DE GRUYTER Adv. Nonlinear Anal. 2019; 8: 1–28

Research Article

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Asymptotic behavior of evolution systems in arbitrary Banach spaces using general almost periodic splittings

https://doi.org/10.1515/anona-2016-0075
Received March 31, 2016; revised August 24, 2016; accepted September 11, 2016

Abstract: We present sufficient conditions on the existence of solutions, with various specific almost periodicity properties, in the context of nonlinear, generally multivalued, non-autonomous initial value differential equations,

$$\frac{du}{dt}(t) \in A(t)u(t), \quad t \geq 0, \qquad u(0) = u_0,$$

and their whole line analogues, $\frac{du}{dt}(t) \in A(t)u(t)$, $t \in \mathbb{R}$, with a family $\{A(t)\}_{t\in\mathbb{R}}$ of ω-dissipative operators $A(t) \subset X \times X$ in a general Banach space X. According to the classical DeLeeuw–Glicksberg theory, functions of various generalized almost periodic types uniquely decompose in a "dominating" and a "damping" part. The second main object of the study – in the above context – is to determine the corresponding "dominating" part $[A(\cdot)]_a(t)$ of the operators $A(t)$, and the corresponding "dominating" differential equation,

$$\frac{du}{dt}(t) \in [A(\cdot)]_a(t)u(t), \quad t \in \mathbb{R}.$$

Keywords: Evolution equations, almost periodicity, limiting equation

MSC 2010: 47J35, 37L05, 335B40

1 Introduction and preliminaries

We consider general closed and translation invariant subspaces of $\mathrm{BUC}(\mathbb{R}, X)$, as in [26]. This will give a framework on how to work with various forms of almost periodicity. Thus, we consider a general splitting of a closed translation invariant subspace,

$$Y \subset \mathrm{BUC}(\mathbb{R}, X), \quad Y = Y_a \oplus Y_0.$$

We answer the question of when a solution of the nonlinear evolution equation

$$\frac{du}{dt}(t) \in A(t)u(t), \quad u(0) = u_0,$$

with $A(t)$ being dissipative, is a member of Y, is asymptotically close to Y, or itself splits in the same manner, $u = u_a + u_0$, where $u_a \in Y_a$ is a generalized solution to

$$\frac{du}{dt}(t) \in [A(\cdot)]_a(t)u(t),$$

where $[A(\cdot)]_a$ denotes the almost periodic part of the family of possibly unbounded nonlinear dissipative

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operators $\{A(t) : t \in \mathbb{R}\}$. We answer questions concerning the perturbations, i.e., of when the following identity holds:

$$[A(\cdot) + B(\cdot)]_a(t) = [A(\cdot)]_a(t) + [B(\cdot)]_a(t).$$

The underlying study is divided into sections where the nonlinearity is studied, which is even new in finite dimensions, and the appendices which show how Eberlein weak almost periodicity (Appendix A) and asymptotic almost periodicity (Appendix B) come into play, or give a hint on how continuous almost automorphy (Appendix C) becomes applicable. In Appendix A we provide the analysis around Eberlein weakly almost periodic functions (EWAP). We show, how Appendices A and B apply to obtain the splittings in Section 4. We give certain conditions on $f: \mathbb{J} \times X \to X$ to obtain Eberlein weak almost periodicity in Banach spaces. For special cases where the family $A(t) \equiv A$ is time-independent, or the family $A(\cdot)$ itself is periodic, cf. [20, 21]. We also refer to [2], where the result has been proved for the special case of classical almost periodic functions on the real line. In [29] the existence of an EWAP solution on the whole line is proved in finite dimensions for a semilinear system of the form

$$y' = Ay + f(t, y),$$

where A denotes a special linear operator and $f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz.

Applying general existence results, even in the finite dimensional case, more general results on sums of dissipative operators are obtained while dispensing with the relationship between the ω-dissipativeness and the Lipschitz continuity of the nonlinear perturbation $f: \mathbb{J} \times X \to X$. In this study general perturbation results for dissipative operators are used (Section 5). Moreover, for these general perturbation results, we also show the connections between the differential equation and the almost periodic part of the weakly almost periodic solution, and, even more abstractly, how the generally defined splitting carries over from the solution to the differential equation.

Given a Banach space X, a function $f \in C_b(\mathbb{J}, X)$, $\mathbb{J} \in \{\mathbb{R}, \mathbb{R}^+ := [0, \infty), [a, \infty)\}$, is said to be weakly almost periodic in the sense of Eberlein (EWAP) if the orbit of f with respect to $\mathbb{J}$, namely,

$$O_{\mathbb{J}}(f) := \{f_r := \{t \mapsto f(t + r)\} : r \in \mathbb{J}\},$$

is relatively compact with respect to the weak topology of the sup-normed Banach space $(C_b(\mathbb{J}, X)\| \cdot \|_\infty)$ (cf. [9, 20, 21, 24, 27]). The space of all such functions will be denoted by $W(\mathbb{J}, X)$. Moreover, $W_0(\mathbb{J}, X)$ denotes the closed subspace of all $f \in W(\mathbb{J}, X)$ such that some sequence $\{f_{s_n}\}_{n\in\mathbb{N}}$ of translates of f is weakly convergent to the zero function (“weakly” referring to the weak topology of $(C_b(\mathbb{J}, X)\| \cdot \|_\infty)$).

Results of DeLeeuw and Glicksberg [5, 6] imply the following decomposition:

$$W(\mathbb{J}, X) = \mathrm{AP}(\mathbb{R}, X)_{|\mathbb{J}} \oplus W_0(\mathbb{J}, X).$$

Here, $\mathrm{AP}(\mathbb{R}, X)$ denotes the space of almost periodic functions, i.e.,

$$\mathrm{AP}(\mathbb{R}, X) := \{f \in C_b(\mathbb{R}, X) : O_{\mathbb{R}}(f) \text{ is relatively compact in } (C_b(\mathbb{R}, X), \| \cdot \|_\infty)\}.$$

An exposition of this result is given in the book of Krengel [17]. For technical reasons we introduce the following spaces on the real line:

$$W^+(\mathbb{R}, X) := \{f \in \mathrm{BUC}(\mathbb{R}, X) : f_{|\mathbb{R}^+} \in W(\mathbb{R}^+, X)\}$$

and

$$W_0^+(\mathbb{R}, X) := \{f \in W^+(\mathbb{R}, X) : f_{|\mathbb{R}^+} \in W_0(\mathbb{R}^+, X)\}.$$

The decomposition of $W(\mathbb{J}, X)$ and the uniqueness of the almost periodic part gives

$$W^+(\mathbb{R}, X) = \mathrm{AP}(\mathbb{R}, X) \oplus W_0^+(\mathbb{R}, X),$$

and by the uniform continuity of Eberlein weakly almost periodic functions, we obtain that for a given $a \in \mathbb{R}$ and $f \in W^+(\mathbb{R}, X)$, we have $f_{|[a,\infty)} \in W([a, \infty), X)$. Additionally, some examples are given in order to point to problems in this class of functions.

2 Preliminaries on integral solutions

In [23] the following two types of equations have been discussed. The initial value problem

$$u'(t) \in A(t)u(t) + \omega u(t), \quad t \in \mathbb{R}^+, \qquad u(0) = u_0, \tag{2.1}$$

and the whole line equation on $\mathbb{R}$,

$$v'(t) \in A(t)v(t) + \omega v(t), \quad t \in \mathbb{R}, \tag{2.2}$$

with $\omega \in \mathbb{R}$. For these equations we define integral solutions. First, some prerequisites.

Let X be a general Banach space. As the given $\omega \in \mathbb{R}$ plays a crucial role, let $\mathbb{J} \in \{\mathbb{R}^+, \mathbb{R}\}$, and throughout the paper assume that $0 < \lambda, \mu < \frac{1}{|\omega|}$. To obtain the solutions of equations (2.1) and (2.2), we follow the approach given in [23]. The assumptions for the family $\{A(t) : t \in \mathbb{J}\}$ are as follows.

Assumption 2.1. The set $\{A(t) : t \in \mathbb{J}\}$ is a family of m-dissipative operators.

In comparison to the assumption given in [15], the uniform continuity on $\mathbb{J}$ for h is added.

Assumption 2.2. There exist $h \in \mathrm{BUC}(\mathbb{J}, X)$ and $L\colon \mathbb{R}^+ \to \mathbb{R}^+$, continuous and monotone non-decreasing, such that for $\lambda > 0$ and $t_1, t_2 \in \mathbb{I}$, we have

$$\|x_1 - x_2\| \le \|x_1 - x_2 - \lambda(y_1 - y_2)\| + \lambda\|h(t_1) - h(t_2)\|L(\|x_2\|)$$

for all $[x_i, y_i] \in A(t_i)$, $i = 1, 2$.

Recalling the study [23], the next assumption is stronger than that in [15] due to the Lipschitz continuity and linearized stability in $\|y_2\|$. This becomes important to obtain uniform convergence on the half line depending on the Lipschitz constant on forthcoming g and $\omega \in \mathbb{R}$.

Assumption 2.3. There exist bounded and Lipschitz continuous functions $g, h\colon \mathbb{I} \to X$, and a continuous and monotone non-decreasing function $L\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that for $\lambda > 0$, and $t_1, t_2 \in \mathbb{I}$, we have

$$\|x_1 - x_2\| \le \|x_1 - x_2 - \lambda(y_1 - y_2)\| + \lambda\|h(t_1) - h(t_2)\|L(\|x_2\|) + \lambda\|g(t_1) - g(t_2)\|\|y_2\|$$

for all $[x_i, y_i] \in A(t_i)$, $i = 1, 2$.

Remark 2.4. Due to Assumptions 2.2 and 2.3, from [23, Definition 2.6, Remark 2.8], we read,

$$D(A(t)) \subset \hat{D} \subset \overline{D(A(t))} = \overline{D_A}.$$

Due to the fact that we consider $(A(t) + \omega I)$ as a perturbation of $A(t)$ by ωI, we have to define the perturbed control functions h^ω and L^ω. We define

$$h^\omega\colon \mathbb{I} \to (X \times X, \|\cdot\|_1), \quad t \mapsto (h(t), |\omega| g(t)),$$

and $L^\omega(t) = L(t) + t$.

Now we are in the situation to define the integral solution to the initial value problem. As the integral solution is defined with the help of Assumptions 2.2 and 2.3, it looks slightly different from the one in [15, Definition 6.18, pp. 217–218], which is due to Benilan.

Definition 2.5. Let $\mathbb{I} = \mathbb{R}^+$ and assume that either Assumption 2.2 or Assumption 2.3 is satisfied for the family $\{A(t) + \omega I : t \in \mathbb{I}\}$. Let also $0 \le a < b$. A continuous function $u\colon [a, b] \to X$ is called an integral solution of (2.1) if $u(0) = u_0$ and

$$\begin{aligned}
&\|u(t) - x\| - \|u(r) - x\| \\
&\quad \le \int_r^t ([y, u(v) - x]_+ + \omega\|u(v) - x\|)\, dv + L^\omega(\|x\|) \int_r^t \|h^\omega(v) - h^\omega(r)\|\, dv + \|y\| \int_r^t \|g(v) - g(r)\|\, dv
\end{aligned}$$

for all $a \le r \le t \le b$, and $[x, y] \in A(r) + \omega I$ (with $g \equiv 0$ in case of Assumption 2.2).

In order to solve the initial value problem (2.1), we use the Yosida approximation of the derivative. This leads to the equation

$$\Big(\frac{d}{dt}\Big)_\lambda u_\lambda(t) \in A(t)u_\lambda(t) + \omega u_\lambda(t), \quad t \in \mathbb{R}^+, \qquad u(0) = u_0, \tag{2.3}$$

with the Yosida approximation

$$\Big(\frac{d}{dt}\Big)_\lambda u(s) := \frac{1}{\lambda}\bigg(u(s) - u_0 - \frac{1}{\lambda}\int_0^s e^{-\frac{\tau}{\lambda}}(u(s-\tau) - u_0)\,d\tau\bigg).$$

Thus, we obtain the solution u_λ to (2.3) as fixpoint (cf. [23, Lemma 3.1]), i.e.,

$$u_\lambda(t) = J_\lambda^\omega(t)\bigg(e^{-\frac{t}{\lambda}}u_0 + \frac{1}{\lambda}\int_0^t e^{-\frac{\tau}{\lambda}}u_\lambda(t-\tau)\,d\tau\bigg),$$

with (see [23, Remark 2.7])

$$J_\lambda^\omega(t)x := J_{\frac{\lambda}{1-\lambda\omega}}(t)\Big(\frac{1}{1-\lambda\omega}x\Big)$$

The results for these approximations on $\mathbb{R}^+$ are the following.

Theorem 2.6 ([23]). *Let $\mathbb{I} = \mathbb{R}^+$, and let $A(t)$ fulfill Assumptions 2.1 and 2.2 with $\omega < 0$, or Assumption 2.3 with $L_g < -\omega$, where L_g is the Lipschitz constant of g. Further, let $u_0 \in \overline{D}_A$, and let u_λ be the corresponding Yosida approximations to equation* (2.3). *Then u_λ converges uniformly on $\mathbb{R}^+$ to the integral solution, as $\lambda \to 0^+$.*

Further, we define an integral solution for the whole line problem (2.2).

Definition 2.7. Let $\mathbb{I} = \mathbb{R}$, and assume that either Assumption 2.2 or Assumption 2.3 is satisfied for the family $\{A(t) + \omega I : t \in \mathbb{I}\}$. A continuous function $u\colon \mathbb{R} \to X$ is called an integral solution on $\mathbb{R}$ if

$$\begin{aligned}&\|u(t) - x\| - \|u(r) - x\|\\ &\quad\le \int_r^t([y, u(v) - x]_+ + \omega\|u(v) - x\|)\,dv + L^\omega(\|x\|)\int_r^t\|h^\omega(v) - h^\omega(r)\|\,dv + \|y\|\int_r^t\|g(v) - g(r)\|\,dv\end{aligned}$$

for all $-\infty < r \le t < \infty$, and $[x, y] \in A(r) + \omega$ (with $g \equiv 0$ in case of Assumption 2.2).

Similar to the initial value case, the Yosida approximation of the derivative was considered and led to the following result:

$$\Big(\frac{d}{dt}\Big)_\lambda u_\lambda(t) \in A(t)u_\lambda(t) + \omega u_\lambda(t), \quad t \in \mathbb{R},$$

with

$$\Big(\frac{d}{dt}\Big)_\lambda u(t) := \frac{1}{\lambda}\bigg(u(t) - \frac{1}{\lambda}\int_0^\infty \exp\Big(-\frac{s}{\lambda}\Big)u(t-s)\,ds\bigg).$$

Again the u_λ are derived by a Banach iteration, i.e.,

$$u_\lambda(t) = J_{\frac{\lambda}{1-\lambda\omega}}(t)\bigg(\frac{1}{1-\lambda\omega}\bigg(\int_0^\infty \frac{1}{\lambda}e^{-\frac{s}{\lambda}}u_\lambda(t-s)\,ds\bigg)\bigg).$$

For these approximants we have the following result.

