DE GRUYTER Adv. Nonlinear Anal. 2019; 8: 101–129

Research Article

Vladimir Bobkov* and Mieko Tanaka

On sign-changing solutions for (p, q)-Laplace equations with two parameters

https://doi.org/10.1515/anona-2016-0172
Received August 2, 2016; accepted September 27, 2016

Abstract: We investigate the existence of nodal (sign-changing) solutions to the Dirichlet problem for a two-parametric family of partially homogeneous (p, q)-Laplace equations $-\Delta_p u - \Delta_q u = \alpha|u|^{p-2}u + \beta|u|^{q-2}u$ where $p \neq q$. By virtue of the Nehari manifolds, the linking theorem, and descending flow, we explicitly characterize subsets of the (α, β)-plane which correspond to the existence of nodal solutions. In each subset the obtained solutions have prescribed signs of energy and, in some cases, exactly two nodal domains. The nonexistence of nodal solutions is also studied. Additionally, we explore several relations between eigenvalues and eigenfunctions of the p- and q-Laplacians in one dimension.

Keywords: (p, q)-Laplacian, p-Laplacian, eigenvalue problem, first eigenvalue, second eigenvalue, nodal solutions, sign-changing solutions, Nehari manifold, linking theorem, descending flow

MSC 2010: 35J62, 35J20, 35P30

1 Introduction

In this article, we study the existence and nonexistence of sign-changing solutions for the problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \alpha|u|^{p-2}u + \beta|u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{GEV; α, β}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain with a sufficiently smooth boundary $\partial\Omega$, and $\alpha, \beta \in \mathbb{R}$ are parameters. The operator $\Delta_r u := \operatorname{div}(|\nabla u|^{r-2}\nabla u)$ is the classical r-Laplacian, $r = \{q, p\} > 1$, and without loss of generality we assume that $q < p$.

Boundary value problems with a combination of several differential operators of different nature (in particular, as in (GEV; α, β)) arise mainly as mathematical models of physical processes and phenomena, and have been extensively studied in the last two decades; see, e.g., [13, 15, 19, 30] and the references below. Among the historically first examples one can mention the Cahn–Hilliard equation [12] describing the process of separation of binary alloys, and the Zakharov equation [33, (1.8)] which describes the behavior of plasma oscillations. Elliptic equations with the $(2, 6)$- and $(2, p)$-Laplacians were considered explicitly in [7, 8] with the aim of obtaining soliton-type solutions (in particular, as a model for elementary particles).

The considered problem (GEV; α, β) attracts special attention due to its symmetric and partially homogeneous structure; cf. [4, 10, 20, 28, 31, 32, 34]. By developing the results of [20, 28, 31], the authors of the present article obtained in [10] a reasonably complete description of the subsets of the (α, β)-plane which

***Corresponding author: Vladimir Bobkov,** Institute of Mathematics, Ufa Scientific Center, Russian Academy of Sciences, Chernyshevsky str. 112, Ufa 450008, Russia; and Department of Mathematics and NTIS, Faculty of Applied Sciences, University of West Bohemia, Univerzitní 8, Plzeň 306 14, Czech Republic, e-mail: bobkov@kma.zcu.cz.
https://orcid.org/0000-0002-4425-0218
Mieko Tanaka, Department of Mathematics, Tokyo University of Science, Kagurazaka 1-3, Shinjyuku-ku, Tokyo 162-8601, Japan, e-mail: miekotanaka@rs.tus.ac.jp

correspond to the existence/nonexistence of *positive* solutions to problem (GEV; α, β). At the same time, to the best of our knowledge, analogous results for *sign-changing* solutions have not been obtained circumstantially so far, although a particular information on the existence can be extracted from [1, 24, 32]. The main reason for this is a crucial dependence of the structure of the solution set to problem (GEV; α, β) on parameters α and β. As a consequence, the existence can not be treated by a unique approach, and various tools have to be used for different parts of the (α, β)-plane.

The aim of the present article is to allocate and characterize the sets of parameters α and β for which problem (GEV; α, β) possesses or does not possess sign-changing solutions (see Figure 1). In this sense, this work can be seen as the second part of the article [10].

1.1 Notations and preliminaries

Before formulating the main results, we introduce several notations. In what follows, $L^r(\Omega)$ with $r \in (1, +\infty)$ and $L^\infty(\Omega)$ stand for the Lebesgue spaces with the norms

$$\|u\|_r := \left(\int_\Omega |u|^r \, dx\right)^{1/r} \quad \text{and} \quad \|u\|_\infty := \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|,$$

respectively, and $W_0^{1,r} := W_0^{1,r}(\Omega)$ denotes the Sobolev space with the norm $\|\nabla u\|_r$. For $u \in W_0^{1,r}$ we define $u^\pm := \max\{\pm u, 0\}$. Note that $u^\pm \in W_0^{1,r}$ and $u = u^+ - u^-$.

By a (weak) solution of (GEV; α, β) we mean function $u \in W_0^{1,p}$ which satisfies

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx + \int_\Omega |\nabla u|^{q-2} \nabla u \nabla \varphi \, dx = \alpha \int_\Omega |u|^{p-2} u \varphi \, dx + \beta \int_\Omega |u|^{q-2} u \varphi \, dx \tag{1.1}$$

for all $\varphi \in W_0^{1,p}$. If u is a solution of (GEV; α, β) and $u^\pm \not\equiv 0$ (a.e. in Ω), then u is called *nodal* or *sign-changing* solution. It is not hard to see that *any* solution of (GEV; α, β) is a critical point of the energy functional $E_{\alpha,\beta} \in C^1(W_0^{1,p}, \mathbb{R})$ defined by

$$E_{\alpha,\beta}(u) := \frac{1}{p} H_\alpha(u) + \frac{1}{q} G_\beta(u),$$

where

$$H_\alpha(u) := \int_\Omega |\nabla u|^p \, dx - \alpha \int_\Omega |u|^p \, dx \quad \text{and} \quad G_\beta(u) := \int_\Omega |\nabla u|^q \, dx - \beta \int_\Omega |u|^q \, dx.$$

Notice that the supports of u^+ and u^- are disjoint for any $u \in W_0^{1,p}$. This fact, together with evenness of the functionals H_α and G_β, easily implies that

$$H_\alpha(u^+) + H_\alpha(u^-) = H_\alpha(u) \quad \text{and} \quad G_\beta(u^+) + G_\beta(u^-) = G_\beta(u).$$

Remark 1.1. Any solution $u \in W_0^{1,p}$ of problem (GEV; α, β) belongs to $C_0^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0, 1)$. In fact, $u \in L^\infty(\Omega)$ by the Moser iteration process; cf. [25, Appendix A]. Furthermore, the regularity up to the boundary in [21, Theorem 1] and [22, p. 320] provides $u \in C_0^{1,\gamma}(\overline{\Omega})$, $\gamma \in (0, 1)$.

Next, we recall several facts related to the eigenvalue problem for the Dirichlet r-Laplacian, $r > 1$. We say that λ is an *eigenvalue* of $-\Delta_r$, if the problem

$$\begin{cases} -\Delta_r u = \lambda |u|^{r-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{EV; r, λ}$$

has a nontrivial (weak) solution. Analogously to the linear case, the set of all eigenvalues of (EV; r, λ) will be denoted as $\sigma(-\Delta_r)$. It is well known that the lowest positive eigenvalue $\lambda_1(r)$ can be obtained through the nonlinear Rayleigh quotient as (cf. [2])

$$\lambda_1(r) := \inf\left\{ \frac{\int_\Omega |\nabla u|^r \, dx}{\int_\Omega |u|^r \, dx} : u \in W_0^{1,r}, \; u \not\equiv 0 \right\}. \tag{1.2}$$

The eigenvalue $\lambda_1(r)$ is simple and isolated, and the corresponding eigenfunction $\varphi_r \in W_0^{1,p}$ (defined up to an arbitrary multiplier) is strictly positive (or strictly negative) in Ω. Moreover, $\lambda_1(r)$ is the unique eigenvalue with a corresponding sign-constant eigenfunction [2]. Note also that any eigenfunction φ of $-\Delta_r$ belongs to $C_0^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0,1)$.

The following lemma directly follows from the definition of $\lambda_1(r)$ and its simplicity.

Lemma 1.2. *Assume that $u \in W_0^{1,p} \setminus \{0\}$. Then we have the following results:*
(i) *Let $\alpha \le \lambda_1(p)$. Then $H_\alpha(u) \ge 0$, and $H_\alpha(u) = 0$ if and only if $\alpha = \lambda_1(p)$ and $u = t\varphi_p$ for some $t \in \mathbb{R} \setminus \{0\}$.*
(ii) *Let $\beta \le \lambda_1(q)$. Then $G_\beta(u) \ge 0$, and $G_\beta(u) = 0$ if and only if $\beta = \lambda_1(q)$ and $u = t\varphi_q$ for some $t \in \mathbb{R} \setminus \{0\}$.*

Although the structure of $\sigma(-\Delta_r)$ is not completely known except for the case $r = 2$ or $N = 1$ (see, e.g., [17, Theorem 3.1]), several unbounded sequences of eigenvalues can be introduced by virtue of minimax variational principles. In what follows, by $\{\lambda_k(r)\}_{k\in\mathbb{N}}$ we denote a sequence of eigenvalues for (EV; r, λ) introduced in [18]. It can be described variationally as

$$\lambda_k(r) := \inf_{h\in\mathscr{F}_k(r)} \max_{z\in S^{k-1}} \|\nabla h(z)\|_r^r, \tag{1.3}$$

where S^{k-1} is the unit sphere in $\mathbb{R}^k$ and

$$\begin{aligned} \mathscr{F}_k(r) &:= \{h \in C(S^{k-1}, S(r)) : h \text{ is odd}\}, \\ S(r) &:= \{u \in W_0^{1,r} : \|u\|_r = 1\}. \end{aligned} \tag{1.4}$$

It is known [18] that $\lambda_k(r) \to +\infty$ as $k \to +\infty$. Moreover, $\lambda_2(r)$ coincides with the second eigenvalue of $-\Delta_r$, i.e.,

$$\lambda_2(r) = \inf\{\lambda \in \sigma(-\Delta_r) : \lambda > \lambda_1(r)\},$$

and it can be alternatively characterized as in [16]:

$$\begin{aligned} \lambda_2(r) &= \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} \|\nabla\gamma(s)\|_r^r, \\ \Gamma &:= \{\gamma \in C([0,1], S(r)) : \gamma(0) = \varphi_r,\ \gamma(1) = -\varphi_r\}, \end{aligned} \tag{1.5}$$

where the first eigenfunction φ_r is normalized such that $\varphi_r \in S(r)$. We denote any eigenfunction corresponding to $\lambda_2(r)$ as $\varphi_{2,r}$. Notice that $\lambda_2(r) > \lambda_1(r)$. Furthermore, in the one-dimensional case the sequence (1.3) describes the whole $\sigma(-\Delta_r)$ (cf. [17, Theorem 4.1], where this result is proved for the Krasnosel'skii-type eigenvalues).

Finally, we introduce the notation for the eigenspace of $-\Delta_r$ at $\lambda \in \mathbb{R}$:

$$\mathrm{ES}(r;\lambda) := \{v \in W_0^{1,r} : v \text{ is a solution of (EV}; r, \lambda)\}. \tag{1.6}$$

It is clear that $\mathrm{ES}(r;\lambda) \neq \{0\}$ if and only if $\lambda \in \sigma(-\Delta_r)$.

1.2 Main results

Let us state the main results of this article. We begin with the nonexistence of nodal solutions for (GEV; α, β).

Theorem 1.3. *Assume that*

$$(\alpha,\beta) \in (-\infty, \lambda_2(p)] \times (-\infty, \lambda_1(q)] \cup (-\infty, \lambda_1(p)] \times (-\infty, \lambda_2(q)].$$

Then (GEV; α, β) *has no nodal solutions.*

In the one-dimensional case Theorem 1.3 can be refined as follows.

Theorem 1.4. *Let $N = 1$. If $(\alpha,\beta) \in (-\infty, \lambda_2(p)] \times (-\infty, \lambda_2(q)]$, then* (GEV; α, β) *has no nodal solutions.*

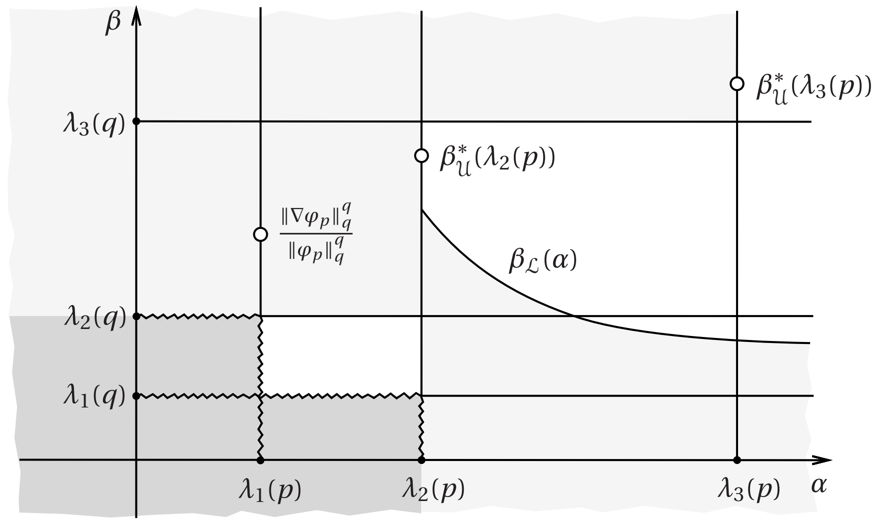

Figure 1: The case $\lambda_2(q) < \lambda_3(q)$, $\lambda_2(p) < \lambda_3(p)$, and $(\lambda_2(p), \lambda_3(p)) \cap \sigma(-\Delta_p) = \emptyset$. Existence (light gray, solid lines), nonexistence (dark gray, zigzag lines), unknown (white, dashed lines).

In the case of general dimensions an additional information on hypothetical nodal solutions to (GEV; α, β) for $\alpha \in (\lambda_1(p), \lambda_2(p)]$ and $\beta \in (\lambda_1(q), \lambda_2(q)]$ is given in Lemma 2.5 below.

Now we formulate the existence result for nodal solutions with a positive energy. Let us define the following "lower" critical value depending on $\alpha \in \mathbb{R}$:

$$\beta_{\mathcal{L}}(\alpha) := \inf\left\{\min\left\{\frac{\int_\Omega |\nabla u^+|^q\,dx}{\int_\Omega |u^+|^q\,dx}, \frac{\int_\Omega |\nabla u^-|^q\,dx}{\int_\Omega |u^-|^q\,dx}\right\} : u \in \mathcal{B}_{\mathcal{L}}(\alpha)\right\}, \tag{1.7}$$

where

$$\mathcal{B}_{\mathcal{L}}(\alpha) := \left\{u \in W_0^{1,p} : u^\pm \not\equiv 0,\ \max\left\{\frac{\int_\Omega |\nabla u^+|^p\,dx}{\int_\Omega |u^+|^p\,dx}, \frac{\int_\Omega |\nabla u^-|^p\,dx}{\int_\Omega |u^-|^p\,dx}\right\} \le \alpha\right\}, \tag{1.8}$$

and put $\beta_{\mathcal{L}}(\alpha) = +\infty$ whenever the admissible set $\mathcal{B}_{\mathcal{L}}(\alpha)$ is empty.

Theorem 1.5. *Let $\alpha > \lambda_2(p)$. Then for all $\beta < \beta_{\mathcal{L}}(\alpha)$ problem* (GEV; α, β) *has a nodal solution u with $E_{\alpha,\beta}(u) > 0$ and precisely two nodal domains.*

Several main properties of the function $\beta_{\mathcal{L}}(\alpha)$ are collected in Lemma 2.11 below. Let us remark that the parametrization by α in (1.7) is different from the parametrization by s of the form $(\alpha, \beta) = (\lambda + s, \lambda)$ which was used in [10] in order to construct a critical curve for the existence of positive solutions. In the context of the present article, the parametrization by α makes problem (GEV; α, β) easier to analyze. We also note that (1.7) is conceptually similar to the characterization of the first nontrivial curve of the Fučík spectrum given in [26]. In Section 2 below, we introduce and study several other critical points besides (1.7), which although are not directly used in the proofs of the main results, increase the understanding of the construction of the (α, β)-plane, and could be employed in further investigations.

Next, we state the existence of negative energy nodal solutions for (GEV; α, β). Consider the "upper" critical value

$$\beta^*_{\mathcal{U}}(\alpha) := \sup\left\{\frac{\int_\Omega |\nabla\varphi|^q\,dx}{\int_\Omega |\varphi|^q\,dx} : \varphi \in \mathrm{ES}(p;\alpha) \setminus \{0\}\right\}, \tag{1.9}$$

where $\alpha \in \mathbb{R}$, and set $\beta^*_{\mathcal{U}}(\alpha) = -\infty$ provided $\alpha \notin \sigma(-\Delta_p)$. Several lower and upper bounds for $\beta^*_{\mathcal{U}}(\alpha)$ are given in Lemmas 3.6 and 3.7 below. Define $k_\alpha := \min\{k \in \mathbb{N} : \alpha < \lambda_{k+1}(p)\}$ and notice that $\lambda_{k_\alpha+1}(q) \ge \lambda_2(q)$ for all $\alpha \in \mathbb{R}$.

Theorem 1.6. *Let $\alpha \in \overline{\mathbb{R} \setminus \sigma(-\Delta_p)}$. Then for all $\beta > \max\{\beta^*_{\mathcal{U}}(\alpha), \lambda_{k_\alpha+1}(q)\}$ problem* (GEV; α, β) *has a nodal solution u satisfying $E_{\alpha,\beta}(u) < 0$.*

Evidently, if $\sigma(-\Delta_p)$ is a discreet set (as it is for $p = 2$ or $N = 1$), then $\overline{\mathbb{R} \setminus \sigma(-\Delta_p)} = \mathbb{R}$. Moreover, $\lambda_1(p)$ and $\lambda_2(p)$ belong to $\overline{\mathbb{R} \setminus \sigma(-\Delta_p)}$ for all $p > 1$ and $N \geq 1$ since $\lambda_1(p)$ is isolated and there are no eigenvalues between $\lambda_1(p)$ and $\lambda_2(p)$ (see Section 1.1).

One of the main ingredients for the proof of Theorem 1.6 is the result on the existence of three nontrivial solutions (positive, negative and sign-changing) to the problem with the (p, q)-Laplacian and a nonlinearity in the general form given by Theorem 3.13 below. This result is of independent interest.

Theorem 1.6 can be refined as follows.

Theorem 1.7. *Assume that*

$$(\alpha, \beta) \in (-\infty, \lambda_2(p)) \times (\lambda_2(q), +\infty) \setminus \{(\lambda_1(p), \|\nabla\varphi_p\|_q^q / \|\varphi_p\|_q^q)\}.$$

Then (GEV; α, β) *has a nodal solution* u *satisfying* $E_{\alpha,\beta}(u) < 0$.

Remark 1.8. In the one-dimensional case we have $\|\varphi_p'\|_q^q / \|\varphi_p\|_q^q < \lambda_2(q)$ (see Lemma A.2 in Appendix A), and hence the assertion of Theorem 1.7 holds for all $(\alpha, \beta) \in (-\infty, \lambda_2(p)) \times (\lambda_2(q), +\infty)$.

Let us note that unlike the case of positive solutions, the structure of the set of nodal solutions for problem (GEV; α, β) is more complicated, and we are not aware of the maximality of the regions obtained in Theorems 1.5 and 1.6.

The article is organized as follows: In Section 2, we apply the method of the Nehari manifold in order to prove Theorem 1.5. In Section 3, by means of linking arguments and the descending flow method, we provide two general existence results which yield, in particular, Theorems 1.6 and 1.7. For the convenience of the reader we collect the proofs of the main theorems in Section 4. In Appendix A, we prove several additional facts on the relation between eigenvalues and eigenfunctions of the p- and q-Laplacians in the one-dimensional case. Finally, in Appendix B, we give a sketch of the proof of Theorem 3.13.

