and such that, for $0 < \theta \le \alpha$, the mapping $\lambda \mapsto w(\lambda) \in C^{2+\theta}(\overline{\Omega})$ is continuous. Moreover, the set $\{(\lambda, w)\}$ of positive solutions of (1.8) around $(\lambda, w) = (0, c^*)$ consists of the union*

$$\left\{(0, c) : c \text{ is a positive constant}, |c - c^*| \le \delta_1\right\} \cup \left\{(\lambda, w(\lambda)) : |\lambda| \le \delta_1\right\}$$

for some $\delta_1 > 0$.

Proof. The proof is similar to the one of [12, Proposition 5.3].

(i) Let w_n be non-negative solutions of (1.8) with $\lambda = \lambda_n$, where $\lambda_n \to 0^+$. By the Green formula we have

$$\int_\Omega w_n^{p-1} + \int_{\partial\Omega} b w_n^{q-1} = 0.$$

Passing to the limit as $n \to \infty$, we deduce the desired conclusion.

(ii) We reduce (1.8) to a bifurcation equation in $\mathbb{R}^2$ by the Lyapunov–Schmidt procedure: we use the usual orthogonal decomposition

$$L^2(\Omega) = \mathbb{R} \oplus V,$$

where $V = \{v \in L^2(\Omega) : \int_\Omega v = 0\}$, and the projection $Q : L^2(\Omega) \to V$ given by

$$v = Qu = u - \frac{1}{|\Omega|} \int_\Omega u.$$

The problem of finding a positive solution of (1.8) then reduces to the following problems:

$$\begin{cases} -\Delta v + \dfrac{\mu}{|\Omega|} \displaystyle\int_{\partial\Omega} (t+v)^{q-1} = \mu Q[(t+v)^{p-1}] & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \mathbf{n}} = \mu b (t+v)^{q-1} & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

and

$$\int_\Omega (t+v)^{p-1} + \int_{\partial\Omega} b(t+v)^{q-1} = 0, \tag{1.10}$$

where

$$\mu = \lambda^{\frac{p-2}{p-q}} \neq 0, \quad t = \frac{1}{|\Omega|} \int_\Omega w, \quad v = w - t.$$

To solve (1.9) in the framework of Hölder spaces, we set

$$Y = \left\{ v \in C^{2+\theta}(\overline{\Omega}) : \int_\Omega v = 0 \right\},$$

$$Z = \left\{ (\phi, \psi) \in C^\theta(\overline{\Omega}) \times C^{1+\theta}(\partial\Omega) : \int_\Omega \phi + \int_{\partial\Omega} \psi = 0 \right\}.$$

Let $c > 0$ be a constant and $U \subset \mathbb{R} \times \mathbb{R} \times Y$ be a small neighborhood of $(0, c, 0)$. The nonlinear mapping $F : U \to Z$ is given by

$$F(\mu, t, v) = \Big(-\Delta v - \mu Q[(t+v)^{p-1}] + \frac{\mu}{|\Omega|}\int_{\partial\Omega} b(t+v)^{q-1},\ \frac{\partial v}{\partial \mathbf{n}} - \mu b(t+v)^{q-1}\Big).$$

The Fréchet derivative F_v of F with respect to v at $(0, c, 0)$ is given by the formula

$$F_v(0, c, 0)v = \Big(-\Delta v,\ \frac{\partial v}{\partial \mathbf{n}}\Big).$$

Since $F_v(0, c, 0)$ is a continuous and bijective linear mapping, the implicit function theorem [15, Theorem 13.3] implies that the set $F(\mu, t, v) = 0$ around $(0, c, 0)$ consists of a unique C^∞ function $v = v(\mu, t)$ in a neighborhood of $(\mu, t) = (0, c)$ which satisfies $v(0, c) = 0$. Now, plugging $v(\mu, t)$ in (1.10), we obtain the bifurcation equation

$$\Phi(\mu, t) = \int_\Omega (t + v(\mu, t))^{p-1} + \int_{\partial\Omega} b(t + v(\mu, t))^{q-1} = 0 \quad \text{for } (\mu, t) \simeq (0, c).$$

It is clear that $\Phi(0, c^*) = 0$. Differentiating Φ with respect to t at $(0, c^*)$, we get

$$\begin{aligned}\Phi_t(0, c^*) &= \int_\Omega (p-1)(c^* + v(0, c^*))^{p-2}(1 + v_t(0, c^*)) \\ &\quad + \int_{\partial\Omega} (q-1)b(c^* + v(0, c^*))^{q-2}(1 + v_t(0, c^*)) \\ &= (p-1)(c^*)^{p-2}\int_\Omega (1 + v_t(0, c^*)) + (q-1)(c^*)^{q-2}\int_{\partial\Omega} b(1 + v_t(0, c^*)).\end{aligned}$$

