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Minimum action solutions of nonhomogeneous Schrödinger equations

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Abstract: In this paper, we are concerned with the qualitative analysis of solutions to a general class of nonlinear Schrödinger equations with lack of compactness. The problem is driven by a nonhomogeneous differential operator with unbalanced growth, which was introduced by Azzollini [1]. The reaction is the sum of a nonautonomous power-type nonlinearity with subcritical growth and an indefinite potential. Our main result establishes the existence of at least one nontrivial solution in the case of low perturbations. The proof combines variational methods, analytic tools, and energy estimates.

Keywords: nonlinear Schrödinger equation; nonhomogeneous differential operator; double phase energy; entire solution; low perturbation

MSC 2010: 35J60 (Primary); 35B20, 47F05, 58E05 (Secondary)

1 Introduction

In a recent paper, Azzollini [1] introduced a new class of quasilinear operators with a variational structure. He considered nonhomogeneous differential operators of the type

$$u \mapsto \operatorname{div}[\varphi'(|\nabla u(x)|^2)\nabla u(x)],$$

where $x \in \mathbb{R}^N$ and $\varphi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is a potential with unbalanced growth near zero and at infinity. For instance, such a behaviour occurs if $\varphi(t) = 2(\sqrt{1+t}-1)$, which corresponds to the prescribed mean curvature operator (capillary surface operator), which is defined by

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right). \tag{1.1}$$

More generally, $\varphi(t)$ behaves like $t^{q/2}$ for small t and $t^{p/2}$ for large t, where $1 < p < q < N$. An example of potential φ of this type is given by

$$\varphi(t) = \frac{2}{p}\,[(1+t^{q/2})^{p/q} - 1].$$

This potential generates the differential operator

$$\operatorname{div}\left[(1+|\nabla u|^q)^{(p-q)/q}|\nabla u|^{q-2}\nabla u\right].$$

Another important example includes the (p, q)-Laplace operator

$$u \mapsto \Delta_p u + \Delta_q u,$$

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which is generated by $\varphi(t) = t^{p/2} + t^{q/2}$ for all $t \geq 0$.

Due to the unbalanced growth of φ, the associated functional is a double-phase energy. The study of non-autonomous variational integrals with double growth has been initiated by Marcellini [15–17]. We also point out the pioneering work by Zhikov [27], in relationship with phenomena arising in nonlinear elasticity and strongly anisotropic materials in the context of homogenisation. These functionals revealed to be important also in the study of duality theory and in the context of the Lavrentiev phenomenon [28]. We recall that Zhikov considered the variational integral

$$\mathcal{P}_{p,q}(u) := \int_{\Omega} (|\nabla u|^p + a(x)|\nabla u|^q)dx, \quad 0 \leq a(x) \leq L,\ 1 < p < q,$$

where the modulating potential $a(\cdot)$ dictates the geometry of the composite made by two differential materials, with hardening exponents p and q, respectively.

In this paper we study a nonlinear Schrödinger equation driven by the operator defined in (1.1) and with lack of compactness. A feature of this paper is the presence of a power-type subcritical reaction and an indefinite potential. Related problems have been studied by Azzollini, d'Avenia and Pomponio [2] and Chorfi and Rădulescu [12].

The study developed in the present paper is motivated by the central role played by the Schrödinger equation in quantum theory, Newton conservation laws in classical mechanics, Bose-Einstein condensates and nonlinear optics, stability of Stokes waves in water, propagation of the electric field in optical fibers, self-focusing and collapse of Langmuir waves in plasma physics, deep water waves and freak waves (or rogue waves) in the ocean, etc.

Recent papers dealing with (isotropic or anisotropic) double-phase energy include the contributions of Bahrouni *et al.* [4–6], Baroni, Colombo and Mingione [7], Cencelj, Rădulescu and Repovš [11], Mingqi, Rădulescu and Zhang [18, 19], Papageorgiou, Rădulescu and Repovš [20–22], Ragusa and Tachikawa [24], and Zhang and Rădulescu [26]. Some of the main abstract methods used in this paper have been developed in the monographs by Brezis [8] and Papageorgiou, Rădulescu and Repovš [23].

2 The main result

In this paper, we are concerned with the study of the following quasilinear Schrödinger equation with lack of compactness

$$-\operatorname{div}[\varphi'(|\nabla u|^2)\nabla u] + \varphi'(u^2)u = a(x)\,|u|^{r-2}u + \lambda f(x) \quad \text{in } \mathbb{R}^N, \tag{2.2}$$

where $N \geq 3$ and λ is a real parameter.

We describe in what follows the main hypotheses on the above considered problem.

