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Research Article

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Solvability of a product-type system of difference equations with six parameters

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Abstract: Closed form formulas for well-defined complex-valued solutions to a product-type system of difference equations of interest with six parameters are presented. The form of the solutions is described in detail in terms of the parameters and initial values.

Keywords: System of difference equations, product-type system, closed form formula

MSC 2010: 39A20, 39A45

1 Introduction

Difference equations and systems have been studied a lot recently (see, e.g., [1–5, 7, 8, 10, 12, 14–38]). About two decades ago Papaschinopoulos and Schinas started investigating concrete symmetric and related systems of difference equations (see, e.g., [14–16]). The investigation later motivated numerous authors to conduct research in this direction (see, e.g., [3, 7, 10, 18, 19, 23, 26–30, 33–38] and the references therein). On the other hand, the problem of solvability of difference equations and systems has re-attracted some recent attention, especially after the publication of our note [20] which explains a formula in [5] and the publication of [21] in which an open problem from population biology was solved by showing the solvability of the difference equation appearing there (see, e.g., [1–3, 17, 23–32, 34–38]). The ideas in [20] were later developed in [24] and [25] for some related equations, and in [23] and [34] for some related systems. In some of these papers, among others, have appeared some new classes of solvable symmetric or closely related systems of difference equations, as well as solvable cyclic ones [30, 34]. Some max-type systems are solvable too [26]. Classical methods for solving difference equations and systems can be found, e.g., in [9, 11, 13].

Some recent papers have also studied equations and systems whose right-hand sides contain product-type expressions (see, e.g., [22, 33] and the references therein). In the study of the equations and systems in [22] and [33] their methodological connection with the product-type ones was essentially noticed. One of the reasons for this is that some product-type equations and systems are obtained by taking the limits of some parameters in the equations and systems in [22] and [33]. It is well known that the product-type systems of difference equations are solvable in some cases (e.g., if initial values are positive), but, in general, this is not always the case. The problem appears if some initial values are not nonnegative, because power functions are multi-valued in the complex plain $\mathbb{C}$ in many cases. These facts have motivated us to investigate the solvability of some concrete product-type systems of difference equations with complex initial values in detail. More precisely, the problem is to find systems for which it is possible to find explicit formulas for all their solutions.

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Recent papers [29, 30, 32, 37, 38] deal with this problem (this problem appeared in [31] too, as some special cases of the equation treated therein). Papers [29, 32, 37, 38] study some two-dimensional product-type systems of difference equations with different delays, while [30] studies a three-dimensional. For the above-mentioned problem connected to multi-valued functions, all these systems could not be dealt with the transformation methods presented in some of our previous papers such as [3, 20, 21, 23–25, 27, 34–36]. Hence, we have been developing some other methods for studying the product-type systems, related to some in [22] and [33], and have some coefficients as the systems in [32] and [38].

Here we continue the study of practical solvability of product-type systems, by investigating the solvability of the following one:

$$z_{n+1} = \alpha z_n^a w_n^b, \quad w_{n+1} = \beta w_{n-1}^c z_n^d, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{C}$ and $z_0, w_{-1}, w_0 \in \mathbb{C}$, in detail.

If $\alpha = 0$ or $\beta = 0$, then solutions of (1.1) are trivial or not well-defined. Hence, the case $\alpha \neq 0 \neq \beta$ is of some interest and it will be treated in the rest of the paper.

Note that the domain of undefinable solutions [27] to (1.1) is a subset of the set

$$\mathcal{U} = \{(z_0, w_{-1}, w_0) \in \mathbb{C}^3 : z_0 = 0 \text{ or } w_{-1} = 0 \text{ or } w_0 = 0\}.$$

If $\min\{a, b, c\} < 0$ or $\min\{b, c, d\} < 0$, then the domain is $\mathcal{U}$. Hence, from now on we will also assume that $z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$, although many obtained formulas will hold on some larger sets or even on the whole $\mathbb{C}^3$ (for example, if $\min\{a, b, c, d\} > 0$).

It is said that the system of the form

$$\begin{aligned} z_n &= f(z_{n-1}, \dots, z_{n-k}, w_{n-1}, \dots, w_{n-l}), \\ w_n &= g(z_{n-1}, \dots, z_{n-s}, w_{n-1}, \dots, w_{n-t}), \quad n \in \mathbb{N}_0, \end{aligned}$$

where $k, l, s, t \in \mathbb{N}$, is *solvable in closed form* if its general solution can be found in terms of initial values z_{-i}, $i = \overline{1, \max\{k, s\}}$, w_{-j}, $j = \overline{1, \max\{l, t\}}$, delays k, l, s, t, and index n only. As usual, if $m > n$ we regard that $\sum_{k=m}^n a_k$ is equal to zero.

2 Auxiliary results

In this section we present two auxiliary results which will be used in the proofs of the main results. The following elementary lemma is known (see, e.g., [13]).

Lemma 2.1. *Let $i \in \mathbb{N}_0$ and*

$$s_n^{(i)}(z) = 1 + 2^i z + 3^i z^2 + \cdots + n^i z^{n-1}, \quad n \in \mathbb{N}, \tag{2.1}$$

where $z \in \mathbb{C}$. Then

$$s_n^{(0)}(z) = \frac{1 - z^n}{1 - z}, \tag{2.2}$$

$$s_n^{(1)}(z) = \frac{1 - (n+1)z^n + nz^{n+1}}{(1 - z)^2}, \tag{2.3}$$

$$s_n^{(2)}(z) = \frac{1 + z - (n+1)^2 z^n + (2n^2 + 2n - 1)z^{n+1} - n^2 z^{n+2}}{(1 - z)^3} \tag{2.4}$$

for every $z \in \mathbb{C} \setminus \{1\}$ and $n \in \mathbb{N}$.

Formula (2.2) is something which should be known to any mathematician, while (2.3) and (2.4) can be obtained by using the following result, which seems not so well known.

Proposition 2.2. *Sequences defined in* (2.1) *satisfy the following recurrent relation:*

$$s_n^{(k)}(z) = \frac{\sum_{i=0}^{k-1} C_i^k (-1)^{k-1-i} s_n^{(i)}(z) - n^k z^n}{1 - z} \tag{2.5}$$

for every $k \in \mathbb{N}$ and $z \in \mathbb{C} \setminus \{0\}$.

Proof. We have

$$\begin{aligned}
s_n^{(k)}(z) - zs_n^{(k)}(z) &= 1 + \sum_{j=2}^{n}(j^k - (j-1)^k)z^{j-1} - n^k z^n \\
&= 1 + \sum_{j=2}^{n}\sum_{i=0}^{k-1} C_i^k(-1)^{k-1-i} j^i z^{j-1} - n^k z^n \\
&= 1 + \sum_{i=0}^{k-1} C_i^k(-1)^{k-1-i} \sum_{j=2}^{n} j^i z^{j-1} - n^k z^n \\
&= C_k^k + \sum_{i=0}^{k-1} C_i^k(-1)^{k-1-i}(s_n^{(i)}(z) - 1) - n^k z^n \\
&= \sum_{i=0}^{k} C_i^k(-1)^{k-i} + \sum_{i=0}^{k-1} C_i^k(-1)^{k-1-i} s_n^{(i)}(z) - n^k z^n \\
&= \sum_{i=0}^{k-1} C_i^k(-1)^{k-1-i} s_n^{(i)}(z) - n^k z^k,
\end{aligned}$$

where we have used the fact $\sum_{i=0}^{k} C_i^k(-1)^{k-i} = 0$, from which (2.5) follows. □

Example 2.3. If $k = 2$, then from (2.5) and some calculations it follows that

$$\begin{aligned}
s_n^{(2)}(z) &= \frac{-C_0^2 s_n^{(0)}(z) + C_1^2 s_n^{(1)}(z) - n^2 z^n}{1-z} = \frac{-\frac{1-z^n}{1-z} + 2\frac{1-(n+1)z^n + nz^{n+1}}{(1-z)^2} - n^2 z^n}{1-z} \\
&= \frac{1 + z - (n+1)^2 z^n + (2n^2 + 2n - 1)z^{n+1} - n^2 z^{n+2}}{(1-z)^3}.
\end{aligned}$$

The following result is a known consequence of the Lagrange interpolation formula, but can be also proved by using some other techniques (see, e.g., [37]).

Lemma 2.4. *Let*

$$P_k(z) = f_k z^k + f_{k-1} z^{k-1} + \cdots + f_1 z + f_0.$$

If the zeros z_j, $j = 1, \ldots, k$, of P_k are mutually different, then

$$\sum_{j=1}^{k} \frac{z_j^l}{P_k'(z_j)} = 0$$

for each $l \in \{0, 1, \ldots, k-2\}$, and

$$\sum_{j=1}^{k} \frac{z_j^{k-1}}{P_k'(z_j)} = \frac{1}{f_k}.$$

3 Main results

The main results in this paper are proved in this section.

Theorem 3.1. *Assume that $b, c, d \in \mathbb{Z}$, $a = 0$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, and $z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then system (1.1) is solvable in closed form.*

Proof. Since $a = 0$, system (1.1) is

$$z_{n+1} = \alpha w_n^b, \quad w_{n+1} = \beta w_{n-1}^c z_n^d, \quad n \in \mathbb{N}_0. \tag{3.1}$$

Using the first equation in (3.1) in the second one, we obtain

$$w_{n+1} = \beta\alpha^d w_{n-1}^{c+bd}, \quad n \in \mathbb{N}. \tag{3.2}$$

Iterating relation (3.2), we can easily show that

$$w_{2n} = (\beta\alpha^d)^{\sum_{j=0}^{n-1}(c+bd)^j} w_0^{(c+bd)^n} = \alpha^{d\sum_{j=0}^{n-1}(c+bd)^j} \beta^{\sum_{j=0}^{n-1}(c+bd)^j} w_0^{(c+bd)^n} \tag{3.3}$$

and

$$\begin{aligned} w_{2n+1} &= (\beta\alpha^d)^{\sum_{j=0}^{n-1}(c+bd)^j} w_1^{(c+bd)^n} \\ &= (\beta\alpha^d)^{\sum_{j=0}^{n-1}(c+bd)^j} (\beta w_{-1}^c z_0^d)^{(c+bd)^n} \\ &= \alpha^{d\sum_{j=0}^{n-1}(c+bd)^j} \beta^{\sum_{j=0}^{n}(c+bd)^j} w_{-1}^{c(c+bd)^n} z_0^{d(c+bd)^n} \end{aligned} \tag{3.4}$$

for every $n \in \mathbb{N}$.

Employing (3.3) and (3.4) into the first equation in (3.1), we get

$$z_{2n+1} = \alpha^{1+bd\sum_{j=0}^{n-1}(c+bd)^j} \beta^{b\sum_{j=0}^{n-1}(c+bd)^j} w_0^{b(c+bd)^n} \tag{3.5}$$

and

$$z_{2n+2} = \alpha^{1+bd\sum_{j=0}^{n-1}(c+bd)^j} \beta^{b\sum_{j=0}^{n}(c+bd)^j} w_{-1}^{bc(c+bd)^n} z_0^{bd(c+bd)^n} \tag{3.6}$$

for every $n \in \mathbb{N}$.

If $c + bd \neq 1$, then from (3.3)–(3.6) and (2.2) we have

$$z_{2n+1} = \alpha^{\frac{1-c-bd(c+bd)^n}{1-c-bd}} \beta^{b\frac{1-(c+bd)^n}{1-c-bd}} w_0^{b(c+bd)^n}, \tag{3.7}$$

$$z_{2n+2} = \alpha^{\frac{1-c-bd(c+bd)^n}{1-c-bd}} \beta^{b\frac{1-(c+bd)^{n+1}}{1-c-bd}} w_{-1}^{bc(c+bd)^n} z_0^{bd(c+bd)^n}, \tag{3.8}$$

$$w_{2n} = \alpha^{d\frac{1-(c+bd)^n}{1-c-bd}} \beta^{\frac{1-(c+bd)^n}{1-c-bd}} w_0^{(c+bd)^n}, \tag{3.9}$$

$$w_{2n+1} = \alpha^{d\frac{1-(c+bd)^n}{1-c-bd}} \beta^{\frac{1-(c+bd)^{n+1}}{1-c-bd}} w_{-1}^{c(c+bd)^n} z_0^{d(c+bd)^n} \tag{3.10}$$

for every $n \in \mathbb{N}$, while if $c + bd = 1$, then we have

$$z_{2n+1} = \alpha^{1+bdn}\beta^{bn}w_0^b, \tag{3.11}$$

$$z_{2n+2} = \alpha^{1+bdn}\beta^{b(n+1)}w_{-1}^{bc}z_0^{bd}, \tag{3.12}$$

$$w_{2n} = \alpha^{dn}\beta^n w_0, \tag{3.13}$$

$$w_{2n+1} = \alpha^{dn}\beta^{n+1}w_{-1}^c z_0^d \tag{3.14}$$

for every $n \in \mathbb{N}$, finishing the proof of the theorem. □

Corollary 3.2. *Consider system* (1.1) *with* $b, c, d \in \mathbb{Z}$, $a = 0$ *and* $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. *Furthermore assume that* $z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true:*

(i) *If* $c + bd \neq 1$, *then the general solution to system* (1.1) *is given by* (3.7)–(3.10).