Theorem 2.8 ([23]). *Let $\mathbb{I} = \mathbb{R}$.*

(i) *If Assumption 2.1, and Assumption 2.2 with $\omega < 0$, are fulfilled, then the Yosida approximants $(u_\lambda : \lambda > 0)$ are Cauchy in* $\mathrm{BUC}(\mathbb{R}, X)$ *when $\lambda \to 0$. The limit $u(t) := \lim_{\lambda\to 0} u_\lambda(t)$ is an integral solution on $\mathbb{R}$.*

(ii) *Let Assumptions 2.1 and 2.3 be fulfilled. Further, assume that the Lipschitz constant L_g of g in Assumption 2.3 is less than $-\omega$. Then the Yosida approximants $(u_\lambda : \lambda > 0)$ are Cauchy in* $\mathrm{BUC}(\mathbb{R}, X)$ *when $\lambda \to 0^+$. The limit $u(t) := \lim_{\lambda\to 0^+} u_\lambda(t)$ is an integral solution on $\mathbb{R}$.*

Remark 2.9 ([22]). (i) The construction of the solutions implies $u(t) \in \overline{D(A(t))} = \overline{D_A}$.
(ii) In the following, with regard to equations (2.1) and (2.2), we always consider the solutions given by Theorem 2.6 and Theorem 2.8, respectively.

Moreover, we have a comparison result between the solution found on $\mathbb{R}$ and $\mathbb{R}^+$.

Corollary 2.10 ([23]). *Let $A(t)$ fulfill Assumption 2.1 and either Assumption 2.2 with $\omega < 0$, or Assumption 2.3 with $L_g < -\omega$, where L_g is the Lipschitz constant of g. Then the solution v of* (2.2) *and the solution u of* (2.1) *satisfy*

$$\|u(t) - v(t)\| \le \exp(\omega t)\|u_0 - v(0)\| \quad \textit{for all } 0 \le t.$$

3 Main definitions and context

We start with the notion of pseudo-resolvents:

Definition 3.1 ([25, Definition 7.1, p. 193]). The members of a family of contractions,

$$\{J_\lambda : D(J_\lambda) \subset X \to X \mid \lambda > 0\},$$

are called pseudo-resolvents if

$$R\Big(\frac{\mu}{\lambda} I + \frac{\lambda - \mu}{\lambda} J_\lambda\Big) \subset D(J_\mu), \quad \lambda, \mu > 0,$$

and

$$J_\lambda u = J_\mu\Big(\frac{\mu}{\lambda} u + \frac{\lambda - \mu}{\lambda} J_\lambda u\Big), \quad \lambda, \mu > 0,\ u \in D(J_\lambda).$$

Lemma 3.2 ([25, Lemma 7.1, pp. 193–194]). *Pseudo-resolvents have the generator*

$$A = \frac{1}{\lambda}(I - J_\lambda^{-1}) \quad \textit{for any } \lambda > 0.$$

Next we motivate a generalization for various types of almost periodicity.

Lemma 3.3. *Let Y be a closed linear subspace of* $\mathrm{BUC}(\mathbb{R}, X)$, *and $\{B(t) : t \in \mathbb{R}\}$ a family of m-dissipative and Lipschitz continuous operators defined on X with the common Lipschitz constant K. Further, if for all $f \in Y$, $\{t \mapsto B(t)f(t)\} \in Y$, then for all $f \in Y$, $\{t \mapsto J_\lambda^B(t)f(t)\} \in Y$, for small $\lambda > 0$.*

Proof. For given $\lambda > 0$ and $x \in Y$, we define

$$T_\lambda : Y \to Y, \quad f \mapsto \{t \mapsto \lambda B(t)(f(t) + x(t))\}.$$

Then, for given $f, g \in Y$, we have

$$\|T_\lambda f - T_\lambda g\|_\infty \le \lambda K \|f - g\|_\infty.$$

Consequently, for $\lambda K < 1$, there exists $f_\lambda \in Y$ such that

$$f_\lambda = \{t \mapsto \lambda B(t)(f_\lambda(t) + x(t))\}$$

and

$$(I - \lambda B(t))(x(t) + f_\lambda(t)) = x(t) + f_\lambda(t) - \lambda B(t)(x(t) + f_\lambda(t)) = x(t).$$

Thus, we found that for $\lambda K < 1$, $(I - \lambda B(t))^{-1} x(t) = x(t) + f_\lambda(t) \in Y$. □

Due to the previous results it becomes straightforward to define types of almost periodicity even for multivalued ω-dissipative operators.

Definition 3.4. Let Y be a closed linear translation-invariant subspace of $\mathrm{BUC}(\mathbb{R}, X)$. An m-dissipative family $\{A(t) : t \in \mathbb{R}\} \subset X \times X$ is called Y invariant if $\{t \mapsto J_\lambda(t)x(t)\} \in Y$ for all $x \in Y$ and $0 < \lambda \le \lambda_0$, and $\lambda\omega < 1$.

Remark 3.5. The examples for Y are $\mathrm{AP}(\mathbb{R}, X)$, $\mathrm{AAP}(\mathbb{R}^+, X)$, $W(\mathbb{R}, X)$, $\mathrm{WRC}(\mathbb{R}, X)$ and $\mathrm{CAA}(\mathbb{R}, X)$. In the case of almost automorphy, we have to add uniform continuity (i.e., $\mathrm{AA}(\mathbb{R}, X) \cap \mathrm{BUC}(\mathbb{R}, X)$). The spaces $\mathrm{AAP}(\mathbb{R}^+, X)$, $W(\mathbb{R}^+, X)$ and $\mathrm{WRC}(\mathbb{R}^+, X)$ are considered in the following way:

$$Y = \{f \in \mathrm{BUC}(\mathbb{R}, X) : f_{|\mathbb{R}^+} \in \mathrm{AAP}(\mathbb{R}^+, X), (\mathrm{WRC}(\mathbb{R}^+, X), W(\mathbb{R}^+, X))\}.$$

This makes sense even for the splitting, as the almost periodic part is uniquely defined on $\mathbb{R}$ while the weak almost periodic function is only given on $\mathbb{R}^+$. This approach allows to consider equations on the real line, which are given only on $\mathbb{R}^+$, while extending $A(t)$ by

$$\tilde{A} := \begin{cases} A(t), & t \geq 0, \\ A(0), & t < 0. \end{cases}$$

The control functions are extended in the same manner.

Due to Lemma 3.3, we can define the almost periodic part of a generally multivalued operator $A(\cdot)$. Therefore, let Y split into the direct sum

$$Y = Y_a \oplus Y_0,$$

with a projection $P_a : Y \to Y_a$ satisfying $\|P_a\| \leq 1$.

Definition 3.6. A linear, closed and translation invariant subspace $Y \subset \mathrm{BUC}(\mathbb{R}, X)$ splits almost periodic if $Y = Y_a \oplus Y_0$, where Y_a, Y_0 are closed translation invariant linear subspaces, the constants are in Y_a ($X \subset Y_a$), the corresponding projection fulfills

$$P_a(Y) \subset Y_a, \quad \text{with } \|P_a\| \leq 1,$$

and, finally, P_a commutes with the translation semigroup.

Lemma 3.7. *Let Y split almost periodic, and let $\{A(t) : t \in \mathbb{R}\} \subset X \times X$ be m-dissipative and Y-invariant so that*

$$J_\lambda(t)x = J_{\lambda,a}(t)x + \phi,$$

with $J_{\lambda,a}(\cdot)x \in Y_a$ and $\phi \in Y_0$. If for a given $f \in Y$,

$$P_a(J_\lambda(\cdot)f(\cdot)) = J_{\lambda,a}(\cdot)P_a f(\cdot), \tag{3.1}$$

then the members of the following family of operators are pseudo-resolvents:

$$J_{\lambda,a}(t) : X \to X, \quad x \mapsto P_a(J_\lambda(\cdot)x)(t)$$

If $\{A(t) : t \in \mathbb{R}\}$ is Y-invariant and (3.1) is fulfilled, we say that $\{A(t) : t \in \mathbb{R}\}$ splits almost periodic with respect to Y_a, Y_0.

Proof. First we prove that $\{J_{\lambda,a}(t) : t \in \mathbb{R}\}$ are contractions:

$$\begin{aligned} \|J_{\lambda,a}(t)x - J_{\lambda,a}(t)y\| &= \|P_a(J_\lambda(\cdot)x)(t) - P_a(J_\lambda(\cdot)y)(t)\| \\ &\leq \|J_\lambda(\cdot)x - J_\lambda(\cdot)y\|_\infty \\ &\leq \|x - y\|. \end{aligned} \tag{3.2}$$

Next the range condition, but this is fulfilled, because we assumed $A(t)$ to be m-dissipative, therefore $X = D(J_\lambda(t)) = D(J_{\lambda,a}(t))$.

As $J_\lambda(t)$ are resolvents, we have the following resolvent equation:

$$J_\lambda(t)u = J_\mu(t)\Big(\frac{\mu}{\lambda}u + \frac{\lambda - \mu}{\lambda}J_\lambda(t)u\Big), \quad \lambda, \mu > 0,\ u \in X.$$

Using, $u(t) \equiv u \in Y_a$, the pseudo-resolvent-equation holds in Y. Hence, we can apply the projection P_a on both sides of the equation. Thus,

$$P_a J_\lambda(\cdot)u = P_a\Big(J_\mu(\cdot)\Big(\frac{\mu}{\lambda}u(\cdot) + \frac{\lambda - \mu}{\lambda}J_\lambda(\cdot)u\Big)\Big), \quad \lambda, \mu > 0,\ u \in X.$$

Applying (3.1), and evaluating at $t \in \mathbb{R}$, serves for the proof. □

Definition 3.8. Let $\{A(t) : t \in \mathbb{R}\} \subset X \times X$ be m-dissipative and split almost periodic with respect to Y, Y_a. The almost periodic part of $A(\cdot)$ is defined as follows:

$$[A(\cdot)]_a := \frac{1}{\lambda}(I - J_{\lambda,a}(\cdot)^{-1}), \quad \lambda > 0.$$

Remark 3.9. By the previous definition, for m-dissipative operators $\{A(t) : t \in \mathbb{R}\} \subset X \times X$ which split almost periodic, we obtain

$$J_\lambda^{A_a}(t)x = J_{\lambda,a}(t)x := (J_\lambda(\cdot)x)_a(t),$$

where $(\cdot)_a$ denotes the Y_a part of a function given in Y.

Proof. Using the fact that $A = \frac{1}{\lambda}(I - (J_\lambda^A)^{-1})$ for A dissipative, we have

$$A = B \iff J_\lambda^A = J_\lambda^B \quad \text{for some } \lambda > 0,$$

and

$$\frac{1}{\lambda}(I - J_\lambda^{A_a}(\cdot)^{-1}) = [A(\cdot)]_a = \frac{1}{\lambda}(I - J_{\lambda,a}^A(\cdot)^{-1}).$$

The first equality comes with the dissipativeness of $[A(\cdot)]_a$ due to (3.2), and the second by definition. □

Remark 3.10. Let $\{A(t) : t \in \mathbb{R}\} \subset X \times X$ split almost periodic with respect to Y_a, Y_0. Due to (3.2), $[A(\cdot)]_a$ is dissipative. From $D(J_{\lambda,a}) = X$, we derive the m-dissipativeness, and the facts that $P_a x = x$ for all $x \in Y_a$, and, together with equation (3.1), that the almost periodic part of $A(\cdot)$ is Y_a-invariant.

Proof. Given $x \in Y_a$, we apply (3.1):

$$J_{\lambda,a}(\cdot)x(\cdot) = J_{\lambda,a}(\cdot)P_a x(\cdot) = P_a J_\lambda(\cdot)x(\cdot) \in Y_a.$$

Thus, $J_{\lambda,a}(\cdot)x(\cdot) \in Y_a$, and therefore $J_{\lambda,a}(\cdot)x(\cdot) = P_a J_{\lambda,a}(\cdot)x(\cdot)$. □

For an example, we recall the definition of demi-closedness [15, Definition 1.15, p. 18]

Definition 3.11. An operator $A \subset X \times X$ is called demi-closed if A is norm-(weakly-sequentially) closed as a subset of $X \times X$, i.e., $\|x_n - x\| \to 0$, $w - \lim_{n\to\infty} y_n = y$, and $[x_n, y_n] \in A$ for all $n \in \mathbb{N}$ implies $[x, y] \in A$.

Example 3.12. Let $a\colon \mathbb{R} \to \mathbb{R}$ be weakly almost periodic and assume that $a(t) \geq q$ for some $q > 0$. Using the splitting $a(t) = a_a(t) + \phi(t)$, with $a_a \in \mathrm{AP}(\mathbb{R})$, for an m-dissipative operator $A \subset X \times X$, we define

$$A(t) := a(t)A.$$

Let $Y_a = \mathrm{AP}(\mathbb{R}, X)$, and either $Y = \mathrm{WRC}(\mathbb{R}^+, X)$ or $Y = W(\mathbb{R}^+, X)$. Let also A be demiclosed and J_λ compact. Then $A(t)$ splits almost periodic, and the Y_a-part is given by $[A(\cdot)]_a(t) = a_a(t)A$. The same result can be obtained using $Y = \mathrm{AAP}(\mathbb{R}^+, X)$, with $a \in \mathrm{AAP}(\mathbb{R}^+, X)$.

Proof. For a given $\lambda > 0$, we have to prove that $\{t \mapsto J_\lambda(t)x\}$ is weakly almost periodic. Note, that $J_\lambda(t)x = J_{\lambda a(t)}x$. Thus, we obtain the result using the following help function with $0 < r < q$:

$$f\colon \mathbb{R} \times \mathbb{R} \to X, \quad (t, p) \mapsto J_{\lambda(|p-r|+r)}x$$

which is Lipschitz continuous, by the resolvent equation ($\lambda(|p - r| + r) \geq \lambda r > \delta > 0$), $f(t, a(t)) = J_{\lambda a(t)}x$, and since $f \in W(\mathbb{R} \times \mathbb{R}, X)$ (cf. Definition A.4), Corollary A.10 applies. To prove $\{t \mapsto J_\lambda(t)x(t)\} \in Y$ for $x \in Y$ in the case where $Y = \mathrm{WRC}(\mathbb{R}^+, X)$, we can use Corollary A.10 with the help function

$$f\colon \mathbb{R} \times (\mathbb{R} \times X) \to X, \quad (t, (p_1, p_2)) \mapsto J_{\lambda(|p_1-r|+r)}p_2,$$

and $f(t, a(t), x(t)) = J_{\lambda a(t)}x(t) = J_\lambda(t)x(t)$. Hence, for $x \in Y = W(\mathbb{R}^+, X)$, by a further use of the previously defined help function and an application of Theorem A.8, it remains to prove that, for $K_1 \subset X$ weakly compact metrizable, the following function is continuous:

$$\iota\colon [a, b] \times (K_1, w) \to (X, \|\cdot\|), \quad (s, x) \mapsto J_{\lambda(|s-r|+r)}x.$$

For a given $\{x_n\}_{n\in\mathbb{N}} \subset K_1$, weakly convergent to $x \in K_1$, and $\{s_n\}_{n\in\mathbb{N}} \subset [a, b]$ convergent to $s \in [r, R]$, we define $\lambda_n := \lambda(|s_n - r| + r) \to \mu \in [\lambda r, R]$ and claim that

$$J_{\lambda_n} x_n \to J_\mu x \quad \text{when } n \to \infty.$$

To this end, we show that any subsequence of $\{J_{\lambda_n} x_n\}_{n\in\mathbb{N}}$ has a subsequence convergent to $J_\mu x$. Let such a subsequence be chosen. Then, without loss of generality, we have $u_n := J_\mu x_n \to u$. This is equivalent to $x_n \in u_n - \mu A u_n$, and the demiclosedness leads to $J_\mu x_n \to J_\mu x$. Using the fact that $\|A_\lambda x\| \le \|A_\mu x\|$ for all $0 < \mu < \lambda$, by the resolvent equation, we have

$$\|J_{\lambda_n} x_n - J_\mu x\| \le \|J_{\lambda_n} x_n - J_\mu x_n\| + \|J_\mu x_n - J_\mu x\| \le |\lambda_n - \mu| \|A_{\lambda r} x_n\| + \|J_\mu x_n - J_\mu x\|.$$

The Lipschitz continuity of $A_{\lambda r}$ proves the boundedness of $\|A_{\lambda r} x_n\|$, which finishes the proof.