2 Nodal solutions with positive energy

The classical Nehari manifold for problem (GEV; α, β) is defined by

$$\mathcal{N}_{\alpha,\beta} := \{u \in W_0^{1,p} \setminus \{0\} : \langle E'_{\alpha,\beta}(u), u\rangle = H_\alpha(u) + G_\beta(u) = 0\}.$$

It can be readily seen that $\mathcal{N}_{\alpha,\beta}$ contains *all* nontrivial solutions of (GEV; α, β). On the other hand, if $u \in W_0^{1,p}$ is a sign-changing solution of (GEV; α, β), then

$$\begin{aligned} 0 &= \langle E'_{\alpha,\beta}(u), u^+\rangle = \langle E'_{\alpha,\beta}(u^+), u^+\rangle = H_\alpha(u^+) + G_\beta(u^+), \\ 0 &= -\langle E'_{\alpha,\beta}(u), u^-\rangle = \langle E'_{\alpha,\beta}(u^-), u^-\rangle = H_\alpha(u^-) + G_\beta(u^-). \end{aligned}$$

These equalities bring us to the definition of the so-called *nodal* Nehari set for (GEV; α, β):

$$\mathcal{M}_{\alpha,\beta} := \{u \in W_0^{1,p} : u^\pm \not\equiv 0,\ H_\alpha(u^\pm) + G_\beta(u^\pm) = 0\} = \{u \in W_0^{1,p} : u^\pm \in \mathcal{N}_{\alpha,\beta}\}. \tag{2.1}$$

By construction, $\mathcal{M}_{\alpha,\beta}$ contains *all* sign-changing solutions of (GEV; α, β), and hence $\mathcal{M}_{\alpha,\beta} \subset \mathcal{N}_{\alpha,\beta}$.

Let us divide $\mathcal{M}_{\alpha,\beta}$ into the following three subsets:

$$\begin{aligned} \mathcal{M}^1_{\alpha,\beta} &:= \{u \in \mathcal{M}_{\alpha,\beta} : H_\alpha(u^+) < 0,\ H_\alpha(u^-) < 0\}, \\ \mathcal{M}^2_{\alpha,\beta} &:= \{u \in \mathcal{M}_{\alpha,\beta} : H_\alpha(u^+) > 0,\ H_\alpha(u^-) > 0\}, \\ \mathcal{M}^3_{\alpha,\beta} &:= \{u \in \mathcal{M}_{\alpha,\beta} : H_\alpha(u^+) \cdot H_\alpha(u^-) \leq 0\}. \end{aligned}$$

Evidently, $\mathcal{M}_{\alpha,\beta} = \mathcal{M}^1_{\alpha,\beta} \cup \mathcal{M}^2_{\alpha,\beta} \cup \mathcal{M}^3_{\alpha,\beta}$ and all $\mathcal{M}^i_{\alpha,\beta}$ are mutually disjoint. The main aim of this section is to prove the existence of nodal solutions for (GEV; α, β) through minimization of $E_{\alpha,\beta}$ over $\mathcal{M}^1_{\alpha,\beta}$ in an appropriate subset of the (α, β)-plane.

2.1 Preliminary analysis

In this subsection, we mainly study the properties of the sets $\mathcal{M}^1_{\alpha,\beta}$, $\mathcal{M}^2_{\alpha,\beta}$, and $\mathcal{M}^3_{\alpha,\beta}$. First of all, we give the following auxiliary lemma, which is in fact analogous to [10, Proposition 6] and can be proved in the same manner.

Lemma 2.1. *Let $u \in W_0^{1,p}$. If $H_\alpha(u) \cdot G_\beta(u) < 0$, then there exists a unique critical point $t(u) > 0$ of $E_{\alpha,\beta}(tu)$ with respect to $t > 0$ and $t(u)u \in \mathcal{N}_{\alpha,\beta}$. In particular, if*

$$H_\alpha(u) < 0 < G_\beta(u),$$

then $t(u)$ is the unique maximum point of $E_{\alpha,\beta}(tu)$ with respect to $t > 0$ and $E_{\alpha,\beta}(t(u)u) > 0$.

We start our consideration of the sets $\mathcal{M}^i_{\alpha,\beta}$ with several simple facts.

Lemma 2.2. *Let $\alpha, \beta \in \mathbb{R}$. The following hold:*
(i) *If $\beta \leq \lambda_1(q)$, then $\mathcal{M}^1_{\alpha,\beta} = \mathcal{M}_{\alpha,\beta}$ and, consequently, $\mathcal{M}^2_{\alpha,\beta}, \mathcal{M}^3_{\alpha,\beta} = \emptyset$.*
(ii) *If $\alpha \leq \lambda_1(p)$, then $\mathcal{M}^2_{\alpha,\beta} = \mathcal{M}_{\alpha,\beta}$ and, consequently, $\mathcal{M}^1_{\alpha,\beta}, \mathcal{M}^3_{\alpha,\beta} = \emptyset$.*

Proof. Let us first prove assertion (i). Assume that $\beta \leq \lambda_1(q)$ and $w \in \mathcal{M}_{\alpha,\beta}$. Then Lemma 1.2 implies that $G_\beta(w^\pm) \geq 0$ and in fact $G_\beta(w^\pm) > 0$, since otherwise $w^\pm = \varphi_q$, which is impossible in view of the strict positivity of φ_q in Ω. Thus, the Nehari constraints $H_\alpha(w^\pm) + G_\beta(w^\pm) = 0$ yield $H_\alpha(w^\pm) < 0$, whence $w \in \mathcal{M}^1_{\alpha,\beta}$. Assertion (ii) can be shown by the same arguments. □

Let us introduce the following sets:

$$\mathcal{B}_1(\alpha) := \{u \in W_0^{1,p} : H_\alpha(u^+) < 0,\ H_\alpha(u^-) < 0\}, \tag{2.2}$$

$$\mathcal{B}_2(\alpha) := \{u \in W_0^{1,p} : H_\alpha(u^+) > 0,\ H_\alpha(u^-) > 0\}. \tag{2.3}$$

Obviously, $\mathcal{M}^1_{\alpha,\beta} \subset \mathcal{B}_1(\alpha)$ and $\mathcal{M}^2_{\alpha,\beta} \subset \mathcal{B}_2(\alpha)$. Moreover, we have the following result.

Lemma 2.3. *Let $\alpha, \beta \in \mathbb{R}$. The following hold:*
(i) *If $\alpha \leq \lambda_2(p)$, then $\mathcal{B}_1(\alpha) = \emptyset$ and, consequently, $\mathcal{M}^1_{\alpha,\beta} = \emptyset$.*
(ii) *If $\beta \leq \lambda_2(q)$, then $\mathcal{B}_2(\alpha) = \emptyset$ and, consequently, $\mathcal{M}^2_{\alpha,\beta} = \emptyset$.*

Proof. We give the proof of assertion (i). The second part can be proved analogously. Suppose, by contradiction, that $\alpha \leq \lambda_2(p)$ and there exists $w \in \mathcal{B}_1(\alpha)$. These assumptions read as

$$\max\left\{\frac{\int_\Omega |\nabla w^+|^p\,dx}{\int_\Omega |w^+|^p\,dx}, \frac{\int_\Omega |\nabla w^-|^p\,dx}{\int_\Omega |w^-|^p\,dx}\right\} < \alpha \leq \lambda_2(p). \tag{2.4}$$

On the other hand, it is shown in [9, Proposition 4.2] that the second eigenvalue $\lambda_2(r)$, $r > 1$, can be characterized as follows:

$$\lambda_2(r) = \inf\left\{\max\left\{\frac{\int_\Omega |\nabla u^+|^r\,dx}{\int_\Omega |u^+|^r\,dx}, \frac{\int_\Omega |\nabla u^-|^r\,dx}{\int_\Omega |u^-|^r\,dx}\right\} : u \in W_0^{1,r},\ u^\pm \not\equiv 0\right\}. \tag{2.5}$$

Comparing (2.4) and (2.5) (with $r = p$), we obtain a contradiction. □

Lemmas 2.2 and 2.3 readily entail the following information about the emptiness of $\mathcal{M}_{\alpha,\beta}$ and, consequently, the nonexistence of nodal solutions for (GEV; α, β).

Lemma 2.4. *If $\alpha \leq \lambda_2(p)$ and $\beta \leq \lambda_1(q)$, or $\alpha \leq \lambda_1(p)$ and $\beta \leq \lambda_2(q)$, then $\mathcal{M}_{\alpha,\beta} = \emptyset$.*

Lemma 2.5. *Let $\alpha \leq \lambda_2(p)$ and $\beta \leq \lambda_2(q)$. If u is a nodal solution of* (GEV; α, β)*, then*

$$\alpha > \lambda_1(p), \quad \beta > \lambda_1(q), \quad u \in \mathcal{M}^3_{\alpha,\beta}.$$

Let us now subsequently treat the emptiness and nonemptiness of $\mathcal{M}^1_{\alpha,\beta}$ and $\mathcal{M}^2_{\alpha,\beta}$. First we consider $\mathcal{M}^1_{\alpha,\beta}$. Introduce the critical value

$$\beta_1(\alpha) := \sup\left\{\min\left\{\frac{\int_\Omega |\nabla u^+|^q\,dx}{\int_\Omega |u^+|^q\,dx}, \frac{\int_\Omega |\nabla u^-|^q\,dx}{\int_\Omega |u^-|^q\,dx}\right\} : u \in \mathcal{B}_1(\alpha)\right\}$$

for each $\alpha \in \mathbb{R}$, where the admissible set $\mathcal{B}_1(\alpha)$ is defined by (2.2), or, equivalently,

$$\mathcal{B}_1(\alpha) = \left\{u \in W_0^{1,p} : u^\pm \not\equiv 0,\ \max\left\{\frac{\int_\Omega |\nabla u^+|^p\,dx}{\int_\Omega |u^+|^p\,dx}, \frac{\int_\Omega |\nabla u^-|^p\,dx}{\int_\Omega |u^-|^p\,dx}\right\} < \alpha\right\}.$$

We assume that $\beta_1(\alpha) = -\infty$ whenever $\mathcal{B}_1(\alpha)$ is empty. Consider also

$$\beta_1^* := \sup\left\{\min\left\{\frac{\int_\Omega |\nabla \varphi^+|^q\,dx}{\int_\Omega |\varphi^+|^q\,dx}, \frac{\int_\Omega |\nabla \varphi^-|^q\,dx}{\int_\Omega |\varphi^-|^q\,dx}\right\} : \varphi \in \mathrm{ES}(p, \lambda_2(p)) \setminus \{0\}\right\}, \tag{2.6}$$

where $\mathrm{ES}(p, \lambda_2(p))$ is the eigenspace of the second eigenvalue $\lambda_2(p)$ defined by (1.6).

The main properties of $\beta_1(\alpha)$ are collected in the following lemma.

Lemma 2.6. *The following assertions hold:*
(i) $\beta_1(\alpha) = -\infty$ *for any* $\alpha \le \lambda_2(p)$, *and* $\beta_1(\alpha) \in [\beta_1^*, +\infty)$ *for all* $\alpha > \lambda_2(p)$.
(ii) $\beta_1(\alpha)$ *is nondecreasing for* $\alpha \in (\lambda_2(p), +\infty)$.
(iii) $\beta_1(\alpha)$ *is left-continuous for* $\alpha \in (\lambda_2(p), +\infty)$.
(iv) $\beta_1(\alpha) \to +\infty$ *as* $\alpha \to +\infty$.
(v) $\mathcal{M}^1_{\alpha,\beta} \ne \emptyset$ *if and only if* $\alpha > \lambda_2(p)$ *and* $\beta < \beta_1(\alpha)$.

Proof. (i) If $\alpha \le \lambda_2(p)$, then $\mathcal{B}_1(\alpha) = \emptyset$ in view of Lemma 2.3, and hence $\beta_1(\alpha) = -\infty$. On the other hand, if $\alpha > \lambda_2(p)$, then any second eigenfunction $\varphi_{2,p}$ satisfies $H_\alpha(\varphi_{2,p}^\pm) < 0$ and, in consequence, it belongs to $\mathcal{B}_1(\alpha)$. This implies that $\mathrm{ES}(p, \lambda_2(p)) \setminus \{0\} \subset \mathcal{B}_1(\alpha)$ and $\beta_1(\alpha) \ge \beta_1^*$.

Consider the set

$$X(\alpha) := \{v \in W_0^{1,p} : \|\nabla v\|_p^p \le \alpha \|v\|_p^p\}. \tag{2.7}$$

It is known that for any $\alpha \in \mathbb{R}$ there exists $C(\alpha) > 0$ such that $\|\nabla v\|_p \le C(\alpha)\|v\|_q$ for all $v \in X(\alpha)$; see [31, Lemma 9]. Therefore, since $u^\pm \in X(\alpha)$ for any $u \in \mathcal{B}_1(\alpha)$, the Hölder inequality yields the existence of a constant $C_1 > 0$ such that

$$C_1\|\nabla u^\pm\|_q \le \|\nabla u^\pm\|_p \le C(\alpha)\|u^\pm\|_q \quad \text{for all } u \in \mathcal{B}_1(\alpha),$$

which gives the boundedness of $\beta_1(\alpha)$ from above.

(ii) If $\lambda_2(p) < \alpha_1 \le \alpha_2$, then $\mathcal{B}_1(\alpha_1) \subset \mathcal{B}_1(\alpha_2)$, which implies the desired monotonicity.

(iii) Let us fix an arbitrary $\alpha_0 > \lambda_2(p)$. Since assertion (ii) readily leads to $\lim_{\alpha\to\alpha_0-0}\beta_1(\alpha) \le \beta_1(\alpha_0)$, it is enough to show that $\lim_{\alpha\to\alpha_0-0}\beta_1(\alpha) \ge \beta_1(\alpha_0)$. By the definition of $\beta_1(\alpha_0)$, for any $\varepsilon > 0$ there exists $u_\varepsilon \in \mathcal{B}_1(\alpha_0)$ such that

$$\beta_1(\alpha_0) - \varepsilon \le \min\left\{\frac{\int_\Omega |\nabla u_\varepsilon^+|^q\,dx}{\int_\Omega |u_\varepsilon^+|^q\,dx}, \frac{\int_\Omega |\nabla u_\varepsilon^-|^q\,dx}{\int_\Omega |u_\varepsilon^-|^q\,dx}\right\}. \tag{2.8}$$

Recalling that $H_{\alpha_0}(u_\varepsilon^\pm) < 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that $H_\alpha(u_\varepsilon^\pm) < 0$ for any $\alpha \in (\alpha_0 - \delta, \alpha_0]$. Therefore, $u_\varepsilon \in \mathcal{B}_1(\alpha)$, and for all $\alpha \in (\alpha_0 - \delta, \alpha_0]$ the definition of $\beta_1(\alpha)$ leads to

$$\min\left\{\frac{\int_\Omega |\nabla u_\varepsilon^+|^q\,dx}{\int_\Omega |u_\varepsilon^+|^q\,dx}, \frac{\int_\Omega |\nabla u_\varepsilon^-|^q\,dx}{\int_\Omega |u_\varepsilon^-|^q\,dx}\right\} \le \beta_1(\alpha). \tag{2.9}$$

Combining (2.8) and (2.9), we obtain the inequality $\lim_{\alpha\to\alpha_0-0}\beta_1(\alpha) \ge \beta_1(\alpha_0)$, since $\varepsilon > 0$ is arbitrary.

(iv) Let $L > \lambda_1(q)$ be an arbitrary positive constant. Recalling that for the variational eigenvalues $\lambda_k(q)$ there holds $\lambda_k(q) \to +\infty$ as $k \to +\infty$, we can find $k_L \ge 2$ such that $\lambda_{k_L}(q) > L$. Take an eigenfunction φ corresponding to $\lambda_{k_L}(q)$. Since $\varphi \in C_0^{1,\gamma}(\overline{\Omega})$ and φ changes its sign in Ω (see Section 1.1), there exists α_L satisfying

$$\max\left\{\frac{\int_\Omega |\nabla \varphi^+|^p\,dx}{\int_\Omega |\varphi^+|^p\,dx}, \frac{\int_\Omega |\nabla \varphi^-|^p\,dx}{\int_\Omega |\varphi^-|^p\,dx}\right\} < \alpha_L.$$

Therefore, $\varphi \in \mathcal{B}_1(\alpha_L)$, and from the definition of $\beta_1(\alpha_L)$ and its monotonicity it follows that

$$\beta_1(\alpha) \geq \beta_1(\alpha_L) \geq \min\left\{\frac{\int_\Omega |\nabla \varphi^+|^q\,dx}{\int_\Omega |\varphi^+|^q\,dx}, \frac{\int_\Omega |\nabla \varphi^-|^q\,dx}{\int_\Omega |\varphi^-|^q\,dx}\right\} = \lambda_{k_L}(q) > L$$

provided $\alpha \geq \alpha_L$. Since L can be chosen arbitrary large, we conclude that $\lim_{\alpha\to+\infty} \beta_1(\alpha) = +\infty$.

(v) If $\alpha > \lambda_2(p)$ and $\beta < \beta_1(\alpha)$, then, by the definition of $\beta_1(\alpha)$, there exists $u \in \mathcal{B}_1(\alpha)$ such that

$$\beta < \min\left\{\frac{\int_\Omega |\nabla u^+|^q\,dx}{\int_\Omega |u^+|^q\,dx}, \frac{\int_\Omega |\nabla u^-|^q\,dx}{\int_\Omega |u^-|^q\,dx}\right\} \leq \beta_1(\alpha). \tag{2.10}$$

This means that $H_\alpha(u^\pm) < 0$ and $G_\beta(u^\pm) > 0$. Hence, by Lemma 2.1 we obtain $t^\pm > 0$ such that $t^\pm u^\pm \in \mathcal{N}_{\alpha,\beta}$, whence $t^+u^+ - t^-u^- \in \mathcal{M}^1_{\alpha,\beta}$.

Suppose now that there exists $u \in \mathcal{M}^1_{\alpha,\beta}$ for some $\alpha, \beta \in \mathbb{R}$. Lemma 2.3 implies that $\alpha > \lambda_2(p)$. On the other hand, $u \in \mathcal{M}^1_{\alpha,\beta} \subset \mathcal{B}_1(\alpha)$. Hence, from the Nehari constraints it follows that $G_\beta(u^\pm) > 0$, and we arrive to (2.10). □

Consider now the set $\mathcal{M}^2_{\alpha,\beta}$. The corresponding critical value, parametrized again by $\alpha \in \mathbb{R}$, appears to be the following:

$$\beta_2(\alpha) := \inf\left\{\max\left\{\frac{\int_\Omega |\nabla u^+|^q\,dx}{\int_\Omega |u^+|^q\,dx}, \frac{\int_\Omega |\nabla u^-|^q\,dx}{\int_\Omega |u^-|^q\,dx}\right\} : u \in \mathcal{B}_2(\alpha)\right\},$$

where the admissible set $\mathcal{B}_2(\alpha)$ is defined by (2.3).

The main properties of $\beta_2(\alpha)$ are similar to those for $\beta_1(\alpha)$ and collected in the following lemma.

Lemma 2.7. *The following assertions hold:*
(i) $\beta_2(\alpha) \in [\lambda_2(q), +\infty)$ *for any* $\alpha \in \mathbb{R}$.
(ii) $\beta_2(\alpha)$ *is nondecreasing for* $\alpha \in \mathbb{R}$, *and* $\beta_2(\alpha) = \beta_2(\lambda_1(p)) = \lambda_2(q)$ *for* $\alpha \leq \lambda_1(p)$.
(iii) $\beta_2(\alpha)$ *is right-continuous for* $\alpha \in \mathbb{R}$.
(iv) $\mathcal{M}^2_{\alpha,\beta} \neq \emptyset$ *if and only if* $\alpha \in \mathbb{R}$ *and* $\beta > \beta_2(\alpha)$.