(ii) *If* $c + bd = 1$, *then the general solution to system* (1.1) *is given by* (3.11)–(3.14).

Theorem 3.3. *Assume that* $a, b, d \in \mathbb{Z}$, $c = 0$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, *and* $z_0, w_0 \in \mathbb{C} \setminus \{0\}$. *Then system* (1.1) *is solvable in closed form.*

Proof. In this case system (1.1) becomes

$$z_{n+1} = \alpha z_n^a w_n^b, \quad w_{n+1} = \beta z_n^d, \quad n \in \mathbb{N}_0. \tag{3.15}$$

The system essentially appeared in [29], but we will give a proof for the sake of completeness. Using the second equation in (3.15) in the first one, we obtain

$$z_{n+1} = \alpha\beta^b z_n^a z_{n-1}^{bd}, \quad n \in \mathbb{N}. \tag{3.16}$$

Now we consider the subcases $bd = 0$ and $bd \neq 0$ separately.

Subcase $bd = 0$. In this case we have

$$z_n = \alpha\beta^b z_{n-1}^a, \quad n \geq 2,$$

from which it follows that

$$z_n = (\alpha\beta^b)^{\sum_{j=0}^{n-2} a^j} z_1^{a^{n-1}} = (\alpha\beta^b)^{\sum_{j=0}^{n-2} a^j} (\alpha z_0^a w_0^b)^{a^{n-1}} = \alpha^{\sum_{j=0}^{n-1} a^j} \beta^{b\sum_{j=0}^{n-2} a^j} z_0^{a^n} w_0^{ba^{n-1}}, \quad n \geq 2. \tag{3.17}$$

Using (3.17) in the second equation in (3.15), we get

$$w_n = \beta(\alpha^{\sum_{j=0}^{n-2} a^j} \beta^{b\sum_{j=0}^{n-3} a^j} z_0^{a^{n-1}} w_0^{ba^{n-2}})^d = \alpha^{d\sum_{j=0}^{n-2} a^j} \beta z_0^{da^{n-1}}, \quad n \geq 3. \tag{3.18}$$

A direct calculation shows that (3.18) holds also for $n = 2$.

From (3.17) and (3.18) we have that

$$z_n = \alpha^{\frac{1-a^n}{1-a}} \beta^{b\frac{1-a^{n-1}}{1-a}} z_0^{a^n} w_0^{ba^{n-1}}, \quad n \geq 2, \tag{3.19}$$

$$w_n = \alpha^{d\frac{1-a^{n-1}}{1-a}} \beta z_0^{da^{n-1}}, \quad n \geq 2, \tag{3.20}$$

in the case $a \neq 1$, while in the case $a = 1$ we have

$$z_n = \alpha^n \beta^{b(n-1)} z_0 w_0^b, \quad n \geq 2, \tag{3.21}$$

$$w_n = \alpha^{d(n-1)} \beta z_0^d, \quad n \geq 2. \tag{3.22}$$

Subcase $bd \neq 0$. Let

$$\gamma := \alpha\beta^b, \quad a_1 := a, \quad b_1 := bd, \quad x_1 := 1.$$

Then (3.16) can be written as

$$z_{n+1} = \gamma^{x_1} z_n^{a_1} z_{n-1}^{b_1}, \quad n \in \mathbb{N}, \tag{3.23}$$

from which it follows that

$$z_{n+1} = \gamma^{x_1}(\gamma z_{n-1}^{a_1} z_{n-2}^{b_1})^{a_1} z_{n-1}^{b_1} = \gamma^{x_1+a_1} z_{n-1}^{a_1a_1+b_1} z_{n-2}^{b_1a_1} = \gamma^{x_2} z_{n-1}^{a_2} z_{n-2}^{b_2} \tag{3.24}$$

for $n \geq 2$, where

$$a_2 := a_1a_1 + b_1, \quad b_2 := b_1a_1, \quad x_2 := x_1 + a_1. \tag{3.25}$$

Assume that for some $k \geq 2$ we have

$$z_{n+1} = \gamma^{x_k} z_{n+1-k}^{a_k} z_{n-k}^{b_k}, \quad n \geq k, \tag{3.26}$$

where

$$a_k := a_1a_{k-1} + b_{k-1}, \quad b_k := b_1a_{k-1}, \quad x_k := x_{k-1} + a_{k-1}. \tag{3.27}$$

Then, using (3.23) with $n \to n-k$ into (3.26), we obtain

$$z_{n+1} = \gamma^{x_k}(\gamma z_{n-k}^{a_1} z_{n-k-1}^{b_1})^{a_k} z_{n-k}^{b_k} = \gamma^{x_k+a_k} z_{n-k}^{a_1a_k+b_k} z_{n-k-1}^{b_1a_k} = \gamma^{x_{k+1}} z_{n-k}^{a_{k+1}} z_{n-k-1}^{b_{k+1}} \tag{3.28}$$

for $n \geq k+1$, where

$$a_{k+1} := a_1a_k + b_k, \quad b_{k+1} := b_1a_k, \quad x_{k+1} := x_k + a_k. \tag{3.29}$$

From (3.24), (3.25), (3.28), (3.29), and the induction we see that (3.26) and (3.27) hold for every k and n such that $2 \leq k \leq n$ ((3.26) also holds for $k = 1$ because of (3.23)).

Now note that from the first two equations in (3.27) we get

$$a_k = a_1a_{k-1} + b_1a_{k-2}, \quad k \geq 3. \tag{3.30}$$

The equalities in (3.27) with $k = 1$ yield

$$a_1 = a_1a_0 + b_0, \quad b_1 = b_1a_0, \quad x_1 = x_0 + a_0. \tag{3.31}$$

Since $b_1 = bd \neq 0$, from the second equation in (3.31) we get $a_0 = 1$. This, along with $x_1 = 1$ and the other two relations in (3.31) implies $b_0 = x_0 = 0$.

From this and (3.27) with $k = 0$, we obtain

$$1 = a_0 = a_1 a_{-1} + b_{-1}, \quad 0 = b_0 = b_1 a_{-1}, \quad 0 = x_0 = x_{-1} + a_{-1}. \tag{3.32}$$

Since $b_1 \neq 0$, from the second equation in (3.32) we get $a_{-1} = 0$. This, along with the other two relations in (3.32), implies $b_{-1} = 1$ and $x_{-1} = 0$.

From this and the second equation in (3.27), we have that $(a_k)_{k\geq -1}$ and $(b_k)_{k\geq -1}$ are solutions to linear equation (3.30) satisfying the initial conditions

$$a_{-1} = 0, \quad a_0 = 1, \quad \text{and} \quad b_{-1} = 1, \quad b_0 = 0,$$

respectively, and that $(x_k)_{k\geq -1}$ satisfies the third recurrent relation in (3.27) and

$$x_{-1} = x_0 = 0, \quad x_1 = 1.$$

From (3.26) with $n \to n-1$ and $k = n-1$, along with the equality $z_1 = \alpha z_0^a w_0^b$ and the first and third relations in (3.27), we have that

$$\begin{aligned} z_n &= \gamma^{x_{n-1}} z_1^{a_{n-1}} z_0^{b_{n-1}} = (\alpha\beta^b)^{x_{n-1}} (\alpha z_0^a w_0^b)^{a_{n-1}} z_0^{b_{n-1}} \\ &= \alpha^{a_{n-1}+x_{n-1}} \beta^{b x_{n-1}} z_0^{a a_{n-1}+b_{n-1}} w_0^{b a_{n-1}} \\ &= \alpha^{x_n} \beta^{b x_{n-1}} z_0^{a_n} w_0^{b a_{n-1}}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{3.33}$$

Using this in the second equation in (3.15), we obtain

$$w_n = \alpha^{d x_{n-1}} \beta^{1+bd x_{n-2}} z_0^{d a_{n-1}} w_0^{bd a_{n-2}}, \quad n \in \mathbb{N}. \tag{3.34}$$

From the third equation in (3.27) and since $x_1 = 1$, we get

$$x_n = 1 + \sum_{j=1}^{n-1} a_j, \quad n \in \mathbb{N},$$

which due to the fact $a_0 = 1$ can be written as

$$x_n = \sum_{j=0}^{n-1} a_j, \quad n \in \mathbb{N}. \tag{3.35}$$

Now note that the characteristic equation associated to difference equation (3.30) is $\lambda^2 - a\lambda - bd = 0$, from which it follows that

$$\lambda_{1,2} = \frac{a \pm \sqrt{a^2 + 4bd}}{2}.$$

Hence if $a^2 + 4bd \neq 0$, then

$$a_n = c_1 \lambda_1^n + c_2 \lambda_2^n.$$

From this and since $a_{-1} = 0$ and $a_0 = 1$, we have that

$$a_n = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}, \tag{3.36}$$

which along with the second equation in (3.27) implies

$$b_n = bd \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}.$$

Using (3.36) in (3.35) for the case when $\lambda_1 \neq 1 \neq \lambda_2$, which is equivalent to $a + bd \neq 1$, we get

$$\begin{aligned} x_n &= \sum_{j=0}^{n-1} \frac{\lambda_1^{j+1} - \lambda_2^{j+1}}{\lambda_1 - \lambda_2} \\ &= \frac{1}{(\lambda_1 - \lambda_2)}\Big(\lambda_1 \frac{\lambda_1^n - 1}{\lambda_1 - 1} - \lambda_2 \frac{\lambda_2^n - 1}{\lambda_2 - 1}\Big) \\ &= \frac{(\lambda_2 - 1)\lambda_1^{n+1} - (\lambda_1 - 1)\lambda_2^{n+1} + \lambda_1 - \lambda_2}{(\lambda_1 - 1)(\lambda_2 - 1)(\lambda_1 - \lambda_2)}, \end{aligned} \tag{3.37}$$

while if $a + bd = 1$, that is, if one of the characteristic roots is equal to one, say λ_2, which implies that $\lambda_1 = -bd$, we get

$$\begin{aligned} x_n &= \sum_{j=0}^{n-1} \frac{\lambda_1^{j+1} - 1}{\lambda_1 - 1} = \frac{1}{(\lambda_1 - 1)}\Big(\lambda_1 \frac{\lambda_1^n - 1}{\lambda_1 - 1} - n\Big) \\ &= \frac{\lambda_1^{n+1} - (n+1)\lambda_1 + n}{(\lambda_1 - 1)^2} = \frac{(-bd)^{n+1} + (n+1)bd + n}{(1 + bd)^2}. \end{aligned} \tag{3.38}$$

If $a^2 + 4bd = 0$, then

$$a_n = (\hat{c}_1 + \hat{c}_2 n)\Big(\frac{a}{2}\Big)^n.$$

From this and since $a_{-1} = 0$ and $a_0 = 1$, we have that

$$a_n = (n+1)\Big(\frac{a}{2}\Big)^n, \tag{3.39}$$

which along with the second equation in (3.27) and $bd = -a^2/4$, implies

$$b_n = bdn\Big(\frac{a}{2}\Big)^{n-1} = -n\Big(\frac{a}{2}\Big)^{n+1}.$$

Using (3.39) in (3.35) and employing (2.3) for the case $a \neq 2$, we get

$$x_n = \sum_{j=0}^{n-1} (j+1)\Big(\frac{a}{2}\Big)^j = \frac{1 - (n+1)(\frac{a}{2})^n + n(\frac{a}{2})^{n+1}}{(1 - \frac{a}{2})^2}, \tag{3.40}$$

while if $a = 2$, we get

$$x_n = \sum_{j=0}^{n-1} (j+1) = \frac{n(n+1)}{2}, \tag{3.41}$$

as desired. □

Corollary 3.4. *Consider system* (1.1) *with* $a, b, d \in \mathbb{Z}$, $c = 0$ *and* $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. *Assume that* $z_0, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true:*

(i) *If* $bd = 0$ *and* $a \neq 1$, *then the general solution to system* (1.1) *is given by* (3.19) *and* (3.20).

(ii) *If* $bd = 0$ *and* $a = 1$, *then the general solution to system* (1.1) *is given by* (3.21) *and* (3.22).

(iii) *If* $bd \neq 0$, $a^2 + 4bd \neq 0$ *and* $a + bd \neq 1$, *then the general solution to system* (1.1) *is given by formulas* (3.33) *and* (3.34), *where the sequences* $(a_n)_{n \geq -1}$ *and* $(x_n)_{n \geq -1}$ *are given by formulas* (3.36) *and* (3.37), *respectively.*

(iv) *If* $bd \neq 0$, $a^2 + 4bd \neq 0$ *and* $a + bd = 1$, *then the general solution to system* (1.1) *is given by formulas* (3.33) *and* (3.34), *where the sequences* $(a_n)_{n \geq -1}$ *and* $(x_n)_{n \geq -1}$ *are given by formulas* (3.36) *and* (3.38), *respectively.*

(v) *If* $bd \neq 0$, $a^2 + 4bd = 0$ *and* $a \neq 2$, *then the general solution to system* (1.1) *is given by formulas* (3.33) *and* (3.34), *where the sequences* $(a_n)_{n \geq -1}$ *and* $(x_n)_{n \geq -1}$ *are given by formulas* (3.39) *and* (3.40), *respectively.*

(vi) *If* $a^2 + 4bd = 0$ *and* $a = 2$, *then the general solution to system* (1.1) *is given by formulas* (3.33) *and* (3.34), *where the sequence* $(a_n)_{n \geq -1}$ *is given by formula* (3.39) *with* $a = 2$, *while* $(x_n)_{n \geq -1}$ *is given by* (3.41).