To compute the almost periodic part of $\{t \mapsto J_{\lambda a(t)} x\}$, apply Corollary A.17. □

To have a comparison between the classical almost periodicity and the generalized one, we provide the following proposition. It shows that under certain conditions $t \to B(t)x \in Y = Y_a \oplus Y_0$ (i.e., it splits canonically), and the splitting is obtained from coinciding resolvents.

Proposition 3.13. *Let $Y \subset \mathrm{BUC}(\mathbb{R}, X)$ split almost periodic, and let $\{B(t) : t \in \mathbb{R}\}$ be a family of uniformly Lipschitz operators defined on X, with Lipschitz constant K. Further, let $B(\cdot)x \in Y$ for all $x \in X$. Due to the splitting, we have $B(t)x = P_a(B(\cdot)x)(t) + \phi(t)$. We define*

$$B^a(t)x := P_a(B(\cdot)x)(t),$$

and assume that

$$P_a(B(\cdot)y(\cdot))(t) = B^a(t)P_a(y(\cdot))(t) \quad \text{for all } y \in Y. \tag{3.3}$$

Then, for $B_a(\cdot)x$, the almost periodic part in the sense of Definition 3.8, we have

$$B^a(\cdot)x = B_a(\cdot)x \quad \text{for all } x \in X.$$

Proof. To prove the equality we show that the resolvents coincide, i.e.,

$$(I - \lambda B^a(t))^{-1} x = P_a(I - \lambda B(t))^{-1} = J^B_{\lambda,a}(t)x. \tag{3.4}$$

The above together with Remark 3.9 will finish the proof. To prove (3.4) we need a representation of the resolvents. In doing this, we consider

$$T^a_\lambda\colon Y_a \to Y_a, \quad f \mapsto \{t \mapsto \lambda B^a(t)(f(t) + x)\}.$$

For given $x, y \in X$, we have

$$\|B^a(t)x - B^a(t)y\| = \|P_a(B(\cdot)x)(t) - P_a(B(\cdot)y)(t)\| \le \|B(\cdot)x - B(\cdot)y)\|_\infty \le K\|x - y\|.$$

The Lipschitz continuity of B^a leads to

$$\|T^a_\lambda f - T^a_\lambda g\|_\infty \le \lambda K \|f - g\|_\infty \quad \text{for } f, g \in Y_a.$$

Consequently, for $\lambda K < 1$, we find a net of fixpoints f^a_λ satisfying

$$\begin{aligned} \lambda B^a(t)(f^a_\lambda(t) + x) &= f^a_\lambda(t), \\ (I - \lambda B^a(t))(f^a_\lambda(t) + x) &= f^a_\lambda(t) + x - f^a_\lambda(t) = x, \\ f^a_\lambda(t) + x &= (I - \lambda B^a(t))^{-1} x. \end{aligned}$$

Next we consider the resolvent of B_a which is defined by the almost periodic part of the resolvent of B. Hence, we need a representation for the resolvent of B and we have to compute the almost periodic part. We define the fixpoint mapping with respect to B:

$$T_\lambda\colon Y \to Y, \quad f \mapsto \{t \mapsto \lambda B(t)(f(t) + x)\}.$$

Using the fact that B is Lipschitz for $\lambda K < 1$, we find a net of fixpoints f_λ satisfying

$$\begin{aligned}\lambda B(t)(f_\lambda(t) + x) &= f_\lambda(t),\\ (I - \lambda B(t))(f_\lambda(t) + x) &= f_\lambda(t) + x - f_\lambda(t) = x,\\ f_\lambda(t) + x &= (I - \lambda B(t))^{-1}x.\end{aligned}$$

Using the representation, we compute the almost periodic part and apply assumption (3.3), to obtain

$$P_a(f_\lambda) + x = P_a(f_\lambda + x) = P_a(\lambda B(\,\cdot\,)(f_\lambda + x) + x = \lambda B^a(\,\cdot\,)(P_a f_\lambda + x) + x.$$

We found that $P_a f_\lambda$ is a fixpoint of T_λ^a, hence $f_\lambda^a = P_a(f_\lambda)$. This leads to

$$(I - \lambda B^a(t))^{-1}x = f_\lambda^a(t) + x = P_a(f_\lambda)(t) + x = P_a(I - \lambda B(\,\cdot\,))^{-1}x = J_\lambda^B(t)x,$$

which finishes the proof. □

Kenmochi and Otani [16] considered nonlinear evolution equations governed by time dependent subdifferential operators in Hilbert space H:

$$u(t) \in -\partial\varphi^t(u(t)), \quad u(0) = u_0.$$

They constructed a not necessarily complete metric space of convex functions (Φ, d), whereby, $d(\varphi^{t_n}, \varphi) \to 0$, implies $\varphi^{t_n} \to \varphi$, in the sense of Mosco, cf. [16, Lemma 4.1, p. 75]. For short, we write $\varphi^{t_n} \xrightarrow{\Phi} \varphi$. Next we show that the notion of almost periodicity on subdifferentials given in [16] is stronger than the one given via resolvents defined in this study, when viewing $-\partial\varphi(t)$ as a dissipative operator.

Proposition 3.14. *Let $\varphi^t\colon H \to \mathbb{R}$ be proper, lower semicontinuous and convex for all $t \in \mathbb{R}$. If*

$$\varphi^{(\,\cdot\,)}\colon \mathbb{R} \to (\Phi, d), \quad t \mapsto \varphi^t,$$

is almost periodic, then

$$J_\lambda^\varphi(\,\cdot\,)x\colon \mathbb{R} \to H, \quad t \mapsto J_\lambda^\varphi(t)x := (I + \lambda\partial\varphi^t)^{-1}x,$$

is almost periodic for all $x \in H$.

Proof. We assume that for $x \in H$, the mapping $\{t \mapsto J_\lambda^\varphi(t)x\}$ is not almost periodic. Consequently, we find a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $\{J_\lambda^\varphi(t + t_n)x\}_{n\in\mathbb{N}}$ is not uniformly Cauchy. Without loss of generality,

$$\varphi^{(t+t_n)} \xrightarrow{\Phi} \psi^t \quad \text{uniformly for } t \in \mathbb{R}.$$

The non-Cauchy assumption on $\{J_\lambda^\varphi(t + t_n)x\}_{n\in\mathbb{N}}$ leads to subsequences $\{t_{n_k}\}_{k\in\mathbb{N}}, \{t_{m_k}\}_{k\in\mathbb{N}} \subset \{t_n\}_{n\in\mathbb{N}}$, and a sequence $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\|J_\lambda^\varphi(s_k + t_{n_k})x - J_\lambda^\varphi(s_k + t_{m_k})x\| \geq \varepsilon \quad \text{for all } k \in \mathbb{N}.$$

Note that $\{t \mapsto \varphi^t\}$ is assumed to be almost periodic, $O(\varphi) := \overline{\{\{t \mapsto \varphi^{t+s}\} : s \in \mathbb{R}\}}$ is a compact subset of $C_b(\mathbb{R}, (\Phi, d))$, and $\psi^{(\,\cdot\,+s)} \in O(\varphi)$ for all $s \in \mathbb{R}$. Consequently, we may assume that

$$\psi^{(t+s_k)} \xrightarrow{\Phi} \phi^{(t)} \quad \text{uniformly for } t \in \mathbb{R}.$$

Clearly,

$$\varphi^{(t+s_k+t_{n_k})} \xrightarrow{\Phi} \phi^t \quad \text{and} \quad \varphi^{(t+s_k+t_{m_k})} \xrightarrow{\Phi} \phi^t$$

uniformly for $t \in \mathbb{R}$. Applying [16, Lemma 4.1, p. 75], we obtain

$$\varphi^{(0+s_k+t_{n_k})} \to \phi^0 \quad \text{and} \quad \varphi^{(0+s_k+t_{m_k})} \to \phi^0,$$

in the sense of Mosco, which implies, by using [1, Theorem 3.26, p. 305],

$$J_\lambda^\varphi(s_k + t_{n_k})x \to J_\lambda^\phi x \quad \text{and} \quad J_\lambda^\varphi(s_k + t_{m_k})x \to J_\lambda^\phi x$$

when $k \to \infty$. A contradiction to the non-Cauchy assumption. □

4 Main results

In this section we show how the previous results apply to evolution equations of the following type:

$$\frac{du}{dt}(t) \in A(t)u(t) + \omega u(t), \quad t \in \mathbb{R}, \tag{4.1}$$

and the corresponding initial value problem

$$\frac{du}{dt}(t) \in A(t)u(t) + \omega u(t), \quad t \in \mathbb{R}^+, \qquad u(0) = u_0, \tag{4.2}$$

where $A(t)$ is a possibly nonlinear multivalued and dissipative operator satisfying a type of almost periodicity defined above. Throughout this section, we assume Y, Y_a, and Y_0 to be closed and translation invariant subspaces of $\mathrm{BUC}(\mathbb{R}, X)$, with $X \subset Y_a$. In the case where $A(t)$ is classically almost periodic and $Y = \mathrm{AP}(\mathbb{R}, X)$, the result in case of Assumption 2.2 is due to [2]. Even in the book [14, equation (C1), p. 153], Hino et al. considered Assumption 2.2. Thus, they were not able to consider operators coming from Example 3.12. The problem is considered in the infinite dimensional case, and is even new for finite dimensions. The ω-dissipativeness ($\omega < 0$) is needed to obtain the uniform convergence of the approximants. Moreover, when $\omega = 0$ and $A(t)$ only m-dissipative, there exists a counterexample for classical almost periodicity in the case of dimension two, see [13, Remark 1.3 (2)].

Definition 4.1. A solution u of (4.2) is called asymptotically Y if there exists $v \in Y$ such that

$$\lim_{t\to\infty} \|u(t) - v(t)\| = 0.$$

The difference to earlier splitting results is that a splitting of the solution is found, but in the case of general dissipative and time dependent operators, the equation fulfilled by the almost periodic part was unknown.

Theorem 4.2. *Let, for a given $Y \subset \mathrm{BUC}(\mathbb{R}, X)$, $A(\cdot)$ be Y-invariant, and $\{A(t) : t \in \mathbb{R}\}$ fulfill Assumption 2.1 and either Assumption 2.2 with $\omega < 0$, or Assumption 2.3 with $L_g < -\omega$, where L_g denotes the Lipschitz constant for the control function g. Then there exists a solution $u \in Y$ to (4.1), and all integral solutions of (4.2) are asymptotically Y.*

Let Y and $A(\cdot)$ split almost periodic, with $Y = Y_a \oplus Y_0$. Then the almost periodic part u^a of the solution $u \in Y$ fulfills $u^a \in Y_a$, and u^a is a generalized solution to the evolution equation

$$\frac{du^a}{dt}(t) \in [A(\cdot)]_a(t)u^a(t) + \omega u^a(t), \quad t \in \mathbb{R}. \tag{4.3}$$

Proof. The first part is a direct consequence of [22, Theorem 4.10]. For the second part, note that

$$u_\lambda(t) = J_{\frac{\lambda}{1-\lambda\omega}}(t)\bigg(\frac{1}{1-\lambda\omega}\bigg(\int_0^\infty \frac{1}{\lambda}e^{-\frac{s}{\lambda}}u(t-s)\,ds\bigg)\bigg),$$

and due to the assumption of an almost periodic splitting, we apply P_a and obtain

$$u_\lambda^a = J^a_{\frac{\lambda}{1-\lambda\omega}}(t)\bigg(\frac{1}{1-\lambda\omega}\bigg(\int_0^\infty \frac{1}{\lambda}e^{-\frac{s}{\lambda}}u_\lambda^a(t-s)\,ds\bigg)\bigg). \tag{4.4}$$

Thus,

$$\left(\frac{d}{dt}\right)_\lambda u_\lambda^a(t) \in [A(\cdot)]_a(t)u_\lambda^a(t) + \omega u_\lambda^a(t), \quad t \in \mathbb{R}. \tag{4.5}$$

Applying Theorem 2.8, $u_\lambda \to u$ uniformly on $\mathbb{R}$, and by the continuity of the projection P_a, we have $u_\lambda^a \to u^a$. We call u^a the generalized solution to (4.3). □

Remark 4.3. (i) The notion of an integral solution in the case of a general splitting is not possible due to the missing control functions for $\{[A(\cdot)]_a(t) : t \in \mathbb{R}\}$. Thus, in general we cannot construct a solution applying existence results of [23] to the equation

$$\frac{du}{dt}(t) \in [A(\cdot)]_a(t)u(t) + \omega u(t),$$

due to lack of Assumptions 2.2 and 2.3.

(ii) If $\{[A(\cdot)]_a(t) : t \in \mathbb{R}\}$ fulfills Assumption 2.2 with a function $h_a \in \mathrm{BUC}(\mathbb{R}, X)$, L_a non-decreasing and $\omega_a < 0$, then u^a is an integral solution, in the sense of Definition 2.7, with respect to h_a^ω, L_a.

(iii) If $\{[A(\cdot)]_a(t) : t \in \mathbb{R}\}$ fulfills Assumption 2.3 with $h_a, g_a : \mathbb{R} \to X$ bounded and Lipschitz, L_a non-decreasing and $L_{g_a} < -\omega_a$, then the generalized solution is an integral solution, in the sense of Definition 2.7, with respect to the control functions h_a^ω, g_a, L_a.

Proof. Equation (4.5) and the fixpoint equation (4.4) show that we are in the situation of approximations used in the study [23, equation (31), p. 1076]. The uniqueness of fixpoints and the corresponding uniform limits (cf. Theorem 2.8) conclude the proof. □

Proposition 4.4. *In the context of Theorem 4.2, let $Y = \mathrm{WRC}(\mathbb{R}, X)$, $Y_a = \mathrm{AP}(\mathbb{R}, X)$ and $Y_0 = \mathrm{WRC}_0(\mathbb{R}, X)$. If the control functions h, g belong to $\mathrm{WRC}(\mathbb{R}, X)$, then the generalized-(almost periodic)-solution is an integral solution with respect to L^ω, and the almost periodic parts of h^ω, g.*

Proof. Clearly, $h^\omega \in \mathrm{WRC}(\mathbb{R}, X \times X)$. From Corollary A.14, we find a sequence $\{s_n\}_{n\in\mathbb{N}}$, with $s_n \to \infty$, such that, due to the compact ranges,

$$h^\omega(\cdot + s_n) \to h_a^\omega, \quad g(\cdot + s_n) \to g_a \quad \text{and} \quad u(\cdot + s_n) \to u_a,$$

pointwise in the norm of X. As the solution on $\mathbb{R}$ is an integral solution (see Definition 2.7), we have

$$\begin{aligned}
&\|u(t) - x\| - \|u(r) - x\| \\
&\quad \le \int_r^t ([y, u(v) - x]_+ + \omega\|u(v) - x\|)\, dv + L^\omega(\|x\|) \int_r^t \|h^\omega(v) - h^\omega(r)\|\, dv + \|y\| \int_r^t \|g(v) - g(r)\|\, dv.
\end{aligned}$$

Therefore, for $t := t + s_n$, $r := r + s_n$, we have

$$\begin{aligned}
&\|u(t + s_n) - x\| - \|u(r + s_n) - x\| \\
&\quad \le \int_r^t ([y, u(v + s_n) - x]_+ + \omega\|u(v + s_n) - x\|)\, dv \\
&\qquad + L^\omega(\|x\|) \int_r^t \|h^\omega(v + s_n) - h^\omega(r + s_n)\|\, dv + \|y\| \int_r^t \|g(v + s_n) - g(r + s_n)\|\, dv. \qquad (4.6)
\end{aligned}$$

Using the fact that $[y, \cdot]_+$ is upper semicontinuous, we have

$$\limsup_{n\to\infty} [y, u(v + s_n) - x]_+ \le [y, u_a(v) - x]_+.$$

Consequently, we may pass to the limit superior on both sides of (4.6), and by Fatou's lemma we obtain the inequality. □

Corollary 4.5. *In the context of Theorem 4.2, let*

$$\begin{aligned}
Y &= \mathrm{AAP}^+(\mathbb{R}, X) := \{f \in \mathrm{BUC}(\mathbb{R}, X) : f_{|\mathbb{R}^+} \in \mathrm{AAP}(\mathbb{R}^+, X)\}, \\
Y_a &= \mathrm{AP}(\mathbb{R}, X) \quad \textit{and} \quad Y_0 = C_0^+(\mathbb{R}, X) := \{f \in \mathrm{BUC}(\mathbb{R}, X) : f_{|\mathbb{R}^+} \in C_0(\mathbb{R}^+, X)\}.
\end{aligned}$$

If the control functions h, g belong to $\mathrm{AAP}^+(\mathbb{R}, X)$, then the generalized-(almost periodic)-solution is an integral solution with respect to L^ω, and the almost periodic parts of h^ω, g.