Proof. (i) It is easy to see that for any $\alpha \in \mathbb{R}$ the admissible set $\mathcal{B}_2(\alpha)$ is nonempty. For example, any eigenfunction corresponding to $\lambda \in \sigma(-\Delta_p)$ belongs to $\mathcal{B}_2(\alpha)$ provided $\lambda > \max\{\alpha, \lambda_1(p)\}$. Hence, $\beta_2(\alpha) < +\infty$. On the other hand, the definition of $\beta_2(\alpha)$ and characterization (2.5) with $r = q$ directly imply that $\beta_2(\alpha) \geq \lambda_2(q)$ for any $\alpha \in \mathbb{R}$ since $\mathcal{B}_2(\alpha) \subset W_0^{1,p} \subset W_0^{1,q}$.

(ii) If $\alpha_1 \leq \alpha_2$, then $\mathcal{B}_2(\alpha_2) \subset \mathcal{B}_2(\alpha_1)$, which leads to the desired monotonicity. Since *any* sign-changing function $w \in W_0^{1,p}$ satisfies $H_{\lambda_1(p)}(w^\pm) > 0$ (see Lemma 1.2), we get $\mathcal{B}_2(\alpha) = \mathcal{B}_2(\lambda_1(p)) = \{u \in W_0^{1,p} : u^\pm \not\equiv 0\}$ for all $\alpha \leq \lambda_1(p)$, and hence $\beta_2(\alpha) = \beta_2(\lambda_1(p))$ for all $\alpha \leq \lambda_1(p)$. In order to show that $\beta_2(\lambda_1(p)) = \lambda_2(q)$, let us recall that any eigenfunction $\varphi_{2,q}$ corresponding to $\lambda_2(q)$ belongs to $C_0^{1,\gamma}(\overline{\Omega})$ (see Section 1.1). Hence, $\varphi_{2,q} \in \mathcal{B}_2(\lambda_1(p))$ and, consequently,

$$\lambda_2(q) = \max\left\{\frac{\int_\Omega |\nabla \varphi^+_{2,q}|^q\,dx}{\int_\Omega |\varphi^+_{2,q}|^q\,dx}, \frac{\int_\Omega |\nabla \varphi^-_{2,q}|^q\,dx}{\int_\Omega |\varphi^-_{2,q}|^q\,dx}\right\} \geq \beta_2(\lambda_1(p)) \geq \lambda_2(q),$$

where the equality follows from (2.5) with $r = q$, and the last inequality is given by assertion (i).

Assertions (iii) and (iv) can be proved in much the same way as in Lemma 2.6. □

For the further proof of the existence of nodal solutions to (GEV; α, β) in $\mathcal{M}^1_{\alpha,\beta}$, let us study the properties of the critical value (1.7) defined as

$$\beta_{\mathcal{L}}(\alpha) := \inf\left\{\min\left\{\frac{\int_\Omega |\nabla u^+|^q\,dx}{\int_\Omega |u^+|^q\,dx}, \frac{\int_\Omega |\nabla u^-|^q\,dx}{\int_\Omega |u^-|^q\,dx}\right\} : u \in \mathcal{B}_{\mathcal{L}}(\alpha)\right\},$$

where the admissible set $\mathcal{B}_{\mathcal{L}}(\alpha)$ is given by (1.8), or, equivalently,

$$\mathcal{B}_{\mathcal{L}}(\alpha) = \{u \in W_0^{1,p} : u^\pm \not\equiv 0,\ H_\alpha(u^+) \leq 0,\ H_\alpha(u^-) \leq 0\}.$$

We put $\beta_{\mathcal{L}}(\alpha) = +\infty$ whenever $\mathcal{B}_{\mathcal{L}}(\alpha) = \emptyset$. Arguing as in the proof of Lemma 2.3, it can be shown that $\mathcal{B}_{\mathcal{L}}(\alpha) = \emptyset$ if and only if $\alpha < \lambda_2(p)$. Note that $\mathcal{M}^1_{\alpha,\beta} \subset \mathcal{B}_1(\alpha) \subset \mathcal{B}_{\mathcal{L}}(\alpha)$.

First we give two auxiliary results.

Lemma 2.8. *Let $\alpha > 0$, $\beta \in \mathbb{R}$, and $\{u_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence in $\mathcal{B}_{\mathcal{L}}(\alpha)$ (or in $\mathcal{M}^1_{\alpha,\beta}$). Denote by $\{v_n\}_{n\in\mathbb{N}}$ a sequence normalized as follows:*

$$v_n := \frac{u_n^+}{\|\nabla u_n^+\|_p} - \frac{u_n^-}{\|\nabla u_n^-\|_p}, \qquad n \in \mathbb{N}. \tag{2.11}$$

Then the following assertions hold:
(i) *$v_n \in \mathcal{B}_{\mathcal{L}}(\alpha)$ (or $v_n \in \mathcal{M}^1_{\alpha,\beta}$) for all $n \in \mathbb{N}$.*
(ii) *v_n converges, up to a subsequence, to some $v_0 \in W_0^{1,p}$ weakly in $W_0^{1,p}$ and strongly in $L^p(\Omega)$.*
(iii) *$v_0^\pm \not\equiv 0$ and $H_\alpha(v_0^\pm) \le 0$, that is, $v_0 \in \mathcal{B}_{\mathcal{L}}(\alpha)$.*

Proof. Obviously, $v_n^\pm = u_n^\pm/\|\nabla u_n^\pm\|_p$, and hence assertion (i) follows from the p-homogeneity of H_α. Assertion (ii) is a consequence of the boundedness of $\{v_n\}_{n\in\mathbb{N}}$ in $W_0^{1,p}$. Since $H_\alpha(v_n^\pm) \le 0$ for all $n \in \mathbb{N}$, we get $\|v_n^\pm\|_p^p \ge 1/\alpha$, whence $v_0^\pm \not\equiv 0$ a.e. in Ω, due to the strong convergence of v_n in $L^p(\Omega)$. Moreover, using the weak lower semicontinuity of the $W_0^{1,p}$-norm, we conclude that $H_\alpha(v_0^\pm) \le \liminf_{n\to+\infty} H_\alpha(v_n^\pm) \le 0$. This is assertion (iii). □

Proposition 2.9. *For any $\alpha \ge \lambda_2(p)$ there exists a minimizer $u_\alpha \in \mathcal{B}_{\mathcal{L}}(\alpha)$ of $\beta_{\mathcal{L}}(\alpha)$.*

Proof. If $\alpha \ge \lambda_2(p)$, then $\mathcal{B}_{\mathcal{L}}(\alpha)$ is nonempty, since $H_\alpha(\varphi_{2,p}^\pm) \le 0$ for any second eigenfunction $\varphi_{2,p}$ corresponding to $\lambda_2(p)$. Thus, there exists a minimizing sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_{\mathcal{L}}(\alpha)$ for $\beta_{\mathcal{L}}(\alpha)$. Consider the corresponding normalized sequence $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_{\mathcal{L}}(\alpha)$ given by (2.11). Lemma 2.8 implies that the limit point $v_0 \in \mathcal{B}_{\mathcal{L}}(\alpha)$, and hence

$$\beta_{\mathcal{L}}(\alpha) \le \min\left\{\frac{\int_\Omega |\nabla v_0^+|^q\,dx}{\int_\Omega |v_0^+|^q\,dx}, \frac{\int_\Omega |\nabla v_0^-|^q\,dx}{\int_\Omega |v_0^-|^q\,dx}\right\} \le \liminf_{n\to+\infty} \min\left\{\frac{\int_\Omega |\nabla v_n^+|^q\,dx}{\int_\Omega |v_n^+|^q\,dx}, \frac{\int_\Omega |\nabla v_n^-|^q\,dx}{\int_\Omega |v_n^-|^q\,dx}\right\} = \beta_{\mathcal{L}}(\alpha),$$

which means that v_0 is a minimizer of $\beta_{\mathcal{L}}(\alpha)$. □

Remark 2.10. The definition (1.7) of $\beta_{\mathcal{L}}(\alpha)$ is equivalent to

$$\beta_{\mathcal{L}}(\alpha) := \inf\left\{\frac{\int_\Omega |\nabla u^+|^q\,dx}{\int_\Omega |u^+|^q\,dx} : u \in \mathcal{B}_{\mathcal{L}}(\alpha)\right\}. \tag{2.12}$$

This can be seen by testing $\beta_{\mathcal{L}}(\alpha)$ either with the corresponding minimizer u_α or with $-u_\alpha$.

Consider now the critical value

$$\beta_{\mathcal{L}}^* := \inf\left\{\frac{\int_\Omega |\nabla \varphi^+|^q\,dx}{\int_\Omega |\varphi^+|^q\,dx} : \varphi \in \mathrm{ES}(p, \lambda_2(p)) \setminus \{0\}\right\}. \tag{2.13}$$

The following lemma contains the main properties of $\beta_{\mathcal{L}}(\alpha)$.

Lemma 2.11. *The following assertions hold:*
(i) *$\beta_{\mathcal{L}}(\alpha) = +\infty$ for any $\alpha < \lambda_2(p)$, and $\beta_{\mathcal{L}}(\alpha) \in (\lambda_1(q), \beta_{\mathcal{L}}^*]$ for any $\alpha \ge \lambda_2(p)$.*
(ii) *$\beta_{\mathcal{L}}(\alpha)$ is nonincreasing for $\alpha \in [\lambda_2(p), +\infty)$.*
(iii) *$\beta_{\mathcal{L}}(\alpha)$ is right-continuous for $\alpha \in [\lambda_2(p), +\infty)$.*
(iv) *$\mathcal{K}_{\alpha,\beta} \ne \emptyset$ if and only if $\alpha \ge \lambda_2(p)$ and $\beta \ge \beta_{\mathcal{L}}(\alpha)$, where $\mathcal{K}_{\alpha,\beta}$ is defined by*

$$\begin{aligned}\mathcal{K}_{\alpha,\beta} &:= \{u \in W_0^{1,p} : u^\pm \not\equiv 0,\ H_\alpha(u^+) \le 0,\ H_\alpha(u^-) \le 0,\ G_\beta(u^+) \le 0\} \\ &= \mathcal{B}_{\mathcal{L}}(\alpha) \cap \{u \in W_0^{1,p} : G_\beta(u^+) \le 0\}.\end{aligned} \tag{2.14}$$

Proof. (i) As stated in the proof of Lemma 2.3, we easily see that $\mathcal{B}_{\mathcal{L}}(\alpha) = \emptyset$ for all $\alpha < \lambda_2(p)$, and hence $\beta_{\mathcal{L}}(\alpha) = +\infty$. If $\alpha \ge \lambda_2(p)$, then $\mathrm{ES}(p, \lambda_2(p)) \setminus \{0\} \subset \mathcal{B}_{\mathcal{L}}(\alpha)$, and using (2.12), we obtain that $\beta_{\mathcal{L}}(\alpha) \le \beta_{\mathcal{L}}^*$. Since any sign-changing function $w \in W_0^{1,p}$ satisfies $\|\nabla w^\pm\|_q^q > \lambda_1(q)\|w^\pm\|_q^q$ (see Lemma 1.2), taking a minimizer u_α of $\beta_{\mathcal{L}}(\alpha)$ (see Proposition 2.9), we conclude that

$$\beta_{\mathcal{L}}(\alpha) = \|\nabla u_\alpha^+\|_q^q / \|u_\alpha^+\|_q^q > \lambda_1(q) \quad \text{for all } \alpha \geq \lambda_2(p).$$

Assertion (ii) can be proved as in Lemma 2.6.

(iii) Due to assertion (ii), it is sufficient to show that $\beta_{\mathcal{L}}(\alpha_0) \leq \lim_{\alpha\to\alpha_0+0} \beta_{\mathcal{L}}(\alpha)$ for all $\alpha_0 \geq \lambda_2(p)$. Since $\beta_{\mathcal{L}}(\alpha)$ is monotone and bounded in a right neighborhood of α_0, for any decreasing sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ such that $\alpha_n \to \alpha_0 + 0$ as $n \to +\infty$ there holds

$$\lim_{n\to+\infty} \beta_{\mathcal{L}}(\alpha_n) = \lim_{\alpha\to\alpha_0+0} \beta_{\mathcal{L}}(\alpha).$$

According to Proposition 2.9, for each $n \in \mathbb{N}$ there exists a minimizer $u_n \in \mathcal{B}_{\mathcal{L}}(\alpha_n)$ of $\beta_{\mathcal{L}}(\alpha_n)$, and we can assume that $\|\nabla u_n^\pm\|_p = 1$. Thus, passing to an appropriate subsequence, u_n converges to some $u_0 \in W_0^{1,p}$ weakly in $W_0^{1,p}$ and strongly in $L^p(\Omega)$. Moreover, $u_0^\pm \not\equiv 0$ in Ω since $H_{\alpha_n}(u_n^\pm) \leq 0$ implies that $\|u_n^\pm\|_p^p \geq 1/\alpha_n$. Furthermore, due to the weak lower semicontinuity of the $W_0^{1,p}$-norm, we have $H_{\alpha_0}(u_0^\pm) \leq 0$, and hence $u_0 \in \mathcal{B}_{\mathcal{L}}(\alpha_0)$. Consequently, using (2.12), we conclude that

$$\beta_{\mathcal{L}}(\alpha_0) \leq \frac{\int_\Omega |\nabla u_0^+|^q\,dx}{\int_\Omega |u_0^+|^q\,dx} \leq \liminf_{n\to+\infty} \frac{\int_\Omega |\nabla u_n^+|^q\,dx}{\int_\Omega |u_n^+|^q\,dx} = \liminf_{n\to+\infty} \beta_{\mathcal{L}}(\alpha_n) = \lim_{\alpha\to\alpha_0+0} \beta_{\mathcal{L}}(\alpha).$$

(iv) Assume that $\alpha \geq \lambda_2(p)$ and $\beta \geq \beta_{\mathcal{L}}(\alpha)$. Let $u \in \mathcal{B}_{\mathcal{L}}(\alpha)$ be a minimizer of $\beta_{\mathcal{L}}(\alpha)$. Then $H_\alpha(u^\pm) \leq 0$ and, in view of (2.12), we may suppose that $G_{\beta_{\mathcal{L}}(\alpha)}(u^+) = 0$. Therefore, $G_\beta(u^+) \leq G_{\beta_{\mathcal{L}}(\alpha)}(u^+) = 0$ and hence $u \in \mathcal{K}_{\alpha,\beta}$.

Suppose now that there exists $u \in \mathcal{K}_{\alpha,\beta}$ for some $\alpha, \beta \in \mathbb{R}$. Since $\mathcal{K}_{\alpha,\beta} \subset \mathcal{B}_{\mathcal{L}}(\alpha)$, assertion (i) implies that $\alpha \geq \lambda_2(p)$. Moreover, since $G_\beta(u^+) \leq 0$, the definition of $\beta_{\mathcal{L}}(\alpha)$ leads to

$$\beta_{\mathcal{L}}(\alpha) \leq \min\left\{\frac{\int_\Omega |\nabla u^+|^q\,dx}{\int_\Omega |u^+|^q\,dx}, \frac{\int_\Omega |\nabla u^-|^q\,dx}{\int_\Omega |u^-|^q\,dx}\right\} \leq \frac{\int_\Omega |\nabla u^+|^q\,dx}{\int_\Omega |u^+|^q\,dx} \leq \beta,$$

which completes the proof. □

In the sequel, it will be convenient to use the notation

$$\Sigma_{\mathcal{L}} := \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha > \lambda_2(p),\ \beta < \beta_{\mathcal{L}}(\alpha)\}. \tag{2.15}$$

Remark 2.12. Due to Lemmas 2.6 and 2.11, the definitions of β_1^* and $\beta_{\mathcal{L}}^*$ (see (2.6) and (2.13)) imply that $\beta_{\mathcal{L}}(\alpha) \leq \beta_{\mathcal{L}}^* \leq \beta_1^* \leq \beta_1(\alpha)$ for all $\alpha > \lambda_2(p)$, and hence $\mathcal{M}_{\alpha,\beta}^1 \neq \emptyset$ for any $(\alpha, \beta) \in \Sigma_{\mathcal{L}}$.

Remark 2.13. In the one-dimensional case we have

$$\beta_1^* = \beta_{\mathcal{L}}^* \in (\lambda_2(q), \lambda_4(q)). \tag{2.16}$$

Indeed, if $\Omega = (0, T)$, then the second eigenfunction $\varphi_{2,p}$ is given explicitly through the first eigenfunction φ_p by $\varphi_{2,p}(x) = \varphi_p(2x)$ for $x \in (0, T/2]$, and $\varphi_{2,p}(x) = -\varphi_p(2x - T)$ for $x \in (T/2, T)$ (see Appendix A). Hence, Lemma A.2 in Appendix A implies that

$$\frac{\|(\varphi_{2,p}^+)'\|_q^q}{\|\varphi_{2,p}^+\|_q^q} = 2^q \frac{\|\varphi_p'\|_q^q}{\|\varphi_p\|_q^q} \in (2^q\lambda_1(q), 2^q\lambda_2(q)) = (\lambda_2(q), \lambda_4(q)),$$

and, consequently, (2.16) holds.

2.2 Existence of positive energy nodal solutions

In this subsection, we prove the existence of nodal solutions in the set $\Sigma_{\mathcal{L}}$ defined by (2.15). To this end, we consider the minimization of the energy functional $E_{\alpha,\beta}$ over the set $\mathcal{M}_{\alpha,\beta}^1$.

First, we prepare the following auxiliary lemma.

Lemma 2.14. *Let $\{u_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence in $\mathcal{M}^1_{\alpha,\beta}$ and let $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{M}^1_{\alpha,\beta}$ be a corresponding normalized sequence given by* (2.11) *in Lemma 2.8. If $\|\nabla u_n^+\|_p \to +\infty$ as $n \to +\infty$, and $\{E_{\alpha,\beta}(u_n^+)\}_{n\in\mathbb{N}}$ is bounded from above, then $G_\beta(v_0^+) \le 0$. Consequently, $v_0 \in \mathcal{K}_{\alpha,\beta}$.*

Proof. Assume that $\{E_{\alpha,\beta}(u_n^+)\}_{n\in\mathbb{N}}$ is bounded from above. Recalling that $-G_\beta(u_n^\pm) = H_\alpha(u_n^\pm) < 0$ by $u_n \in \mathcal{M}^1_{\alpha,\beta}$ and noting that the equalities

$$E_{\alpha,\beta}(u) = \frac{p-q}{pq} G_\beta(u) = -\frac{p-q}{pq} H_\alpha(u) \tag{2.17}$$

hold for all $u \in \mathcal{N}_{\alpha,\beta}$, we get the boundedness of $G_\beta(u_n^+)$:

$$0 < \frac{p-q}{pq} G_\beta(u_n^+) = E_{\alpha,\beta}(u_n^+) \le \sup_{l\in\mathbb{N}} E_{\alpha,\beta}(u_l^+) < +\infty.$$

Consequently, the weak lower semicontinuity and the assumption that $\|\nabla u_n^+\|_p \to +\infty$ as $n \to +\infty$ imply

$$G_\beta(v_0^+) \le \liminf_{n\to+\infty} G_\beta(v_n^+) = \liminf_{n\to+\infty} \frac{G_\beta(u_n^+)}{\|\nabla u_n^+\|_p^q} = 0.$$

Combining this inequality with the fact that $v_0 \in \mathcal{B}_{\mathcal{L}}(\alpha)$ (see Lemma 2.8), we conclude that $v_0 \in \mathcal{K}_{\alpha,\beta}$. □

From Remark 2.12 we know that $\mathcal{M}^1_{\alpha,\beta} \ne \emptyset$ for any $(\alpha,\beta) \in \Sigma_{\mathcal{L}}$. Hence, there exists a minimizing sequence for $E_{\alpha,\beta}$ over $\mathcal{M}^1_{\alpha,\beta}$. Moreover, this minimizing sequence, in fact, converges.