Theorem 3.5. *Assume that* $a, c, d \in \mathbb{Z}$, $b = 0$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, *and* $z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then system* (1.1) *is solvable in closed form.*

Proof. In this case system (1.1) becomes

$$z_{n+1} = \alpha z_n^a, \quad w_{n+1} = \beta w_{n-1}^c z_n^d, \quad n \in \mathbb{N}_0. \tag{3.42}$$

From the first equation in (3.42) we get

$$z_n = \alpha^{\sum_{j=0}^{n-1} a^j} z_0^{a^n}, \quad n \in \mathbb{N}, \tag{3.43}$$

which for the case $a \neq 1$ implies that

$$z_n = \alpha^{\frac{1-a^n}{1-a}} z_0^{a^n}, \quad n \in \mathbb{N}, \tag{3.44}$$

while for the case $a = 1$ it implies that

$$z_n = \alpha^n z_0, \quad n \in \mathbb{N}. \tag{3.45}$$

Employing (3.43) in the second equation in (3.42), we obtain

$$w_{n+1} = \beta \alpha^{d \sum_{j=0}^{n-1} a^j} z_0^{da^n} w_{n-1}^c, \quad n \in \mathbb{N}. \tag{3.46}$$

Using (3.46) twice, we obtain

$$\begin{aligned} w_{2n} &= \beta \alpha^{d \sum_{j=0}^{2n-2} a^j} z_0^{da^{2n-1}} w_{2n-2}^c \\ &= \beta \alpha^{d \sum_{j=0}^{2n-2} a^j} z_0^{da^{2n-1}} \left(\beta \alpha^{d \sum_{j=0}^{2n-4} a^j} z_0^{da^{2n-3}} w_{2n-4}^c\right)^c \\ &= \beta^{1+c} \alpha^{d \sum_{j=0}^{2n-2} a^j + dc \sum_{j=0}^{2n-4} a^j} z_0^{da^{2n-1} + dca^{2n-3}} w_{2n-4}^{c^2}, \end{aligned} \tag{3.47}$$

$$\begin{aligned} w_{2n+1} &= \beta \alpha^{d \sum_{j=0}^{2n-1} a^j} z_0^{da^{2n}} w_{2n-1}^c \\ &= \beta \alpha^{d \sum_{j=0}^{2n-1} a^j} z_0^{da^{2n}} \left(\beta \alpha^{d \sum_{j=0}^{2n-3} a^j} z_0^{da^{2n-2}} w_{2n-3}^c\right)^c \\ &= \beta^{1+c} \alpha^{d \sum_{j=0}^{2n-1} a^j + dc \sum_{j=0}^{2n-3} a^j} z_0^{da^{2n} + dca^{2n-2}} w_{2n-3}^{c^2}, \quad n \geq 2. \end{aligned} \tag{3.48}$$

Assume that for a natural number k the equations

$$w_{2n} = \beta^{\sum_{j=0}^{k-1} c^j} \alpha^{d \sum_{i=0}^{k-1} c^i \sum_{j=0}^{2n-2i-2} a^j} z_0^{d \sum_{j=0}^{k-1} c^j a^{2n-2j-1}} w_{2n-2k}^{c^k}, \tag{3.49}$$

$$w_{2n+1} = \beta^{\sum_{j=0}^{k-1} c^j} \alpha^{d \sum_{i=0}^{k-1} c^i \sum_{j=0}^{2n-2i-1} a^j} z_0^{d \sum_{j=0}^{k-1} c^j a^{2n-2j}} w_{2n-2k+1}^{c^k} \tag{3.50}$$

for every $n \geq k$ have been proved.

By using (3.46) with $n \to 2n - 2k - 1$ and $n \to 2n - 2k$ in (3.49) and (3.50), we get

$$\begin{aligned} w_{2n} &= \beta^{\sum_{j=0}^{k-1} c^j} \alpha^{d \sum_{i=0}^{k-1} c^i \sum_{j=0}^{2n-2i-2} a^j} z_0^{d \sum_{j=0}^{k-1} c^j a^{2n-2j-1}} \left(\beta \alpha^{d \sum_{j=0}^{2n-2k-2} a^j} z_0^{da^{2n-2k-1}} w_{2n-2k-2}^c\right)^{c^k} \\ &= \beta^{\sum_{j=0}^{k} c^j} \alpha^{d \sum_{i=0}^{k} c^i \sum_{j=0}^{2n-2i-2} a^j} z_0^{d \sum_{j=0}^{k} c^j a^{2n-2j-1}} w_{2n-2k-2}^{c^{k+1}}, \end{aligned} \tag{3.51}$$

$$\begin{aligned} w_{2n+1} &= \beta^{\sum_{j=0}^{k-1} c^j} \alpha^{d \sum_{i=0}^{k-1} c^i \sum_{j=0}^{2n-2i-1} a^j} z_0^{d \sum_{j=0}^{k-1} c^j a^{2n-2j}} \left(\beta \alpha^{d \sum_{j=0}^{2n-2k-1} a^j} z_0^{da^{2n-2k}} w_{2n-2k-1}^c\right)^{c^k} \\ &= \beta^{\sum_{j=0}^{k} c^j} \alpha^{d \sum_{i=0}^{k} c^i \sum_{j=0}^{2n-2i-1} a^j} z_0^{d \sum_{j=0}^{k} c^j a^{2n-2j}} w_{2n-2k-1}^{c^{k+1}} \end{aligned} \tag{3.52}$$

for every $n \geq k + 1$.

From (3.47), (3.48), (3.51), (3.52), and induction it follows that (3.49) and (3.50) hold for all $k, n \in \mathbb{N}$ such that $1 \leq k \leq n$. Taking $k = n$ in (3.49) and (3.50), we get

$$\begin{aligned} w_{2n} &= \beta^{\sum_{j=0}^{n-1} c^j} \alpha^{d \sum_{i=0}^{n-1} c^i \sum_{j=0}^{2n-2i-2} a^j} z_0^{d \sum_{j=0}^{n-1} c^j a^{2n-2j-1}} w_0^{c^n}, \\ w_{2n+1} &= \beta^{\sum_{j=0}^{n-1} c^j} \alpha^{d \sum_{i=0}^{n-1} c^i \sum_{j=0}^{2n-2i-1} a^j} z_0^{d \sum_{j=0}^{n-1} c^j a^{2n-2j}} w_1^{c^n} \\ &= \beta^{\sum_{j=0}^{n-1} c^j} \alpha^{d \sum_{i=0}^{n-1} c^i \sum_{j=0}^{2n-2i-1} a^j} z_0^{d \sum_{j=0}^{n-1} c^j a^{2n-2j}} (\beta w_{-1}^c z_0^d)^{c^n} \\ &= \beta^{\sum_{j=0}^{n} c^j} \alpha^{d \sum_{i=0}^{n-1} c^i \sum_{j=0}^{2n-2i-1} a^j} z_0^{d \sum_{j=0}^{n} c^j a^{2n-2j}} w_{-1}^{c^{n+1}} \end{aligned}$$

for $n \in \mathbb{N}$.

Subcase $a \neq 1 \neq c$, $c \neq a^2$. By multiple use of (2.2) and some calculation, we have

$$\begin{aligned} w_{2n} &= \beta^{\frac{1-c^n}{1-c}} \alpha^{d\sum_{i=0}^{n-1} c^i \frac{1-a^{2n-2i-1}}{1-a}} z_0^{da\frac{a^{2n}-c^n}{a^2-c}} w_0^{c^n} \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{\frac{d}{1-a}(\frac{1-c^n}{1-c}-a\frac{a^{2n}-c^n}{a^2-c})} z_0^{da\frac{a^{2n}-c^n}{a^2-c}} w_0^{c^n} \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{\frac{d(a^2-c+(a+c)(1-a)c^n-(1-c)a^{2n+1})}{(1-a)(1-c)(a^2-c)}} z_0^{da\frac{a^{2n}-c^n}{a^2-c}} w_0^{c^n}, \quad n \in \mathbb{N}, \end{aligned} \tag{3.53}$$

$$\begin{aligned} w_{2n+1} &= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{d\sum_{i=0}^{n-1} c^i \frac{1-a^{2n-2i}}{1-a}} z_0^{d\frac{a^{2n+2}-c^{n+1}}{a^2-c}} w_{-1}^{c^{n+1}} \\ &= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{\frac{d}{1-a}(\frac{1-c^n}{1-c}-a^2\frac{a^{2n}-c^n}{a^2-c})} z_0^{d\frac{a^{2n+2}-c^{n+1}}{a^2-c}} w_{-1}^{c^{n+1}} \\ &= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{\frac{d(a^2-c+(1-a^2)c^{n+1}-(1-c)a^{2n+2})}{(1-a)(1-c)(a^2-c)}} z_0^{d\frac{a^{2n+2}-c^{n+1}}{a^2-c}} w_{-1}^{c^{n+1}} \end{aligned} \tag{3.54}$$

for every $n \in \mathbb{N}_0$.

Subcase $a \neq 1 \neq c$, $c = a^2 \neq 0$. In this case we have

$$\begin{aligned} w_{2n} &= \beta^{\sum_{j=0}^{n-1} a^{2j}} \alpha^{d\sum_{i=0}^{n-1} a^{2i} \sum_{j=0}^{2n-2i-2} a^j} z_0^{d\sum_{j=0}^{n-1} a^{2j} a^{2n-2j-1}} w_0^{a^{2n}} \\ &= \beta^{\frac{1-a^{2n}}{1-a^2}} \alpha^{d\sum_{i=0}^{n-1} a^{2i} \frac{1-a^{2n-2i-1}}{1-a}} z_0^{dna^{2n-1}} w_0^{a^{2n}} \\ &= \beta^{\frac{1-a^{2n}}{1-a^2}} \alpha^{\frac{d}{1-a}(\frac{1-a^{2n}}{1-a^2}-na^{2n-1})} z_0^{dna^{2n-1}} w_0^{a^{2n}} \\ &= \beta^{\frac{1-a^{2n}}{1-a^2}} \alpha^{\frac{d(1-na^{2n-1}-a^{2n}+na^{2n+1})}{(a-1)^2(a+1)}} z_0^{dna^{2n-1}} w_0^{a^{2n}}, \quad n \in \mathbb{N}, \end{aligned} \tag{3.55}$$

$$\begin{aligned} w_{2n+1} &= \beta^{\sum_{j=0}^{n} a^{2j}} \alpha^{d\sum_{i=0}^{n-1} a^{2i} \sum_{j=0}^{2n-2i-1} a^j} z_0^{d\sum_{j=0}^{n} a^{2j} a^{2n-2j}} w_{-1}^{a^{2n+2}} \\ &= \beta^{\frac{1-a^{2n+2}}{1-a^2}} \alpha^{d\sum_{i=0}^{n-1} a^{2i} \frac{1-a^{2n-2i}}{1-a}} z_0^{d(n+1)a^{2n}} w_{-1}^{a^{2n+2}} \\ &= \beta^{\frac{1-a^{2n+2}}{1-a^2}} \alpha^{\frac{d}{1-a}(\frac{1-a^{2n}}{1-a^2}-na^{2n})} z_0^{d(n+1)a^{2n}} w_{-1}^{a^{2n+2}} \\ &= \beta^{\frac{1-a^{2n+2}}{1-a^2}} \alpha^{\frac{d(1-(n+1)a^{2n}+na^{2n+2})}{(a+1)(a-1)^2}} z_0^{d(n+1)a^{2n}} w_{-1}^{a^{2n+2}} \end{aligned} \tag{3.56}$$

for every $n \in \mathbb{N}_0$.

Subcase $a^2 \neq 1 = c$. In this case we have

$$\begin{aligned} w_{2n} &= \beta^n \alpha^{d\sum_{i=0}^{n-1} \frac{1-a^{2n-2i-1}}{1-a}} z_0^{ad\frac{a^{2n}-1}{a^2-1}} w_0 \\ &= \beta^n \alpha^{\frac{d}{1-a}(n-a\frac{a^{2n}-1}{a^2-1})} z_0^{ad\frac{a^{2n}-1}{a^2-1}} w_0 \\ &= \beta^n \alpha^{\frac{d(a^{2n+1}+n(1-a^2)-a)}{(a-1)^2(a+1)}} z_0^{ad\frac{a^{2n}-1}{a^2-1}} w_0, \quad n \in \mathbb{N}, \end{aligned} \tag{3.57}$$

$$\begin{aligned} w_{2n+1} &= \beta^{\sum_{j=0}^{n} 1} \alpha^{d\sum_{i=0}^{n-1} \sum_{j=0}^{2n-2i-1} a^j} z_0^{d\sum_{j=0}^{n} a^{2n-2j}} w_{-1} \\ &= \beta^{n+1} \alpha^{d\sum_{i=0}^{n-1} \frac{1-a^{2n-2i}}{1-a}} z_0^{d\frac{a^{2n+2}-1}{a^2-1}} w_{-1} \\ &= \beta^{n+1} \alpha^{\frac{d}{1-a}(n-a^2\frac{a^{2n}-1}{a^2-1})} z_0^{d\frac{a^{2n+2}-1}{a^2-1}} w_{-1} \\ &= \beta^{n+1} \alpha^{\frac{d(a^{2n+2}+n(1-a^2)-a^2)}{(a-1)^2(a+1)}} z_0^{d\frac{a^{2n+2}-1}{a^2-1}} w_{-1} \end{aligned} \tag{3.58}$$

for every $n \in \mathbb{N}_0$.