Proof. Apply inequality (4.6), the existence of a sequence $\{s_n\}_{n\in\mathbb{N}}$ with $s_n \to \infty$, and the pointwise-norm convergence. □

Theorem 4.6. *Let $\{A(t) : t \in \mathbb{R}\}, \{B(t) : t \in \mathbb{R}\} \subset X \times X$ split almost periodic to $Y = Y_a \oplus Y_0$. Consider the equations*

$$\frac{du}{dt}(t) \in A(t)u(t) + \omega u(t), \quad t \in \mathbb{R}^+, \qquad u(0) = u_0 \in \overline{D_A},$$

and

$$\frac{dv}{dt}(t) \in B(t)v(t) + \omega v(t), \quad t \in \mathbb{R}^+, \qquad v(0) = v_0 \in \overline{D_B}.$$

If the families $\{A(t) : t \in \mathbb{R}\}, \{B(t) : t \in \mathbb{R}\}$ satisfy Assumption 2.1 and either Assumption 2.2 with $\omega < 0$, or Assumption 2.3 with $L_g < -\omega$, where L_g denotes the Lipschitz constant for the control function g, then

$$\{t \mapsto J_\lambda^A(t)x - J_\lambda^B(t)x\} \in Y_0 \quad \text{for all } x \in X, \tag{4.7}$$

implies that

$$\{t \mapsto U_{A+\omega I}(t,s)x - U_{B+\omega I}(t,s)y\}$$

is asymptotically Y_0 for all $x \in \overline{D_A}$, $y \in \overline{D_B}$.

Proof. It is sufficient to prove that the almost periodic parts of $U_{A+\omega I}(\,\cdot\,,s)x$ and $U_{B+\omega I}(\,\cdot\,,s)y$ coincide. Due to the uniform convergence of the approximants, it suffices to prove it for the approximants. Moreover, asymptotically they are close to their corresponding bounded solution on the whole line [23, Corollary 2.16]. Therefore, let u_λ, v_λ be the approximants to the whole line solution given by [23, Theorem 2.18]. Due to condition (4.7),

$$(J_\lambda^A(\,\cdot\,))^a x = (J_\lambda^B(\,\cdot\,))^a x.$$

Thus, using the fixpoint equations for the approximants u_λ and v_λ, we have

$$\begin{aligned}(u_\lambda(t))^a &= (J^A_{\frac{\lambda}{1-\lambda\omega}})^a(t)\Big(\frac{1}{1-\lambda\omega}\Big(\int_0^\infty \frac{1}{\lambda}e^{-\frac{s}{\lambda}}u_\lambda^a(t-s)\,ds\Big)\Big)\\ &= (J^B_{\frac{\lambda}{1-\lambda\omega}})^a(t)\Big(\frac{1}{1-\lambda\omega}\Big(\int_0^\infty \frac{1}{\lambda}e^{-\frac{s}{\lambda}}u_\lambda^a(t-s)\,ds\Big)\Big),\end{aligned}$$

Thus, $(u_\lambda(t))^a$ is the fixpoint of the strict contraction defining the solution $(v_\lambda(t))^a$. Consequently, by the uniqueness of the fixpoint, $(u_\lambda(t))^a = (v_\lambda(t))^a$. Thus, the almost periodic parts of the solutions of (2.2) coincide. Thanks to Corollary 2.10, the solutions of the initial value problem (2.1) are asymptotically close. □

Corollary 4.7 ([22]). *Let*

$$Y = \mathrm{AAP}^+(\mathbb{R},X) := \{f \in \mathrm{BUC}(\mathbb{R},X) : f_{|\mathbb{R}^+} \in \mathrm{AAP}(\mathbb{R}^+,X)\}, \qquad Y_a = \mathrm{AP}(\mathbb{R},X),$$

and let $\{A(t) : t \in \mathbb{R}\}, \{B(t) : t \in \mathbb{R}\} \subset X \times X$ split almost periodic with $Y = Y_a \oplus Y_0$. Consider the equations

$$\frac{du}{dt}(t) \in A(t)u(t) + \omega u(t), \quad t \in \mathbb{R}^+, \qquad u(0) = u_0 \in \overline{D_A},$$

and

$$\frac{dv}{dt}(t) \in B(t)v(t) + \omega v(t), \quad t \in \mathbb{R}^+, \qquad u(0) = v_0 \in \overline{D_B}.$$

Let also the families $\{A(t) : t \in \mathbb{R}\}, \{B(t) : t \in \mathbb{R}\}$ satisfy Assumption 2.1 and either Assumption 2.2 with $\omega < 0$, or Assumption 2.3 with $L_g < -\omega$, where L_g denotes the Lipschitz constant for the control function g. If

$$\lim_{t\to\infty} \|J_\lambda^A(t)x - J_\lambda^B(t)x\| = 0 \quad \text{for all } x \in X,\ \lambda \text{ small},$$

then, for the corresponding evolution systems $U_{A+\omega I}(t,s)$ and $U_{B+\omega I}(t,s)$, we have

$$\lim_{t\to\infty} \|U_{A+\omega I}(t,0)x - U_{B+\omega I}(t,0)y\| = 0 \quad \text{for all } x \in \overline{D_A},\ y \in \overline{D_B},$$

i.e., they are asymptotically equivalent.

In the case of Eberlein weak almost periodicity, we obtain the following corollary.

Corollary 4.8. *Let $Y = W(\mathbb{R}, X)$, $Y_a = \mathrm{AP}(\mathbb{R}, X)$, $Y_0 = W_0(\mathbb{R}, X)$, and let $\{A(t) : t \in \mathbb{R}\}, \{B(t) : t \in \mathbb{R}\} \subset X \times X$ split almost periodic with $Y = Y_a \oplus Y_0$. Consider the equations*

$$\frac{du}{dt}(t) \in A(t)u(t) + \omega u(t), \quad t \in \mathbb{R}^+, \qquad u(0) = u_0 \in \overline{D_A},$$

and

$$\frac{dv}{dt}(t) \in B(t)v(t) + \omega v(t), \quad t \in \mathbb{R}^+, \qquad u(0) = v_0 \in \overline{D_B}.$$

Let also the families $\{A(t) : t \in \mathbb{R}\}, \{B(t) : t \in \mathbb{R}\}$ satisfy Assumption 2.1 and either Assumption 2.2 with $\omega < 0$, or Assumption 2.3 with $L_g < -\omega$, where L_g denotes the Lipschitz constant for the control function g. If for all $\alpha \in \mathbb{R}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \exp(-i\alpha\tau)(J_\lambda^A(t+\tau)x - J_\lambda^B(t+\tau)x)\, d\tau = 0$$

for all λ small, $x \in X$, uniformly in $t \in \mathbb{R}^+$, then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \exp(-i\alpha\tau)\big(U_{A+\omega I}(t+\tau, 0)x - U_{B+\omega I}(t+\tau, 0)y\big)\, d\tau = 0$$

for all $x \in \overline{D_A}$, $y \in \overline{D_B}$, and $\alpha \in \mathbb{R}$, uniformly in $t \in \mathbb{R}^+$.

Proof. From [5, 6] we learn that $Y = Y_a \oplus Y_0$, and that Y_0 is characterized by a zero mean. □

In the case of ordinary differential equations the result of Theorem 4.6 extends as follows.

Proposition 4.9. *Let $Y \subset \mathrm{BUC}(\mathbb{R}, X)$, let $f, g\colon \mathbb{R} \times X \to X$ be Lipschitz, and let*

$$\{t \mapsto f(t, x(t)\}, \{t \mapsto g(t, x(t)\} \in Y \quad \text{for all } x \in Y.$$

Further, let

$$f^a(\,\cdot\,, z) = P_a f(\,\cdot\,, z), \quad g^a(\,\cdot\,, z) = P_a g(\,\cdot\,, z) \quad \text{for all } z \in X,$$

and

$$P_a f(\,\cdot\,, x(\,\cdot\,)) = f^a(\,\cdot\,, P_a x(\,\cdot\,)), \quad P_a g(\,\cdot\,, x(\,\cdot\,)) = g^a(\,\cdot\,, P_a x(\,\cdot\,)) \quad \text{for all } x \in Y.$$

Then

$$\{t \mapsto f(t, x) - g(t, x)\} \in Y_0 \quad \text{for all } x \in X, \tag{4.8}$$

if and only if

$$\{t \mapsto J_\lambda^f(t)x - J_\lambda^g(t)x\} \in Y_0 \quad \text{for all } x \in X. \tag{4.9}$$

Proof. Due (4.8), the almost periodic parts of f, g coincide. Therefore, with respect to the proof of Proposition 3.13, the resolvents come with a common fixpoint of T_λ, which proves (4.9). If (4.9) holds we have

$$J_{\lambda,a}^f = J_{\lambda,a}^g,$$

which gives $f_a = g_a$. An application of Proposition 3.13 concludes the proof. □

5 Perturbations

In this section we put some general perturbation theorems for dissipative operators into the context of almost periodic splittings. This extends the theory of semilinear operators like $A(t) + f(t, \cdot)$, as they are considered in [3], and in [31] for the case of almost automorphy. We start with a Lipschitz perturbation.

Theorem 5.1. *Let $\{A(t) : t \in \mathbb{R}\}$ split almost periodic with respect to Y, Y_a. Further, let $\{B(t) : t \in \mathbb{R}\}$ be uniformly Lipschitz with constant K, satisfying $P_a(B(\cdot)f = B_a(\cdot)P_a f$ for all $f \in Y$, where $B_a(\cdot)x := P_a B(\cdot)x$ for all $x \in X$. Then $A(t) + B(t)$ is ω-dissipative with at least $\omega = K$, and splits almost periodic with respect to Y_a, Y_0. Moreover,*

$$\{A(\cdot) + B(\cdot)\}_a = A_a(\cdot) + B_a(\cdot). \tag{5.1}$$

Proof. We start with proving the ω-dissipativeness. Let $t \in \mathbb{R}$ and $y_i \in A(t)x_i$ for $i = 1, 2$. Then

$$\|x_1 - x_2\| \le \|x_y - x_2 - \lambda(y_1 - y_2)\| \le \big\|x_1 - x_2 - \lambda(y_1 + B(t)x_1 - y_2 - B(t)x_2)\big\| + \lambda K\|x_1 - x_2\|.$$

Thus, $A(t) + B(t)$ is dissipative with $\omega = K$. From the uniform Lipschitz condition of $\{B(t) : t \in \mathbb{R}\}$, we obtain the uniform Lipschitz condition of $\{B_a(t) : t \in \mathbb{R}\}$. For given $x, y \in X$, we have

$$\|B_a(t)x - B_a(t)y\| \le \|B_a(\cdot)x - B_a(\cdot)y\|_\infty \le \|B(\cdot)x - B(\cdot)y\|_\infty \le K\|x - y\|.$$

Consequently, we obtain that $A_a(t) + B_a(t)$ is ω-dissipative with $\omega = K$.

To prove the m-dissipativeness and to obtain a representation of the resolvent of $A(\cdot) + B(\cdot)$, for given $\phi \in Y$, we define

$$T_\lambda \colon Y \to Y, \quad f \mapsto \Big\{t \mapsto J_\lambda^A(t)\{\phi(t) + \lambda B(t)(f(t) + \phi(t))\} - \phi(t)\Big\},$$

with $J_\lambda^A(t) = (I - \lambda A(t))^{-1}$.

An estimation gives that T_λ is a strict contraction for $\lambda K < 1$. Thus, the Banach fixpoint principle leads to a $f_\lambda \in Y$ such that

$$f_\lambda(t) = J_\lambda^A(t)\{\phi(t) + \lambda B(t)(f_\lambda(t) + \phi(t))\} - \phi(t).$$

We have

$$\begin{aligned}
&\phi(t) + f_\lambda(t) = J_\lambda^A(t)\{\phi(t) + \lambda B(t)(f_\lambda(t) + \phi(t))\},\\
&(I - \lambda A(t))(\phi(t) + f_\lambda(t)) \ni \phi(t) + \lambda B(t)(\phi(t) + f_\lambda(t)),\\
&(I - \lambda A(t) - \lambda B(t))(\phi(t) + f_\lambda(t)) \ni \phi(t),\\
&(\phi(t) + f_\lambda(t)) = (I - \lambda A(t) - \lambda B(t))^{-1}\phi(t).
\end{aligned}$$

As the constants are contained in $Y_a \subset Y$, the m-dissipativeness of $A(t) + B(t)$ is proved. We also proved $(I - \lambda(A(t) + B(t)))^{-1}\phi(t) = \phi(t) + f_\lambda(t) \in Y$. Therefore, it remains to prove the perturbation result (5.1), which is equivalent to

$$J_{\lambda,a}^{A+B}(t)x = J_\lambda^{A_a+B_a}(t)x.$$

From the previous step, we have $(I - \lambda(A(t) + B(t))^{-1}x = x + f_\lambda(t)$ for $\phi \equiv x$. This leads to

$$\begin{aligned}
P_a(J_\lambda^{A+B}(\cdot)x)(t) &= P_a(x + f_\lambda)(t)\\
&= x + P_a(f_\lambda)(t)\\
&= x + P_a(J_\lambda^A(t)(x + \lambda B(t)(f_\lambda(t) + x)) - x)\\
&= x + J_{\lambda,a}^A(t)\big(x + \lambda B_a(t)(f_\lambda^a(t) + x)\big) - x.
\end{aligned}$$

Noting that $J_\lambda^{A_a}(t)x = J_{\lambda,a}^A(t)x$, we define

$$T_\lambda^a \colon Y_a \to Y_a, \quad f \mapsto \{t \mapsto J_{\lambda,a}^A(t)(x + \lambda B_a(t)(f(t) + x)) - x\}.$$

Let f_λ^a be the net of fixpoints for $\lambda K < 1$. Then we have

$$\begin{aligned}
(I - \lambda(A_a(t) + B_a(t)))^{-1}x &= x + f_\lambda^a\\
&= J_{\lambda,a}^A(t)\big(x + \lambda B_a(t)(f_\lambda^a(t) + x)\big)\\
&= P_a(J_\lambda^{A+B}(\cdot)x)(t).
\end{aligned}$$

As a consequence, we obtain the same contraction mapping, and the uniqueness of the fixpoint concludes the proof. □

Remark 5.2. In many studies the existence of solutions to equations like

$$y'(t) \in A(t)y(t) + B(t)y(t), \quad t \in \mathbb{R}, \tag{5.2}$$

need the precondition $L + \omega_A < 0$, where L denotes the Lipschitz constant of B. Theorem 5.1 can be viewed as a split of the assumptions on m-dissipativity (i.e., $R(I + \lambda(A(t) + B(t)) = X)$ and the dissipativity constant ω_{A+B} of $\{A(t) + B(t) + \omega I : t \in \mathbb{R}\} \subset X \times X$. For the m-dissipativity the Lipschitz constant on B is needed. To obtain a bounded solution to (5.2) by the methods of Section 2, we obtain that $\omega_{A+B} < 0$ in case of Assumption 2.2, or $L_g < \omega_{A+B}$ in case of Assumption 2.3, is sufficient. Consequently, the direct connection between the Lipschitz constant on $x \to B(t)x$ and ω_A to obtain a bounded solution on $\mathbb{R}$ is cut.