Theorem 2.15. *Let $(\alpha,\beta) \in \Sigma_{\mathcal{L}}$. Then there exists a minimizer $u \in \mathcal{M}^1_{\alpha,\beta}$ of $E_{\alpha,\beta}$ over $\mathcal{M}^1_{\alpha,\beta}$.*

Proof. Assume $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{M}^1_{\alpha,\beta}$ to be a minimizing sequence for $E_{\alpha,\beta}$ over $\mathcal{M}^1_{\alpha,\beta}$. Equalities (2.17) imply that $E_{\alpha,\beta}(u_n^\pm) > 0$, and hence $\{E_{\alpha,\beta}(u_n)\}_{n\in\mathbb{N}}$ and $\{E_{\alpha,\beta}(u_n^\pm)\}_{n\in\mathbb{N}}$ are bounded. Applying Lemma 2.14, we conclude that if $\|\nabla u_n^+\|_p \to +\infty$ as $n \to +\infty$, then the limit v_0 of a normalized sequence (2.11) belongs to the set $\mathcal{K}_{\alpha,\beta}$ defined by (2.14). However, $\mathcal{K}_{\alpha,\beta} = \emptyset$ for all $(\alpha,\beta) \in \Sigma_{\mathcal{L}}$, due to Lemma 2.11 (iv). This is a contradiction. Thus, $\{u_n^+\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}$. Since $\{-u_n\}_{n\in\mathbb{N}}$ is also a minimizing sequence for $E_{\alpha,\beta}$ over $\mathcal{M}^1_{\alpha,\beta}$, we apply the same arguments to derive that $(-u_n)^+ \equiv u_n^-$ is bounded in $W_0^{1,p}$, which finally yields the boundedness of the whole sequence $\{u_n\}_{n\in\mathbb{N}}$.

Let us now show that $\|\nabla u_n^+\|_p$ and $\|\nabla u_n^-\|_p$ do not converge to zero. Applying assertions (ii) and (iii) of Lemma 2.8 to the corresponding normalized sequence $\{v_n\}_{n\in\mathbb{N}}$ given by (2.11), we see that its limit point v_0 belongs to $\mathcal{B}_{\mathcal{L}}(\alpha)$. Suppose, by contradiction, that $\|\nabla u_n^+\|_p \to 0$ as $n \to +\infty$. Then, using the Nehari constraints, we get

$$0 < G_\beta(v_n^+) = -\|\nabla u_n^+\|_p^{p-q} H_\alpha(v_n^+) \to 0 \quad \text{as } n \to +\infty$$

since H_α is bounded on a bounded set and $\|\nabla v_n^+\|_p = 1$. Consequently, $G_\beta(v_0^+) \le \liminf_{n\to+\infty} G_\beta(v_n^\pm) = 0$ and $H_\alpha(v_0^\pm) \le \liminf_{n\to+\infty} H_\alpha(v_n^\pm) \le 0$, i.e., $v_0 \in \mathcal{K}_{\alpha,\beta}$, and we obtain a contradiction as above. In the case $\|\nabla u_n^-\|_p \to 0$, we consider $-u_n$ instead of u_n, and again obtain a contradiction. As a result, there holds

$$\delta^+ := \inf_{n\in\mathbb{N}} \|\nabla u_n^+\|_p^p > 0 \quad \text{and} \quad \delta^- := \inf_{n\in\mathbb{N}} \|\nabla u_n^-\|_p^p > 0. \tag{2.18}$$

Now, choosing an appropriate subsequence, we get $u_n \rightharpoonup u_0$ weakly in $W_0^{1,p}$ and $u_n \to u_0$ strongly in $L^p(\Omega)$, where $u_0 \in W_0^{1,p}$. Inequalities (2.18) together with $H_\alpha(u_n^\pm) < 0$ imply that $\|u_n^\pm\|_p^p \ge \delta^\pm/\alpha$ for all $n \in \mathbb{N}$, and hence $u_0^\pm \not\equiv 0$. At the same time, the weak lower semicontinuity yields

$$H_\alpha(u_0^\pm) \le \liminf_{n\to+\infty} H_\alpha(u_n^\pm) \le 0. \tag{2.19}$$

Let us show that

$$H_\alpha(u_0^+) < 0 < G_\beta(u_0^+) \quad \text{and} \quad H_\alpha(u_0^-) < 0 < G_\beta(u_0^-). \tag{2.20}$$

Indeed, since $\mathcal{K}_{\alpha,\beta}$ is empty for $(\alpha,\beta) \in \Sigma_{\mathcal{L}}$, we see that $u_0^+ - u_0^- \notin \mathcal{K}_{\alpha,\beta}$ and $u_0^- - u_0^+ \notin \mathcal{K}_{\alpha,\beta}$. This leads to $G_\beta(u_0^\pm) > 0$ since $H_\alpha(u_0^\pm) \le 0$ by (2.19). Finally, from the Nehari constraints and the weak lower semicontinuity we derive that

$$H_\alpha(u_0^\pm) + G_\beta(u_0^\pm) \le \liminf_{n\to+\infty} \big(H_\alpha(u_n^\pm) + G_\beta(u_n^\pm)\big) = 0.$$

This means that $H_\alpha(u_0^\pm) \le -G_\beta(u_0^\pm) < 0$, and hence (2.20) is shown.

According to (2.20), Lemma 2.1 implies the existence of unique maximum points $t_0^+ > 0$ of $E_{\alpha,\beta}(tu_0^+)$ and $t_0^- > 0$ of $E_{\alpha,\beta}(tu_0^-)$ with respect to $t > 0$, and $t_0^\pm u_0^\pm \in \mathcal{N}_{\alpha,\beta}$. Accordingly, we conclude from (2.20) that $t_0^+ u_0^+ - t_0^- u_0^- \in \mathcal{M}^1_{\alpha,\beta}$. Therefore,

$$\begin{aligned}\inf_{\mathcal{M}^1_{\alpha,\beta}} E_{\alpha,\beta} \le E_{\alpha,\beta}(t_0^+ u_0^+ - t_0^- u_0^-) &\le \liminf_{n\to+\infty} E_{\alpha,\beta}(t_0^+ u_n^+ - t_0^- u_n^-)\\ &= \liminf_{n\to+\infty}(E_{\alpha,\beta}(t_0^+ u_n^+) + E_{\alpha,\beta}(t_0^- u_n^-))\\ &\le \liminf_{n\to+\infty}(E_{\alpha,\beta}(u_n^+) + E_{\alpha,\beta}(u_n^-)) = \liminf_{n\to+\infty} E_{\alpha,\beta}(u_n) = \inf_{\mathcal{M}^1_{\alpha,\beta}} E_{\alpha,\beta}.\end{aligned}$$

The last inequality in this formula is due to the fact that $\max_{t>0} E_{\alpha,\beta}(tu_n^\pm) = E_{\alpha,\beta}(u_n^\pm)$; see Lemma 2.1. Consequently, $t_0^+ u_0^+ - t_0^- u_0^- \in \mathcal{M}^1_{\alpha,\beta}$ is the minimizer of $E_{\alpha,\beta}$ over $\mathcal{M}^1_{\alpha,\beta}$. □

Lemma 2.16. *Let $(\alpha, \beta) \in \Sigma_{\mathcal{L}}$. If $u \in \mathcal{M}^1_{\alpha,\beta}$ is a minimizer of $E_{\alpha,\beta}$ over $\mathcal{M}^1_{\alpha,\beta}$, then u is a critical point of $E_{\alpha,\beta}$ on $W_0^{1,p}$.*

Proof. The proof can be handled in much the same way as the proof of [9, Lemma 3.2], where a variant of the deformation lemma was used in a framework of the problem with indefinite nonlinearities; see also [6, Proposition 3.1]. □

2.3 Qualitative properties

In this subsection, we show that *any* minimizer u of $E_{\alpha,\beta}$ over $\mathcal{M}^1_{\alpha,\beta}$ for $(\alpha, \beta) \in \Sigma_{\mathcal{L}}$ has exactly two nodal domains (that is, connected components of $\Omega \setminus u^{-1}(0)$).

Lemma 2.17. *Let $(\alpha, \beta) \in \Sigma_{\mathcal{L}}$ and let $u \in \mathcal{M}^1_{\alpha,\beta}$ be a minimizer of $E_{\alpha,\beta}$ over $\mathcal{M}^1_{\alpha,\beta}$. Then u has exactly two nodal domains.*

Proof. Suppose, contrary to our claim, that there exists a minimizer $u \in \mathcal{M}^1_{\alpha,\beta}$ of $E_{\alpha,\beta}$ over $\mathcal{M}^1_{\alpha,\beta}$ with (at least) three nodal domains. We decompose u such that $u = u_1 + u_2 + u_3$, where $u_i \not\equiv 0$ for $i = 1, 2, 3$, and each u_i is of a constant sign on its support. Note that each $u_i \in \mathcal{N}_{\alpha,\beta}$. Indeed, $u_i \in W_0^{1,p}$ (cf. [16, Lemma 5.6]), and since u is a solution of (GEV; α, β) by Lemma 2.16, we obtain

$$0 = \langle E'_{\alpha,\beta}(u), u_i\rangle = H_\alpha(u_i) + G_\beta(u_i) \quad \text{for } i = 1, 2, 3. \tag{2.21}$$

Assume, without loss of generality, that $u^+ = u_1 + u_2$ and $u^- = -u_3$. Since $u \in \mathcal{M}^1_{\alpha,\beta}$, we have

$$H_\alpha(u^+) = H_\alpha(u_1) + H_\alpha(u_2) < 0 \quad \text{and} \quad H_\alpha(u^-) = H_\alpha(-u_3) = H_\alpha(u_3) < 0.$$

Moreover, we may assume that $H_\alpha(u_2) \le H_\alpha(u_1)$, whence $H_\alpha(u_2) < 0$. This assumption splits into the following four cases:
(i) $H_\alpha(u_2) \le H_\alpha(u_1) < 0$.
(ii) $H_\alpha(u_2) < H_\alpha(u_1) = 0$.
(iii) $H_\alpha(u_2) < 0 < H_\alpha(u_1)$ and $H_\alpha(u_1) + H_\alpha(u_3) \ge 0$.
(iv) $H_\alpha(u_2) < 0 < H_\alpha(u_1)$ and $H_\alpha(u_1) + H_\alpha(u_3) < 0$.
Now we will subsequently show a contradiction for each case.

(i) It is easy to see that $u_1 + u_3 \in \mathcal{M}^1_{\alpha,\beta}$. Since $H_\alpha(u_2) < 0$ leads to $E_{\alpha,\beta}(u_2) > 0$, we have a contradiction by the following inequality:

$$\inf_{\mathcal{M}^1_{\alpha,\beta}} E_{\alpha,\beta} \le E_{\alpha,\beta}(u_1 + u_3) < E_{\alpha,\beta}(u_1 + u_3) + E_{\alpha,\beta}(u_2) = E_{\alpha,\beta}(u_1 + u_2 + u_3) = \inf_{\mathcal{M}^1_{\alpha,\beta}} E_{\alpha,\beta}.$$

(ii) Since $H_\alpha(u_1) = 0$, we can derive from (2.21) that $G_\beta(u_1) = 0$. Recalling that $H_\alpha(u_2) < 0$, we get $u_1 - u_2 \in \mathcal{K}_{\alpha,\beta}$, which contradicts assertion (iv) of Lemma 2.11.

(iii) Recall $H_\alpha(u_3) < 0$ and set

$$1 \leq t_0^p := -\frac{H_\alpha(u_1)}{H_\alpha(u_3)} = -\frac{G_\beta(u_1)}{G_\beta(u_3)}.$$

Since $u_1, u_3 \in \mathcal{N}_{\alpha,\beta}$, we obtain

$$H_\alpha(u_1 - t_0 u_3) = H_\alpha(u_1) + t_0^p H_\alpha(u_3) = G_\beta(u_1) + t_0^p G_\beta(u_3) = 0.$$

On the other hand, since $G_\beta(u_3) > 0$, $t_0 \geq 1$, and $p > q$, we have

$$0 = G_\beta(u_1) + t_0^p G_\beta(u_3) \geq G_\beta(u_1) + t_0^q G_\beta(u_3) = G_\beta(u_1 - t_0 u_3).$$

Consequently, $H_\alpha(u_1 - t_0 u_3) = 0$ and $G_\beta(u_1 - t_0 u_3) \leq 0$. Considering a function $w = u_1 - t_0 u_3 - u_2$, we get $w^+ = u_1 - t_0 u_3$ and $w^- = u_2$, which implies that $w \in \mathcal{K}_{\alpha,\beta}$. This is again a contradiction to the emptiness of $\mathcal{K}_{\alpha,\beta}$.

(iv) Consider a function $w = u_1 - u_3 - u_2$. Then $w^+ = u_1 - u_3$ and $w^- = u_2$. By the assumptions, we have $H_\alpha(w^\pm) < 0$. Therefore, $w \in \mathcal{M}^1_{\alpha,\beta}$ and

$$E_{\alpha,\beta}(w) = E_{\alpha,\beta}(u_1 - u_3 - u_2) = E_{\alpha,\beta}(u_1) + E_{\alpha,\beta}(u_3) + E_{\alpha,\beta}(u_2) = E_{\alpha,\beta}(u) = \inf_{\mathcal{M}^1_{\alpha,\beta}} E_{\alpha,\beta},$$

that is, w is also a minimizer of $E_{\alpha,\beta}$ over $\mathcal{M}^1_{\alpha,\beta}$ and hence a weak solution of (GEV; α, β) in view of Lemma 2.16. This implies that for any $\xi \in W_0^{1,p}$ there holds

$$\begin{aligned}\int_\Omega |\nabla w|^{p-2}\nabla(u_1 - u_3 - u_2)\nabla\xi\,dx &+ \int_\Omega |\nabla w|^{q-2}\nabla(u_1 - u_3 - u_2)\nabla\xi\,dx \\ &= \alpha\int_\Omega |w|^{p-2}(u_1 - u_3 - u_2)\xi\,dx + \beta\int_\Omega |w|^{q-2}(u_1 - u_3 - u_2)\xi\,dx. \end{aligned} \tag{2.22}$$

On the other hand, since $u = u_1 + u_2 + u_3$ is also a weak solution of (GEV; α, β), we obtain

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2}\nabla(u_1 + u_3 + u_2)\nabla\xi\,dx &+ \int_\Omega |\nabla u|^{q-2}\nabla(u_1 + u_3 + u_2)\nabla\xi\,dx \\ &= \alpha\int_\Omega |u|^{p-2}(u_1 + u_3 + u_2)\xi\,dx + \beta\int_\Omega |u|^{q-2}(u_1 + u_3 + u_2)\xi\,dx \end{aligned} \tag{2.23}$$

for all $\xi \in W_0^{1,p}$. Summarizing (2.22) and (2.23) and noting that $|u| \equiv |w|$ and $|\nabla u| \equiv |\nabla w|$, we get

$$\int_\Omega |\nabla u_1|^{p-2}\nabla u_1\nabla\xi\,dx + \int_\Omega |\nabla u_1|^{q-2}\nabla u_1\nabla\xi\,dx = \alpha\int_\Omega |u_1|^{p-2}u_1\xi\,dx + \beta\int_\Omega |u_1|^{q-2}u_1\xi\,dx$$

for each $\xi \in W_0^{1,p}$, since the supports of u_i are mutually disjoint. This means that u_1 is a nonnegative solution of (GEV; α, β) *in* Ω. However, the strong maximum principle implies that $u_1 > 0$ in Ω; cf. [10, Remark 1, p. 3284]. Hence, $u_2 \equiv 0$ and $u_3 \equiv 0$, which is a contradiction. □

3 Nodal solutions with negative energy

In this section, we provide the main ingredients for the proofs of Theorems 1.6 and 1.7.

3.1 Auxiliary results

Consider the set

$$Y(\lambda) := \{u \in W_0^{1,p} : \|\nabla u\|_p^p \geq \lambda\|u\|_p^p\},$$

where $\lambda \geq 0$. Hereinafter, by S_+^k we denote the closed unit upper hemisphere in $\mathbb{R}^{k+1}$ with the boundary S^{k-1}. We begin with the following linking lemma.

Lemma 3.1. *Let $k \in \mathbb{N}$. Then $h(S_+^k) \cap Y(\lambda_{k+1}(p)) \neq \emptyset$ for any $h \in C(S_+^k, W_0^{1,p})$ provided $h|_{S^{k-1}}$ is odd.*

Proof. Fix any $h \in C(S_+^k, W_0^{1,p})$ such that $h|_{S^{k-1}}$ is odd. If $\|u\|_p = 0$ for some $u \in h(S_+^k)$, then, obviously, $u \in Y(\lambda_{k+1}(p))$. Thus, we may assume that $\|u\|_p > 0$ for every $u \in h(S_+^k)$. Define the map

$$\tilde{h}(z) := \begin{cases} h(z)/\|h(z)\|_p & \text{if } z \in S_+^k, \\ -h(-z)/\|h(-z)\|_p & \text{if } z \in S_-^k. \end{cases}$$

It is not hard to see that $\tilde{h} \in \mathscr{F}_{k+1}(p)$, where $\mathscr{F}_{k+1}(p)$ is the set given by (1.4) with $r = p$. By the definition (1.3) of $\lambda_{k+1}(p)$, there exists $z_0 \in S^k$ such that $\|\nabla \tilde{h}(z_0)\|_p^p \geq \lambda_{k+1}(p)$. Since $\tilde{h}(z_0) \in S(p)$, we have $\|\nabla \tilde{h}(z_0)\|_p^p \geq \lambda_{k+1}(p)\|\tilde{h}(z_0)\|_p^p$. Moreover, since $\tilde{h}$ is odd, we may suppose that $z_0 \in S_+^k$. Consequently, we obtain $h(z_0) \in Y(\lambda_{k+1}(p))$. □

Lemma 3.2. *Let $\alpha, \beta \in \mathbb{R}$ and let $\lambda > \max\{\alpha, 0\}$. Then $E_{\alpha,\beta}$ is bounded from below on $Y(\lambda)$.*

Proof. Assume that $u \in Y(\lambda)$ with $\lambda > \max\{\alpha, 0\}$. Using the Hölder inequality, we obtain

$$E_{\alpha,\beta}(u) \geq \frac{\lambda - \alpha}{p\lambda}\|\nabla u\|_p^p - \frac{\beta}{q}|\Omega|^{\frac{p-q}{p}}\|u\|_p^q \geq \frac{\lambda - \alpha}{p\lambda}\|\nabla u\|_p^p - \frac{\beta}{q(\lambda_1(p))^{q/p}}|\Omega|^{\frac{p-q}{p}}\|\nabla u\|_p^q,$$

which implies the desired conclusion since $q < p$. □

Lemma 3.3. *Assume $\alpha, \beta \in \mathbb{R}$ and let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $W_0^{1,p}$ which satisfies $\|\nabla u_n\|_p \to +\infty$ and $E'_{\alpha,\beta}(u_n)/\|\nabla u_n\|_p^{p-1} \to 0$ in $(W_0^{1,p})^*$ as $n \to +\infty$. Then $v_n := u_n/\|\nabla u_n\|_p$ has a subsequence strongly convergent in $W_0^{1,p}$ to some $v_0 \in \mathrm{ES}(p; \alpha) \setminus \{0\}$, that is, $\alpha \in \sigma(-\Delta_p)$.*

Proof. Since $\|\nabla v_n\|_p = 1$ for any $n \in \mathbb{N}$, passing to an appropriate subsequence, we may assume that v_n converges to some v_0 weakly in $W_0^{1,p}$ and strongly in $L^p(\Omega)$. In particular, $\langle H'_\alpha(v_0), v_n\rangle \to \langle H'_\alpha(v_0), v_0\rangle$ as $n \to +\infty$. Moreover,

$$\frac{|\langle E'_{\alpha,\beta}(u_n), v_n - v_0\rangle|}{\|\nabla u_n\|_p^{p-1}} \leq \frac{\|E'_{\alpha,\beta}(u_n)\|_{(W_0^{1,p})^*}}{\|\nabla u_n\|_p^{p-1}}\|\nabla(v_n - v_0)\|_p \leq 2\frac{\|E'_{\alpha,\beta}(u_n)\|_{(W_0^{1,p})^*}}{\|\nabla u_n\|_p^{p-1}} \to 0$$

as $n \to +\infty$, by the assumption. Using these facts, we get

$$\begin{aligned} o(1) &= \left\langle \frac{E'_{\alpha,\beta}(u_n)}{\|\nabla u_n\|_p^{p-1}} - H'_\alpha(v_0), v_n - v_0 \right\rangle \\ &= \int_\Omega (|\nabla v_n|^{p-2}\nabla v_n - |\nabla v_0|^{p-2}\nabla v_0)(\nabla v_n - \nabla v_0)\,dx - \alpha \int_\Omega (|v_n|^{p-2}v_n - |v_0|^{p-2}v_0)(v_n - v_0)\,dx \\ &\quad + \frac{1}{\|\nabla u_n\|_p^{p-q}} \int_\Omega |\nabla v_n|^{q-2}\nabla v_n(\nabla v_n - \nabla v_0)\,dx - \frac{\beta}{\|\nabla u_n\|_p^{p-q}} \int_\Omega |v_n|^{q-2}v_n(v_n - v_0)\,dx \\ &= \int_\Omega (|\nabla v_n|^{p-2}\nabla v_n - |\nabla v_0|^{p-2}\nabla v_0)(\nabla v_n - \nabla v_0)\,dx + o(1) \\ &\geq (\|\nabla v_n\|_p^{p-1} - \|\nabla v_0\|_p^{p-1})(\|\nabla v_n\|_p - \|\nabla v_0\|_p) + o(1), \end{aligned}$$

where the last inequality is obtained by Hölder's inequality. Hence, $\|\nabla v_n\|_p \to \|\nabla v_0\|_p = 1$ as $n \to +\infty$, and the uniform convexity of $W_0^{1,p}$ implies that v_n converges to v_0 strongly in $W_0^{1,p}$.