Subcase $a = -1$, $c = 1$. In this case we have

$$\begin{aligned} w_{2n} &= \beta^{\sum_{j=0}^{n-1} 1} \alpha^{d\sum_{i=0}^{n-1} \sum_{j=0}^{2n-2i-2} (-1)^j} z_0^{d\sum_{j=0}^{n-1} (-1)^{2n-2j-1}} w_0 \\ &= \beta^n \alpha^{d\sum_{i=0}^{n-1} \frac{1-(-1)^{2n-2i-1}}{2}} z_0^{d\sum_{j=0}^{n-1} (-1)^{-1}} w_0 \\ &= \beta^n \alpha^{nd} z_0^{-dn} w_0 \end{aligned} \tag{3.59}$$

for every $n \in \mathbb{N}$, and

$$\begin{aligned} w_{2n+1} &= \beta^{\sum_{j=0}^{n} 1} \alpha^{d \sum_{i=0}^{n-1} \sum_{j=0}^{2n-2i-1} (-1)^j} z_0^{d \sum_{j=0}^{n} (-1)^{2n-2j}} w_{-1} \\ &= \beta^{n+1} \alpha^{d \sum_{i=0}^{n-1} \frac{1-(-1)^{2n-2i}}{2}} z_0^{d \sum_{j=0}^{n} 1} w_{-1} \\ &= \beta^{n+1} z_0^{d(n+1)} w_{-1} \end{aligned} \tag{3.60}$$

for every $n \in \mathbb{N}_0$.

Subcase $a = 1$, $c \neq 1$. In this case, using (2.3), we have

$$\begin{aligned} w_{2n} &= \beta^{\frac{1-c^n}{1-c}} \alpha^{d \sum_{i=0}^{n-1} (2n-2i-1) c^i} z_0^{d \frac{1-c^n}{1-c}} w_0^{c^n} \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{d((2n-1)\frac{1-c^n}{1-c} - 2c \frac{1-nc^{n-1}+(n-1)c^n}{(1-c)^2})} z_0^{d \frac{1-c^n}{1-c}} w_0^{c^n} \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{\frac{d(2n-1-(2n+1)c+c^n+c^{n+1})}{(1-c)^2}} z_0^{d \frac{1-c^n}{1-c}} w_0^{c^n} \end{aligned} \tag{3.61}$$

for $n \in \mathbb{N}$, and

$$\begin{aligned} w_{2n+1} &= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{d \sum_{i=0}^{n-1} (2n-2i) c^i} z_0^{d \frac{1-c^{n+1}}{1-c}} w_{-1}^{c^{n+1}} \\ &= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{d(2n \frac{1-c^n}{1-c} - 2c \frac{1-nc^{n-1}+(n-1)c^n}{(1-c)^2})} z_0^{d \frac{1-c^{n+1}}{1-c}} w_{-1}^{c^{n+1}} \\ &= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{\frac{2d(n-(n+1)c+c^{n+1})}{(1-c)^2}} z_0^{d \frac{1-c^{n+1}}{1-c}} w_{-1}^{c^{n+1}} \end{aligned} \tag{3.62}$$

for every $n \in \mathbb{N}_0$.

Subcase $a = c = 1$. In this case we have

$$\begin{aligned} w_{2n} &= \beta^n \alpha^{d \sum_{i=0}^{n-1} (2n-2i-1)} z_0^{dn} w_0 \\ &= \beta^n \alpha^{dn^2} z_0^{dn} w_0 \end{aligned} \tag{3.63}$$

for $n \in \mathbb{N}$, and

$$\begin{aligned} w_{2n+1} &= \beta^{n+1} \alpha^{d \sum_{i=0}^{n-1} (2n-2i)} z_0^{d(n+1)} w_{-1} \\ &= \beta^{n+1} \alpha^{dn(n+1)} z_0^{d(n+1)} w_{-1} \end{aligned} \tag{3.64}$$

for every $n \in \mathbb{N}_0$. □

Corollary 3.6. *Consider system* (1.1) *with* $a, c, d \in \mathbb{Z}$, $b = 0$ *and* $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. *Furthermore assume that* $z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true:*

(i) *If* $a \neq 1 \neq c$ *and* $c \neq a^2$, *then the general solution to system* (1.1) *is given by* (3.44), (3.53) *and* (3.54).
(ii) *If* $a \neq 1 \neq c$ *and* $c = a^2 \neq 0$, *then the general solution to system* (1.1) *is given by* (3.44), (3.55) *and* (3.56).
(iii) *If* $a^2 \neq 1 = c$, *then the general solution to system* (1.1) *is given by* (3.44), (3.57) *and* (3.58).
(iv) *If* $a = -1$ *and* $c = 1$, *then the general solution to system* (1.1) *is given by* (3.44), (3.59) *and* (3.60).
(v) *If* $a = 1$ *and* $c \neq 1$, *then the general solution to system* (1.1) *is given by* (3.45), (3.61) *and* (3.62).
(vi) *If* $a = c = 1$, *then the general solution to system* (1.1) *is given by* (3.45), (3.63) *and* (3.64).

Theorem 3.7. *Assume that* $a, b, c \in \mathbb{Z}$, $d = 0$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, *and* $z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then system* (1.1) *is solvable in closed form.*

Proof. In this case system (1.1) becomes

$$z_{n+1} = \alpha z_n^a w_n^b, \quad w_{n+1} = \beta w_{n-1}^c, \quad n \in \mathbb{N}_0. \tag{3.65}$$

From the second equation in (3.65) it easily follows that

$$w_{2n} = \beta^{\sum_{j=0}^{n-1} c^j} w_0^{c^n}, \quad w_{2n+1} = \beta^{\sum_{j=0}^{n} c^j} w_{-1}^{c^{n+1}}, \quad n \in \mathbb{N}_0, \tag{3.66}$$

which for the case $c \neq 1$ implies that

$$w_{2n} = \beta^{\frac{1-c^n}{1-c}} w_0^{c^n}, \quad w_{2n+1} = \beta^{\frac{1-c^{n+1}}{1-c}} w_{-1}^{c^{n+1}}, \quad n \in \mathbb{N}_0, \tag{3.67}$$

while for the case $c = 1$ we have

$$w_{2n} = \beta^n w_0, \quad w_{2n+1} = \beta^{n+1} w_{-1}, \quad n \in \mathbb{N}_0. \tag{3.68}$$

Employing (3.66) in the first equation in (3.65), we obtain

$$z_{2n} = \alpha\beta^{b\sum_{j=0}^{n-1} c^j} w_{-1}^{bc^n} z_{2n-1}^a, \tag{3.69}$$

$$z_{2n+1} = \alpha\beta^{b\sum_{j=0}^{n-1} c^j} w_0^{bc^n} z_{2n}^a, \quad n \in \mathbb{N}. \tag{3.70}$$

Combining (3.69) and (3.70), we have that

$$\begin{aligned} z_{2n} &= \alpha\beta^{b\sum_{j=0}^{n-1} c^j} w_{-1}^{bc^n} (\alpha\beta^{b\sum_{j=0}^{n-2} c^j} w_0^{bc^{n-1}} z_{2n-2}^a)^a \\ &= \alpha^{1+a}\beta^{b\sum_{j=0}^{n-1} c^j + ab\sum_{j=0}^{n-2} c^j} (w_{-1}^{bc} w_0^{ab})^{c^{n-1}} z_{2n-2}^{a^2} \end{aligned} \tag{3.71}$$

for $n \geq 2$, and

$$\begin{aligned} z_{2n+1} &= \alpha\beta^{b\sum_{j=0}^{n-1} c^j} w_0^{bc^n} (\alpha\beta^{b\sum_{j=0}^{n-1} c^j} w_{-1}^{bc^n} z_{2n-1}^a)^a \\ &= \alpha^{1+a}\beta^{b(1+a)\sum_{j=0}^{n-1} c^j} (w_{-1}^{ab} w_0^b)^{c^n} z_{2n-1}^{a^2} \end{aligned} \tag{3.72}$$

for every $n \in \mathbb{N}$.

Assume that for some natural number k we have proved that

$$z_{2n} = \alpha^{(1+a)\sum_{j=0}^{k-1} a^{2j}} \beta^{b\sum_{i=0}^{k-1} a^{2i}(\sum_{j=0}^{n-i-1} c^j + a\sum_{j=0}^{n-i-2} c^j)} (w_{-1}^{bc} w_0^{ab})^{\sum_{j=0}^{k-1} a^{2j} c^{n-j-1}} z_{2n-2k}^{a^{2k}} \tag{3.73}$$

and

$$z_{2n+1} = \alpha^{(1+a)\sum_{j=0}^{k-1} a^{2j}} \beta^{b(1+a)\sum_{i=0}^{k-1} a^{2i}\sum_{j=0}^{n-i-1} c^j} (w_{-1}^{ab} w_0^b)^{\sum_{j=0}^{k-1} a^{2j} c^{n-j}} z_{2n-2k+1}^{a^{2k}} \tag{3.74}$$

for every $n \geq k$.

Using (3.71) with $n \to n-k$ into (3.73), and (3.72) with $n \to n-k$ into (3.74), we have that

$$\begin{aligned} z_{2n} &= \alpha^{(1+a)\sum_{j=0}^{k-1} a^{2j}} \beta^{b\sum_{i=0}^{k-1} a^{2i}(\sum_{j=0}^{n-i-1} c^j + a\sum_{j=0}^{n-i-2} c^j)} (w_{-1}^{bc} w_0^{ab})^{\sum_{j=0}^{k-1} a^{2j} c^{n-j-1}} \\ &\quad\times (\alpha^{1+a}\beta^{b\sum_{j=0}^{n-k-1} c^j + ab\sum_{j=0}^{n-k-2} c^j} (w_{-1}^{bc} w_0^{ab})^{c^{n-k-1}} z_{2n-2k-2}^{a^2})^{a^{2k}} \\ &= \alpha^{(1+a)\sum_{j=0}^{k} a^{2j}} \beta^{b\sum_{i=0}^{k} a^{2i}(\sum_{j=0}^{n-i-1} c^j + a\sum_{j=0}^{n-i-2} c^j)} (w_{-1}^{bc} w_0^{ab})^{\sum_{j=0}^{k} a^{2j} c^{n-j-1}} z_{2n-2k-2}^{a^{2k+2}} \end{aligned} \tag{3.75}$$

and

$$\begin{aligned} z_{2n+1} &= \alpha^{(1+a)\sum_{j=0}^{k-1} a^{2j}} \beta^{b(1+a)\sum_{i=0}^{k-1} a^{2i}\sum_{j=0}^{n-i-1} c^j} (w_{-1}^{ab} w_0^b)^{\sum_{j=0}^{k-1} a^{2j} c^{n-j}} (\alpha^{1+a}\beta^{b(1+a)\sum_{j=0}^{n-k-1} c^j} (w_{-1}^{ab} w_0^b)^{c^{n-k}} z_{2n-2k-1}^{a^2})^{a^{2k}} \\ &= \alpha^{(1+a)\sum_{j=0}^{k} a^{2j}} \beta^{b(1+a)\sum_{i=0}^{k} a^{2i}\sum_{j=0}^{n-i-1} c^j} (w_{-1}^{ab} w_0^b)^{\sum_{j=0}^{k} a^{2j} c^{n-j}} z_{2n-2k-1}^{a^{2k+2}} \end{aligned} \tag{3.76}$$

for every $n \geq k+1$.

From (3.71), (3.72), (3.75), (3.76), and induction we have that (3.73) and (3.74) hold for all $k, n \in \mathbb{N}$ such that $1 \leq k \leq n$.

If we choose $k = n$ in (3.73) and (3.74), we get

$$\begin{aligned} z_{2n} &= \alpha^{(1+a)\sum_{j=0}^{n-1} a^{2j}} \beta^{b\sum_{i=0}^{n-1} a^{2i}(\sum_{j=0}^{n-i-1} c^j + a\sum_{j=0}^{n-i-2} c^j)} (w_{-1}^{bc} w_0^{ab})^{\sum_{j=0}^{n-1} a^{2j} c^{n-j-1}} z_0^{a^{2n}} \\ z_{2n+1} &= \alpha^{(1+a)\sum_{j=0}^{n-1} a^{2j}} \beta^{b(1+a)\sum_{i=0}^{n-1} a^{2i}\sum_{j=0}^{n-i-1} c^j} (w_{-1}^{ab} w_0^b)^{\sum_{j=0}^{n-1} a^{2j} c^{n-j}} z_1^{a^{2n}} \\ &= \alpha^{(1+a)\sum_{j=0}^{n-1} a^{2j}} \beta^{b(1+a)\sum_{i=0}^{n-1} a^{2i}\sum_{j=0}^{n-i-1} c^j} (w_{-1}^{ab} w_0^b)^{\sum_{j=0}^{n-1} a^{2j} c^{n-j}} (\alpha z_0^a w_0^b)^{a^{2n}} \\ &= \alpha^{\sum_{j=0}^{2n} a^j} \beta^{b(1+a)\sum_{i=0}^{n-1} a^{2i}\sum_{j=0}^{n-i-1} c^j} w_{-1}^{ab\sum_{j=0}^{n-1} a^{2j} c^{n-j}} w_0^{b\sum_{j=0}^{n} a^{2j} c^{n-j}} z_0^{a^{2n+1}} \end{aligned}$$

for every $n \in \mathbb{N}$.