If X^* is strictly convex, then the duality mapping

$$F\colon X \setminus \{0\} \to X^*, \quad x \mapsto \{y^* : y^*(x) = \|x\|^2 = \|y^*\|^2\},$$

is single-valued and, in case of a uniform convex dual, uniformly continuous on bounded subsets of X, cf. [15, Proposition 1.1, p. 2]. Note that a dissipative operator, in this case, is always strictly dissipative (i.e., $[\cdot,\cdot]_- = [\cdot,\cdot]_+$), due to the single valuedness of F. Consequently, a sum of dissipative operators is dissipative. Next we consider the case where X^* is uniformly convex.

Lemma 5.3. *Let X^* be uniformly convex, let $\mathbb{I} \subset \mathbb{R}$, and let the families $\{A(t) : t \in \mathbb{I}\}, \{B(t) : t \in \mathbb{I}\} \subset X \times X$ be m-dissipative. If there exist $R > 0$ and $x_1(t)\colon \mathbb{I} \to X$ such that for all $t \in \mathbb{I}$,*

$$x_1(t) \in D(A(t)) \cap D(B(t)) \cap K(0, R) \quad \text{and} \quad |A(t)x_1(t)|, |B(t)x_1(t)| \le R, \tag{5.3}$$

then, for $\{y(t)\}_{t\in\mathbb{I}} \in X$ bounded, there exist $C > 0$ and a unique $x_\lambda(t) \in D(A(t))$, $\|x_\lambda(t)\| \le C$, such that

$$x_\lambda(t) - A(t)x_\lambda(t) - B_\lambda(t)x_\lambda(t) \ni y(t). \tag{5.4}$$

Further, if for $C_1 \ge 0$, $\|B_\lambda(t)x_\lambda(t)\| \le C_1$ uniformly in $\lambda > 0$, $t \in \mathbb{I}$, then, there exists a function $x\colon \mathbb{I} \to X$ such that $x(t) \in D(A(t)) \cap D(B(t)) \cap K(0, C)$ and

$$x(t) - A(t)x(t) - B(t)x(t) \ni y(t) \quad \text{for all } t \in \mathbb{I}. \tag{5.5}$$

Additionally,

$$\lim_{\lambda\to 0} x_\lambda(t) = x(t) \quad \text{uniformly in } \mathbb{I}.$$

Proof. As $A(t) + B_\lambda(t)$ is m-dissipative, the solution $x_\lambda(t)$ of (5.4) is unique for every t. As we need a representation for $x_\lambda(t)$, from the proof of [15, Lemma 1.23, Theorem 1.24], we read

$$x_\lambda(t) = \Big(I - \frac{\lambda}{\lambda+1}A(t)\Big)^{-1}\Big(\frac{\lambda}{\lambda+1}y(t) + \frac{1}{\lambda+1}(I - \lambda B(t))^{-1}x_\lambda(t)\Big).$$

Using Banach's fixpoint theorem, we will find $x_\lambda(t) \in D(A(t))$. From the precondition (5.3), we find uniformly bounded pairs $(x_1(t), y_{1,\lambda}(t))$ such that

$$y_{1,\lambda}(t) \in x_1(t) - A(t)x_1(t) - B_\lambda(t)x_1(t).$$

Using $\|B_\lambda(t)x_1(t)\| \le |B(t)x_1(t)|$, we can choose $y_{1,\lambda}$ so that

$$\|x_\lambda(t) - x_1(t)\|^2 \le \big\langle y(t) - y_{1,\lambda}(t), F(x_\lambda(t) - x_1(t))\big\rangle \le \|y(t) - y_{1,\lambda}(t)\|\|x_\lambda(t) - x_1(t)\|. \tag{5.6}$$

Thus, $\|x_\lambda(t)\|$ is uniformly bounded. Using (5.4) for $\lambda, \mu > 0$, and the fact that $A(t)$ is dissipative, we have

$$\|x_\lambda(t) - x_\mu(t)\|^2 \le \big\langle B_\lambda(t)x_\lambda(t) - B_\mu(t)x_\mu(t), F(x_\lambda(t) - x_\mu(t))\big\rangle.$$

Using $B_\lambda(t)x_\lambda(t) \in B(t)J_\lambda^B(t)x_\lambda(t)$ and the dissipativeness of B, we have

$$\big\langle B_\lambda(t)x_\lambda(t) - B_\mu(t)x_\mu(t), F\big(J_\lambda^B(t)x_\lambda(t) - J_\mu^B(t)x_\mu(t)\big)\big\rangle \le 0.$$

In sum this leads with the boundedness of $\{B_\lambda(t)x_\lambda(t)\}_{t\in\mathbb{I},\lambda>0}$ to some $C_1 > 0$, so that

$$\begin{aligned}\|x_\lambda(t)-x_\mu(t)\|^2 &\le \langle B_\lambda(t)x_\lambda(t)-B_\mu(t)x_\mu(t), F(x_\lambda(t)-x_\mu(t)) - F(J_\lambda^B(t)x_\lambda(t) - J_\mu^B(t)x_\mu(t))\rangle\\ &\le 2C_1\big\|F(x_\lambda(t)-x_\mu(t)) - F(J_\lambda^B(t)x_\lambda(t)-J_\mu^B(t)x_\mu(t))\big\|.\end{aligned} \tag{5.7}$$

Recalling that $\lambda B_\lambda(t)x_\lambda(t) = J_\lambda^B(t)x_\lambda(t) - x_\lambda(t)$, the uniform continuity on bounded sets of the duality map yields that $\{x_\lambda\}_{\lambda>0}$ is uniformly Cauchy when $\lambda\to 0$. The completeness of X gives uniform convergence in $B(\mathbb{I},X)$, cf. [8, p. 258]. We define

$$x(t) = \lim_{\lambda\to 0} x_\lambda(t).$$

For a given $t\in\mathbb{I}$, we can conclude the proof with the arguments used in the proof of [15, Theorem 1.24, pp. 25–26, and Theorem 1.17 (ii), p. 18]. Using $\|B_\lambda(s)x_\lambda(s)\|\le C_1$ for all $s\in\mathbb{I}$, $\lambda>0$, we have $x(t)\in D(B(t))$. The reflexivity gives $B_{\lambda_n}(t)x_{\lambda_n}(t)\to w(t)$ weakly, with $w(t)\in B(t)x(t)$. Applying (5.4), we find a bounded selection $y_\lambda(t)\in A(t)x_\lambda(t)$ satisfying

$$x_\lambda(t) - y_\lambda(t) - B_\lambda(t)x_\lambda(t) = y(t).$$

Thus, by the demiclosedness of $A(t)$ and convergence of $\{y_{\lambda_n}(t) : n\in\mathbb{N}\}$ we have

$$y_{\lambda_n}(t)\to z(t) = y(t) - x(t) + w(t)$$

weakly, with $z(t)\in A(t)x(t)$. Summarizing the previous, we have

$$y(t) = x_{\lambda_n}(t) - y_{\lambda_n}(t) - B_{\lambda_n}(t)x_{\lambda_n}(t) \to x(t) - z(t) - w(t)$$

weakly in X, which concludes the proof. □

Theorem 5.4. *Let X^* be uniformly convex, let $\mathbb{I}\subset\mathbb{R}$, and let the families $\{A(t) : t\in\mathbb{I}\}$, $\{B(t) : t\in\mathbb{I}\}\subset X\times X$ be m-dissipative. Assume that there exist $R>0$ and $x_1(t) : \mathbb{I}\to X$ such that for all $t\in\mathbb{I}$,*

$$x_1(t)\in D(A(t))\cap D(B(t))\cap K(0,R) \quad \textit{and} \quad |A(t)x_1(t)|, |B(t)x_1(t)|\le R.$$

Further, let $\langle y, F(B_\lambda(t)x)\rangle \ge 0$ for all $[x,y]\in A(t)$, $t\in\mathbb{I}$, $\lambda>0$. Then $A(t)+B(t)$ is m-dissipative and the solution of (5.4) *converges uniformly on $\mathbb{I}$ to the solution of* (5.5).

Proof. The proof is analogous to that of [15, Corollary 1.25], proving the boundedness of $B_\lambda(t)x_\lambda(t)$. □

Remark 5.5. To study the resolvents $J_\mu^{A+B}(t) = (I+\mu(A(t)+B(t)))^{-1}$, the same methods, as in Lemma 5.3, apply. Therefore, assume the condition (5.3). This selection gives a bounded pair $(x_1(t), y_{1,\lambda,\mu}(t))$ satisfying

$$y_{1,\lambda,\mu}(t) := (1-\mu)x_1(t) + \mu y_{1,\lambda}(t) \in x_1(t) - \mu A(t)x_1(t) - \mu B_\lambda(t)x_1(t).$$

Considering, $y\in B(\mathbb{I},X)$ and

$$y(t)\in x_\lambda^\mu(t) - \mu A(t)x_\lambda^\mu(t) - \mu B_\lambda(t)x_\lambda^\mu(t),$$

we find that the solutions are fixpoints of the contractions

$$x_\lambda^\mu(t) = \Big(I - \frac{\lambda\mu}{\lambda+\mu}A(t)\Big)^{-1}\Big(\frac{\lambda}{\lambda+\mu}y(t) + \frac{\mu}{\lambda+\mu}J_\lambda^B(t)x_\lambda^\mu(t)\Big).$$

Moreover, applying the bounded selection $(x_1(t), y_{1,\lambda,\mu}(t))$, we have, similar to (5.6), some $C_2>0$ such that $\|x_\lambda^\mu(t)\|\le C_2$ uniformly in $\lambda>0$, $t\in\mathbb{I}$. Thus, if for a given $\mu>0$,

$$\|B_\lambda(t)x_\lambda^\mu(t)\|\le C_1 \quad \text{for all } t\in\mathbb{I},\ \lambda>0, \tag{5.8}$$

then, by an observation similar to (5.7), we have

$$x_\mu(t) := \lim_{\lambda\to 0} x_\lambda^\mu(t) \quad \text{uniformly for } t\in\mathbb{I}.$$

The uniform convexity of X^* and the demiclosedness of $A(t)$ and $B(t)$ imply that $x_\mu(t)$ is a bounded solution of

$$x_\mu(t) - \mu A(t)x_\mu(t) - \mu B(t)x_\mu(t) \ni y(t),$$

which gives

$$J_\mu^{A+B}(t)y(t) = \lim_{\lambda\to 0} x_\lambda^\mu(t) \quad \text{uniformly for } t\in\mathbb{I}.$$

Theorem 5.6. *Let X^* be uniformly convex, and let $Y \subset \mathrm{BUC}(\mathbb{R}, X)$, $Y = Y_a \oplus Y_0$. Further, let $\{A(t) : t \in \mathbb{R}\}$ and $\{B(t) : t \in \mathbb{R}\}$ be families of dissipative operators, which split almost periodic with respect to Y_a, Y_0. Under the assumptions* (5.3) *and* (5.8) *of Remark 5.5, we have*

$$(A(\cdot) + B(\cdot))_a = A_a(\cdot) + B_a(\cdot).$$

Proof. We have to show that the almost periodic part of $J_\lambda^{A+B}(t)x$ equals $J_\lambda^{A_a+B_a}(t)x$. To prove that the fixpoints are in Y, note that $X \subset Y_a \subset Y$, and that J_λ^A and J_λ^B leave Y invariant. Thus, for $y \in X$, we have the strict contraction

$$T\colon Y \to Y, \quad x \mapsto \Big\{t \mapsto J^A_{\frac{\lambda\mu}{\lambda+\mu}}(t)\Big(\frac{\lambda}{\lambda+\mu}y + \frac{\mu}{\lambda+\mu}J_\lambda^B(t)x(t)\Big)\Big\}.$$

From the Remark 5.5, for a given $\mu > 0$, we obtain the convergence when $\lambda \to 0$ uniformly for $t \in \mathbb{R}$. In doing so, we use the representations of the fixpoints given in Remark 5.5,

$$\begin{aligned} P_a x_\lambda^\mu(\cdot) &= P_a\Big(J^A_{\frac{\lambda\mu}{\lambda+\mu}}(\cdot)\Big(\frac{\lambda}{\lambda+\mu}y + \frac{\mu}{\lambda+\mu}J_\lambda^B(\cdot)x_\lambda^\mu(\cdot)\Big)\Big) \\ &= (J^A_{\frac{\lambda\mu}{\lambda+\mu}}(\cdot))^a\Big(\frac{\lambda}{\lambda+\mu}y + \frac{\mu}{\lambda+\mu}(J_\lambda^B(\cdot))^a P_a x_\lambda^\mu(\cdot)\Big). \end{aligned}$$

As $P_a\colon Y \to Y$ is bounded, we obtain the uniform convergence of $P_a x_\lambda^\mu$ as well. For $A_a + B_a$, we have the strict contraction

$$T_a\colon Y_a \to Y_a, \quad x \mapsto \Big\{t \mapsto (J^A_{\frac{\lambda\mu}{\lambda+\mu}}(\cdot))^a(t)\Big(\frac{\lambda}{\lambda+\mu}y + \frac{\mu}{\lambda+\mu}(J_\lambda^B(\cdot))^a(t)x(t)\Big)\Big\}.$$

Thus, the projection gives the fixpoint $x_{\lambda,a}^\mu := P_a x_\lambda^\mu$ and uniform convergence when $\lambda \to 0$. From (5.8), we conclude

$$\|(B_\lambda(\cdot))^a(t)x_{\lambda,a}^\mu(t)\| \le \|P_a B_\lambda(\cdot)x_\lambda^\mu(\cdot)\|_\infty \le \|B_\lambda(\cdot)x_\lambda^\mu(\cdot)\|_\infty \le C_1 \quad \text{uniformly in } \lambda > 0.$$

Hence, we are in the situation of Remark 5.5, and can redo the final steps, similar to the proof of Lemma 5.3. Consequently, for a given $t \in \mathbb{R}$, there exists a sequence $\{\lambda_n : n \in \mathbb{N}\}$ such that

$$y = x^\mu_{\lambda_n,a}(t) - \mu y^\mu_{\lambda_n,a}(t) - \mu B_{\lambda_n,a}(t)x^\mu_{\lambda_n,a}(t) \to x_a^\mu(t) - \mu z_a^\mu(t) - \mu w_a^\mu(t)$$

weakly when $n \to \infty$, with $z_a^\mu(t) \in A_a(t)x^\mu(t)$ and $w_a^\mu(t) \in B_a(t)x^\mu(t)$. This concludes the proof. □

In the next example we show how the previous results apply to perturbations to the linear Dirichlet problem (Δ_0) on bounded domains Ω.