On the other hand, for any $\xi \in W_0^{1,p}$ the following equality holds:

$$\begin{aligned} \left\langle \frac{E'_{\alpha,\beta}(u_n)}{\|\nabla u_n\|_p^{p-1}}, \xi \right\rangle &= \int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla \xi\,dx - \alpha \int_\Omega |v_n|^{p-2}v_n \xi\,dx \\ &\quad + \frac{1}{\|\nabla u_n\|_p^{p-q}} \int_\Omega |\nabla v_n|^{q-2}\nabla v_n \nabla \xi\,dx - \frac{\beta}{\|\nabla u_n\|_p^{p-q}} \int_\Omega |v_n|^{q-2}v_n \xi\,dx. \end{aligned}$$

Therefore, passing to the limit as $n \to +\infty$, we derive

$$\int_\Omega |\nabla v_0|^{p-2} \nabla v_0 \nabla \xi \, dx - \alpha \int_\Omega |v_0|^{p-2} v_0 \xi \, dx = 0$$

for all $\xi \in W_0^{1,p}$, that is, $v_0 \in \mathrm{ES}(p;\alpha) \setminus \{0\}$. □

Lemma 3.4. *If $\alpha \notin \sigma(-\Delta_p)$, then $E_{\alpha,\beta}$ satisfies the Palais–Smale condition.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p}$ be a Palais–Smale sequence for $E_{\alpha,\beta}$, that is,

$$E_{\alpha,\beta}(u_n) \to c \quad \text{and} \quad \|E'_{\alpha,\beta}(u_n)\|_{(W_0^{1,p})^*} \to 0$$

as $n \to +\infty$, where c is a constant. Due to the (S_+)-property for the operator $-\Delta_p - \Delta_q$ (see Remark 3.5 below), it is sufficient to show that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}$. If we suppose, by contradiction, that $\|\nabla u_n\|_p \to +\infty$ as $n \to +\infty$, then Lemma 3.3 implies that $\alpha \in \sigma(-\Delta_p)$, which contradicts the assumption of the lemma. □

Remark 3.5. For the reader's convenience we show that the operator $-\Delta_p - \Delta_q$ has the (S_+)-property, namely, any sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p}$ converging to some u_0 weakly in $W_0^{1,p}$ and satisfying

$$\limsup_{n\to+\infty} \langle -\Delta_p u_n - \Delta_q u_n, u_n - u_0 \rangle \le 0 \tag{3.1}$$

converges strongly in $W_0^{1,p}$. Let $u_n \rightharpoonup u_0$ in $W_0^{1,p}$ as $n \to +\infty$, and let (3.1) hold. Then the Hölder inequality yields

$$\begin{aligned}
&\langle -\Delta_p u_n - \Delta_q u_n, u_n - u_0 \rangle + o(1) \\
&\quad = \langle -\Delta_p u_n - \Delta_q u_n, u_n - u_0 \rangle - \langle -\Delta_p u_0 - \Delta_q u_0, u_n - u_0 \rangle \\
&\quad = \int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_0|^{p-2}\nabla u_0)(\nabla u_n - \nabla u_0)\, dx + \int_\Omega (|\nabla u_n|^{q-2}\nabla u_n - |\nabla u_0|^{q-2}\nabla u_0)(\nabla u_n - \nabla u_0)\, dx \\
&\quad \ge (\|\nabla u_n\|_p^{p-1} - \|\nabla u_0\|_p^{p-1})(\|\nabla u_n\|_p - \|\nabla u_0\|_p) + (\|\nabla u_n\|_q^{q-1} - \|\nabla u_0\|_q^{q-1})(\|\nabla u_n\|_q - \|\nabla u_0\|_q) \ge 0,
\end{aligned}$$

which implies that $\|\nabla u_n\|_p \to \|\nabla u_0\|_p$ and $\|\nabla u_n\|_q \to \|\nabla u_0\|_q$ as $n \to +\infty$. Due to the uniform convexity of $W_0^{1,p}$, we conclude that u_n converges to u_0 strongly in $W_0^{1,p}$.

Recall the definition (1.9):

$$\beta_{\mathcal{U}}^*(\alpha) := \sup\left\{ \frac{\int_\Omega |\nabla \varphi|^q \, dx}{\int_\Omega |\varphi|^q \, dx} : \varphi \in \mathrm{ES}(p;\alpha) \setminus \{0\} \right\}. \tag{3.2}$$

Lemma 3.6. *If $\alpha \in \sigma(-\Delta_p)$, then $\lambda_1(q) \le \beta_{\mathcal{U}}^*(\alpha) < +\infty$.*

Proof. Let $\alpha \in \sigma(-\Delta_p)$. Recall that [31, Lemma 9] implies the existence of a constant $C(\alpha) > 0$ such that $\|\nabla u\|_p \le C(\alpha)\|u\|_q$ for any $u \in X(\alpha)$, where $X(\alpha)$ is defined by (2.7). Thus, applying the Hölder inequality, we get

$$\int_\Omega |\nabla u|^q \, dx \le |\Omega|^{\frac{p-q}{p}} \Big(\int_\Omega |\nabla u|^p \, dx \Big)^{q/p} \le |\Omega|^{\frac{p-q}{p}} C(\alpha)^q \int_\Omega |u|^q \, dx$$

for any $u \in X(\alpha)$. Therefore, $\beta_{\mathcal{U}}^*(\alpha) < +\infty$ since $\mathrm{ES}(p;\alpha) \subset X(\alpha)$. On the other hand, it is clear that $\beta_{\mathcal{U}}^*(\alpha) \ge \lambda_1(q)$ provided $\mathrm{ES}(p;\alpha) \setminus \{0\} \ne \emptyset$. □

In the one-dimensional case we can clarify the bounds for $\beta_{\mathcal{U}}^*(\alpha)$ as follows.

Lemma 3.7. *Let $N = 1$ and $\alpha = \lambda_k(p)$, $k \in \mathbb{N}$. Then*

$$\lambda_{k+1}(q)\Big(\frac{k}{k+1}\Big)^q = k^q \lambda_1(q) < \beta_{\mathcal{U}}^*(\alpha) = k^q \frac{\|\varphi_p'\|_q^q}{\|\varphi_p\|_q^q} < k^q \lambda_2(q) = \lambda_{k+1}(q)\Big(\frac{2k}{k+1}\Big)^q. \tag{3.3}$$

Proof. Let $\Omega = (0, T)$, $T > 0$, and $\alpha = \lambda_k(p)$ for some $k \in \mathbb{N}$. It is known that $\lambda_k(r) = (r-1)(\frac{k\pi_r}{T})^p$ for any $r > 1$ and $k \in \mathbb{N}$ (cf. Appendix A), and hence the first and third equalities in (3.3) are satisfied.

Note that the eigenspace $\mathrm{ES}(p; \lambda_k(p))$ is one-dimensional, as it follows from [17, Proposition 2.1]. Denoting the corresponding eigenfunction as φ_k, we directly get $\beta^*_{\mathcal{U}}(\lambda_k(p)) = \|\varphi_k'\|_q^q / \|\varphi_k\|_q^q$. On the other hand, φ_k has exactly k nodal domains of equivalent length (see Appendix A), and hence the standard scaling yields $\beta^*_{\mathcal{U}}(\lambda_k(p)) = k^q \|\varphi_p'\|_q^q / \|\varphi_p\|_q^q$, where φ_p is the first eigenfunction of $-\Delta_p$. The inequalities in (3.3) follow from Lemma A.2 below. □

The following lemma ensues readily from the definition (3.2).

Lemma 3.8. *Let $\alpha \in \sigma(-\Delta_p)$ and $\beta > \beta^*_{\mathcal{U}}(\alpha)$. Then $G_\beta(\varphi) < 0$ for all $\varphi \in \mathrm{ES}(p; \alpha) \setminus \{0\}$.*

Lemma 3.9. *Let $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$. If $\beta > \lambda_{k+1}(q)$, then there exist an odd map $h_0 \in C(S^k, W_0^{1,p})$ and $t_0 > 0$ such that*

$$\max_{z \in S^k} E_{\alpha,\beta}(t_0 h_0(z)) < 0.$$

Proof. Let $\beta > \lambda_{k+1}(q)$ and choose $\varepsilon \in \mathbb{R}$ satisfying

$$0 < \varepsilon < \frac{1}{2} \quad \text{and} \quad \frac{\lambda_{k+1}(q) + 2\varepsilon}{(1-2\varepsilon)^q} < \beta - \varepsilon. \tag{3.4}$$

By the definition of $\lambda_{k+1}(q)$, there exists a map $h_1 \in \mathscr{F}_{k+1}(q)$ such that

$$\max_{z \in S^k} \|\nabla h_1(z)\|_q^q < \lambda_{k+1}(q) + \varepsilon. \tag{3.5}$$

Note that by taking $t > 0$ small enough it is easy to get $\max_{z \in S^k} E_{\alpha,\beta}(t h_1(z)) < 0$. However, $h_1 \in C(S^k, S(q))$, and we do not know a priori that $h_1 \in C(S^k, W_0^{1,p})$. Hence the arguments below are needed.

Since $C_0^\infty(\Omega)$ is a dense subset of $W_0^{1,q}$ and h_1 is odd, for any $z \in S^k$ we can find $u_z \in C_0^\infty(\Omega)$ such that

$$u_{-z} = -u_z, \quad \big| \|\nabla h_1(z)\|_q^q - \|\nabla u_z\|_q^q \big| < \varepsilon, \quad \|h_1(z) - u_z\|_q < \varepsilon. \tag{3.6}$$

By the continuity of h_1, for any $z \in S^k$ there exists $\delta(z) \in (0, 1)$ such that

$$\|h_1(z) - h_1(y)\|_q < \varepsilon \quad \text{for all } y \in S^k \text{ with } |z - y| < \delta(z). \tag{3.7}$$

Considering $\min\{\delta(z), \delta(-z)\}$ instead of $\delta(z)$, we may assume that δ is even. Note that (3.6) and (3.7) lead to

$$\|u_z - h_1(y)\|_q < 2\varepsilon \quad \text{for all } y \in S^k \text{ such that } |z - y| < \delta(z). \tag{3.8}$$

Due to the compactness of S^k, we may choose a finite number of points $z_i \in S^k$, $i = 1, 2, \dots, m$, such that

$$S^k \subset \bigcup_{i=1}^m [B(z_i, \delta(z_i)) \cup B(-z_i, \delta(-z_i))],$$

where $B(z_i, \delta(z_i)) \subset \mathbb{R}^{k+1}$ is a ball of radius $\delta(z_i)$ centered at the point z_i. Now, for each $i = 1, 2, \dots, m$ we take a function $\rho_i \in C_0(\mathbb{R}^{k+1})$ such that

$$\operatorname{supp} \rho_i = \overline{B(z_i, \delta(z_i))} \quad \text{and} \quad \rho_i > 0 \text{ in } B(z_i, \delta(z_i)).$$

Note that $B(z_i, \delta(z_i)) \cap B(-z_i, \delta(-z_i)) = \emptyset$ for all $i = 1, 2, \dots, m$ since $\delta(z_i) = \delta(-z_i) < 1$. Thus, $\rho_i(-z) = 0$ whenever $\rho_i(z) > 0$. Define

$$\tilde{\rho}_i(z) := \frac{\rho_i(z)}{\sum_{j=1}^m (\rho_j(z) + \rho_j(-z))} \quad \text{for } z \in S^k.$$

Since $\{B(z_i, \delta(z_i)) \cup B(-z_i, \delta(-z_i))\}_{i=1}^m$ is an open covering of S^k, it is easy to see that $\tilde{\rho}_i \in C(S^k)$ for all $i = 1, 2, \dots, m$. Moreover,

$$0 \le \tilde{\rho}_i \le 1, \quad \tilde{\rho}_i(-z) = 0 \quad \text{provided } \tilde{\rho}_i(z) > 0 \text{ and } \sum_{j=1}^m (\tilde{\rho}_j(z) + \tilde{\rho}_j(-z)) = 1 \tag{3.9}$$

for all $z \in S^k$ and $i = 1, 2, \dots, m$. That is, $\{\tilde{\rho}_i\}_{i=1}^m$ forms a partition of unity of S^k. Set

$$h_0(z) := \sum_{i=1}^m \big(\tilde{\rho}_i(z) u_{z_i} + \tilde{\rho}_i(-z) u_{-z_i}\big) \equiv \sum_{i=1}^m u_{z_i} \big(\tilde{\rho}_i(z) - \tilde{\rho}_i(-z)\big) \quad \text{for } z \in S^k.$$

Evidently, h_0 is odd, and the continuity of $\tilde{\rho}_i$ implies that $h_0 \in C(S^k, W_0^{1,p})$.

Let us show that $\max_{z \in S^k} E_{\alpha,\beta}(t h_0(z)) < 0$ for sufficiently small $t > 0$. First, for all $z \in S^k$ there holds

$$\begin{aligned} \|\nabla h_0(z)\|_q &\le \sum_{i=1}^m \|\nabla u_{z_i}\|_q \big(\tilde{\rho}_i(z) + \tilde{\rho}_i(-z)\big) \\ &< (\lambda_{k+1}(q) + 2\varepsilon)^{1/q} \sum_{i=1}^m (\tilde{\rho}_i(z) + \tilde{\rho}_i(-z)) = (\lambda_{k+1}(q) + 2\varepsilon)^{1/q}, \end{aligned} \tag{3.10}$$

where we used that $\|\nabla u_{z_i}\|_q^q < \lambda_{k+1}(q) + 2\varepsilon$, by virtue of (3.6) and (3.5). Moreover, $h_0(z) \neq 0$ for all $z \in S^k$. Indeed, using the convexity of $\|\cdot\|_q^q$, the oddness of h_1, (3.9) and (3.8), we derive

$$\begin{aligned} \|h_1(z) - h_0(z)\|_q^q &= \Big\| \sum_{i=1}^m \big(\tilde{\rho}_i(z)(h_1(z) - u_{z_i}) + \tilde{\rho}_i(-z)(h_1(z) - u_{-z_i})\big) \Big\|_q^q \\ &\le \sum_{i=1}^m \big(\tilde{\rho}_i(z) \|h_1(z) - u_{z_i}\|_q^q + \tilde{\rho}_i(-z) \|u_{z_i} - h_1(-z)\|_q^q\big) < 2^q \varepsilon^q \end{aligned}$$

since $\tilde{\rho}_i(-z) > 0$ if and only if $-z \in B(z_i, \delta(z_i))$. Hence, $\|h_0(z)\|_q \ge \|h_1(z)\|_q - 2\varepsilon = 1 - 2\varepsilon > 0$ for every $z \in S^k$. Now using (3.10) and (3.4), we get

$$\frac{\|\nabla h_0(z)\|_q^q}{\|h_0(z)\|_q^q} < \frac{\lambda_{k+1}(q) + 2\varepsilon}{\|h_0(z)\|_q^q} \le \frac{\lambda_{k+1}(q) + 2\varepsilon}{(1 - 2\varepsilon)^q} < \beta - \varepsilon$$

for all $z \in S^k$. Thus, for sufficiently small $t > 0$ and any $z \in S^k$ we obtain

$$\begin{aligned} E_{\alpha,\beta}(t h_0(z)) &= \frac{t^p}{p} \big(\|\nabla h_0(z)\|_p^p - \alpha \|h_0(z)\|_p^p\big) + \frac{t^q}{q} \big(\|\nabla h_0(z)\|_q^q - \beta \|h_0(z)\|_q^q\big) \\ &\le \frac{t^p}{p} \max_{z \in S^k} \big(\|\nabla h_0(z)\|_p^p - \alpha \|h_0(z)\|_p^p\big) - \frac{t^q (1 - 2\varepsilon)^q \varepsilon}{q} < 0 \end{aligned}$$

since $q < p$. This is the desired conclusion. □

In the sequel, we will also need the following variant of the deformation lemma. We refer the reader to [14, Theorem 3.2] for the proof.

Lemma 3.10. *Let Ψ be a C^1-functional on a Banach space W, let Ψ satisfy the Palais–Smale condition at any level $c \in [a, b]$ and let Ψ have no critical values in (a, b). Assume that either $K_a := \{u \in W : \Psi'(u) = 0,\ \Psi(u) = a\}$ consists only of isolated points, or $K_a = \emptyset$. Define $\Psi^c := \{u \in W : \Psi(u) \le c\}$. Then there exists $\eta \in C([0, 1] \times W, W)$ such that the following hold:*

(i) *$\Psi(\eta(s, u))$ is nonincreasing in s for every $u \in W$.*
(ii) *$\eta(s, u) = u$ for any $u \in \Psi^a$, $s \in [0, 1]$.*
(iii) *$\eta(0, u) = u$ and $\eta(1, u) \in \Psi^a$ for any $\Psi^b \setminus K_b$.*
(iv) *If Ψ is even, then $\eta(s, \cdot)$ is odd for all $s \in [0, 1]$.*

That is, Ψ^a is a strong deformation retract of $\Psi^b \setminus K_b$.

3.2 General existence result via minimax arguments

In this subsection, we prove a result on the existence of an *abstract nontrivial* solution to (GEV; α, β). Let us emphasize that this result does not guarantee that the obtained solution is *sign-changing*. (However, it is shown in [10] that for sufficiently large α and β problem (GEV; α, β) has no sign-constant solutions).

Recall that we denote $k_\alpha := \min\{k \in \mathbb{N} : \alpha < \lambda_{k+1}(p)\}$.

Theorem 3.11. *Assume that $\alpha \in \overline{\mathbb{R} \setminus \sigma(-\Delta_p)}$. Then for any $\beta > \max\{\beta^*_{\mathcal{U}}(\alpha), \lambda_{k_\alpha+1}(q)\}$ problem* (GEV; α, β) *has a nontrivial solution u with $E_{\alpha,\beta}(u) < 0$, where $\beta^*_{\mathcal{U}}(\alpha)$ is defined by* (1.9).

Proof. Since $\alpha \in \overline{\mathbb{R} \setminus \sigma(-\Delta_p)}$, we need to investigate two cases:
(i) $\alpha \notin \sigma(-\Delta_p)$.
(ii) $\alpha \in \sigma(-\Delta_p)$ and there exists a sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{R} \setminus \sigma(-\Delta_p)$ such that $\lim_{n\to+\infty} \alpha_n = \alpha$.