Subcase $c \neq a^2 \neq 1 \neq c$. In this case we have

$$\begin{aligned} z_{2n} &= \alpha^{\frac{1-a^{2n}}{1-a}} \beta^{b\sum_{i=0}^{n-1} a^{2i}(\frac{1-c^{n-i}}{1-c}+a\frac{1-c^{n-i-1}}{1-c})} (w_{-1}^{bc} w_0^{ab})^{\frac{c^n-a^{2n}}{c-a^2}} z_0^{a^{2n}} \\ &= \alpha^{\frac{1-a^{2n}}{1-a}} \beta^{\frac{b}{1-c}(\frac{1-a^{2n}}{1-a}-(c+a)\frac{c^n-a^{2n}}{c-a^2})} (w_{-1}^{bc} w_0^{ab})^{\frac{c^n-a^{2n}}{c-a^2}} z_0^{a^{2n}} \\ &= \alpha^{\frac{1-a^{2n}}{1-a}} \beta^{\frac{b(c-a^2+(a-1)(a+c)c^n+(1-c)a^{2n+1})}{(1-a)(1-c)(c-a^2)}} (w_{-1}^{bc} w_0^{ab})^{\frac{c^n-a^{2n}}{c-a^2}} z_0^{a^{2n}}, \end{aligned} \tag{3.77}$$

$$\begin{aligned} z_{2n+1} &= \alpha^{\frac{1-a^{2n+1}}{1-a}} \beta^{b(1+a)\sum_{i=0}^{n-1} a^{2i}\frac{1-c^{n-i}}{1-c}} w_{-1}^{abc\frac{c^n-a^{2n}}{c-a^2}} w_0^{b\frac{c^{n+1}-a^{2n+2}}{c-a^2}} z_0^{a^{2n+1}} \\ &= \alpha^{\frac{1-a^{2n+1}}{1-a}} \beta^{\frac{b(1+a)}{1-c}(\frac{1-a^{2n}}{1-a^2}-c\frac{c^n-a^{2n}}{c-a^2})} w_{-1}^{abc\frac{c^n-a^{2n}}{c-a^2}} w_0^{b\frac{c^{n+1}-a^{2n+2}}{c-a^2}} z_0^{a^{2n+1}} \\ &= \alpha^{\frac{1-a^{2n+1}}{1-a}} \beta^{\frac{b(1+a)(c-a^2+(a^2-1)c^{n+1}+(1-c)a^{2n+2})}{(1-c)(1-a^2)(c-a^2)}} w_{-1}^{abc\frac{c^n-a^{2n}}{c-a^2}} w_0^{b\frac{c^{n+1}-a^{2n+2}}{c-a^2}} z_0^{a^{2n+1}} \end{aligned} \tag{3.78}$$

for every $n \in \mathbb{N}$.

Subcase $a^2 \neq 1 \neq c$, $c = a^2$. In this case we have

$$\begin{aligned} z_{2n} &= \alpha^{\frac{1-a^{2n}}{1-a}} \beta^{b\sum_{i=0}^{n-1} a^{2i}(\frac{1-a^{2n-2i}}{1-a^2}+a\frac{1-a^{2n-2i-2}}{1-a^2})} (w_{-1}^{ba^2} w_0^{ab})^{na^{2n-2}} z_0^{a^{2n}} \\ &= \alpha^{\frac{1-a^{2n}}{1-a}} \beta^{\frac{b}{1-a^2}(\frac{1-a^{2n}}{1-a}-(a+1)na^{2n-1})} (w_{-1}^{ba^2} w_0^{ab})^{na^{2n-2}} z_0^{a^{2n}} \\ &= \alpha^{\frac{1-a^{2n}}{1-a}} \beta^{\frac{b(1-na^{2n-1}-a^{2n}+na^{2n+1})}{(1-a)^2(1+a)}} (w_{-1}^{ba^2} w_0^{ab})^{na^{2n-2}} z_0^{a^{2n}}, \end{aligned} \tag{3.79}$$

$$\begin{aligned} z_{2n+1} &= \alpha^{\frac{1-a^{2n+1}}{1-a}} \beta^{b(1+a)\sum_{i=0}^{n-1} a^{2i}\frac{1-a^{2n-2i}}{1-a^2}} w_{-1}^{bna^{2n+1}} w_0^{b(n+1)a^{2n}} z_0^{a^{2n+1}} \\ &= \alpha^{\frac{1-a^{2n+1}}{1-a}} \beta^{\frac{b(1+a)(1-(n+1)a^{2n}+na^{2n+2})}{(1-a^2)^2}} w_{-1}^{bna^{2n+1}} w_0^{b(n+1)a^{2n}} z_0^{a^{2n+1}} \end{aligned} \tag{3.80}$$

for every $n \in \mathbb{N}$.

Subcase $a^2 \neq 1 = c$. In this case, using (2.3), we have

$$\begin{aligned} z_{2n} &= \alpha^{(1+a)\sum_{j=0}^{n-1} a^{2j}} \beta^{b\sum_{i=0}^{n-1} a^{2i}(\sum_{j=0}^{n-i-1} 1+a\sum_{j=0}^{n-i-2} 1)} (w_{-1}^{b} w_0^{ab})^{\sum_{j=0}^{n-1} a^{2j}} z_0^{a^{2n}} \\ &= \alpha^{\frac{1-a^{2n}}{1-a}} \beta^{b\sum_{i=0}^{n-1} a^{2i}((1+a)n-a-(1+a)i)} (w_{-1}^{b} w_0^{ab})^{\frac{1-a^{2n}}{1-a^2}} z_0^{a^{2n}} \\ &= \alpha^{\frac{1-a^{2n}}{1-a}} \beta^{b(((1+a)n-a)\frac{1-a^{2n}}{1-a^2}-\frac{a^2-na^{2n}+(n-1)a^{2n+2}}{(1-a^2)(1-a)})} (w_{-1}^{b} w_0^{ab})^{\frac{1-a^{2n}}{1-a^2}} z_0^{a^{2n}} \\ &= \alpha^{\frac{1-a^{2n}}{1-a}} \beta^{\frac{b(n-a-na^2+a^{2n+1})}{(1-a^2)(1-a)}} (w_{-1}^{b} w_0^{ab})^{\frac{1-a^{2n}}{1-a^2}} z_0^{a^{2n}}, \quad n \in \mathbb{N}, \end{aligned} \tag{3.81}$$

$$\begin{aligned} z_{2n+1} &= \alpha^{\sum_{j=0}^{2n} a^j} \beta^{b(1+a)\sum_{i=0}^{n-1} a^{2i}(n-i)} w_{-1}^{ab\sum_{j=0}^{n-1} a^{2j}} w_0^{b\sum_{j=0}^{n} a^{2j}} z_0^{a^{2n+1}} \\ &= \alpha^{\frac{1-a^{2n+1}}{1-a}} \beta^{b(1+a)(n\frac{1-a^{2n}}{1-a^2}-a^2\frac{1-na^{2n-2}+(n-1)a^{2n}}{(1-a^2)^2})} w_{-1}^{ab\frac{1-a^{2n}}{1-a^2}} w_0^{b\frac{1-a^{2n+2}}{1-a^2}} z_0^{a^{2n+1}} \\ &= \alpha^{\frac{1-a^{2n+1}}{1-a}} \beta^{\frac{b(1+a)(n-(n+1)a^2+a^{2n+2})}{(1-a^2)^2}} w_{-1}^{ab\frac{1-a^{2n}}{1-a^2}} w_0^{b\frac{1-a^{2n+2}}{1-a^2}} z_0^{a^{2n+1}} \end{aligned} \tag{3.82}$$

for every $n \in \mathbb{N}_0$.

Subcase $a = -1$, $c = 1$. In this case we have

$$\begin{aligned} z_{2n} &= \beta^{b\sum_{i=0}^{n-1}(-1)^{2i}(\sum_{j=0}^{n-i-1} 1-\sum_{j=0}^{n-i-2} 1)} (w_{-1}^{b} w_0^{-b})^{\sum_{j=0}^{n-1}(-1)^{2j}} z_0^{(-1)^{2n}} \\ &= \beta^{bn}(w_{-1}^{b} w_0^{-b})^n z_0, \end{aligned} \tag{3.83}$$

$$\begin{aligned} z_{2n+1} &= \alpha^{\sum_{j=0}^{2n}(-1)^j} w_{-1}^{-b\sum_{j=0}^{n-1}(-1)^{2j}} w_0^{b\sum_{j=0}^{n}(-1)^{2j}} z_0^{(-1)^{2n+1}} \\ &= \alpha w_{-1}^{-bn} w_0^{b(n+1)} z_{-1}^{-1} \end{aligned} \tag{3.84}$$

for every $n \in \mathbb{N}$.

Subcase $a = 1 \neq c$. In this case we have

$$\begin{aligned} z_{2n} &= \alpha^{2n}\beta^{b\sum_{i=0}^{n-1}\left(\frac{1-c^{n-i}}{1-c}+\frac{1-c^{n-i-1}}{1-c}\right)}(w_{-1}^{bc}w_0^b)^{\frac{1-c^n}{1-c}}z_0 \\ &= \alpha^{2n}\beta^{\frac{b}{1-c}\sum_{i=0}^{n-1}(2n-(c+1)\frac{1-c^n}{1-c})}(w_{-1}^{bc}w_0^b)^{\frac{1-c^n}{1-c}}z_0 \\ &= \alpha^{2n}\beta^{\frac{b(2n-1-(2n+1)c+c^n+c^{n+1})}{(1-c)^2}}(w_{-1}^{bc}w_0^b)^{\frac{1-c^n}{1-c}}z_0, \end{aligned} \tag{3.85}$$

$$\begin{aligned} z_{2n+1} &= \alpha^{2n+1}\beta^{2b\sum_{i=0}^{n-1}\frac{1-c^{n-i}}{1-c}}w_{-1}^{bc\frac{1-c^n}{1-c}}w_0^{b\frac{1-c^{n+1}}{1-c}}z_0 \\ &= \alpha^{2n+1}\beta^{\frac{2b(n-(n+1)c+c^{n+1})}{(1-c)^2}}w_{-1}^{bc\frac{1-c^n}{1-c}}w_0^{b\frac{1-c^{n+1}}{1-c}}z_0 \end{aligned} \tag{3.86}$$

for every $n \in \mathbb{N}$.

Subcase $a = c = 1$. In this case we have

$$\begin{aligned} z_{2n} &= \alpha^{2\sum_{j=0}^{n-1}1}\beta^{b\sum_{i=0}^{n-1}(\sum_{j=0}^{n-i-1}1+\sum_{j=0}^{n-i-2}1)}(w_{-1}^b w_0^b)^{\sum_{j=0}^{n-1}1}z_0 \\ &= \alpha^{2n}\beta^{b\sum_{i=0}^{n-1}(2n-2i-1)}(w_{-1}^b w_0^b)^n z_0 \\ &= \alpha^{2n}\beta^{bn^2}(w_{-1}w_0)^{bn}z_0, \end{aligned} \tag{3.87}$$

$$\begin{aligned} z_{2n+1} &= \alpha^{\sum_{j=0}^{2n}1}\beta^{2b\sum_{i=0}^{n-1}\sum_{j=0}^{n-i-1}1}w_{-1}^{b\sum_{j=0}^{n-1}1}w_0^{b\sum_{j=0}^{n}1}z_0 \\ &= \alpha^{2n+1}\beta^{2b\sum_{i=0}^{n-1}(n-i)}w_{-1}^{bn}w_0^{b(n+1)}z_0 \\ &= \alpha^{2n+1}\beta^{bn(n+1)}w_{-1}^{bn}w_0^{b(n+1)}z_0 \end{aligned} \tag{3.88}$$

for every $n \in \mathbb{N}$, completing the proof. □

Corollary 3.8. *Consider system* (1.1) *with* $a, b, c \in \mathbb{Z}$, $d = 0$ *and* $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. *Furthermore assume that* $z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true:*
(i) *If* $c \neq a^2 \neq 1 \neq c$, *then the general solution to system* (1.1) *is given by* (3.67), (3.77) *and* (3.78).
(ii) *If* $c = a^2 \neq 1 \neq c$, *then the general solution to system* (1.1) *is given by* (3.67), (3.79) *and* (3.80).
(iii) *If* $a^2 \neq 1 = c$, *then the general solution to system* (1.1) *is given by* (3.68), (3.81) *and* (3.82).
(iv) *If* $a = -1$ *and* $c = 1$, *then the general solution to system* (1.1) *is given by* (3.68), (3.83) *and* (3.84).
(v) *If* $a = 1$ *and* $c \neq 1$, *then the general solution to system* (1.1) *is given by* (3.67), (3.85) *and* (3.86).
(vi) *If* $a = c = 1$, *then the general solution to system* (1.1) *is given by* (3.68), (3.87) *and* (3.88).