Example 5.7. Let $\Omega \subset \mathbb{R}^n$ be open, bounded and with smooth boundary, let $H = L^2(\Omega)$,

$$\Delta\colon H_0^1(\Omega) \cap H^2(\Omega) \subset L^2(\Omega) \to L^2(\Omega),$$

with ω its dissipativity constant, and let $\varphi\colon \mathbb{J} \times \mathbb{R} \to \mathbb{R}$ be a proper, lower semicontinuous and convex function, satisfying $\partial\varphi(t)0 \ni 0$ for all $t \in \mathbb{J}$. Defining, for all $t \in \mathbb{J}$,

$$\partial\phi(t) := \{[u, v] \in L^2(\Omega) \times L^2(\Omega) : u(x) \in D(\partial\varphi(t)) \text{ a.e.}, \varphi(t)(u(\cdot)) \in L^1(\Omega), \text{and } v(x) \in \partial\varphi(t)(u(x)) \text{ a.e.}\},$$

we have that $\Delta - \partial\phi(t)$ is m-dissipative and $\partial\phi(t)0 \ni 0$. If additionally, $-\partial\varphi$ and $-\partial\phi$ split almost periodic, with respect to $Y_{a,\mathbb{R}}$, $Y_{0,\mathbb{R}} \subset \mathrm{BUC}(\mathbb{R})$ and Y_{a,L^2}, $Y_{0,L^2} \subset \mathrm{BUC}(\mathbb{R}, L^2(\Omega))$, respectively, with

$$P_{a,Y_{L^2}}(I + \lambda\partial\phi(\cdot_t))^{-1}u = \{x \mapsto P_{a,Y_\mathbb{R}}(I + \lambda\partial\varphi(\cdot_t))^{-1}u(x)\} \quad \text{a.e. for all } u \in L^2(\Omega), \tag{5.9}$$

then we have

$$(\Delta - \partial\phi(\cdot_t))_a = \Delta - \partial\phi_a(\cdot_t) = \Delta - (\partial\varphi_a(\cdot_t)), \tag{5.10}$$

where $-(\partial\varphi(\cdot))_a$ and $-\partial\phi(\cdot)_a$ are the generalized almost periodic parts. Moreover, if the Dirichlet problem with respect to $\Delta - \partial\varphi(t)(\cdot)$ fulfills either Assumption 2.2 (see [4, Example, pp. 88–90]), or Assumption 2.3, with $L_g < -\omega$, then the solution of equation

$$u'(t) = \Delta u(t) - \partial\phi(t)u(t) + f(t), \quad t \in \mathbb{R},$$

with $f \in Y_{L^2}$, has an almost periodic splitting as well.

Proof. From [15, Example 1.60 and Lemma 1.61, pp. 53–54], we obtain that $\partial\phi$ is a subdifferential, and therefore $-\partial\phi$ is m-dissipative. Further, [15, Proposition 1.63, p. 55] proves the m-dissipativeness of $\Delta - \partial\phi$ on $L^2(\Omega)$. As $f(t) \in \Delta 0 + \partial\phi(t)(0) + f(t)$, we verified condition (5.3). The claim (5.10) is an application of Theorem 5.6, Theorem 4.2 and assumption (5.9). To consider right-hand sides $f \in Y$, we use the fact that the mollified $f_\varepsilon \in Y$ is Lipschitz and approximates f uniformly due to the uniform continuity of f. Applying [23, Proposition 6.1], we obtain, with f_ε, an additional Lipschitz and bounded control function. Finally, [23, inequality (43), p. 1084] finishes the proof. □

To view the case of Eberlein weak almost periodicity we have the following.

Example 5.8. Let X^* be uniformly convex and $A \subset X \times X$ an m-dissipative operator with a compact resolvent. Further, let $\{B(t)\}_{t\in\mathbb{J}} \subset X \times X$ be m-dissipative, and let $J_\lambda^B(\cdot)z \in W(\mathbb{J}, X)$ for all $z \in X$. For a given $v \in X$, we define

$$F_{\lambda,\mu}^v : \mathbb{J} \times X \to X, \quad (t, x) \mapsto J^A_{\frac{\lambda\mu}{\lambda+\mu}}\Big(\frac{\lambda}{\lambda+\mu}v + \frac{\mu}{\lambda+\mu}J_\lambda^B(t)x\Big).$$

Then $F_{\lambda,\mu}^v$ fulfills Definition A.4 with $D = (X, \|\cdot\|)$. If A, $B(t)$ fulfill the assumption of Theorem 5.4, then

$$J_\lambda^{A+B}(\cdot)w(\cdot) \in \mathrm{WRC}(\mathbb{J}, X) \quad \text{for all } w \in W(\mathbb{J}, X).$$

Proof. Following the Definition A.4 and Theorem A.10, for given $g \in W(\mathbb{J}, X)$, we have to verify that

$$\Big\{t \mapsto J^A_{\frac{\lambda\mu}{\lambda+\mu}} g(t)\Big\} \in \mathrm{WRC}(\mathbb{J}, X).$$

The Eberlein weak almost periodicity is a consequence of the proof of Example 3.12, and the relative compact range comes with the compactness of J_λ^A. For given $w \in W(\mathbb{J}, X)$, choosing

$$g(t) = \frac{\lambda}{\lambda+\mu}w(t) + \frac{\mu}{\lambda+\mu}J_\lambda^B(t)x,$$

it remains to prove that

$$\iota : K \to \mathrm{WRC}(\mathbb{J}, X), \quad x \mapsto \Big(t \mapsto J^A_{\frac{\lambda\mu}{\lambda+\mu}}\Big(\frac{\lambda}{\lambda+\mu}w(t) + \frac{\mu}{\lambda+\mu}J_\lambda^B(t)x\Big)\Big),$$

is continuous, which comes with the inequality

$$\|\iota(x) - \iota(y)\|_\infty \le \frac{\mu}{\lambda+\mu}\|x - y\|.$$

From Lemma 5.3 and Remark 5.5, we obtain the resolvent $J_\mu^{A+B}(\cdot)w(t)$ as uniform limits of fixpoints defined by $\mathrm{WRC}(\mathbb{J}, X) \ni x_{\lambda,\mu}(t) = F_{\lambda,\mu}^v(t, x_{\lambda,\mu}(t))$, i.e.,

$$J^A_{\frac{\lambda\mu}{\lambda+\mu}}\Big(\frac{\lambda}{\lambda+\mu}w(t) + \frac{\mu}{\lambda+\mu}J_\lambda^B(t)x_{\lambda,\mu}(t)\Big) \to J_\mu^{A+B}w(t) \quad \text{when } \lambda \to 0,$$

uniformly for $t \in \mathbb{J}$, and the proof is finished. □

Remark 5.9. From Appendices B and C, we can obtain similar results for asymptotically almost periodic or continuous almost automorphic functions.

From Remark 3.9 and the previous perturbation results, we obtain the following corollary.

Corollary 5.10. *Let X^* be uniformly convex, and let $\{A(t) : t \in \mathbb{R}\}$ and $\{C(t) : t \in \mathbb{R}\}$ split almost periodic with respect to $Y = Y_a \oplus Y_0$, and fulfill the assumptions of Theorem 5.6. Consider the equations*

$$\frac{du}{dt}(t) \in A(t)u(t) + \omega u(t), \quad t \in \mathbb{R}^+, \qquad u(0) = u_0 \in \overline{D_A},$$

and

$$\frac{dv}{dt}(t) \in A(t)v(t) + C(t)v(t) + \omega v(t), \quad t \in \mathbb{R}^+, \qquad u(0) = v_0 \in \overline{D_{A+C}}.$$

If the families $\{A(t) : t \in \mathbb{R}\}$, $\{A(t) + C(t) : t \in \mathbb{R}\}$ satisfy either Assumption 2.2 with $\omega < 0$, or Assumption 2.3 with $L_g < -\omega$, where L_g denotes the Lipschitz constant for the control function g, then $\{t \mapsto C_a(t)\} = 0$ implies that $\{t \mapsto U_{A+C+\omega}(t, s)y - U_{A+\omega}(t, s)x\}$ is asymptotically Y_0 for all $x \in \overline{D_A}$ and $y \in \overline{D_{A+C}}$.

Proof. From the assumptions of Theorem 5.6 we obtain the m-dissipativeness for every operator $\{A(t) + C(t)\}$, i.e., Assumption 2.1. Next we apply Remark 3.9 and the perturbation result for $A + C$, which give

$$P_a(J_\lambda^{A+C}(\cdot)x - J_\lambda^{A}(\cdot)x)(t) = J_\lambda^{A_a+C_a}(t)x - J_\lambda^{A_a}(t)x = 0.$$

Finally, an application of Theorem 4.6 concludes the proof. □

A Eberlein weak almost periodicity

This and the remaining appendices discuss a number of closed and translation invariant subspaces of $\mathrm{BUC}(\mathbb{R}, X)$ and $(\mathrm{BUC}(\mathbb{R}^+, X))$, which under certain conditions satisfy the invariance conditions for the resolvent $J_\lambda(t)$. Moreover, we give some examples for almost periodic splittings. We mainly restrict our attention to weak almost periodicity, in the sense of Eberlein, and to asymptotically almost periodic functions. These results demonstrate the use of the above general results in a variety of special cases.

Proposition A.1. (i) *If $f \in C_b(\mathbb{J}, X)$, then $f \in W(\mathbb{J}, X)$ if and only if for $\{(t_m, x_m^*)\}_{m\in\mathbb{N}} \subset \mathbb{J} \times \operatorname{ext} B_{X^*}$ and for all $\{\omega_n\}_{n\in\mathbb{N}} \subset \mathbb{J}$, the following double limits condition holds:*

$$\lim_{n\to\infty} \lim_{m\to\infty} \langle f(\omega_n + t_m), x_m^*\rangle = \lim_{m\to\infty} \lim_{n\to\infty} \langle f(\omega_n + t_m), x_m^*\rangle,$$

whenever the iterated limits exist, see [27].

(ii) *If $f \in W(\mathbb{J}, X)$, then*

$$\|\cdot\| - \lim_{R\to\infty} \frac{1}{R} \int_0^R f(t + r)\, dt = z \in X$$

exists uniformly over $r \in \mathbb{J}$. Moreover, if $f \in W_0(\mathbb{R}^+, X)$, then the ergodic limit z is equal to $0 \in X$ (cf. [28]*).*

Proposition A.2. *Let $\{a_{n,m}\}_{n,m\in\mathbb{N}} \subset K$ with K compact metrizable. Then there exist subsequences $\{n_k\}_{k\in\mathbb{N}}$ and $\{m_l\}_{l\in\mathbb{N}}$ such that the following limits exist:*

$$\lim_{k\to\infty} \lim_{l\to\infty} a_{n_k,m_l}, \quad \lim_{l\to\infty} \lim_{k\to\infty} a_{n_k,m_l}.$$

Proof. Let $m = 1$. Then we find a subsequence $\{n(1, k)\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} a_{n(1,k),1}$ exists. Thus, we may assume that for a given $m \in \mathbb{N}$, we can find $\{n(m, k)\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} a_{n(m,k),l}$ exist for $1 \le l \le m$. For the step “$m \to m + 1$”, choose an appropriate subsequence $\{n(m + 1, k)\}_{k\in\mathbb{N}} \subset \{n(m, k)\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} a_{n(m+1,k),m+1}$ exists. After this recursive construction, we define $n_k := n(k, k)$, and since $n(k, k) < n(k, k+1) \le n(k+1, k+1)$, we have $\{n(k, k)\}_{k\in\mathbb{N},k\ge m} \subset \{n(m, k)\}_{k\in\mathbb{N},k\ge m}$ for all $m \in \mathbb{N}$. Computing the limits, we obtain $b_m := \lim_{k\to\infty} a_{n_k,m}$. Passing to an appropriate $\{m_l\}_{l\in\mathbb{N}}$, we are done for the first double limit. Redoing the previous steps on the found subsequence $\{a_{n_k,m_l}\}_{k,l\in\mathbb{N}}$, for the interchanged limits we will find $\{m_{l_j}\}_{j\in\mathbb{N}}$ and $\{n_{k_i}\}_{i\in\mathbb{N}}$, the desired subsequences. □

The proposition above allows to assume, without loss of generality, that the limits in Proposition A.1 exist. Recall that only the equality of the limits has to be proven.

In this section we provide an extension of Traple’s result to infinite dimensions.

Theorem A.3 ([29]). *If X has finite dimension, $g \in W(\mathbb{R}, X)$ and $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, with the property that*

$$\iota\colon K \to W(\mathbb{R}, \mathbb{R}^n), \quad p \mapsto f(\cdot, p),$$

is continuous for every compact $K \subseteq \mathbb{R}^n$, then $\{t \mapsto f(t, g(t))\} \in W(\mathbb{R}, \mathbb{R}^n)$.

Traple’s proof does not extend to the infinite dimensional case, for he used both the algebraic structure of $W(\mathbb{R})$ and the fact that the coordinate-wise polynomials are dense in $C(K, \mathbb{R}^n)$ for $K \subset \mathbb{R}^n$ closed and bounded. Clearly, for general Banach spaces such structure is missing.

In order to give sufficient conditions for the extension of Traple's result, hence existence of Eberlein weakly almost periodic solutions to nonlinear differential equations in infinite dimensions, we need the following definition, where (D,τ) is assumed to be a Hausdorff topological space. Moreover, in this study, D is X with the weak topology or X with norm topology.

Definition A.4. A function $f\colon \mathbb{J}\times D\to X$ is called DEWAP if
(i) $f(\cdot,p)\in W(\mathbb{J},X)$ for all $p\in D$,
(ii) for every compact and metric subset $K\subset D$, the following map is continuous:

$$\iota\colon K\to W(\mathbb{J},X),\quad p\mapsto f(\cdot,p).$$

We denote these functions by

$$W(\mathbb{J}\times D,X):=\{f\in C(\mathbb{J}\times D,X):f\text{ is DEWAP}\}.$$

With the aim to find a representation for the almost periodic part, we recall some technical results.

Proposition A.5 ([28]). *Eberlein weakly almost periodic functions are uniformly continuous.*

Thus, $W(\mathbb{R},X)$ fulfills the assumptions made on Y in the previous sections.

Remark A.6. Let $f\colon \mathbb{J}\times D\to X$ fulfill Definition A.4, and let $K\subset D$ be compact metric. Then $f\in \mathrm{BUC}(\mathbb{J}\times K,X)$.

Proof. As K is compact metric, for given $\varepsilon>0$, we find $\delta>0$ such that for all $x,y\in K$ with $d_K(x,y)<\delta$, we have $\|\iota(x)-\iota(y)\|_\infty<\varepsilon$. From the compactness of K we find $\{x_i\}_{i=1}^n$ with the following properties:
(i) $\sup_{z\in K}\inf_{1\le k\le n} d_K(z,x_k)<\delta$.
(ii) There exists $\delta_1>0$ such that for $|t-s|<\delta_1$, we have $\sup_{1\le k\le n}\|f(t,x_k)-f(s,x_k)\|<\varepsilon$.
(iii) From Definition A.4 (i), we obtain some $K>0$ with $\max_{1\le k\le n}\|f(\cdot,x_k)\|_\infty<K$.