Differentiating (1.9) with respect to t and plugging $(\mu, t) = (0, c^*)$ into it, we have $v_t(0, c^*) = 0$. Hence,

$$\Phi_t(0, c^*) = (p-1)(c^*)^{p-2}|\Omega| + (q-1)(c^*)^{q-2}\int_{\partial\Omega} b = |\Omega|(p-q)(c^*)^{p-2} > 0.$$

By the implicit function theorem, the function

$$w(\lambda) = t(\mu) + v(\mu, t(\mu)) \quad \text{with } \mu = \lambda^{\frac{p-2}{p-q}}$$

satisfies the desired assertion. □

Remark 1.19. Combining Remark 1.15 and the uniqueness result in Proposition 1.18 (ii) for a smooth curve of bifurcating positive solutions of (1.8) at $(0, c^*)$, we infer that the positive solution $\lambda^{-1/(p-q)}w(\lambda)$ of (P_λ) constructed with the bifurcating positive solution $w(\lambda)$ of (1.8) coincides with $u_{2,\lambda}$ for sufficiently small $\lambda > 0$. We summarize our results in Table 1.

1.5 Positivity in the case $N = 1$

We now show the positivity of nontrivial non-negative weak solutions for the one-dimensional case of (P_λ). We take $\Omega = I = (0, 1)$ and show that nontrivial non-negative solutions satisfy $u > 0$ on $\overline{I}$. More precisely, we consider nontrivial non-negative weak solutions of the problem

$$\begin{cases} -u'' = u^{p-1} \text{ in } I, \\ -u'(0) = \lambda b_0 u(0)^{q-1}, \\ u'(1) = \lambda b_1 u(1)^{q-1}, \end{cases} \tag{1.11}$$

where $1 < q < 2 < p$ and $b_0, b_1 \in \mathbb{R}$. A non-negative function $u \in H^1(I)$ is a non-negative weak solution

Solution	Approach	Asymptotic order as $\lambda \to 0^+$	Positivity	Proposition
$u_{1,\lambda}$	variational (in N_λ^+)	$\sim \lambda^{\frac{1}{2-q}}$ (up to a subsequence)	in $\Omega \cup \{x \in \partial\Omega : b(x) > 0\}$	1.12
$u_{2,\lambda}$	variational (in N_λ^-)	$\sim \lambda^{\frac{1}{p-q}}$	in $\overline{\Omega}$	1.14
U_λ	sub- and supersolutions	$\sim \lambda^{\frac{1}{2-q}}$ (at most)	in $\Omega \cup \{x \in \partial\Omega : b(x) > 0\}$	1.17
$\lambda^{\frac{1}{p-q}} w(\lambda)$	bifurcation	$\sim \lambda^{\frac{1}{p-q}}$	in $\overline{\Omega}$	1.18

Table 1: Results on nontrivial non-negative solutions of (P_λ).

of (1.11) if it satisfies

$$\int_I u'\phi' = \lambda\big(b_0 u(0)^{q-1}\phi(0) + b_1 u(1)^{q-1}\phi(1)\big) + \int_I u^{p-1}\phi \quad \text{for all } \phi \in H^1(I).$$

Proposition 1.20. *Let $b_0, b_1 \in \mathbb{R}$ be arbitrary. Then any nontrivial non-negative weak solution u of* (1.11) *satisfies $u > 0$ in $\overline{I}$.*

Proof. If u is a non-negative weak solution of (1.11), then, thanks to the inclusion $H^1(I) \subset C(\overline{I})$ (see [6]), we have $u \in C(\overline{I})$. Moreover, we claim that $u \in H^2(I)$, so that $u \in C^1(\overline{I})$. Indeed, from the definition we derive

$$\int_I u'\phi' = \int_I u^{p-1}\phi \quad \text{for all } \phi \in C_c^1(I).$$

This implies that $(u')' = -u^{p-1}$ in I in the distribution sense. By the chain rule we obtain $u^{p-1} \in H^1(I)$. By definition we infer that $u \in H^2(I)$. From the inclusion $H^2(I) \subset C^1(\overline{I})$ it follows that $u \in C^1(\overline{I})$.

In fact, by a bootstrap argument and elliptic regularity, we have $u \in C^2(I)$. Hence, it follows that $u \in C^1(\overline{I}) \cap C^2(I)$, and we infer that $u > 0$ in I by the strong maximum principle. In order to show that $u(0) > 0$, we assume by contradiction that $u(0) = 0$. Then the boundary point lemma yields $-u'(0) < 0$. However, the boundary condition in (1.11) is understood in the classical sense under the condition $u \in C^1(\overline{I}) \cap C^2(I)$, and thus $u'(0) = 0$, which is a contradiction. Likewise, we can show that $u(1) > 0$. □

Remark 1.21. Using the same argument as in Proposition 1.20, we infer that in the case $N = 1$ nontrivial non-negative solutions of (P_w) satisfy $w > 0$ on $\overline{\Omega}$.

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