Theorem 3.9. *Assume that* $a, b, c, d \in \mathbb{Z}$, $abcd \neq 0$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, *and suppose that* $z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then system* (1.1) *is solvable in closed form.*

Proof. The conditions $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$ along with (1.1) imply $z_n \neq 0 \neq w_n$, $n \in \mathbb{N}$. From the first equation in (1.1), for such a solution, we have

$$w_n^b = \frac{z_{n+1}}{\alpha z_n^a}, \quad n \in \mathbb{N}_0, \tag{3.89}$$

while by taking the second equation in (1.1) to the b-th power it follows that

$$w_{n+1}^b = \beta^b w_{n-1}^{bc} z_n^{bd}, \quad n \in \mathbb{N}_0. \tag{3.90}$$

Putting (3.89) into (3.90), we easily obtain

$$z_{n+2} = \alpha^{1-c}\beta^b z_{n+1}^a z_n^{bd+c} z_{n-1}^{-ac}, \quad n \in \mathbb{N}, \tag{3.91}$$

which is a third order product-type difference equation.

Note also that

$$z_1 = \alpha z_0^a w_0^b \quad \text{and} \quad z_2 = \alpha(\alpha z_0^a w_0^b)^a(\beta w_{-1}^c z_0^d)^b = \alpha^{1+a}\beta^b z_0^{a^2+bd} w_0^{ab} w_{-1}^{bc}. \tag{3.92}$$

Let

$$\delta := \alpha^{1-c}\beta^{b}, \quad a_1 = a, \quad b_1 = bd + c, \quad c_1 = -ac, \quad y_1 = 1. \tag{3.93}$$

Then equation (3.91) can be written as

$$z_{n+2} = \delta^{y_1} z_{n+1}^{a_1} z_n^{b_1} z_{n-1}^{c_1}, \quad n \in \mathbb{N}. \tag{3.94}$$

Putting (3.94) with $n \to n-1$ into itself, we get

$$\begin{aligned} z_{n+2} &= \delta^{y_1}(\delta z_n^{a_1} z_{n-1}^{b_1} z_{n-2}^{c_1})^{a_1} z_n^{b_1} z_{n-1}^{c_1} \\ &= \delta^{y_1+a_1} z_n^{a_1a_1+b_1} z_{n-1}^{b_1a_1+c_1} z_{n-2}^{c_1a_1} \\ &= \delta^{y_2} z_n^{a_2} z_{n-1}^{b_2} z_{n-2}^{c_2} \end{aligned} \tag{3.95}$$

for $n \geq 2$, where

$$a_2 := a_1a_1 + b_1, \quad b_2 := b_1a_1 + c_1, \quad c_2 := c_1a_1, \quad y_2 = y_1 + a_1. \tag{3.96}$$

Assume that for a $k \geq 2$ we have proved that

$$z_{n+2} = \delta^{y_k} z_{n+2-k}^{a_k} z_{n+1-k}^{b_k} z_{n-k}^{c_k} \tag{3.97}$$

for $n \geq k$, and that

$$a_k = a_1a_{k-1} + b_{k-1}, \quad b_k = b_1a_{k-1} + c_{k-1}, \quad c_k = c_1a_{k-1}, \tag{3.98}$$

$$y_k = y_{k-1} + a_{k-1}, \quad k \geq 2. \tag{3.99}$$

Then, putting relation (3.94) with $n \to n-k$ into (3.97), we obtain

$$\begin{aligned} z_{n+2} &= \delta^{y_k}(\delta z_{n+1-k}^{a_1} z_{n-k}^{b_1} z_{n-k-1}^{c_1})^{a_k} z_{n+1-k}^{b_k} z_{n-k}^{c_k} \\ &= \delta^{y_k+a_k} z_{n+1-k}^{a_1a_k+b_k} z_{n-k}^{b_1a_k+c_k} z_{n-k-1}^{c_1a_k} \\ &= \delta^{y_{k+1}} z_{n+1-k}^{a_{k+1}} z_{n-k}^{b_{k+1}} z_{n-k-1}^{c_{k+1}} \end{aligned} \tag{3.100}$$

for $n \geq k+1$, where

$$a_{k+1} := a_1a_k + b_k, \quad b_{k+1} := b_1a_k + c_k, \quad c_{k+1} := c_1a_k, \quad y_{k+1} = y_k + a_k. \tag{3.101}$$

From (3.95), (3.96), (3.100), (3.101), and using induction, we have that (3.97)–(3.99) hold for every $k, n \in \mathbb{N}$ such that $2 \leq k \leq n$. Moreover, due to (3.94), equation (3.97) holds also for $k = 1$.

Choosing $k = n$ in (3.97) and using (3.92), (3.98) and (3.99), we have

$$\begin{aligned} z_{n+2} &= \delta^{y_n} z_2^{a_n} z_1^{b_n} z_0^{c_n} \\ &= (\alpha^{1-c}\beta^{b})^{y_n}(\alpha^{1+a}\beta^{b} z_0^{a^2+bd} w_0^{ab} w_{-1}^{bc})^{a_n}(\alpha z_0^{a} w_0^{b})^{b_n} z_0^{c_n} \\ &= \alpha^{(1-c)y_n+(1+a)a_n+b_n}\beta^{b(y_n+a_n)} z_0^{(a^2+bd)a_n+ab_n+c_n} w_{-1}^{bca_n} w_0^{aba_n+bb_n} \\ &= \alpha^{y_{n+2}-cy_n}\beta^{by_{n+1}} z_0^{a_{n+2}-ca_n} w_{-1}^{bca_n} w_0^{ba_{n+1}} \end{aligned} \tag{3.102}$$

for $n \in \mathbb{N}$.

From (3.98) we have that $(a_k)_{k\geq 4}$ satisfies the following recurrent relation:

$$a_k = a_1a_{k-1} + b_1a_{k-2} + c_1a_{k-3}. \tag{3.103}$$

Since $b_{k-1} = a_k - a_1a_{k-1}$ and $c_k = c_1a_{k-1}$, noting that equation (3.103) is linear, we have that $(b_k)_{k\in\mathbb{N}}$ and $(c_k)_{k\in\mathbb{N}}$ also satisfy this equation.

Moreover, a_k, b_k and c_k satisfy (3.103) for $k \geq -2$. Indeed, from (3.101) with $k = 0$ we get

$$a_1 = a_1a_0 + b_0, \quad b_1 = b_1a_0 + c_0, \quad c_1 = c_1a_0, \quad 1 = y_1 = y_0 + a_0. \tag{3.104}$$

Since $c_1 = -ac \neq 0$, from the third equation in (3.104) we get $a_0 = 1$. Using this fact in the other three equalities in (3.104), we get $b_0 = c_0 = y_0 = 0$.

From this and by (3.101) with $k = -1$ we get

$$\begin{aligned} 1 &= a_0 = a_1 a_{-1} + b_{-1}, \quad & 0 &= b_0 = b_1 a_{-1} + c_{-1}, \\ 0 &= c_0 = c_1 a_{-1}, & 0 &= y_0 = y_{-1} + a_{-1}. \end{aligned} \tag{3.105}$$

Since $c_1 \neq 0$, from the third equation in (3.105) we get $a_{-1} = 0$. Using this fact in the other three equalities in (3.105), we get $b_{-1} = 1$, $c_{-1} = y_{-1} = 0$.

From this and by (3.101) with $k = -2$ we get

$$\begin{aligned} 0 &= a_{-1} = a_1 a_{-2} + b_{-2}, \quad & 1 &= b_{-1} = b_1 a_{-2} + c_{-2}, \\ 0 &= c_{-1} = c_1 a_{-2}, & 0 &= y_{-1} = y_{-2} + a_{-2}. \end{aligned} \tag{3.106}$$

Since $c_1 \neq 0$, from the third equation in (3.106) we get $a_{-2} = 0$. Using this fact in the other three equalities in (3.106), we get $b_{-2} = y_{-2} = 0$ and $c_{-2} = 1$

Hence, the sequences $(a_k)_{k\geq -2}$, $(b_k)_{k\geq -2}$ and $(c_k)_{k\geq -2}$ are solutions to linear difference equation (3.103) satisfying the initial conditions

$$\begin{aligned} a_{-2} &= 0, \quad a_{-1} = 0, \quad a_0 = 1, \\ b_{-2} &= 0, \quad b_{-1} = 1, \quad b_0 = 0, \\ c_{-2} &= 1, \quad c_{-1} = 0, \quad c_0 = 0, \end{aligned} \tag{3.107}$$

respectively, and $(y_k)_{k\geq -2}$ satisfies recurrent relation (3.99) and

$$y_{-2} = y_{-1} = y_0 = 0, \quad y_1 = 1.$$

From (3.99) and since $a_0 = 1$, we have that

$$y_k = 1 + \sum_{j=1}^{k-1} a_j = \sum_{j=0}^{k-1} a_j, \quad k \in \mathbb{N}_0. \tag{3.108}$$

Since difference equation (3.103) is solvable, it follows that closed form formulas for $(a_k)_{k\geq -2}$, $(b_k)_{k\geq -2}$ and $(c_k)_{k\geq -2}$ can be found. From this fact, (3.102), (3.108), and since the sum $\sum_{j=0}^{m} a_j$, $m \in \mathbb{N}_0$, can be calculated by using Lemma 2.1, we see that equation (3.91) is solvable too.

From the second equation in (1.1) we have that

$$z_n^d = \frac{w_{n+1}}{\beta w_{n-1}^c}, \quad n \in \mathbb{N}_0, \tag{3.109}$$

for every well-defined solution, while by taking the first equation in (1.1) to the d-th power, it follows that

$$z_{n+1}^d = \alpha^d z_n^{ad} w_n^{bd}, \quad n \in \mathbb{N}_0. \tag{3.110}$$

Putting (3.109) into (3.110), we easily obtain

$$w_{n+2} = \alpha^d \beta^{1-a} w_{n+1}^a w_n^{bd+c} w_{n-1}^{-ac}, \quad n \in \mathbb{N}_0, \tag{3.111}$$

which is a related difference equation to (3.91) (only with a different coefficient). Note also that $(w_n)_{n\in\mathbb{N}_0}$ satisfies the following initial condition:

$$w_1 = \beta w_{-1}^c z_0^d. \tag{3.112}$$

Hence, the above presented procedure for the sequence z_n can be repeated and we obtain that for every $k \in \mathbb{N}$ such that $1 \leq k \leq n$ we have

$$w_{n+2} = \eta^{\hat{y}_k} w_{n+2-k}^{a_k} w_{n+1-k}^{b_k} w_{n-k}^{c_k}, \quad n \geq k-1, \tag{3.113}$$

where $\eta = \alpha^d \beta^{1-a}$ and $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$ and $(c_k)_{k\in\mathbb{N}}$ satisfy recurrent relations (3.98) with initial conditions (3.93), and

$$\hat{y}_{k+1} = \hat{y}_k + a_k, \quad k \in \mathbb{N},$$

with $\hat{y}_1 = 1$, from which it follows that $\hat{y}_k$ is also given by formula (3.108).

From (3.113) with $k = n + 1$ and by using (3.112), we get

$$\begin{aligned} w_{n+2} &= \eta^{\hat{y}_{n+1}} w_1^{a_{n+1}} w_0^{b_{n+1}} w_{-1}^{c_{n+1}} \\ &= (\alpha^d \beta^{1-a})^{\hat{y}_{n+1}} (\beta w_{-1}^c z_0^d)^{a_{n+1}} w_0^{b_{n+1}} w_{-1}^{c_{n+1}} \\ &= \alpha^{d\hat{y}_{n+1}} \beta^{\hat{y}_{n+2} - a\hat{y}_{n+1}} z_0^{da_{n+1}} w_0^{a_{n+2} - aa_{n+1}} w_{-1}^{ca_{n+1} - aca_n}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{3.114}$$

Note that $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$ and $(c_k)_{k\in\mathbb{N}}$ satisfy (3.103), and, as above, since $c_1 \neq 0$, they can be prolonged for $k = -2, -1, 0$, respectively, so that they satisfy (3.107), and $(\hat{y}_k)_{k\in\mathbb{N}}$ can be prolonged also for $k = -2, -1, 0$ and $\hat{y}_{-2} = \hat{y}_{-1} = \hat{y}_0 = 0$.

The solvability of equation (3.103) shows that closed form formulas for $(a_k)_{k\geq -2}$, $(b_k)_{k\geq -2}$ and $(c_k)_{k\geq -2}$ can be found, from which, along with (3.108) and by Lemma 2.1, closed form formulas for $(\hat{y}_k)_{k\geq -2}$ can be found. These facts along with (3.114) imply that equation (3.111) is solvable too. Direct but time-consuming calculation shows that the sequences z_n and w_n given by formulas (3.102) and (3.114) are solutions to system (1.1). Hence, system (1.1) is also solvable in this case, finishing the proof of the theorem. □

Corollary 3.10. *Consider system* (1.1) *with* $a, b, c, d \in \mathbb{Z}$, $abcd \neq 0$ *and* $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. *Assume further that* $z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the general solution to system* (1.1) *is given by* (3.102) *and* (3.114), *where the sequence* $(a_k)_{k\geq -2}$ *satisfies the difference equation* (3.103) *with initial conditions* (3.107), *and the sequences* $(y_k)_{k\geq -2}$ *and* $(\hat{y}_k)_{k\geq -2}$ *are both given by* (3.108).