First we prove the boundedness. For $z\in K$,

$$\begin{aligned}\|f(t,z)\| &\le \inf_{1\le k\le n}\|f(t,z)-f(t,x_k)\|+\max_{1\le k\le n}\|f(t,x_k)\|\\ &\le \inf_{1\le k\le n}\|\iota(z)-\iota(x_k)\|_\infty+\max_{1\le k\le n}\|f(\cdot,x_k)\|_\infty\\ &\le \varepsilon+K.\end{aligned}$$

To prove the uniform continuity, let $t,s\in\mathbb{J}$, $|t-s|<\delta_1$ and $x,y\in K$, with $d(x,y)<\delta$. Then, we obtain

$$\begin{aligned}\|f(t,x)-f(s,y)\| &\le \|f(t,x)-f(s,x)\|+\|f(s,x)-f(s,y)\|\\ &\le \|f(t,x)-f(t,x_k)\|+\|f(t,x_k)-f(s,x_k)\|+\|f(s,x_k)-f(s,x)\|+\|\iota(x)-\iota(y)\|_\infty\\ &\le 2\sup_{z\in K}\inf_{1\le k\le n}\|\iota(z)-\iota(x_k)\|_\infty+\|\iota(x)-\iota(y)\|_\infty+\sup_{1\le k\le n}\|f(t,x_k)-f(s,x_k)\|\\ &\le 2\varepsilon+\varepsilon+\varepsilon,\end{aligned}$$

which finishes the proof. □

Remark A.7. From the previous remark we conclude that if $f\colon \mathbb{J}\times D\to X$ fulfills Definition A.4, and $K\subset D$ is compact metric, then the following map is continuous:

$$S\colon \mathbb{J}\times K\to W(\mathbb{J},X),\quad (t,x)\mapsto\{s\mapsto f(t+s,x)\}.$$

Note that the continuity of ι does not imply a norm compact range for every $f(\cdot,z)$ if z is given. For example, one may take $f(\cdot,z)\equiv g$ for some g, which is Eberlein weakly almost periodic. Then ι is clearly continuous, but the range of g is not necessarily compact. If $D=(X,\text{weak})$, then for every $g\in C(\mathbb{J}\times D,X)$, lipschitzian in the second variable, with $g(\cdot,x)$ Eberlein weakly almost periodic for all $x\in X$, and $C\colon X\to X$ compact linear,

$$f\colon \mathbb{J}\times D\to X,\quad (t,x)\mapsto g(t,Cx),$$

fulfills the previous definition. Consequently, for each $A \in W(\mathbb{R}, L(X))$ and $C\colon X \to X$ compact linear,

$$f\colon \mathbb{J} \times D \to X, \quad (t, x) \mapsto A(t)Cx,$$

will satisfy the assumptions of Definition A.4. An example for $A \in W(\mathbb{R}^+, L(X))$ is given by $A(t) = S(t)CT(t)$, provided the underlying space is reflexive, $\{S(t)\}_{t\in\mathbb{R}^+}$, $\{T(t)\}_{t\in\mathbb{R}^+}$ are bounded C_0-semigroups, and C is a compact linear operator. A proof of this fact can be found in [19].

The restriction on K to be compact metric comes with the range of a function $f \in W(\mathbb{J}, X)$, since f is continuous, $\overline{\operatorname{span}}\{f(\mathbb{J})\}$ is separable and $\overline{f(\mathbb{J})}$ is weakly compact. The fact that the weak topology on weakly compact sets in separable Banach spaces is metrizable, see [8, Section V.6.3., pp. 434], motivates the restriction.

Theorem A.8. *If $D = (X, weak)$, $g \in W(\mathbb{J}, X)$ and $f \in W(\mathbb{J} \times D, X)$, then $\{t \mapsto f(t, g(t))\} \in W(\mathbb{J}, X)$.*

For the proof, the following technical lemma is needed.

Lemma A.9. *Let (D, τ) be a topological Hausdorff space, let $f\colon \mathbb{J} \times D \to X$ with $f(\cdot, z) \in W(\mathbb{J}, X)$ for all $z \in D$, and let*

$$\iota\colon D \to W(\mathbb{J}, X), \quad p \mapsto f(\cdot, p),$$

be continuous. Further, let a given sequence $\{x_{n,m}\}_{n,m\in\mathbb{N}} \subset D$ satisfy the following double limits condition:

$$\lim_{n\to\infty} \lim_{m\to\infty} x_{n,m} = \lim_{m\to\infty} \lim_{n\to\infty} x_{n,m} = z.$$

Then the interchanged limits are equal for $\langle f(t_m + \omega_n, x_{n,m}), x_m^*\rangle$, whenever

$$\{(t_m, x_m^*)\}_{m\in\mathbb{N}} \subseteq \mathbb{J} \times B_{X^*}, \quad \{\omega_n\}_{n\in\mathbb{N}} \subseteq \mathbb{J},$$

and the iterated limits exist.

Proof. Applying Proposition A.2 to bounded set on $\mathbb{R}$ we may assume that the following iterated limits in the situation of the above lemma exist:

$$\begin{aligned} b_{n,m} &:= \langle f(t_m + \omega_n, x_{n,m}), x_m^*\rangle, \\ b &:= \lim_{n\to\infty} \lim_{m\to\infty} b_{n,m}, \\ \tilde{b} &:= \lim_{m\to\infty} \lim_{n\to\infty} b_{n,m}, \\ a_{n,m} &:= \langle f(t_m + \omega_n, z), x_m^*\rangle. \end{aligned}$$

By our hypothesis, we have $\{t \mapsto f(t, z)\}$ is EWAP, thus $\{a_{n,m}\}_{n,m\in\mathbb{N}}$ satisfies the double limits condition, i.e.,

$$a = \lim_{n\to\infty} \lim_{m\to\infty} a_{n,m} = \lim_{m\to\infty} \lim_{n\to\infty} a_{n,m}.$$

Now,

$$\begin{aligned} |b - a| &\le |b - b_{n,m}| + |b_{n,m} - a_{n,m}| + |a_{n,m} - a| \\ &\le |b - b_{n,m}| + \|\iota(x_{n,m}) - \iota(z)\| + |a_{n,m} - a|. \end{aligned}$$

As the double limits on the right-hand side exist and are equal to 0, we proved $b = a$. As the same routine works for $\tilde{b}$, we finished the proof. □

Proof of Theorem A.8. To verify the double limits condition, we apply the previous lemma for given

$$\{(t_m, x_m^*)\}_{m\in\mathbb{N}} \subseteq \mathbb{J} \times B_{X^*} \quad \text{and} \quad \{\omega_n\}_{n\in\mathbb{N}} \subseteq \mathbb{J},$$

to

$$D := (X, w) \quad \text{and} \quad K := \overline{\{g(t_m + \omega_n)\}}^{w}_{n,m\in\mathbb{N}}.$$

From [8, Section V.6.3, p. 434], we recall that the weak topology on weakly compact subsets in separable B-spaces is a metric topology. Noting that continuous images of separable spaces are separable, we obtain that $Y := \overline{\operatorname{span}}\{g(\mathbb{J})\}$, for $g \in W(\mathbb{J}, X)$ is separable, hence the weak topology on $K := \overline{g(\mathbb{J})}^{w}$ is metric, where w denotes the weak topology. By an application of Proposition A.2, we may assume that for $x_{n,m} = g(t_{k_m} + \omega_{l_n})$, the double limits exist, and $g \in W(\mathbb{J}, X)$ implies that they have to coincide.

Thus, we are in the situation of the previous lemma and our claim is proved. □

From the proof of Lemma A.9, and using that only local continuity is needed, we give the corollary for

$$\mathrm{WRC}(\mathbb{J}, X) := \{f \in W(\mathbb{J}, X) : f(\mathbb{J}) \text{ is relative compact in } X\},$$

which was introduced by Goldberg and Irwin [12].

Corollary A.10. *Let, for a Banach space Y, $D = (Y, \|\cdot\|)$, and $f \in W(\mathbb{J} \times D, X)$. Then, for any given $g\colon \mathbb{J} \to Y$, Eberlein weakly almost periodic with a relatively compact range, we have $\{t \mapsto f(t, g(t))\} \in W(\mathbb{J}, X)$. Moreover, if $\{t \mapsto f(t, x)\} \in \mathrm{WRC}(\mathbb{J}, X)$ for all $x \in Y$, then $\{t \mapsto f(t, g(t))\} \in \mathrm{WRC}(\mathbb{J}, X)$.*

Proof. The reader will have no difficulty to apply the previous theorem to $K := \overline{g(\mathbb{J})}$, since $(K, w) = (K, \|\cdot\|)$, hence obtain the first part.

For the second part, it remains to prove the compactness of $\{f(t, g(t)) : t \in \mathbb{J}\}$. Thus, for a given sequence $\{t_n\}_{n\in\mathbb{N}}$, we have to find a subsequence $\{t_{n_k}\}_{n_k\in\mathbb{N}}$ such that $\{f(t_{n_k}, g(t_{n_k}))\}_{n_k\in\mathbb{N}}$ is convergent in X. Since g has compact range, without loss of generality, $g(t_n) \to x$. For this $x \in Y$, we may choose a subsequence such that $f(t_{n_k}, x) \to y$ for some $y \in X$. From the continuity of ι, we obtain $f(\cdot, g(t_{n_k})) \to f(\cdot, x)$ uniformly on $\mathbb{J}$. Thus,

$$\begin{aligned}\|f(t_{n_k}, g(t_{n_k})) - y\| &\le \|f(t_{n_k}, g(t_{n_k})) - f(t_{n_k}, x)\| + \|f(t_{n_k}, x) - y\| \\ &\le \|f(\cdot, g(t_{n_k})) - f(\cdot, x)\|_\infty + \|f(t_{n_k}, x) - y\|,\end{aligned}$$

and the proof is complete. □

In [18, Example 2.17, p. 17], it is shown that the compactness assumption on the range of g is essential.

Corollary A.11. *Let $A \in \mathrm{WRC}(\mathbb{J}, L(X))$. Then $\{t \mapsto A(t)g(t)\}$ is Eberlein weakly almost periodic for all $g \in W(\mathbb{J}, X)$.*

Proof. Letting $D := (L(X), \|\cdot\|_\infty)$, $g \in W(\mathbb{J}, X)$, and

$$f\colon \mathbb{J} \times D \to X, \quad (t, B) \mapsto f(t, B) := Bg(t),$$

the previous corollary serves for the proof. □

From Corollary A.10, we also obtain the next result of Goldberg and Irwin [12].

Corollary A.12. *If $f \in \mathrm{WRC}(\mathbb{J}, X)$, then $\|f(\cdot)\| \in W(\mathbb{J})$.*

Theorem A.8 gives a condition on f such that $\{t \mapsto f(t, x(t))\}$ is Eberlein weakly almost periodic. Noting that every $f \in W_0(\mathbb{J}, X)$ satisfies

$$\lim_{T\to\infty} \left\| \frac{1}{T} \int_0^T f_\tau \, d\tau \right\|_\infty = 0,$$

it is also of interest when $\{t \to f(t, x(t))\} \in W_0(\mathbb{J}, X)$ for a given $x \in W_0(\mathbb{J}, X)$. More generally, we have,

$$\lim_{T\to\infty} \left\| \frac{1}{T} \int_0^T (f_\tau - f_\tau^a) \, d\tau \right\|_\infty = 0,$$

where f^a denotes the almost periodic part of f. Thus, the question arises how the almost periodic part of the map $\{t \mapsto f(t, x(t)\}$ looks like. In order to discuss these problems, we introduce the projection on the almost periodic part:

$$P_a\colon W(\mathbb{J}, X) \to W(\mathbb{J}, X).$$

Proposition A.13. *For the decomposition*

$$W(\mathbb{R}^+, X) = \mathrm{AP}(\mathbb{R}, X)_{|\mathbb{R}^+} \oplus W_0(\mathbb{R}^+, X),$$

we have that the projection P_a onto $\mathrm{AP}(\mathbb{R}, X)_{|\mathbb{R}^+}$ has norm less than or equal to one.

Proof. For $f \in W(\mathbb{R}^+, X)$, we find $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ such that $f_{s_n} \rightharpoonup f_a$. Consequently, for given $x^* \in B_{X^*}$ and $t \in \mathbb{R}^+$, we have

$$|x^*(f_a(t))| = \lim_{s_n\to\infty} |x^*(f(t+s_n)| \le \|f\|_\infty,$$

which leads to the claim. □

Corollary A.14. *Any two functions $f, g \in W(\mathbb{J}, X)$ have a common sequence $\{t_n\}_{n\in\mathbb{N}}$, such that the translates $\{f_{t_n}\}_{n\in\mathbb{N}}$ and $\{g_{t_n}\}_{n\in\mathbb{N}}$ are weakly convergent to the almost periodic part of f and g, respectively.*

Proof. First we consider the case where the almost periodic parts of f and g are equal to zero, and let $\{u_n\}_{n\in\mathbb{N}}$ and $\{s_n\}_{n\in\mathbb{N}}$, be chosen such that

$$f_{u_m} \rightharpoonup 0 \quad \text{and} \quad g_{s_n} \rightharpoonup 0.$$

By the proposition above, we have that every Eberlein weakly almost periodic function is uniformly continuous, hence the semigroup of translations $\{T(t)\}_{t\in\mathbb{J}}$ is strongly continuous. Since $O(f) = T(\mathbb{J})f$, $\overline{O(f)}$ is a weakly compact closure of translates of a uniformly continuous function, hence $\overline{O(f)}$ is compact metrizable in the weak topology of $\mathrm{BUC}(\mathbb{J}, X)$. As a consequence of Proposition A.2, we may pass to subsequences of $\{u_m\}_{m\in\mathbb{N}}$ and $\{s_n\}_{n\in\mathbb{N}}$, such that the iterated limits of $\{f_{s_n+u_m}\}_{n,m\in\mathbb{N}}$ exist in the weak topology of $\mathrm{BUC}(\mathbb{J}, X)$, and, without loss of generality, the sequences are chosen in this way. From the interchangeable double limits condition, we obtain

$$w-\lim_{m\to\infty} w-\lim_{n\to\infty} f_{s_n+u_m} = w-\lim_{n\to\infty} w-\lim_{m\to\infty} f_{s_n+u_m}.$$

Thus, if $H := T(\mathbb{J})f \cup T(\mathbb{J})g$, and $d\colon \overline{H}\times\overline{H} \to [0,\infty)$ denotes the metric which induces the weak topology, then we can repeat the arguments on g, and, without loss of generality, we have

$$\lim_{n\to\infty}\lim_{m\to\infty}\big(d(f_{s_n+u_m}, 0) + d(g_{s_n+u_m}, 0)\big) = 0.$$

Thus, the desired result is a consequence of the classical diagonal process for the double sequence

$$k_{n,m} = d(f_{s_n+u_m}, 0) + d(g_{s_n+u_m}, 0).$$

If $f, g \in W(\mathbb{J}, X)$, then for the function h, given by

$$h\colon \mathbb{J} \to X\times X, \quad t \mapsto (f(t), g(t)),$$

we find, by the double limits criterion and the representation for the dual of $X\times X$, that it is Eberlein weakly almost periodic if for given sequences $\{t_m\}_{m\in\mathbb{N}}, \{\omega_n\}_{n\in\mathbb{N}} \subset \mathbb{J}$ and $\{x_m^*\}_{m\in\mathbb{N}}, \{y_m^*\}_{m\in\mathbb{N}} \subset B_{X^*}$,

$$\lim_{n\to\infty}\lim_{m\to\infty}\{\langle f(t_m+\omega_n), x_m^*\rangle + \langle g(t_m+\omega_n), y_m^*\rangle\} = \lim_{m\to\infty}\lim_{n\to\infty}\{\langle f(t_m+\omega_n), x_m^*\rangle + \langle g(t_m+\omega_n), y_m^*\rangle\},$$

whenever the iterated limits exist. However, by a successive diagonalisation of the double sequences

$$\{\langle f(t_m+\omega_n), x_m^*\rangle\}_{n,m\in\mathbb{N}} \quad \text{and} \quad \{\langle g(t_m+\omega_n), y_m^*\rangle\}_{n,m\in\mathbb{N}},$$

we may assume that their individual iterated limits exist and are equal, hence h is Eberlein weakly almost periodic. Thus, h has a unique decomposition into an almost periodic and a W_0 part:

$$h_{ap} = (f_{ap}, g_{ap}), \quad h_0 = (f_0, g_0).$$

Clearly, from the decomposition of f and g, we obtain

$$h = (f_{ap}, g_{ap}) + (f_0, g_0).$$

Further, by the observation in the first part of this proof, we have that (f_0, g_0) is W_0, and choosing subsequences two times will prove that (f_{ap}, g_{ap}) is almost periodic, hence the claim follows from the uniqueness of the decomposition. Hence, the sequence $\{t_n\}_{n\in\mathbb{N}}$, for which

$$h_{t_n} \rightharpoonup h_{ap} \quad \text{weakly as } n \to \infty,$$

is the desired one. □

Lemma A.15. *For every $K \subseteq D$ compact metric, and $f \in W(\mathbb{J} \times D, X)$, there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ such that*

$$f_{s_n}(\cdot\,, x) = f(\cdot + s_n, x) \rightharpoonup f_a(\cdot\,, x)$$

for every $x \in K$, where $f_a(\cdot\,, x)$ denotes the almost periodic part of $f(\cdot\,, x)$.