Let p, q and r be real numbers such that

$$1 < p < q < r < p^\star := \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N. \end{cases} \tag{2.3}$$

As in Azzollini, d'Avenia and Pomponio [2], we assume that the potential $\varphi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ satisfies the following hypotheses:

(φ_1) $\varphi(0) = 0$;

(φ_2) there exists $c_1 > 0$ such that $\varphi(t) \geq c_1 t^{p/2}$ if $t \geq 1$ and $\varphi(t) \geq c_1 t^{q/2}$ if $0 \leq t \leq 1$;

(φ_3) there exists $c_2 > 0$ such that $\varphi(t) \leq c_2 t^{p/2}$ if $t \geq 1$ and $\varphi(t) \leq c_2 t^{q/2}$ if $0 \leq t \leq 1$;

(φ_4) there exists $\mu \in (0, 1)$ such that $2t\varphi'(t) \leq r\mu\varphi(t)$ for all $t \geq 0$;

(φ_5) the mapping $t \mapsto \varphi(t^2)$ is strictly convex.

These hypotheses imply that

$$\varphi(|\nabla u|^2) \simeq \begin{cases} |\nabla u|^p, & \text{if } |\nabla u| \gg 1; \\ |\nabla u|^q, & \text{if } |\nabla u| \ll 1. \end{cases}$$

Since our hypotheses allow that φ' approaches 0, problem (2.2) is both degenerate and non-uniformly elliptic.

We assume that the weight $a : \mathbb{R}^N \to \mathbb{R}$ in problem (2.2) satisfies

(a) $a \in L^\infty(\mathbb{R}^N)$ and $\operatorname{essinf}_{x\in\mathbb{R}^N} a(x) = a_0 > 0$.

We describe in what follows the abstract setting corresponding to problem (2.2).

Let $\|\cdot\|_r$ denote the Lebesgue norm for all $1 \le r \le \infty$ and $C_c^\infty(\mathbb{R}^N)$ the space of all C^∞ functions with a compact support.

Definition 1. *We define the function space $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ as the completion of $C_c^\infty(\mathbb{R}^N)$ in the norm*

$$\|u\|_{L^p+L^q} := \inf\{\|v\|_p + \|w\|_q;\ v \in L^p(\mathbb{R}^N),\ w \in L^q(\mathbb{R}^N),\ u = v + w\}.$$

In the same way we can define the space $L^p(\Omega) + L^q(\Omega)$ for an arbitrary open set $\Omega \subset \mathbb{R}^N$. However, the space $L^p(\Omega) + L^q(\Omega)$ is of interest only if $p < q$ and the set Ω has infinite measure. Indeed, if either $|\Omega| < +\infty$ or $p = q$, then $L^q(\Omega) \subset L^p(\Omega)$, hence $L^p(\Omega) + L^q(\Omega) = L^q(\Omega)$.

As established in [2, Proposition 2.2], the space $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ is reflexive and $(L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N))' = L^{p'}(\mathbb{R}^N) + L^{q'}(\mathbb{R}^N)$. For more properties of the Orlicz space $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ we refer to Badiale, Pisani, and Rolando [3, Sect. 2].

On $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ we can also consider the equivalent norm $\|\cdot\|^\star$ given by

$$\|u\|^\star := \inf_{v+w=u} \max\{\|v\|_p, \|w\|_q\}.$$

Consider the space $L^{p'}(\mathbb{R}^N) \cap L^{q'}(\mathbb{R}^N)$ endowed with the norm

$$\|u\|_{L^{p'}\cap L^{q'}} := \|u\|_{p'} + \|u\|_{q'}.$$

According to [3, Lemma 2.9], for all $f \in L^{p'}(\mathbb{R}^N)\cap L^{q'}(\mathbb{R}^N)$ and $u \in L^p(\mathbb{R}^N)+L^q(\mathbb{R}^N)$, the following Hölder-type inequality holds:

$$\int_{\mathbb{R}^N} fu\,dx \le \|f\|_{L^{p'}\cap L^{q'}} \|u\|^\star. \tag{2.4}$$

We define the function space $\mathcal{X}$ as the completion of $C_c^\infty(\mathbb{R}^N)$ in the norm

$$\|u\|_{p,q} := \|u\|_{L^p+L^q} + \|\nabla u\|_{L^p+L^q}.$$

Definition 2. *Assume that $f \in L^{p'}(\mathbb{R}^N) \cap L^{q'}(\mathbb{R}^N)$. A weak solution of problem* (2.2) *is a function $u \in \mathcal{X} \setminus \{0\}$ such that for all $v \in \mathcal{X}$*

$$\int_{\mathbb{R}^N} \left[\varphi'(|\nabla u|^2)\nabla u \nabla v + \varphi'(u^2)uv\right] dx = \int_{\mathbb{R}^N} a(x)\,|u|^{r-2}uv\,dx + \lambda \int_{\mathbb{R}^N} f(x)v\,dx.$$

According to the terminology introduced by Fučik, Nečas, Souček and Souček [13, p. 117] we can say that u is an eigenfunction of the nonlinear problem (2.2) corresponding to the eigenvalue λ.

The main result of this paper establishes the following existence property in the case of 'low perturbations'.