Remark 3.11. Note that if $ac \neq 0$, then by using the recurrent relations in (3.98) and (3.99) it can be shown, similar as in the proof of Theorem 3.9, that the sequences a_k, b_k, c_k can be prolonged for every negative index k.

The characteristic polynomial associated to equation (3.103) is

$$P_3(\lambda) = \lambda^3 - a_1\lambda^2 - b_1\lambda - c_1, \tag{3.115}$$

where a_1, b_1 and c_1 are defined in (3.93). In the case $ac \neq 0$, polynomial (3.115) is of the third degree, so by the Cardano formula (see, e.g., [6]) the zeros of (3.115) are

$$\begin{aligned} \lambda_1 &= \frac{a}{3} + \frac{\sqrt[3]{B - \sqrt{4A^3 + B^2}}}{3\sqrt[3]{2}} + \frac{\sqrt[3]{B + \sqrt{4A^3 + B^2}}}{3\sqrt[3]{2}}, \\ \lambda_2 &= \frac{a}{3} - \frac{(1 + i\sqrt{3})\sqrt[3]{B - \sqrt{4A^3 + B^2}}}{6\sqrt[3]{2}} - \frac{(1 - i\sqrt{3})\sqrt[3]{B + \sqrt{4A^3 + B^2}}}{6\sqrt[3]{2}}, \\ \lambda_3 &= \frac{a}{3} - \frac{(1 - i\sqrt{3})\sqrt[3]{B - \sqrt{4A^3 + B^2}}}{6\sqrt[3]{2}} - \frac{(1 + i\sqrt{3})\sqrt[3]{B + \sqrt{4A^3 + B^2}}}{6\sqrt[3]{2}}, \end{aligned}$$

where

$$A := -(a^2 + 3bd + 3c), \quad B := 2a^3 + 9abd - 18ac.$$

Let

$$\Delta := 4A^3 + B^2.$$

Then if $\Delta > 0$, one zero of (3.115) is real and the other two are complex conjugate. If $\Delta = 0$, all the zeros are real and at least two of them are equal, while if $\Delta < 0$, all the zeros are real and different (see, e.g., [6]).

Case $\Delta \neq 0$. Since $\Delta \neq 0$, all the zeros λ_i, $i = \overline{1,3}$, of polynomial (3.115) are mutually different, and the general solution to (3.103) has the form

$$u_n = \alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \alpha_3\lambda_3^n, \quad n \in \mathbb{N}, \tag{3.116}$$

where α_i, $i = \overline{1,3}$, are arbitrary constants. Since for the case $c_1 \neq 0$ the solution can be prolonged for every nonpositive index, we may assume that formula (3.116) holds also for $n \geq -3$.

From Lemma 2.4 with $P_3(t) = \prod_{j=1}^{3}(t-\lambda_j)$ we have

$$\sum_{j=1}^{3} \frac{\lambda_j^l}{P_3'(\lambda_j)} = 0 \quad \text{for } l = 0, 1,$$

and

$$\sum_{j=1}^{3} \frac{\lambda_j^2}{P_3'(\lambda_j)} = 1.$$

This along with $a_{-2} = a_{-1} = 0$ and $a_0 = 1$ implies that

$$a_n = \frac{\lambda_1^{n+2}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\lambda_2^{n+2}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{\lambda_3^{n+2}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} \tag{3.117}$$

for $n \ge -2$.

On the other hand, from (3.98) we get

$$b_n = a_{n+1} - a_1 a_n, \tag{3.118}$$

$$c_n = c_1 a_{n-1} \tag{3.119}$$

for $n \ge -2$.

Putting (3.117) and (3.93) into (3.118), we obtain

$$b_n = \sum_{j=1}^{3} \frac{\lambda_j - a}{P_3'(\lambda_j)} \lambda_j^{n+2}$$

for $n \ge -2$.

Putting (3.117), which also holds for $n = -3$, and (3.93) into (3.119), we obtain

$$c_n = -\sum_{j=1}^{3} \frac{ac}{P_3'(\lambda_j)} \lambda_j^{n+1}$$

for $n \ge -2$.

From (3.108) and (3.117) we have

$$y_n = \sum_{i=0}^{n-1} a_i = \sum_{i=0}^{n-1} \left(\frac{\lambda_1^{i+2}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\lambda_2^{i+2}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{\lambda_3^{i+2}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} \right) \tag{3.120}$$

for every $n \in \mathbb{N}$.

Recall also that

$$\widehat{y}_n = y_n, \quad n \in \mathbb{N}. \tag{3.121}$$

Now assume that $\lambda_i \ne 1$, $i = 1, 2, 3$. Then, from formula (3.120), it follows that

$$y_n = \widehat{y}_n = R_n^{(1)}(\lambda_1, \lambda_2, \lambda_3) \tag{3.122}$$

for $n \in \mathbb{N}$, where

$$R_n^{(1)}(\lambda_1, \lambda_2, \lambda_3) = \frac{\lambda_1^2(\lambda_1^n - 1)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-1)} + \frac{\lambda_2^2(\lambda_2^n - 1)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-1)} + \frac{\lambda_3^2(\lambda_3^n - 1)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-1)}.$$

It is easy to verify that formula (3.122) holds also for every $n \ge -2$.

If one of the zeros is equal to one, say λ_3, then $1 \ne \lambda_1 \ne \lambda_2 \ne 1$, and we have

$$y_n = \widehat{y}_n = R_n^{(2)}(\lambda_1, \lambda_2) \tag{3.123}$$

for $n \in \mathbb{N}$, where

$$R_n^{(2)}(\lambda_1, \lambda_2) = \frac{\lambda_1^2(\lambda_1^n - 1)}{(\lambda_1-\lambda_2)(\lambda_1-1)^2} + \frac{\lambda_2^2(\lambda_2^n - 1)}{(\lambda_2-\lambda_1)(\lambda_2-1)^2} + \frac{n}{(\lambda_1-1)(\lambda_2-1)}.$$

It is also easy to verify that formula (3.122) holds also for every $n \ge -2$.

From the above consideration and Theorem 3.9 we obtain the following corollary for the case $abcd \ne 0$ and $\Delta \ne 0$.

Corollary 3.12. *Consider system* (1.1) *with* $a, b, c, d \in \mathbb{Z}$ *and* $abcd \neq 0$. *Assume* $z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$ *and* $\Delta \neq 0$. *Then the following statements are true:*

(i) *If none of the zeros of characteristic polynomial* (3.115) *is equal to one, i.e., if* $P_3(1) \neq 0$, *then the general solution to* (1.1) *is given by formulas* (3.102) *and* (3.114), *where the sequence* $(a_n)_{n\geq -2}$, *is given by* (3.117), *while* $(y_n)_{n\geq -2}$ *and* $(\hat{y}_n)_{n\geq -2}$ *are given by* (3.122).

(ii) *If only one of the zeros of characteristic polynomial* (3.115) *is equal to one, say* λ_3, *i.e., if* $P_3(1) = 0 \neq P_3'(1)$, *then the general solution to* (1.1) *is given by formulas* (3.102) *and* (3.114), *where the sequence* $(a_n)_{n\geq -2}$ *is given by* (3.117), *while* $(y_n)_{n\geq -2}$ *and* $(\hat{y}_n)_{n\geq -2}$ *are given by* (3.123).

Remark 3.13. Equation (3.115) will have a zero equal to one if

$$P_3(1) = 1 - a - bd - c + ac = 0,$$

that is, if $(a-1)(c-1) = bd$ so that

$$P_3(\lambda) = \lambda^3 - a\lambda^2 - (ac - a + 1)\lambda + ac.$$

If $bd = 0$, then $b = 0$ or $d = 0$, which implies $a = 1$ or $c = 1$. Assume that $a = 1$ and $c \neq 1$. Then

$$\Delta = -4(1+3c)^3 + 4(1-9c)^2 = -108c(c-1)^2$$

and

$$P_3(\lambda) = \lambda^3 - \lambda^2 - c\lambda + c = (\lambda - 1)(\lambda^2 - c).$$

Thus, if $a = 1$, $b = 0$, $c \in \mathbb{Z} \setminus \{0, 1\}$, $d \in \mathbb{Z}$, we have that $\Delta \neq 0$, so the conditions of Corollary 3.12 (ii) are satisfied, only one zero of polynomial (3.115) is equal to one and all three zeros are mutually different. Moreover, if $c > 0$ we have that $\Delta < 0$ and all three zeros are real and different, while if $c < 0$ we have that $\Delta > 0$ and all three zeros are different but two are complex conjugate.

Case $\Delta = 0$. If $\Delta = 0$ and $ac \neq 0$, then at least two zeros of characteristic polynomial (3.115) are equal, say, λ_2 and λ_3. It is easy to see that the polynomial would have three equal zeros only if $B = 0$, which along with $\Delta = 0$ would also imply $A = 0$. In this case there must be $P_3(\lambda_1) = P_3'(\lambda_1) = P_3''(\lambda_1) = 0$. Hence, from the equality $P_3''(\lambda_1) = 0$, it follows that $\lambda_j = a/3$, $j = 1, 2, 3$, and consequently

$$P_3(a/3) = -\frac{2a^3 + 9abd - 18ac}{27} = 0 \quad \text{and} \quad P_3'(a/3) = -\frac{a^2 + 3bd + 3c}{3} = 0,$$

which immediately implies $2a^3 + 9abd - 18ac = a^2 + 3bd + 3c = 0$. From these two relations it easily follows that $bd = 8c$, and consequently $a^2 = -27c$. Thus, we have

$$P_3(\lambda) = \lambda^3 - a\lambda^2 + \frac{a^2}{3}\lambda - \frac{a^3}{27} = \left(\lambda - \frac{a}{3}\right)^3.$$

Hence, if $a = 9\hat{a}$ for some $\hat{a} \in \mathbb{Z} \setminus \{0\}$, we have that $c = -3\hat{a}^2 \in \mathbb{Z} \setminus \{0\}$, and $b, d \in \mathbb{Z} \setminus \{0\}$ can be chosen so that $bd = 8c = -24\hat{a}^2$. Thus, there are polynomials of the form in (3.115) such that all three zeros are equal.

For $a = c = 1$, $b = 0$, $d \in \mathbb{Z}$ we have that $\Delta = 0$ and

$$P_3(\lambda) = (\lambda + 1)(\lambda - 1)^2,$$

so that there are polynomials with three real zeros such that two of them are equal and different from the third one.

If $\lambda_1 \neq \lambda_2 = \lambda_3$, then the general solution of (3.103) has the form

$$a_n = \hat{\alpha}_1 \lambda_1^n + (\hat{\alpha}_2 + \hat{\alpha}_3 n)\lambda_2^n, \quad n \in \mathbb{N}, \tag{3.124}$$

where $\hat{\alpha}_1$, $\hat{\alpha}_2$ and $\hat{\alpha}_3$ are arbitrary constants. Since in our case the conditions $a_{-2} = a_{-1} = 0$ and $a_0 = 1$ must be satisfied, the solution $(a_n)_{n\geq -2}$ of equation (3.103) can be found by letting $\lambda_3 \to \lambda_2$ in (3.117).

We have

$$a_n = \lim_{\lambda_3\to\lambda_2}\Big(\frac{\lambda_1^{n+2}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\lambda_2^{n+2}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{\lambda_3^{n+2}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\Big)$$
$$= \frac{\lambda_1^{n+2} - (n+2)\lambda_1\lambda_2^{n+1} + (n+1)\lambda_2^{n+2}}{(\lambda_2-\lambda_1)^2}$$

for $n \ge -2$, that is,

$$a_n = \frac{\lambda_1^{n+2} + (\lambda_2 - 2\lambda_1 + n(\lambda_2-\lambda_1))\lambda_2^{n+1}}{(\lambda_2-\lambda_1)^2} \tag{3.125}$$

for $n \ge -2$.