Proof. Given any $\epsilon > 0$, we find an $n(\epsilon)$ and $\{x_i\}_{i=1}^{n(\epsilon)}$, such that

$$K \subseteq \bigcup_{i=1}^{n(\epsilon)} \{x : \|f(\cdot\,, x) - f(\cdot\,, x_i)\|_\infty < \epsilon\}.$$

Since $\iota(K)$ is compact metric (therefore separable),

$$S\colon \mathbb{J} \times K \to W(\mathbb{J}, X), \quad (t, x) \mapsto T(t)f(\cdot\,, x),$$

is continuous, and $S(\mathbb{J}, K)$ separable, by Remark A.7, where $\{T(t)\}_{t\in\mathbb{J}}$ denotes the semigroup of translations. Therefore,

$$L := \{f(t + \cdot\,, x) : t \in \mathbb{J},\ x \in K\} \subset S(\mathbb{J}, K).$$

Consequently, L is a subset of a closed and separable subspace Y of $C_b(\mathbb{J}, X)$.

By the fact that

$$L \subset \bigcup_{i=1}^{n(\epsilon)} H(f(\cdot\,, x_i)) + \epsilon B_{C_b(\mathbb{J},X)},$$

we obtain the relative weak compactness for L. Hence, the weak topology on L is metrizable, and we may choose a metric of the form

$$d(f, g) = \sum_{i=0}^{\infty} 2^{-i} \frac{|\langle f - g, x_i^*\rangle|}{1 + |\langle f - g, x_i^*\rangle|}, \quad \text{with } \{x_i^*\}_{i\in\mathbb{N}} \subset B_{C_b(\mathbb{J},X)^*}.$$

Choosing $\epsilon = \frac{1}{k}$, we obtain, by the way of the first observation, elements $\{x_1^k, \dots, x_{n(\frac{1}{k})}^k\}$, and by setting

$$\{y_1, y_2, \dots\} := \{(x_1^1, \dots, x_{n(1)}^1, x_1^2, \dots, x_{n(\frac{1}{2})}^2, \dots\},$$

we construct a dense sequence $\{y_i\}_{i\in\mathbb{N}}$. As a consequence of Corollary A.14 and by a simple induction, we find, for all $n \in \mathbb{N}$, a sequence $\{s_k^n\}_{k\in\mathbb{N}}$ such that

$$f(s_k^n + \cdot\,, y_i) \to f_a(\cdot\,, y_i)$$

converges weakly as $k \to \infty$ for all $1 \le i \le n$ in $\mathrm{BUC}(\mathbb{J}, X)$. This, together with the existence of a metric, implies that for all $n \in \mathbb{N}$, there exists $l_n \in \mathbb{N}$ such that

$$d(f(s_l^n + \cdot\,, y_i), f_a(\cdot\,, y_i)) < \frac{1}{n}$$

for all $l \ge l_n$ and $1 \le i \le n$.

Now, for a given $x \in K$ and y_k, we have

$$\begin{aligned} d(f(\cdot + s_{l_n}^n, x), f_a(\cdot\,, x)) &\le d(f(\cdot + s_{l_n}^n, x), f(\cdot + s_{l_n}^n, y_k)) + d(f(\cdot + s_{l_n}^n, y_k), f_a(\cdot\,, y_k)) + d(f_a(\cdot\,, y_k), f_a(\cdot\,, x)) \\ &\le 2\|f(\cdot\,, x) - f(\cdot\,, y_k)\|_\infty + d(f(\cdot + s_{l_n}^n, y_k), f_a(\cdot\,, y_k)), \end{aligned}$$

where the last inequality follows from the definition of the metric and the fact that the norm of the projection on the almost periodic part is less or equal to one. This completes the proof. □

Theorem A.16. *Let $D := (X, w)$, $y \in W(\mathbb{J}, X)$ and $f \in W(\mathbb{J} \times D, X)$. Then the following identity holds:*

$$P_a f(\cdot, y(\cdot))(t) = \{t \mapsto f_a(t, y_a(t))\},$$

where $f_a(\cdot, x)$ and y_a denote the almost periodic parts of $f(\cdot, x)$ and y, respectively.

Proof. From the notation of the theorem, we find that $K := \overline{\{y_a(t) : t \in \mathbb{J}\}}$ is a norm compact set. Now let $K_1 := K \cup \{v, w\}$, $v \neq w$, $v, w \notin K$. Since v, w are two discrete points attached to K_1 with a positive distance to K we have that K_1 is still norm compact. Let

$$F(t, z) := \begin{cases} y(t), & z = v, \\ f(t, y(t)), & z = w, \\ f(t, z), & z \in K. \end{cases}$$

By Theorem A.8, F fulfills the hypothesis of Lemma A.15. Note that F is a function on discrete points $\{x, y\}$ plus $f(t, z)$ on K, hence ι is continuous on K_1. Consequently, we find a sequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$\begin{aligned} f_{s_n}(\cdot, x) &\rightharpoonup f_a(\cdot, x) \quad \text{for every } x \in K, \\ y_{s_n} &\rightharpoonup y_a, \\ f(\cdot + s_n, y(\cdot + s_n)) &\rightharpoonup P_a f(\cdot, y(\cdot)). \end{aligned}$$

Using that all the sequences are convergent, it remains to compute the limit, which can be done in the pointwise weak topology as follows:

$$x^* f(t + s_n, y(t + s_n)) = x^*\{f(t + s_n, y(t + s_n)) - f(t + s_n, y_a(t))\} + x^* f(t + s_n, y_a(t)).$$

Since $f \in W(\mathbb{J} \times D, X)$, the first term on the right-hand side tends to zero as n tends to infinity, and the theorem is proved. □

Corollary A.17. *If for a Banach space Y, $D = (Y, \|\cdot\|)$ and $f \in W(\mathbb{J} \times D, X)$, then, for every given $g \in \mathrm{WRC}(\mathbb{J}, Y)$,*

$$P_a(f(\cdot, g(\cdot))) = f_a(\cdot, g_a(\cdot)).$$

Proof. Using that on $K := \overline{g(\mathbb{J})}$ the norm and the weak topology coincide leads to the given result. □

Remark A.18. We consider the context of Example 5.7 with $X = L^2(\Omega)$,

$$Y_{L^2} = W(\mathbb{R}, L^2(\Omega)), \quad Y_{a,L^2} = \mathrm{AP}(\mathbb{R}, L^2(\Omega)), \quad Y_{0,L^2} = W_0(\mathbb{R}, L^2(\Omega))$$

and

$$Y_{\mathbb{R}} = W(\mathbb{R}), \quad Y_{a,\mathbb{R}} = \mathrm{AP}(\mathbb{R}), \quad Y_{0,\mathbb{R}} = W_0(\mathbb{R}).$$

Then, for $u \in L^2(\Omega)$, we have

$$P_{a,Y_{L^2}}(I + \lambda\partial\phi(\cdot_t))^{-1}u = \{x \mapsto P_{a,Y_{\mathbb{R}}}((I + \lambda\partial\varphi(\cdot_t))^{-1}u(x))\} \quad \text{a.e.}$$

Proof. We give a proof for $u \in C_0^\infty(\Omega)$, and note that $\overline{C_0^\infty(\Omega)} = L^2(\Omega)$. If $u \in C_0^\infty(\Omega)$, we have that $\overline{u(\Omega)}$ is compact. As $((I + \lambda\partial\varphi(\cdot_t))^{-1}$ is a contraction, we can apply Lemma A.15 to

$$f : \mathbb{R} \times \overline{u(\Omega)} \to \mathbb{R} \times L^2(\Omega), \quad \begin{pmatrix} t \\ y \end{pmatrix} \mapsto \begin{pmatrix} (I + \lambda\partial\varphi(t))^{-1}y \\ (I + \lambda\partial\phi(t))^{-1}u \end{pmatrix},$$

which leads to a single sequence $\{s_n\}_{s\in\mathbb{N}}$ with

$$\begin{aligned} (I + \lambda\partial\varphi(\cdot + s_n))^{-1}y &\rightharpoonup J_{\lambda,a}^{-\partial\varphi}(\cdot)y \quad \text{for all } y \in \overline{u(\Omega)}, \\ (I + \lambda\partial\phi(\cdot + s_n))^{-1}u &\rightharpoonup J_{\lambda,a}^{-\partial\phi}(\cdot)u \end{aligned}$$

weakly in BUC($\mathbb{R}$) and BUC($\mathbb{R}, L^2(\Omega)$), respectively. Applying the weaker pointwise and pointwise weak topology, we have, for $t \in \mathbb{R}$ and $v \in L^2(\Omega)$,

$$\lim_{n\to\infty} (I + \lambda\partial\varphi(t+s_n))^{-1}u(x) = J_{\lambda,a}^{-\partial\varphi}(t)u(x) \quad \text{a.e.},$$
$$|(I + \lambda\partial\varphi(t+s_n))^{-1}u(x)| \le |u(x)| \quad \text{a.e.},$$

and

$$\langle J_{\lambda,a}^{-\partial\phi}(t)u, v\rangle = \lim_{n\to\infty} \langle J_\lambda^{-\partial\phi}(t+s_n)u, v\rangle = \int_\Omega (I + \lambda\partial\varphi(t+s_n))^{-1}u(x)v(x)\,dx = \int_\Omega J_{\lambda,a}^{-\partial\varphi}(t)u(x)v(x)\,dx,$$

which concludes the proof. □

Remark A.19. Note that the methods apply in a similar way, when $W(\mathbb{J}, X)$ is substituted by $W^+(\mathbb{R}, X)$ and $W_0(\mathbb{J}, X)$ by $W_0^+(\mathbb{R}, X)$, since the weak relative compactness of orbit on a positive half line serves for $W(\mathbb{J}, X)$, $W_0(\mathbb{J}, X) \subset \mathrm{BUC}(\mathbb{J}, X)$. Thus, $W^+(\mathbb{R}, X)_{|\mathbb{J}} = W(\mathbb{J}, X)$ and $W_0^+(\mathbb{R}, X)_{|\mathbb{J}} = W_0(\mathbb{J}, X)$, hence the proofs are similar.

B Asymptotically almost periodic functions

The classical concepts are due to Frechet [10, 11]. By [5, 6, 10, 11], we have

$$\begin{aligned}\mathrm{AAP}(\mathbb{R}^+, X) &:= \{f \in \mathrm{BUC}(\mathbb{R}^+, X) : O_{\mathbb{R}^+}(f) \text{ is relative compact in } \mathrm{BUC}(\mathbb{R}^+, X)\}\\ &= \{f \in \mathrm{BUC}(\mathbb{R}^+, X) : f = g_{|\mathbb{R}^+} + \phi,\ g \in \mathrm{AP}(\mathbb{R}, X) \text{ and } \phi \in C_0(\mathbb{R}^+, X)\}.\end{aligned}$$

In consequence, we have the following proposition.

Proposition B.1. *For the decomposition*

$$\mathrm{AAP}(\mathbb{R}^+, X) = \mathrm{AP}(\mathbb{R}, X)_{|\mathbb{R}^+} \oplus C_0(\mathbb{R}^+, X)$$

we have that the projection P_a onto $\mathrm{AP}(\mathbb{R}, X)_{\mathbb{R}^+}$ has norm less than or equal to one.

Using compactness methods we have the following theorem.

Theorem B.2. *Let $f\colon \mathbb{R} \times X \mapsto X$ be such that $f(\cdot, x) \in \mathrm{AAP}(\mathbb{R}^+, X)$, with $f(t, \cdot)$ being uniformly Lipschitz with a constant L, and $f_a(\cdot, x)$ its almost periodic part. Further, let $g \in \mathrm{AAP}(\mathbb{R}^+, X)$, with g_a its the almost periodic part. Then*

$$\{t \mapsto f(t, g(t))\} \in \mathrm{AAP}(\mathbb{R}^+, X), \quad \{t \mapsto f(t, g(t))\}_a = \{t \mapsto f_a(t, g_a(t))\}.$$

Proof. As $g(\mathbb{R}^+)$ is relative compact, Lemma A.15 serves for the needed norm convergent subsequence, so that for all $x \in \overline{g(\mathbb{R}^+)}$, $f(\cdot + s_n, x) \rightharpoonup f_a(\cdot, x)$ weakly. The relative compactness $O(f(\cdot, x))$ serves for the norm convergence. The rest of the proof is straightforward. □

C Almost automorphic functions

Bochner introduced the notion of almost automorphy.

Definition C.1. A function $f \in C(\mathbb{R}, X)$ is said to be almost automorphic if for any real sequence $\{s_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} f(t + s_{n_k}) = g(t) \quad \text{for all } t \in \mathbb{R},$$

and

$$\lim_{k\to\infty} g(t - s_{n_k}) = f(t) \quad \text{for all } t \in \mathbb{R}.$$

We define

$$\mathrm{AA}(\mathbb{R}, X) = \{f \in C(\mathbb{R}, X) : f \text{ almost automorphic}\}.$$

If the limit g is continuous, then f is called continuous (Bochner)-almost automorphic. We define

$$\mathrm{CAA}(\mathbb{R}, X) = \{f \in C(\mathbb{R}, X) : f \text{ continuous almost automorphic}\}.$$

Noting, that for $f \in \mathrm{AA}(\mathbb{R}, X)$, $f(\mathbb{R})$ is relatively compact, clearly, we have $\mathrm{AA}(\mathbb{R}, X) \subset C_b(\mathbb{R}, X)$, and that $\mathrm{AA}(\mathbb{R}, X)$ is translation invariant.

The following Theorem is due to [30, Lemma 4.1.1, p. 742].

Theorem C.2. *Continuous almost automorphic functions are uniformly continuous, i.e.,*

$$\mathrm{CAA}(\mathbb{R}, X) \subset \mathrm{BUC}(\mathbb{R}, X).$$

Remark C.3. In [7], the asymptotically almost automorphic functions $\mathrm{AAA}(\mathbb{R}^+, X)$ are discussed. By definition, we have

$$\mathrm{AAA}(\mathbb{R}^+, X) = \mathrm{AA}(\mathbb{R}, X)_{|\mathbb{R}^+} \oplus C_0(\mathbb{R}^+, X).$$

For suitable $f\colon \mathbb{R} \times X \to X$, the almost automorphic part

$$P_a\{t \mapsto f(t, x(t))\} = \{t \mapsto f_a(t, x_a(t))\}$$

was computed. Thus, the underlying study becomes applicable, when switching from almost automorphy to continuous almost automorphy, and adding for $f(\cdot_t, \cdot_x)$ the uniform continuity and the Lipschitz continuity, in the first and second variable, respectively. As for $u = u_a + u_0$,

$$\overline{\{u_a(t) : t \in \mathbb{R}\}} \subset \overline{\{u(t) : t \in \mathbb{R}^+\}},$$

and the projection has a norm less than one.

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