Case (i): Let $\beta > \lambda_{k_\alpha+1}(q) = \max\{\beta^*_{\mathcal{U}}(\alpha), \lambda_{k_\alpha+1}(q)\}$. Then Lemma 3.9 guarantees the existence of an odd $h_0 \in C(S^{k_\alpha}, W_0^{1,p})$ and of $t_0 > 0$ such that

$$\rho := \max_{z\in S^{k_\alpha}} E_{\alpha,\beta}(t_0 h_0(z)) < 0.$$

Moreover, by the definition of k_α we have $\alpha < \lambda_{k_\alpha+1}(p)$, and hence Lemma 3.2 implies that $E_{\alpha,\beta}$ is bounded from below on $Y(\lambda_{k_\alpha+1}(p))$, that is,

$$\delta_0 := \inf\{E_{\alpha,\beta}(u) : u \in Y(\lambda_{k_\alpha+1}(p))\} > -\infty. \tag{3.11}$$

Since $t_0 h_0(\cdot)$ is odd and $E_{\alpha,\beta}$ is even, Lemma 3.1 justifies that $E_{\alpha,\beta}(t_0 h_0(z_0)) \geq \delta_0$ for some $z_0 \in S^{k_\alpha}_+$, and hence $\delta_0 \leq \rho$. We are going to show that $E_{\alpha,\beta}$ has at least one critical value in $[\delta_0 - 1, \rho]$. Suppose, by contradiction, that $E_{\alpha,\beta}$ has no critical values in $[\delta_0 - 1, \rho]$. Recall that $E_{\alpha,\beta}$ satisfies the Palais–Smale condition by Lemma 3.4 because we are assuming that $\alpha \notin \sigma(-\Delta_p)$. Then, due to Lemma 3.10, there exists $\eta \in C([0,1] \times W_0^{1,p}, W_0^{1,p})$ such that $\eta(s,\cdot)$ is odd for every $s \in [0,1]$ and

$$E_{\alpha,\beta}(\eta(1, t_0 h_0(z))) \leq \delta_0 - 1 \quad \text{for all } z \in S^{k_\alpha}. \tag{3.12}$$

On the other hand, noting that $\eta(1, t_0 h_0(\cdot))|_{S^{k_\alpha}_+} \in C(S^{k_\alpha}_+, W_0^{1,p})$ and $\eta(1, t_0 h_0(\cdot))|_{S^{k_\alpha-1}}$ is odd, Lemma 3.1 guarantees the existence of a point $z_1 \in S^{k_\alpha}_+$ such that

$$\eta(1, t_0 h_0(z_1)) \in Y(\lambda_{k_\alpha+1}(p)),$$

whence $\delta_0 \leq E_{\alpha,\beta}(\eta(1, t_0 h_0(z_1)))$ by the definition of δ_0 (see (3.11)). However, this contradicts (3.12).

Case (ii): Let $\beta > \max\{\beta^*_{\mathcal{U}}(\alpha), \lambda_{k_\alpha+1}(q)\}$. As in the former case, according to Lemma 3.9, there exist an odd map $h_0 \in C(S^{k_\alpha}, W_0^{1,p})$ and $t_0 > 0$ such that

$$\rho := \max_{z\in S^{k_\alpha}} E_{\alpha,\beta}(t_0 h_0(z)) < 0. \tag{3.13}$$

Recalling that $\alpha < \lambda_{k_\alpha+1}(p)$ and discarding, if necessary, a finite number of terms of the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$, we may suppose that $\alpha_n < \lambda_{k_\alpha+1}(p)$ and

$$\rho_n := \max_{z\in S^{k_\alpha}} E_{\alpha_n,\beta}(t_0 h_0(z)) \leq \rho + t_0^p \frac{|\alpha_n - \alpha|}{p} \max_{z\in S^{k_\alpha}} \|h_0(z)\|_p^p < 0 \tag{3.14}$$

for all $n \in \mathbb{N}$. Since $\alpha_n \notin \sigma(-\Delta_p)$, we apply the proof of case (i) to each $\alpha_n < \lambda_{k_\alpha+1}(p)$ and $\beta > \lambda_{k_\alpha+1}(q)$, and hence obtain a sequence of critical values c_n of $E_{\alpha_n,\beta}$ such that

$$\delta_n - 1 \leq c_n \leq \rho_n, \quad \text{where} \quad \delta_n := \inf\{E_{\alpha_n,\beta}(u) : u \in Y(\lambda_{k_\alpha+1}(p))\} > -\infty. \tag{3.15}$$

Let $u_n \in W_0^{1,p}$ be a critical point of $E_{\alpha_n,\beta}$ corresponding to the level c_n, i.e., $E_{\alpha_n,\beta}(u_n) = c_n$. We proceed to show that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}$. Suppose, by contradiction, that $\|\nabla u_n\|_p \to +\infty$ as $n \to +\infty$. Set $v_n := u_n / \|\nabla u_n\|_p$ and note that

$$\|E'_{\alpha,\beta}(u_n)\|_{(W_0^{1,p})^*} = \|E'_{\alpha,\beta}(u_n) - E'_{\alpha_n,\beta}(u_n)\|_{(W_0^{1,p})^*} \leq \frac{|\alpha_n - \alpha|}{\lambda_1(p)} \|\nabla u_n\|_p^{p-1} = o(1)\|\nabla u_n\|_p^{p-1} \tag{3.16}$$

as $n \to +\infty$. Thus, due to Lemma 3.3, we have that v_n converges strongly in $W_0^{1,p}$, up to a subsequence, to some $v_0 \in \mathrm{ES}(p,\alpha) \setminus \{0\}$. Let us prove that $G_\beta(v_0) = 0$. By (3.14), we have

$$\Big(\frac{1}{q} - \frac{1}{p}\Big) G_\beta(v_n) = \frac{1}{\|\nabla u_n\|_p^q} \Big(E_{\alpha_n,\beta}(u_n) - \frac{1}{p}\langle E'_{\alpha_n,\beta}(u_n), u_n\rangle\Big) = \frac{c_n}{\|\nabla u_n\|_p^q} \leq \frac{\rho_n}{\|\nabla u_n\|_p^q} < 0. \tag{3.17}$$

To obtain a converse estimate, we show that δ_n is bounded from below. Since $\lim_{n\to+\infty}\alpha_n = \alpha < \lambda_{k_\alpha+1}(p)$, we can choose α_0 such that $\alpha_n < \alpha_0 < \lambda_{k_\alpha+1}(p)$ for all sufficiently large $n \in \mathbb{N}$. Thus, Lemma 3.2 implies that $E_{\alpha_0,\beta}$ is bounded from below on $Y(\lambda_{k_\alpha+1}(p))$. Noting that $E_{\alpha_n,\beta}(u) \geq E_{\alpha_0,\beta}(u)$ for any $u \in W_0^{1,p}$, we get $\delta_n \geq \inf\{E_{\alpha_0,\beta}(u) : u \in Y(\lambda_{k_\alpha+1}(p))\} > -\infty$ for all $n \in \mathbb{N}$ large enough, which is the desired boundedness. Using this fact, the two equalities in (3.17), and (3.15), we derive that

$$0 > \Big(\frac{1}{q} - \frac{1}{p}\Big)G_\beta(v_n) \geq \frac{\delta_n - 1}{\|\nabla u_n\|_p^q} \to 0$$

as $n \to +\infty$, which leads to $G_\beta(v_0) = 0$, because $v_n \to v_0$ strongly in $W_0^{1,p}$. On the other hand, since $\alpha \in \sigma(-\Delta_p)$ and $\beta > \beta^*_{\mathcal{U}}(\alpha)$, we get

$$G_\beta(\varphi) = \|\nabla\varphi\|_q^q - \beta\|\varphi\|_q^q \neq 0 \quad \text{for all } \varphi \in \mathrm{ES}(p;\alpha) \setminus \{0\}; \tag{3.18}$$

see Lemma 3.8. Hence, we obtain a contradiction since $G_\beta(v_0) = 0$ and $v_0 \in \mathrm{ES}(p,\alpha) \setminus \{0\}$. Thus, from (3.16) it follows that $\{u_n\}_{n\in\mathbb{N}}$ is a *bounded* Palais–Smale sequence for $E_{\alpha,\beta}$. Then the (S_+)-property of the operator $-\Delta_p - \Delta_q$ (see Remark 3.5) implies that u_n converges strongly in $W_0^{1,p}$, up to a subsequence, to some critical point u_0 of $E_{\alpha,\beta}$. Furthermore, u_0 is nontrivial and its energy is negative since

$$E_{\alpha,\beta}(u_0) = \limsup_{n\to+\infty} E_{\alpha_n,\beta}(u_n) = \limsup_{n\to+\infty} c_n \leq \limsup_{n\to+\infty} \rho_n \leq \rho + o(1) < 0$$

by (3.13) and (3.14). □

Remark 3.12. Note that the proof of case (ii) gives more. Namely, if $\alpha \in \sigma(-\Delta_p)$ and $\lim_{n\to+\infty}\alpha_n = \alpha$ for some sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{R} \setminus \sigma(-\Delta_p)$, and $\beta > \lambda_{k_\alpha+1}(q)$ is such that (3.18) holds, then there exists a nontrivial solution to (GEV; α,β).

3.3 General existence result via the descending flow

In the last part of this section, we use the descending flow method to provide an existence result for (p,q)-Laplace equations with a nonlinearity in the general form.

Suppose that $h\colon \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying $h(x,0) = 0$ for a.e. $x \in \Omega$ and there exists $C > 0$ such that

$$|h(x,s)| \leq C(1 + |s|^{p-1}) \quad \text{for every } s \in \mathbb{R} \text{ and a.e. } x \in \Omega. \tag{3.19}$$

Under (3.19), we define a C^1-functional J on $W_0^{1,p}$ by

$$J(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{q}\int_\Omega |\nabla u|^q\,dx - \int_\Omega \int_0^{u(x)} h(x,s)\,ds\,dx. \tag{3.20}$$

For simplicity, we denote the positive cone in $C_0^1(\overline{\Omega})$ by

$$P := \{u \in C_0^1(\overline{\Omega}) : u(x) > 0 \text{ for all } x \in \Omega\}. \tag{3.21}$$

The following result can be proved by the same arguments as [27, Theorem 11]. For the reader's convenience, we give a sketch of the proof in Appendix B.

Theorem 3.13. *Assume that the following conditions hold:*

(A1) *There exists $\lambda_0 > 0$ such that*

$$h(x,u)u + \lambda_0(|u|^q + |u|^p) \geq 0 \quad \text{for every } u \in \mathbb{R} \text{ and a.e. } x \in \Omega.$$

(A2) *There exists $\gamma \in C([0,1], C_0^1(\overline{\Omega}))$ such that $\gamma(0) \in P$, $\gamma(1) \in -P$ and $\max_{s\in[0,1]} J(\gamma(s)) < 0$.*

If, moreover, J is coercive on $W_0^{1,p}$, then J has at least three critical points $w_1 \in \operatorname{int} P$, $w_2 \in -\operatorname{int} P$, and $w_3 \in C_0^1(\overline{\Omega}) \setminus (P \cup -P)$ such that $J(w_i) \leq \max_{s\in[0,1]} J(\gamma(s)) < 0$ for $i = 1,2,3$. Here

$$\operatorname{int} P := \{u \in P : \partial u(x)/\partial\nu < 0 \text{ for all } x \in \partial\Omega\},$$

and ν denotes the unit outer normal vector to $\partial\Omega$.

We say that $v \in W_0^{1,p}$ is a (weak) super-solution of (GEV; α, β) whenever for all nonnegative $\varphi \in W_0^{1,p}$ there holds

$$\int_\Omega |\nabla v|^{p-2} \nabla v \nabla \varphi \, dx + \int_\Omega |\nabla v|^{q-2} \nabla v \nabla \varphi \, dx \geq \alpha \int_\Omega |v|^{p-2} v \varphi \, dx + \beta \int_\Omega |v|^{q-2} v \varphi \, dx.$$

Applying Theorem 3.13 to a truncated functional corresponding to $E_{\alpha,\beta}$, we show the following result on the existence of nodal solutions to (GEV; α, β) with a negative energy.

Proposition 3.14. *Let $\alpha \in \mathbb{R}$ and $\beta > \lambda_2(q)$. If there exists a super-solution of* (GEV; α, β) *which belongs to* $\operatorname{int} P$, *then* (GEV; α, β) *has a nodal solution u such that $E_{\alpha,\beta}(u) < 0$.*

Proof. Let $v \in \operatorname{int} P$ be a super-solution of (GEV; α, β) with $\alpha \in \mathbb{R}$ and $\beta > \lambda_2(q)$. Note that $-v$ becomes a negative sub-solution of (GEV; α, β). Using v, we truncate the right-hand side of (GEV; α, β) as follows:

$$f(x, s) := \begin{cases} \alpha v(x)^{p-1} + \beta v(x)^{q-1} & \text{if } s > v(x), \\ \alpha |s|^{p-2} s + \beta |s|^{q-2} s & \text{if } -v(x) \leq s \leq v(x), \\ -\alpha v(x)^{p-1} - \beta v(x)^{q-1} & \text{if } s < -v(x). \end{cases}$$

It is easy to see that f is the Carathéodory function and $f(x, 0) = 0$ for all $x \in \Omega$. Moreover, f satisfies (3.19) and, taking $\lambda_0 = \max\{|\alpha|, |\beta|\}$, it satisfies assumption (A1) of Theorem 3.13.

Define a corresponding *truncated* C^1-functional I on $W_0^{1,p}$ by

$$I(u) := \frac{1}{p} \int_\Omega |\nabla u|^p \, dx + \frac{1}{q} \int_\Omega |\nabla u|^q \, dx - \int_\Omega \int_0^{u(x)} f(x, s) \, ds \, dx.$$

Note that the boundedness of v in Ω implies the boundedness of f, and therefore I is coercive on $W_0^{1,p}$. To apply Theorem 3.13 it remains to show that (A2) holds. To this end, let us construct an appropriate path γ_0. Choose $\varepsilon > 0$ satisfying $\lambda_2(q) + 2\varepsilon < \beta$. By the characterization (1.5) of $\lambda_2(q)$, there exists $\gamma \in C([0, 1], S(q))$ such that

$$\gamma(0) = \varphi_q \in \operatorname{int} P, \quad \gamma(1) = -\varphi_q \in -\operatorname{int} P, \quad \max_{s \in [0,1]} \|\nabla \gamma(s)\|_q^q < \lambda_2(q) + \varepsilon.$$

Using the density arguments (as in the proof of Lemma 3.9), we can obtain a path $\tilde{\gamma} \in C([0, 1], C_0^1(\overline{\Omega}) \setminus \{0\})$ such that $\tilde{\gamma}(0) \in P$, $\tilde{\gamma}(1) \in -P$, and

$$\|\nabla \tilde{\gamma}(s)\|_q^q \leq (\lambda_2(q) + 2\varepsilon) \|\tilde{\gamma}(s)\|_q^q$$

for every $s \in [0, 1]$. Since $v \in \operatorname{int} P$ and $\tilde{\gamma} \in C([0, 1], C_0^1(\overline{\Omega}) \setminus \{0\})$, we get for any $t > 0$ small enough, $s \in [0, 1]$ and $x \in \Omega$ that

$$-v(x) \leq t\tilde{\gamma}(s)(x) \leq v(x),$$

and hence

$$f(x, t\tilde{\gamma}(s)) = t^{p-1} \alpha |\tilde{\gamma}(s)|^{p-2} \tilde{\gamma}(s) + t^{q-1} \beta |\tilde{\gamma}(s)|^{q-2} \tilde{\gamma}(s).$$

Therefore,

$$\begin{aligned} I(t\tilde{\gamma}(s)) &= \frac{t^p}{p} (\|\nabla \tilde{\gamma}(s)\|_p^p - \alpha \|\tilde{\gamma}(s)\|_p^p) + \frac{t^q}{q} (\|\nabla \tilde{\gamma}(s)\|_q^q - \beta \|\tilde{\gamma}(s)\|_q^q) \\ &\leq t^q \Big(\frac{t^{p-q}}{p} \big[\max_{s \in [0,1]} \|\nabla \tilde{\gamma}(s)\|_p^p + |\alpha| \max_{s \in [0,1]} \|\tilde{\gamma}(s)\|_p^p \big] + \frac{\lambda_2(q) + 2\varepsilon - \beta}{q} \min_{s \in [0,1]} \|\tilde{\gamma}(s)\|_q^q \Big) < 0 \end{aligned}$$

for sufficiently small $t > 0$ since

$$q < p, \quad \min_{s \in [0,1]} \|\tilde{\gamma}(s)\|_q^q > 0, \quad \lambda_2(q) + 2\varepsilon < \beta.$$

Thus, for such a small $t > 0$ the path $\gamma_0(s) := t\tilde{\gamma}(s)$ satisfies assumption (A2) of Theorem 3.13.

As a result, according to Theorem 3.13, we obtain a sign-changing critical point $u \in C_0^1(\overline{\Omega}) \setminus (P \cup -P)$ of I satisfying $I(u) \leq \max_{t \in [0,1]} I(\gamma_0(t)) < 0$. By the standard argument, we can show that $-v \leq u \leq v$ in Ω.

In fact, recalling that v is a super-solution of (GEV; α, β) and taking $(u - v)^+ \in W_0^{1,p}$ as a test function for $I'(u) - E'_{\alpha,\beta}(v)$, we obtain

$$0 \le \int\limits_{u>v} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)(\nabla u - \nabla v)\,dx + \int\limits_{u>v} (|\nabla u|^{q-2}\nabla u - |\nabla v|^{q-2}\nabla v)(\nabla u - \nabla v)\,dx$$

$$\le \int\limits_{\Omega} f(x, u)(u - v)^+\,dx - \int\limits_{\Omega} (\alpha v^{p-1} + \beta v^{q-1})(u - v)^+\,dx = 0,$$

which implies that $(u - v)^+ \equiv 0$ and hence $u \le v$ in Ω. Similarly, taking $-(u - (-v))^-$ as a test function, we get $u \ge -v$. Therefore, u is a nodal solution of (GEV; α, β) and $E_{\alpha,\beta}(u) = I(u) \le \max_{s\in[0,1]} I(\gamma_0(s)) < 0$. □

4 Proofs of the main results

In this section, we collect the proofs of our main results stated in Section 1.2.

Proof of Theorem 1.3. Recall that any sign-changing solution of (GEV; α, β) belongs to the nodal Nehari set $\mathcal{M}_{\alpha,\beta}$ defined by (2.1). At the same time, $\mathcal{M}_{\alpha,\beta}$ is empty under the assumptions of the theorem, as is shown in Lemma 2.4, which completes the proof. □

Proof of Theorem 1.5. The desired conclusion follows directly from the combination of Theorem 2.15 and Lemmas 2.16 and 2.17. □

Proof of Theorem 1.6. Note that problem (GEV; α, β) possesses an abstract nontrivial solution $u \in W_0^{1,p}$ with $E_{\alpha,\beta}(u) < 0$ for any

$$\alpha \in \overline{\mathbb{R} \setminus \sigma(-\Delta_p)} \quad \text{and} \quad \beta > \max\{\beta^*_{\mathcal{U}}(\alpha), \lambda_{k_\alpha+1}(q)\} \ge \lambda_2(q)$$

by Theorem 3.11. If u is a nodal solution, then we are done. If u is a nontrivial nonnegative solution, then $u \in \operatorname{int} P$ (see, e.g., [10, Remark 1, p. 3284]), and hence Proposition 3.14 guarantees the existence of a nodal solution v of (GEV; α, β) such that $E_{\alpha,\beta}(v) < 0$. □

Proof of Theorem 1.7. If $\alpha < \lambda_1(p)$ or $\lambda_1(p) < \alpha < \lambda_2(p)$, then for all $\beta > \lambda_2(q)$ there exists a nodal solution, as follows from Theorem 1.6. If $\alpha = \lambda_1(p)$, then, as noted in Remark 3.12, Theorem 3.11 implies the existence of an abstract nontrivial negative energy solution of (GEV; α, β) for any $\beta > \lambda_2(q)$ such that $G_\beta(\varphi_p) \ne 0$. Since the first eigenfunction φ_p of $-\Delta_p$ is unique, up to a multiplier, we derive the existence under the assumption $\beta \ne \|\nabla\varphi_p\|_q^q / \|\varphi_p\|_q^q$. If the obtained solution changes its sign, then we are done. Otherwise, we apply Proposition 3.14 and obtain the existence of a nodal solution with a negative energy. □

Finally, we will prove the nonexistence result in the one-dimensional case.