Note that $a_{-2} = a_{-1} = 0$ and $a_0 = 1$, and that (3.125) is of the form (3.124) with

$$\hat\alpha_1 = \frac{\lambda_1^2}{(\lambda_2-\lambda_1)^2}, \quad \hat\alpha_2 = \frac{\lambda_2^2 - 2\lambda_1\lambda_2}{(\lambda_2-\lambda_1)^2}, \quad \hat\alpha_3 = \frac{\lambda_2}{\lambda_2-\lambda_1}.$$

Using relations (3.125) in (3.118) and (3.119), we get

$$b_n = \frac{(\lambda_1 - a)\lambda_1^{n+2}}{(\lambda_2-\lambda_1)^2} + \frac{(\lambda_2(2\lambda_2-3\lambda_1) - a(\lambda_2-2\lambda_1) + n(\lambda_2-\lambda_1)(\lambda_2-a))\lambda_2^{n+1}}{(\lambda_2-\lambda_1)^2},$$
$$c_n = -ac\frac{\lambda_1^{n+1} + (-\lambda_1 + n(\lambda_2-\lambda_1))\lambda_2^n}{(\lambda_2-\lambda_1)^2}, \quad n \ge -2.$$

From (3.108) and (3.125) we have

$$y_n = \sum_{j=0}^{n-1} a_j = \sum_{j=0}^{n-1}\frac{\lambda_1^{j+2} + (\lambda_2 - 2\lambda_1 + j(\lambda_2-\lambda_1))\lambda_2^{j+1}}{(\lambda_2-\lambda_1)^2}, \quad n \in \mathbb{N}. \tag{3.126}$$

If we assume $\lambda_1 \ne 1 \ne \lambda_2 = \lambda_3$, then (3.126), Lemma 2.1 and (3.121) imply

$$y_n = \hat{y}_n = R_n^{(3)}(\lambda_1, \lambda_2) \tag{3.127}$$

for every $n \in \mathbb{N}$, where

$$R_n^{(3)}(\lambda_1,\lambda_2) = \frac{\lambda_1^2(\lambda_1^n - 1)}{(\lambda_2-\lambda_1)^2(\lambda_1-1)} + \frac{(\lambda_2-2\lambda_1)\lambda_2(\lambda_2^n-1)}{(\lambda_2-\lambda_1)^2(\lambda_2-1)} + \frac{\lambda_2^2(1 - n\lambda_2^{n-1} + (n-1)\lambda_2^n)}{(\lambda_2-\lambda_1)(\lambda_2-1)^2}.$$

A direct calculation shows that formulas (3.127) hold also for every $n \ge -2$.

If we assume that $\lambda_1 \ne 1$ and $\lambda_2 = \lambda_3 = 1$, then from (3.126) it follows that

$$y_n = \hat{y}_n = R_n^{(4)}(\lambda_1) \tag{3.128}$$

for every $n \in \mathbb{N}$, where

$$R_n^{(4)}(\lambda_1) = \frac{\lambda_1^2(\lambda_1^n - 1)}{(\lambda_1-1)^3} + \frac{(1-2\lambda_1)n}{(\lambda_1-1)^2} + \frac{(n-1)n}{2(1-\lambda_1)}.$$

A direct calculation shows that formulas (3.128) hold also for every $n \ge -2$.

If we assume that $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 \ne 1$, then from (3.126) it follows that

$$y_n = \hat{y}_n = dR_n^{(5)}(\lambda_2) \tag{3.129}$$

for every $n \in \mathbb{N}$, where

$$R_n^{(5)}(\lambda_2) = \frac{n}{(\lambda_2-1)^2} + \frac{(\lambda_2-2)\lambda_2(\lambda_2^n-1)}{(\lambda_2-1)^3} + \frac{\lambda_2^2(1 - n\lambda_2^{n-1} + (n-1)\lambda_2^n)}{(\lambda_2-1)^3}.$$

A direct calculation shows that formulas (3.129) hold also for every $n \ge -2$.

If $\lambda_1 = \lambda_2 = \lambda_3$, then the general solution of (3.103) has the form

$$a_n = (\hat{\beta}_1 + \hat{\beta}_2 n + \hat{\beta}_3 n^2)\lambda_1^n, \quad n \in \mathbb{N}, \tag{3.130}$$

where $\hat{\beta}_1$, $\hat{\beta}_2$ and $\hat{\beta}_3$ are arbitrary constants.

To find the solution of equation (3.103) satisfying the conditions $a_{-2} = a_{-1} = 0$ and $a_0 = 1$, we will let $\lambda_2 \to \lambda_1$ in formula (3.125). We have

$$\begin{aligned} a_n &= \lim_{\lambda_2\to\lambda_1} \frac{\lambda_1^{n+2} - (n+2)\lambda_1\lambda_2^{n+1} + (n+1)\lambda_2^{n+2}}{(\lambda_2-\lambda_1)^2} \\ &= \lim_{\lambda_2\to\lambda_1} \frac{(\lambda_2-\lambda_1)((n+1)\lambda_2^{n+1} - \lambda_1\sum_{i=0}^n \lambda_2^i\lambda_1^{n-i})}{(\lambda_2-\lambda_1)^2} \\ &= \lim_{\lambda_2\to\lambda_1} \frac{\sum_{i=0}^n (\lambda_2^{n+1} - \lambda_2^i\lambda_1^{n+1-i})}{\lambda_2-\lambda_1} \\ &= \lim_{\lambda_2\to\lambda_1} \frac{\sum_{i=0}^n \lambda_2^i(\lambda_2^{n+1-i} - \lambda_1^{n+1-i})}{\lambda_2-\lambda_1} \\ &= \lim_{\lambda_2\to\lambda_1} \sum_{i=0}^n \lambda_2^i \sum_{j=0}^{n-i} \lambda_2^j\lambda_1^{n-i-j} = \frac{(n+1)(n+2)}{2}\lambda_1^n \end{aligned} \tag{3.131}$$

for $n \ge -2$.

Note that $a_{-2} = a_{-1} = 0$ and $a_0 = 1$, and that (3.131) is of the form (3.130) with

$$\hat{\beta}_1 = 1, \quad \hat{\beta}_2 = \frac{3}{2}, \quad \hat{\beta}_3 = \frac{1}{2}.$$

Using relations (3.131), as well as the condition $c = -a^2/27$ in (3.118) and (3.119), we get

$$\begin{aligned} b_n &= \frac{n+2}{2}(3\lambda_1 - a + n(\lambda_1 - a))\lambda_1^n = -n(n+2)\left(\frac{a}{3}\right)^{n+1}, \\ c_n &= -ac\frac{n(n+1)}{2}\lambda_1^{n-1} = \frac{n(n+1)}{2}\left(\frac{a}{3}\right)^{n+2} \end{aligned}$$

for $n \ge -2$.

From (3.108) and (3.131) we have

$$y_n = \sum_{j=0}^{n-1} a_j = \sum_{j=0}^{n-1} \frac{(j+1)(j+2)}{2}\lambda_1^j \tag{3.132}$$

for every $n \in \mathbb{N}$.

If we assume that $\lambda_1 = \lambda_2 = \lambda_3 \ne 1$, then from (3.132), Lemma 2.1 and (3.121) it follows that

$$\begin{aligned} y_n = \widehat{y}_n &= \frac{2s_n^{(0)} + 3\lambda_1 s_{n-1}^{(1)} + \lambda_1 s_{n-1}^{(2)}}{2} \\ &= \frac{2 - (n+1)(n+2)\lambda_1^n + 2n(n+2)\lambda_1^{n+1} - n(n+1)\lambda_1^{n+2}}{2(1-\lambda_1)^3} \end{aligned} \tag{3.133}$$

for every $n \in \mathbb{N}$. A simple calculation shows that (3.133) also holds for $n = -2, -1, 0$.

If we assume $\lambda_1 = \lambda_2 = \lambda_3 = 1$, then from (3.132) and (3.121) it easily follows that

$$y_n = \widehat{y}_n = R_n^{(6)}(1) := \sum_{j=0}^{n-1} \frac{(j+1)(j+2)}{2} = \frac{n(n+1)(n+2)}{6} \tag{3.134}$$

for every $n \in \mathbb{N}$.

Note that from (3.134) it immediately follows that $R_j^{(6)}(1) = 0$, $j = -2, -1, 0$.

From the above consideration and Theorem 3.9 we obtain the following corollary for the case $abcd \ne 0$ and $\Delta = 0$.

Corollary 3.14. *Consider system* (1.1) *with* $a, b, c, d \in \mathbb{Z}$ *and* $abcd \neq 0$. *Assume that* $z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$, $abcd \neq 0$ *and* $\Delta = 0$. *Then the following statements are true:*

(i) *If none of the zeros of characteristic polynomial* (3.115) *is equal to one, i.e., if* $P_3(1) \neq 0$, *and if it has two different zeros, then the general solution to* (1.1) *is given by formulas* (3.102) *and* (3.114), *where the sequence* $(a_n)_{n\geq -2}$ *is given by* (3.125), *while* $(y_n)_{n\geq -2}$ *and* $(\hat{y}_n)_{n\geq -2}$ *are given by* (3.127).

(ii) *If exactly two of the zeros of characteristic polynomial* (3.115) *are equal to one, say* λ_2 *and* λ_3, *i.e., if* $P_3(1) = P_3'(1) = 0 \neq P_3''(1)$, *then the general solution to system* (1.1) *is given by formulas* (3.102) *and* (3.114), *where the sequence* $(a_n)_{n\geq -2}$ *is given by* (3.125), *while* $(y_n)_{n\geq -2}$ *and* $(\hat{y}_n)_{n\geq -2}$ *are given by* (3.128).

(iii) *If all three zeros of characteristic polynomial* (3.115) *are equal, then the general solution to system* (1.1) *is given by formulas* (3.102) *and* (3.114), *where the sequence* $(a_n)_{n\geq -2}$ *is given by* (3.131), *while* $(y_n)_{n\geq -2}$ *and* $(\hat{y}_n)_{n\geq -2}$ *are given by* (3.133).

Remark 3.15. As we know, polynomial (3.115) has a zero equal to one if $(a-1)(c-1) = bd$. If $\lambda = 1$ is a double zero of (3.115), then it must be $P_3'(1) = 3 - 2a - c - bd = 0$, which is possible only if $ac = 2 - a$, that is, if $a \in \{\pm 1, \pm 2\}$ and $c = (2/a) - 1$. Note that in this case

$$P_3(\lambda) = \lambda^3 - a\lambda^2 + (2a-3)\lambda + 2 - a.$$

If $a = 1$, then $c = 1$ and $bd = 0$, which is not a case treated in Theorem 3.9 (note that in this case we have $P_3(\lambda) = (\lambda-1)^2(\lambda+1)$). If $a = 2$, then $c = 0$, which is also not a case treated in Theorem 3.9. If $a = -1$, then $c = -3$, $bd = 8 \neq 0$, $\Delta = 0$, and $P_3(\lambda) = (\lambda-1)^2(\lambda+3)$. If $a = -2$, then $c = -2$, $bd = 9 \neq 0$, $\Delta = 0$, and $P_3(\lambda) = (\lambda-1)^2(\lambda+4)$. Hence, there are two cases in which polynomial (3.115) has exactly two zeros equal to one, that is, the conditions of Corollary 3.10 (ii) are satisfied.

Remark 3.16. For the case of our system it is not possible that all three zeros of (3.115) are equal to one. Namely, in this case there must be $\lambda_i = a/3 = 1$, $i = 1, 2, 3$, which implies $a = 3$. On the other hand, $c = -a^2/27$, which in this case implies $c = -1/3 \notin \mathbb{Z}$.

Remark 3.17. Assume that only one of the zeros of characteristic polynomial (3.115) is equal to one, say λ_1, $abcd \neq 0$ and that $\Delta = 0$. Then the other two zeros of (3.115) are equal and different from one, i.e., $\lambda_2 = \lambda_3 \neq 1$ and

$$\lambda_1 = \frac{a}{3} + \frac{2\sqrt[3]{B}}{3\sqrt[3]{2}} = 1,$$

which implies that $\sqrt[3]{4B} = 3 - a \in \mathbb{Z}$. Hence $B = 2q^3$ for some $q \in \mathbb{Z} \setminus \{0\}$ (if $q = 0$, i.e., $B = 0$, the condition $\Delta = 0$ would imply $A = 0$, and consequently all three zeros of (3.115) would be equal). Moreover, we have $q = (a-3)/2$, which also implies that a is an odd number. From this and since $4A^3 = -B^2$, we would have that $A^3 = -q^6$, i.e., $A = -q^2$. Recall that since polynomial (3.115) has a zero equal to one, it must be $(a-1)(c-1) = bd$. Hence, we have

$$\frac{(a-3)^2}{4} = q^2 = a^2 + 3bd + 3c = a^2 + 3ac - 3a + 3,$$

which is equivalent to $(a-1)^2 = -4ac$. This, along with the fact that $a = 2m + 1$ for some $m \in \mathbb{Z}$, implies that

$$c = -\frac{(a-1)^2}{4a} = -\frac{m^2}{2m+1},$$

so that

$$k := \frac{m^2}{2m+1}$$

must be an integer.

Since $m^2 - 2km - k = 0$, we have that $m = k \pm \sqrt{k^2 + k}$, and consequently $k^2 + k$ must be a perfect square, i.e., $k(k+1) = r^2$, for some $r \in \mathbb{Z}$. This would mean that k and $k+1$ divide r. From this and since k and $k+1$ are mutually prime numbers if $k \neq -1, 0$, it would follow that $r = ck(k+1)$ for some $c \in \mathbb{Z}$, and consequently $(ck(k+1))^2 = k(k+1)$, which is equivalent to

$$k(k+1)(c^2k(k+1) - 1) = 0.$$

The relation $c^2k(k+1) = 1$ is not possible since 1 is divided only by two integers ± 1. If $k = 0$, then $c = 0$, which would contradict the condition $abcd \neq 0$. If $k = -1$, then $c = 1$, and consequently $bd = 0$, which would also contradict the condition $abcd \neq 0$. Hence c cannot be an integer, so that polynomial (3.115) cannot have only one zero equal to one under the conditions of Theorem 3.9.

Remark 3.18. The formulas presented in this paper can be used in the investigation of the asymptotic behavior of solutions to system (1.1). We leave the problem to the reader.

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