Proof of Theorem 1.4. Let $N = 1$ and $\Omega = (0, T)$, $T > 0$. We temporarily denote by $\lambda_k(r, S)$ the kth eigenvalue of $-\Delta_r$ on $(0, S)$ subject to zero Dirichlet boundary conditions, $r > 1$, $S > 0$ (see Appendix A). Suppose, by contradiction, that $\alpha \le \lambda_2(p, T)$ and $\beta \le \lambda_2(q, T)$, but there exists a nodal solution u for (GEV; α, β). Evidently, there is at least one nodal domain of u which length S is less than or equal to $T/2$. Using, if necessary, the translation of the coordinate axis, we may assume that u is a constant-sign solution of (GEV; α, β) on interval $(0, S)$. Define $v := u$ on $(0, S)$ and $v = 0$ on $[S, T/2]$. Clearly, $v \in W_0^{1,p}(0, S) \subset W_0^{1,p}(0, T/2)$. Moreover, it is not hard to see that

$$\lambda_2(r, T) = \lambda_1(r, T/2) = \Big(\frac{2S}{T}\Big)^r \lambda_1(r, S) \le \lambda_1(r, S)$$

for any $r > 1$. Thus, (1.2) and the assumption $S \le T/2$ lead to the inequalities

$$\alpha \le \lambda_2(p, T) \le \lambda_1(p, S) \le \frac{\int_0^S |v'|^p\,dt}{\int_0^S |v|^p\,dt} \quad \text{and} \quad \beta \le \lambda_2(q, T) \le \lambda_1(q, S) \le \frac{\int_0^S |v'|^q\,dt}{\int_0^S |v|^q\,dt}. \tag{4.1}$$

Taking now v as a test function for (1.1), we arrive at

$$0 \le \int_0^S |v'|^p\,dt - \alpha \int_0^S |v|^p\,dt = \beta \int_0^S |v|^q\,dt - \int_0^S |v'|^q\,dt \le 0,$$

and hence we have equalities in (4.1). On the other hand, the simplicity of $\lambda_1(r, S)$ implies that v is the first eigenfunction corresponding to $\lambda_1(p, S)$ and $\lambda_1(q, S)$, simultaneously. However, this is a contradiction, since φ_p and φ_q are linearly independent for $N = 1$ (see [20, Lemma 4.3] or Lemma A.1 below). □

A Appendix A

In this section, we show some relations between eigenvalues and eigenfunctions of the p- and q-Laplacians in the one-dimensional case. Consider the eigenvalue problem

$$\begin{cases} -(|u'|^{r-2}u')' = \lambda |u|^{r-2}u & \text{in } (0, T), \\ u(0) = u(T) = 0, \end{cases}$$

where $r > 1$ and $T > 0$. It is known (cf. [17, Theorem 3.1]) that $\sigma(-\Delta_r)$ is exhausted by eigenvalues

$$\lambda_k(r) = (r-1)\Big(\frac{k\pi_r}{T}\Big)^r, \quad \text{where } \pi_r = \frac{2\pi}{r\sin(\pi/r)}.$$

(It is not hard to see that π_r is a decreasing function of $r > 1$.) The corresponding eigenfunctions are denoted by $\sin_r(\frac{k\pi_r t}{T})$, where $\sin_r(t)$ is the inverse function of $\int_0^x (1 - s^r)^{-1/r}\,ds$, $x \in [0, 1]$, extended periodically and anti-periodically from $[0, \pi_r/2]$ to the whole $\mathbb{R}$ (see also [11]). By construction, $\sin_r(\frac{k\pi_r t}{T})$ has exactly k nodal domains of the length T/k on $(0, T)$. As usual, we denote the first eigenfunction $\sin_r(\frac{\pi_r t}{T})$ as φ_r.

For the convenience of the reader we briefly prove that the first eigenfunctions φ_p and φ_q are linearly independent; see also [20, Lemma 4.3] for a different proof.

Lemma A.1. *Let $N = 1$ and $q \neq p$. Then φ_p and φ_q are linearly independent.*

Proof. Suppose, by contradiction, that $\varphi_p(t) = \varphi_q(t)$ for all $t \in [0, T]$. In particular, we have

$$\sin_p\Big(\frac{\pi_p t}{T}\Big) = \sin_q\Big(\frac{\pi_q t}{T}\Big)$$

for all $t \in [0, T/2]$. By the definitions of $\sin_p$ and $\sin_q$, we obtain

$$\frac{1}{\pi_p}\int_0^x (1 - s^p)^{-1/p}\,ds = \frac{1}{\pi_q}\int_0^x (1 - s^q)^{-1/q}\,ds \quad \text{for all } x \in [0, 1].$$

Using a Taylor series, we get $(1 - s^p)^{-1/p} = 1 + O(s^p)$ and $(1 - s^q)^{-1/q} = 1 + O(s^q)$ in a neighborhood of $s = 0$. Thus,

$$\int_0^x \Big[\Big(\frac{1}{\pi_p} - \frac{1}{\pi_q}\Big) + O(s^p) + O(s^q)\Big]\,ds = 0$$

for sufficiently small $x > 0$, which implies that $\pi_p = \pi_q$ since $p, q > 1$. However, this contradicts the monotonicity of π_r with respect to $r > 1$. □

Next, we prove the main result of the section.

Lemma A.2. *Let $N = 1$ and $1 < q < p < +\infty$. Then*

$$\lambda_1(q) < \frac{\|\varphi_p'\|_q^q}{\|\varphi_p\|_q^q} < \lambda_2(q).$$

Proof. The first inequality is trivial because the first eigenvalue $\lambda_1(q)$ is simple and $\varphi_p \neq \varphi_q$ (see [20] or Lemma A.1). Let us prove by direct calculations that

$$\frac{\|\varphi_p'\|_q^q}{\|\varphi_p\|_q^q} < \lambda_2(q) \quad \text{for } q < p.$$

Note that

$$\frac{\|\varphi_p'\|_q^q}{\|\varphi_p\|_q^q} = \frac{\int_0^T |\sin_p'(\frac{\pi_p t}{T})|^q\,dt}{\int_0^T |\sin_p(\frac{\pi_p t}{T})|^q\,dt} = \frac{\pi_p^q}{T^q}\frac{\int_0^T |\cos_p(\frac{\pi_p t}{T})|^q\,dt}{\int_0^T |\sin_p(\frac{\pi_p t}{T})|^q\,dt} = \frac{\pi_p^q}{T^q}\frac{\int_0^{\pi_p} |\cos_p x|^q\,dx}{\int_0^{\pi_p} |\sin_p x|^q\,dx} = \frac{\pi_p^q}{T^q}\frac{\int_0^{\pi_p/2} \cos_p^q x\,dx}{\int_0^{\pi_p/2} \sin_p^q x\,dx}. \tag{A.1}$$

Using the formulas

$$\int_0^{\pi_p/2} \sin_p^q x\,dx = \frac{1}{p}B\Big(\frac{q+1}{p}, \frac{p-1}{p}\Big) \quad \text{and} \quad \int_0^{\pi_p/2} \cos_p^q x\,dx = \frac{1}{p}B\Big(\frac{1}{p}, 1 + \frac{q-1}{p}\Big)$$

from [11, Proposition 3.1], where $B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$ is the beta function with real $x, y > 0$, it becomes sufficient to prove that

$$\frac{B(\frac{1}{p}, 1 + \frac{q-1}{p})}{B(\frac{q+1}{p}, \frac{p-1}{p})} < \frac{\lambda_2(q)T^q}{\pi_p^q} \equiv (q-1)\Big(\frac{2\pi_q}{\pi_p}\Big)^q. \tag{A.2}$$

We will subsequently simplify (A.2), to obtain an easier sufficient condition. Note that, by definition,

$$B\Big(\frac{1}{p}, 1 + \frac{q-1}{p}\Big) = \int_0^1 t^{\frac{1}{p}-1}(1-t)^{\frac{q-1}{p}}\,dt < \int_0^1 t^{\frac{1}{p}-1}\,dt = B\Big(\frac{1}{p}, 1\Big) = p.$$

Note that

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(y)$ is the gamma function; cf. [3, Theorem 1.1.4]. Hence, combining the Euler reflection formula $\Gamma(y)\Gamma(1-y) = \frac{\pi}{\sin \pi y}$ (see, e.g., [3, p. 9]) with the identity $x\Gamma(x) = \Gamma(x+1)$, we obtain

$$B(x,y)\cdot B(x+y, 1-y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\cdot\frac{\Gamma(x+y)\Gamma(1-y)}{\Gamma(x+1)} = \frac{\Gamma(x)}{\Gamma(x+1)}\Gamma(y)\Gamma(1-y) = \frac{\pi}{x\sin \pi y}. \tag{A.3}$$

Applying (A.3) to $B(\frac{q+1}{p}, \frac{p-1}{p})$ with $x = q/p$ and $y = 1/p$, we get

$$B\Big(\frac{q+1}{p}, \frac{p-1}{p}\Big) = \frac{p\pi}{q\sin(\frac{\pi}{p})}\cdot\frac{1}{B(\frac{q}{p}, \frac{1}{p})}.$$

Therefore, using the estimate

$$B\Big(\frac{q}{p}, \frac{1}{p}\Big) = \int_0^1 t^{\frac{q}{p}-1}(1-t)^{\frac{1}{p}-1}\,dt < \int_0^1 t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}-1}\,dt = B\Big(\frac{1}{p}, \frac{1}{p}\Big),$$

we arrive at

$$\frac{B(\frac{1}{p}, 1 + \frac{q-1}{p})}{B(\frac{q+1}{p}, \frac{p-1}{p})} < \frac{q\sin(\frac{\pi}{p})}{\pi}B\Big(\frac{1}{p}, \frac{1}{p}\Big) = \frac{2q}{p\pi_p}B\Big(\frac{1}{p}, \frac{1}{p}\Big). \tag{A.4}$$

Thus, comparing the right-hand sides of (A.2) and (A.4), we get the following sufficient condition for the assertion of the lemma:

$$\frac{1}{p}B\Big(\frac{1}{p}, \frac{1}{p}\Big) \leq 2^{q-1}\frac{q-1}{q}\frac{\pi_q^q}{\pi_p^{q-1}}. \tag{A.5}$$

To prove this inequality, we first obtain an appropriate upper bound for its left-hand side. From [3, p. 8] we know that

$$\frac{1}{p}B\Big(\frac{1}{p},\frac{1}{p}\Big)=\frac{1}{p}2p\prod_{n=1}^{\infty}\frac{1+\frac{2}{np}}{(1+\frac{1}{np})^2}<2\frac{1+\frac{2}{p}}{(1+\frac{1}{p})^2}=\frac{2p(p+2)}{(p+1)^2} \tag{A.6}$$

since for all $n\in\mathbb{N}$ there holds

$$\frac{1+\frac{2}{np}}{(1+\frac{1}{np})^2}=\frac{1+\frac{2}{np}}{1+\frac{2}{np}+(\frac{1}{np})^2}<1.$$

Next, we will get a suitable lower bound for the right-hand side of (A.5). Since π_r is a decreasing function of $r>1$ (in fact $d\pi_r/dr<0$), we have $\pi_q/\pi_p>1$ for $q<p$. Hence,

$$2^{q-1}\frac{q-1}{q}\frac{\pi_q^q}{\pi_p^{q-1}}>2^{q-1}\frac{q-1}{q}\pi_q=\frac{2^q\pi(q-1)}{q^2\sin(\frac{\pi}{q})}=\frac{2^q}{q}\cdot\frac{\frac{\pi}{q}(q-1)}{\sin(\frac{\pi}{q}(q-1))}>\frac{2^q}{q} \tag{A.7}$$

since $\sin x<x$ for all $x>0$.

Let us consider three cases. Assume first that $1<q<p\le 2$. By a direct analysis, the minimum value of the right-hand side $2^q/q$ of (A.7) is greater than $16/9$. Since the right-hand side

$$\frac{2p(p+2)}{(p+1)^2}$$

of (A.6) is strictly increasing with respect to $p>1$, it is easy to see that

$$\frac{2p(p+2)}{(p+1)^2}\le\frac{16}{9}\quad\text{for all } 1<p\le 2.$$

Combining these facts, we prove that (A.5) holds for $1<q<p\le 2$.

Secondly, assume that $2\le q<p$. Noting that $2^q/q$ is, in fact, strictly increasing for $q\ge 2$, we obtain

$$\frac{2^q}{q}\ge\frac{2^r}{r}\Big|_{r=2}=2>\frac{2p(p+2)}{(p+1)^2}=2\frac{p^2+2p}{p^2+2p+1}$$

for all $q\ge 2$ and $p>1$. Thus, (A.6) and (A.7) yield (A.5) for $2\le q<p$.

Finally, we assume that $1<q<2\le p$. Since π_r is decreasing, $p\ge 2$ implies that $\pi_p\le\pi$, and we refine inequality (A.7) in the following way:

$$2^{q-1}\frac{q-1}{q}\frac{\pi_q^q}{\pi_p^{q-1}}\ge\frac{2^{q-1}}{\pi^{q-1}}\frac{q-1}{q}\frac{2^q\pi^q}{q^q\sin^q(\frac{\pi}{q})}\ge\frac{2^{2q-1}}{q^q}\frac{q-1}{q}\frac{\pi}{\sin(\frac{\pi}{q})}=\frac{2^{2q-1}}{q^q}\cdot\frac{\frac{\pi}{q}(q-1)}{\sin(\frac{\pi}{q}(q-1))}>\frac{2^{2q-1}}{q^q}.$$

It is not hard to check that

$$\frac{2^{2q-1}}{q^q}>2>\frac{2p(p+2)}{(p+1)^2}\quad\text{for all } q\in(1,2),$$

which again implies (A.5).

Therefore, (A.5) holds for all $1<q<p<+\infty$, which completes the proof. □

If we swap p and q in Lemma A.2, then an opposite situation occurs.

Lemma A.3. *Let $N=1$. Then for any $k\in\mathbb{N}$ there exist $1<q_0<p_0$ such that*

$$\frac{\|\varphi_q'\|_p^p}{\|\varphi_q\|_p^p}>\lambda_k(p)\quad\textit{for all } 1<q<q_0 \textit{ and } p>p_0.$$

Proof. The case $k=1$ is obvious. Let $k\ge 2$. Similar to (A.1) and (A.2), it is sufficient to show that

$$\frac{B(\frac{1}{q},1+\frac{p-1}{q})}{B(\frac{p+1}{q},\frac{q-1}{q})}>(p-1)\Big(\frac{k\pi_p}{\pi_q}\Big)^p. \tag{A.8}$$

Note first that

$$B\Big(\frac{1}{q}, 1+\frac{p-1}{q}\Big) = \int_0^1 t^{\frac{1}{q}-1}(1-t)^{\frac{p-1}{q}}\,dt > \int_0^1 (1-t)^{\frac{p-1}{q}}\,dt = B\Big(1, 1+\frac{p-1}{q}\Big) = \frac{q}{p+q-1},$$

and for $q < p$ there holds

$$B\Big(\frac{p+1}{q}, \frac{q-1}{q}\Big) = \int_0^1 t^{\frac{p+1}{q}-1}(1-t)^{-\frac{1}{q}}\,dt < \int_0^1 (1-t)^{-\frac{1}{q}}\,dt = B\Big(1, \frac{q-1}{q}\Big) = \frac{q}{q-1}.$$

Therefore, (A.8) can be simplified as

$$\Big(\frac{q-1}{p+q-1}\Big)^{\frac{1}{p}} > \frac{kq(p-1)^{\frac{1}{p}}}{p}\frac{\sin(\frac{\pi}{q})}{\sin(\frac{\pi}{p})}.$$

Note that $\sin(\frac{\pi}{q}) = \sin(\frac{\pi}{q}(q-1)) < \frac{\pi}{q}(q-1)$. Hence, using the estimates

$$(p+q-1)^{1/p} < 2p^{1/p} \quad \text{and} \quad (p-1)^{1/p} < p^{1/p},$$

we arrive at the following sufficient inequality:

$$\sin\Big(\frac{\pi}{p}\Big) > \frac{2\pi k(q-1)^{\frac{p-1}{p}}}{p^{\frac{p-2}{p}}} = p^{\frac{2}{p}} \cdot \frac{2\pi k(q-1)^{\frac{p-1}{p}}}{p}.$$

At the same time, $(q-1)^{\frac{p-1}{p}} \le (q-1)^{\frac{1}{2}}$ for $1 < q < 2 < p$, and, choosing $p_1 > 2$ large enough, we obtain $2 \ge p^{2/p}$ for any $p \ge p_1$. Therefore, to prove (A.8) it is sufficient to show that

$$\frac{4\pi k(q-1)^{\frac{1}{2}}}{p} < \sin\Big(\frac{\pi}{p}\Big) = \frac{\pi}{p} + o\Big(\frac{\pi}{p}\Big). \tag{A.9}$$

However, (A.9) is obviously satisfied for any $q < 1 + \frac{1}{(4k)^2}$ and sufficiently large $p > p_1$. □

B Appendix B: Sketch of the proof of Theorem 3.13

Let us consider a map $T_\lambda : W_0^{1,p} \to (W_0^{1,p})^*$ defined for $\lambda > 0$ by

$$\langle T_\lambda(u), v\rangle = \int_\Omega (|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u \nabla v\,dx + \lambda\int_\Omega (|u|^{p-2} + |u|^{q-2})uv\,dx$$

for $u, v \in W_0^{1,p}$. The following properties of T_λ can be proved in much the same way as in the proof of [27, Propositions 9, 10].

Lemma B.1. *T_λ is invertible and $T_\lambda^{-1} : (W_0^{1,p})^* \to W_0^{1,p}$ is continuous. Moreover, if $1 < p \le N$ and $r > N/p$, then there exists a constant $D_0 > 0$ such that for all $u \in L^r(\Omega)$ we have*

$$\|T_\lambda^{-1}(u)\|_\infty \le D_0\|u\|_r^{1/(p-1)}.$$

Let us define $\psi(u) := |u|^{p-2}u + |u|^{q-2}u$ and a map $B_\lambda\colon W_0^{1,p} \to W_0^{1,p}$ by

$$B_\lambda(u) := T_\lambda^{-1}(h(\,\cdot\,, u) + \lambda\psi(u))$$

for $u \in W_0^{1,p}$ and $\lambda > 0$. According to Lemma B.1 and assumption (3.19), we see that B_λ is well-defined and continuous. Moreover, critical points of the energy function J given by (3.20) correspond to fixed points of B_λ, see [27, Remark 12]. Throughout this section, $K := \{u \in W_0^{1,p} : J'(u) = 0\}$ is the set of critical points of J, and, to shorten notation, we write $\|u\|$ instead of $\|\nabla u\|_p$ for $u \in W_0^{1,p}$.

By the standard calculations, we have the following facts (cf. [5, Lemmas 3.7 and 3.8] for details).

Lemma B.2. *Let $\lambda > 0$. Then there exist constants $d_i = d_i(\lambda) > 0$, $i = 1, 2, \dots, 6$ such that for all $u \in W_0^{1,p}$ the following assertions hold:*
(i) $\langle J'(u), u - B_\lambda(u)\rangle \geq d_1\|u - B_\lambda(u)\|^2((\|u\| + \|B_\lambda(u)\|)^{p-2} + (\|u\| + \|B_\lambda(u)\|)^{q-2})$ *for* $1 < q < p \leq 2$.
(ii) $\langle J'(u), u - B_\lambda(u)\rangle \geq d_2(\|u - B_\lambda(u)\|^p + \|u - B_\lambda(u)\|^q)$ *for* $2 \leq q < p$.
(iii) $\langle J'(u), u - B_\lambda(u)\rangle \geq d_3\|u - B_\lambda(u)\|^2(\|u\| + \|B_\lambda(u)\|)^{q-2} + d_3\|u - B_\lambda(u)\|^p$ *for* $1 < q \leq 2 \leq p$.
(iv) $\|J'(u)\|_{(W_0^{1,p})^*} \leq d_4(\|u - B_\lambda(u)\|^{p-1} + \|u - B_\lambda(u)\|^{q-1})$ *for* $1 < q < p \leq 2$.
(v) $\|J'(u)\|_{(W_0^{1,p})^*} \leq d_5\|u - B_\lambda(u)\|((\|u\| + \|B_\lambda(u)\|)^{p-2} + (\|u\| + \|B_\lambda(u)\|)^{q-2})$ *for* $2 \leq q < p$.
(vi) $\|J'(u)\|_{(W_0^{1,p})^*} \leq d_6\|u - B_\lambda(u)\|(\|u\| + \|B_\lambda(u)\|)^{p-2} + d_6\|u - B_\lambda(u)\|^{q-1}$ *for* $1 < q \leq 2 \leq p$.

Then similar arguments as in [27, Lemma 17] (see also [5, Lemma 4.1]) can be applied to prove the following result on the existence of a locally Lipschitz continuous pseudo-gradient vector field in order to produce an invariant descending flow with respect to the positive and negative cones $\pm P$ defined by (3.21).

Lemma B.3. *Let $\lambda > \lambda_0$, where $\lambda_0 > 0$ is given by assumption* (A1) *of Theorem 3.13. Then, there exists a locally Lipschitz continuous operator $V_\lambda\colon C_0^1(\overline{\Omega}) \setminus K \to C_0^1(\overline{\Omega})$ such that the following hold:*
(i) *For any $u \in C_0^1(\overline{\Omega}) \setminus K$ we have*

$$\begin{aligned}
&\langle J'(u), u - V_\lambda(u)\rangle \geq \frac{d_1}{2}\|u - B_\lambda(u)\|^2\{(\|u\| + \|B_\lambda(u)\|)^{p-2} + (\|u\| + \|B_\lambda(u)\|)^{q-2}\} && \text{for } 1 < q < p \leq 2,\\
&\langle J'(u), u - V_\lambda(u)\rangle \geq \frac{d_2}{2}(\|u - B_\lambda(u)\|^p + \|u - B_\lambda(u)\|^q) && \text{for } 2 \leq q < p,\\
&\langle J'(u), u - V_\lambda(u)\rangle \geq \frac{d_3}{2}\|u - B_\lambda(u)\|^2(\|u\| + \|B_\lambda(u)\|)^{q-2} + \frac{d_3}{2}\|u - B_\lambda(u)\|^p && \text{for } 1 < q \leq 2 \leq p,\\
&\frac{1}{2}\|u - B_\lambda(u)\| \leq \|u - V_\lambda(u)\| \leq 2\|u - B_\lambda(u)\|.
\end{aligned}$$

Here d_1, d_2, and d_3 are the positive constants from Lemma B.2.
(ii) *$V_\lambda(u) \in \pm \operatorname{int} P$ for every $u \in \pm P \setminus K$, respectively.*
(iii) *Let $p^* := \frac{Np}{N-p}$ for $N > p$, and $p^* := p + 1$ otherwise. Set $r_0 := p^*$ and define a sequence $\{r_n\}_{n\in\mathbb{N}}$ inductively as follows:*

$$r_{n+1} := p^* r_n/p = (p^*/p)^{n+1}p^*.$$

Then for any $n \in \mathbb{N}$ there exists a constant $C_n^* > 0$ such that

$$\|V_\lambda(u)\|_{r_{n+1}} \leq C_{n+1}^*(2 + |\Omega| + \|u\|_{r_n}) \quad \text{for all } u \in C_0^1(\overline{\Omega}) \setminus K.$$

(iv) *If $N \geq p$ and $r > \max\{N/p, 1/(p-1)\}$, then there exists a constant $D_1 > 0$ such that*

$$\|V_\lambda(u)\|_\infty \leq D_1(\|u\|_{r(p-1)} + 2 + |\Omega|) \quad \text{for all } u \in C_0^1(\overline{\Omega}) \setminus K.$$

(v) *There exists a constant $D_2 > 0$ such that*

$$\|V_\lambda(u)\|_\infty \leq D_2(2 + \|u\|_\infty) \quad \text{for all } u \in C_0^1(\overline{\Omega}) \setminus K.$$

(vi) *For every $R > 0$ there exist $\gamma \in (0, 1)$ and $M > 0$ such that*

$$\|V_\lambda(u)\|_{C_0^{1,\gamma}(\overline{\Omega})} \leq M \quad \text{for all } u \in C_0^1(\overline{\Omega}) \setminus K \text{ with } \|u\|_\infty \leq R.$$

Now, we will give the proof of Theorem 3.13.

Proof of Theorem 3.13. Note first that the boundary of $\pm P$ in $C_0^1(\overline{\Omega})$ does not intersect with $K \setminus \{0\}$ since any nonnegative (resp. nonpositive) and nontrivial solution of corresponding equation is strictly positive (resp. negative) in Ω and $\partial u/\partial \nu < 0$ (resp. > 0) on $\partial\Omega$ under assumption (A1) of the theorem, due to the strong maximum principle and boundary point lemma (see [29, Theorem 5.3.1 and Theorem 5.5.1]).

Take $\lambda > \lambda_0$ and let V_λ be a locally Lipschitz continuous operator given by Lemma B.3. Consider the following initial value problem in $C_0^1(\overline{\Omega})$:

$$\begin{cases} \dfrac{d\eta}{dt}(t) = -\eta(t) + V_\lambda(\eta(t)),\\ \eta(0) = u. \end{cases}$$

Denote by $\eta(t,u) \in C_0^1(\overline{\Omega})$ its unique solution on the right maximal interval $[0,\tau(u))$. According to assertion (ii) of Lemma B.3, $\eta(t,u)$ is the *invariant descending flow* with respect to the positive cone P and the negative cone $-P$, namely, $\eta(t,u) \in \pm \operatorname{int} P$ for all $0 < t < \tau(u)$ provided $u \in \pm P \setminus K$ (see [23, Lemma 3.2]). Define the sets

$$Q_\pm := \{u \in C_0^1(\overline{\Omega}) \setminus K : \eta(t,u) \in \pm \operatorname{int} P \text{ for some } t \in [0,\tau(u))\} \cup (\pm \operatorname{int} P).$$

It is known that $Q_\pm$ are open subsets of $C_0^1(\overline{\Omega})$ invariant for the descending flow η, and $\partial Q_\pm$ are closed subsets of $C_0^1(\overline{\Omega})$ invariant for η; see [23, Lemma 2.3].

Choose a constant c satisfying $\max_{s\in[0,1]} J(\gamma(s)) < c < 0$, where γ is the continuous path given by assumption (A2) of the theorem. Since $\gamma(0) \in Q_+$, $\gamma(1) \in Q_-$, and $Q_\pm$ are open in $C_0^1(\overline{\Omega})$, there exist $0 < s_+ \le s_- < 1$ such that $\gamma(s_+) \in \partial Q_+$ and $\gamma(s_-) \in \partial Q_-$. Put $u_1 := \gamma(0)$, $u_2 := \gamma(1)$, and $u_3 := \gamma(s_+)$. Due to assertion (i) of Lemma B.3, we know that

$$\frac{d}{dt} J(\eta(t,u_i)) = -\langle J'(\eta(t,u_i)), \eta(t,u_i) - V_\lambda(\eta(t,u_i))\rangle \le 0, \quad i = 1,2,3,$$

which implies that

$$-\infty < \inf_{W_0^{1,p}} J \le J(\eta(t,u_i)) \le c < 0 \quad \text{for every } t \in [0,\tau(u_i)).$$

Hence, the coercivity of J guarantees the existence of $R > 0$ such that for all $t \in [0,\tau(u_i))$ we have

$$\|\eta(t,u_i)\| \le R \quad \text{and} \quad \|B_\lambda(\eta(t,u_i))\| \le R. \tag{B.1}$$

Therefore, if $\tau(u_i) < \infty$ for $i = 1,2,3$, then for every $0 < t_1 < t_2 < \tau(u_i) < \infty$ we have

$$\begin{aligned}\|\eta(t_1,u_i) - \eta(t_2,u_i)\| &\le \int_{t_1}^{t_2} \|\eta(s,u_i) - V_\lambda(\eta(s,u_i))\|\, ds \\ &\le 2\int_{t_1}^{t_2} \|\eta(s,u_i) - B_\lambda(\eta(s,u_i))\|\, ds \le 4R(t_2 - t_1)\end{aligned}$$

by assertion (i) of Lemma B.3 and (B.1). Thus, $\eta(t,u_i)$ converges to some w_i in $W_0^{1,p}$ as $t \to \tau(u_i) - 0$ whenever $\tau(u_i) < \infty$. On account of Lemma B.3 and [27, Lemma 18 (ii)], it is not hard to prove that $w_i \in K$ and $\eta(t,u_i)$ converges to w_i in $C_0^1(\overline{\Omega})$ as $t \to \tau(u_i) - 0$. Recalling now that $Q_\pm$ and $\partial Q_\pm$ are invariant, we see that $J(w_i) \le J(u_i) \le c < 0$, $i = 1,2,3$, and $w_1 \in \operatorname{int} P$, $w_2 \in -\operatorname{int} P$, $w_3 \in \partial Q_+$. Since $\partial Q_+ \cap (\pm P \setminus \{0\}) = \emptyset$ (note that $\pm P \setminus \{0\} \subset Q_\pm$), our conclusion is proved provided $\tau(u_i) < \infty$ for $i = 1,2,3$.

Assume that $\tau(u_i) = \infty$ for some $i \in \{1,2,3\}$. In this case, we can prove the existence of a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ such that

$$t_n \to +\infty \quad \text{and} \quad J'(\eta(t_n,u_i)) \to 0 \quad \text{in } (W_0^{1,p})^* \text{ as } n \to +\infty. \tag{B.2}$$

Note that there exists a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ such that $t_n \to +\infty$ and $\frac{d}{dt}J(\eta(t_n,u_i)) \to 0$ as $n \to +\infty$ since

$$-\infty < \inf_{W_0^{1,p}} J \le J(\eta(t,u_i)) \le c \quad \text{for all } t \ge 0$$

and $J(\eta(t,u_i))$ is nondecreasing in t. Let us show that this sequence satisfies (B.2). If $1 < q < p \le 2$, then Lemma B.2 (iv), Lemma B.3 (i), and (B.1) imply

$$\begin{aligned}-\frac{d}{dt}J(\eta(t,u_i)) &\ge \frac{d_1}{2}\frac{\|\eta(t,u_i) - B_\lambda(\eta(t,u_i))\|^2}{(\|\eta(t,u_i)\| + \|B_\lambda(\eta(t,u_i))\|)^{2-p} + (\|\eta(t,u_i)\| + \|B_\lambda(\eta(t,u_i))\|)^{2-q}} \\ &\ge \frac{d_1}{2}\frac{\|\eta(t,u_i) - B_\lambda(\eta(t,u_i))\|^2}{(2R)^{2-p} + (2R)^{2-q}} \\ &\ge \frac{d_1}{2d_4^{2/(q-1)}(1+(2R)^{p-q})^{2/(q-1)}\{(2R)^{2-p} + (2R)^{2-q}\}}\|J'(\eta(t,u_i))\|_{(W_0^{1,p})^*}^{2/(q-1)}\end{aligned}$$

for all $t > 0$. Hence,

$$\|J'(\eta(t_n, u_i))\|_{(W_0^{1,p})^*} \to 0 \quad \text{as } n \to +\infty.$$

The cases $2 \le q < p$ and $1 < q \le 2 \le p$ can be handled in a similar way using the estimates of Lemma B.2 and Lemma B.3 (i).

Combining now (B.2) with (B.1), we conclude that $\{\eta(t_n, u_i)\}_{n\in\mathbb{N}}$ is a bounded Palais–Smale sequence to J. At the same time, it is not hard to show that J satisfies the Palais–Smale condition because the coercivity of J implies the boundedness of any Palais–Smale sequence (see Lemma 3.4). Thus, there exists $w_i \in W_0^{1,p} \cap K$ such that $\lim_{n\to+\infty} \eta(t_n, u_i) = w_i$ in $W_0^{1,p}$, up to an appropriate subsequence. Furthermore, arguing as in the proof of [27, Lemma 18 (iii)], using Lemma B.3 (iii)–(vi) and (B.1), we see that $\{\eta(t, u_i) : t \ge 0\}$ is bounded in $C_0^{1,\nu}(\overline{\Omega})$ for some $\nu \in (0, 1)$. Thus, the compactness of $C_0^{1,\nu}(\overline{\Omega}) \hookrightarrow C_0^1(\overline{\Omega})$ and $\lim_{n\to+\infty} \eta(t_n, u_i) = w_i$ in $W_0^{1,p}$ imply that $\lim_{n\to+\infty} \eta(t_n, u_i) = w_i$ in $C_0^1(\overline{\Omega})$. Therefore, $w_1 \in \operatorname{int} P$, $w_2 \in -\operatorname{int} P$ and $w_3 \in C_0^1(\overline{\Omega}) \setminus (P \cup -P)$. □

Acknowledgment: The first author wishes to thank the Tokyo University of Science, where the main constructions and results of the article were obtained, for the invitation and hospitality.

Funding: This work was supported by JSPS KAKENHI grant number 15K17577. The work of the first author was also supported by the project LO1506 of the Czech Ministry of Education, Youth and Sports.

References

[1] S. Aizicovici, N. S. Papageorgiou and V. Staicu, Nodal solutions for $(p, 2)$-equations, *Trans. Amer. Math. Soc.* **367** (2015), 7343–7372.
[2] A. Anane, Simplicité et isolation de la premiere valeur propre du p-Laplacien avec poids, *C. R. Acad. Sci. Paris Sér. I* **305** (1987), no. 16, 725–728.
[3] G. E. Andrews, R. Askeyand and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.
[4] S. Barile and G. M. Figueiredo, Some classes of eigenvalues problems for generalized $p\&q$-Laplacian type operators on bounded domains, *Nonlinear Anal.* **119** (2015), 457–468.
[5] T. Bartsch and Z. Liu, On a superlinear elliptic p-Laplacian equation, *J. Differential Equations* **198** (2004), no. 1, 149–175.
[6] T. Bartsch, T. Weth and M. Willem, Partial symmetry of least energy nodal solutions to some variational problems, *J. Anal. Math.* **96** (2005), 1–18.
[7] V. Benci, P. D'Avenia, D. Fortunato and L. Pisani, Solitons in several space dimensions: Derrick's problem and infinitely many solutions, *Arch. Ration. Mech. Anal.* **154** (2000), no. 4, 297–324.
[8] V. Benci, D. Fortunato and L. Pisani, Soliton like solutions of a Lorentz invariant equation in dimension 3, *Rev. Math. Phys.* **10** (1998), no. 3, 315–344.
[9] V. Bobkov, Least energy nodal solutions for elliptic equations with indefinite nonlinearity, *Electron. J. Qual. Theory Differ. Equ.* **2014** (2014), Paper No. 56.
[10] V. Bobkov and M. Tanaka, On positive solutions for (p, q)-Laplace equations with two parameters, *Calc. Var. Partial Differential Equations* **54** (2015), no. 3, 3277–3301.
[11] P. J. Bushell and D. E. Edmunds, Remarks on generalised trigonometric functions, *Rocky Mountain J. Math.* **42** (2012), no. 1, 25–57.
[12] J. W. Cahn and J. E. Hilliard, Free energy of a nonuniform system. I. Interfacial free energy, *J. Chem. Phys.* **28** (1958), no. 2, 258–267.
[13] P. Candito, S. A. Marano and K. Perera, On a class of critical (p, q)-Laplacian problems, *NoDEA Nonlinear Differential Equations Appl.* **22** (2015), no. 6, 1959–1972.
[14] K. C. Chang, *Infinite-Dimensional Morse Theory and Multiple Solution Problems*, Birkhäuser, Boston, 1993.
[15] L. Cherfils and Y. Il'yasov, On the stationary solutions of generalized reaction diffusion equations with $p\&q$-Laplacian, *Comm. Pure Appl. Math.* **4** (2005), no. 1, 9–22.
[16] M. Cuesta, D. de Figueiredo and J.-P. Gossez, The beginning of the Fučik spectrum for the p-Laplacian, *J. Differential Equations* **159** (1999), no. 1, 212–238.
[17] P. Drábek and R. Manásevich, On the closed solution to some nonhomogeneous eigenvalue problems with p-Laplacian, *Differential Integral Equations* **12** (1999), no. 6, 773–788.
[18] P. Drábek and S. B. Robinson, Resonance problems for the p-Laplacian, *J. Funct. Anal.* **169** (1999), no. 1, 189–200.
[19] T. He, H. Yan, Z. Sun and M. Zhang, On nodal solutions for nonlinear elliptic equations with a nonhomogeneous differential operator, *Nonlinear Anal.* **118** (2015), 41–50.

[20] R. Kajikiya, M. Tanaka and S. Tanaka, Bifurcation of positive solutions for the one dimensional (p, q)-Laplace equation, submitted.

[21] G. M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Anal.* **12** (1988), no. 11, 1203–1219.

[22] G. M. Lieberman, The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations, *Comm. Partial Differential Equations* **16** (1991), no. 2–3, 311–361.

[23] Z. L. Liu and J. X. Sun, Invariant sets of descending flow in critical point theory with applications to nonlinear differential equations, *J. Differential Equations* **172** (2001), 257–299.

[24] S. A. Marano and N. S. Papageorgiou, Constant-sign and nodal solutions of coercive (p, q)-Laplacian problems, *Nonlinear Anal.* **77** (2013), 118–129.

[25] S. Miyajima, D. Motreanu and M. Tanaka, Multiple existence results of solutions for the Neumann problems via super-and sub-solutions, *J. Funct. Anal.* **262** (2012), no. 4, 1921–1953.

[26] R. Molle and D. Passaseo, Variational properties of the first curve of the Fučík spectrum for elliptic operators, *Calc. Var. Partial Differential Equations* **54** (2015), no. 4, 3735–3752.

[27] D. Motreanu and M. Tanaka, Multiple existence results of solutions for quasilinear elliptic equations with a nonlinearity depending on a parameter, *Ann. Mat. Pura Appl. (4)* **193** (2014), no. 5, 1255–1282.

[28] D. Motreanu and M. Tanaka, On a positive solution for (p, q)-Laplace equation with indefinite weight, *Minimax Theory Appl.* **1** (2016), no. 1, 1–20.

[29] P. Pucci and J. Serrin, *The maximum Principle*, Progr. Nonlinear Differential Equations Appl. 73, Birkhäuser, Basel, 2007.

[30] H. R. Quoirin, An indefinite type equation involving two p-Laplacians, *J. Math. Anal. Appl.* **387** (2012), no. 1, 189–200.

[31] M. Tanaka, Generalized eigenvalue problems for (p, q)-Laplacian with indefinite weight, *J. Math. Anal. Appl.* **419** (2014), no. 2, 1181–1192.

[32] M. Tanaka, Uniqueness of a positive solution and existence of a sign-changing solution for (p, q)-Laplace equation, *J. Nonlinear Funct. Anal.* **2014** (2014), 1–15.

[33] V. E. Zakharov, Collapse of Langmuir waves, *Sov. J. Exp. Theoret. Phys.* **35** (1972), no. 5, 908–914.

[34] A. A. Zerouali and B. Karim, Existence and nonexistence of a positive solution for (p, q)-Laplacian with singular weight, *Bol. Soc. Parana. Mat. (3)* **34** (2016), 147–167.