Theorem 1. *Let $f \in L^{p'}(\mathbb{R}^N) \cap L^{q'}(\mathbb{R}^N)$ and assume that hypotheses* (2.3), *(φ_1)-(φ_5) and (a) are fulfilled. Then there exists a positive number $\lambda_\star$ such that problem* (2.2) *has at least one solution for all $|\lambda| < \lambda_\star$.*

We point out that the study of these problems was initiated by Azzollini, d'Avenia and Pomponio [2]. They studied the problem

$$-\operatorname{div}[\varphi'(|\nabla u|^2)\nabla u] + |u|^{\alpha-2} = |u|^{s-2}u \quad \text{in } \mathbb{R}^N \ (N \geq 3).$$

The potential φ satisfies the same hypotheses as above, while $1 < p < q < N$, $1 < \alpha \leq p^\star q'/p'$, and $\max\{q, \alpha\} < s < p^\star$. The lack of compactness due to the unboundedness of the Euclidean space is handled in [2] by restricting the study to the case of radially symmetric weak solutions. In the framework developed in [2], a central role is played by the compact embedding of a related function space with radial symmetry into a related Lebesgue space. This abstract setting does not hold in this work. Due to the lack of symmetry of the problem, we develop a general approach that cannot be reduced to the radial case as in [2].

Finally, we point out that with similar arguments we can treat the more general problem

$$-\operatorname{div}[\varphi'(|\nabla u|^2)\nabla u] + \varphi'(u^2)u = a(x)\, h(x,u) + \lambda f(x) \quad \text{in } \mathbb{R}^N \ (N \geq 3)$$

where $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following conditions:

(h_1) $h(x,u) = o(u^{q-1})$ as $u \to 0^+$, uniformly for a.e. $x \in \mathbb{R}^N$;

(h_2) $h(x,u) = O(u^{r-1})$ as $u \to \infty$, uniformly for a.e. $x \in \mathbb{R}^N$;

(h_3) there exists $\theta > r$ such that $0 < \theta H(x,u) \leq uh(x,u)$ for all $u > 0$, a.e. $x \in \mathbb{R}^N$, where $H(x,u) = \int_0^u h(x,t)dt$.

3 Auxiliary results

The energy functional associated to problem (2.2) is $\mathcal{J} : \mathcal{X} \to \mathbb{R}$ defined by

$$\mathcal{J}(u) := \frac{1}{2}\int_{\mathbb{R}^N} \varphi(|\nabla u|^2)dx + \frac{1}{2}\int_{\mathbb{R}^N} \varphi(u^2)dx - \frac{1}{r}\int_{\mathbb{R}^N} a(x)|u|^r dx - \lambda \int_{\mathbb{R}^N} f(x)u dx.$$

By standard arguments we obtain that $\mathcal{J}$ is well-defined and is of class C^1; we refer to Azzollini [2, Theorem 2.5] for more details. Moreover, for all $u, v \in \mathcal{X}$ its Gâteaux directional derivative is given by

$$\mathcal{J}'(u)(v) = \int_{\mathbb{R}^N} \left[\varphi'(|\nabla u|^2)\nabla u \nabla v - \varphi'(u^2)uv - a(x)\,|u|^{r-2}uv - \lambda f(x)v\right] dx.$$

An important role in our arguments will be played by the following version of the mountain pass theorem, which is due to Brezis and Nirenberg [9].

Theorem 2. *Let X be a real Banach space and assume that $\mathcal{J} : X \to \mathbb{R}$ is a C^1-functional that satisfies the following geometric hypotheses:*
(i) $\mathcal{J}(0) = 0$ and there exist positive numbers a and ρ such that $\mathcal{J}(u) \geq a$ for all $u \in X$ with $\|u\| = \rho$;
(ii) there exists $e \in X$ with $\|e\| > \rho$ such that $\mathcal{J}(e) < 0$.

Set

$$\mathcal{P} := \{p \in C([0,1]; X);\ p(0) = 0,\ p(1) = e\}$$

and

$$c := \inf_{p \in \mathcal{P}} \sup_{t \in [0,1]} \mathcal{J}(p(t)).$$

Then there exists a sequence $(u_n) \subset X$ such that

$$\lim_{n\to\infty} \mathcal{J}(u_n) = c \quad \text{and} \quad \lim_{n\to\infty} \|\mathcal{J}'(u_n)\|_{X^*} = 0.$$

Moreover, if $\mathcal{J}$ satisfies the Palais-Smale condition at the level c, then c is a critical value of $\mathcal{J}$.

We prove in what follows that the mountain pass geometry described in the above hypotheses (i) and (ii) are fulfilled.

Lemma 1. *For any $R > 0$ large enough, there exists $\zeta \in \mathcal{X}$ such that $\mathcal{J}(\zeta) < 0$ and $\|\zeta\| \geq R$.*

Proof. Fix $\psi \in \mathcal{X} \setminus \{0\}$. We find ζ of the form $\zeta = t\psi$ for $t > 0$ large enough.

By our hypotheses, we have

$$\begin{aligned}\mathcal{J}(t\psi) \leq\ & \frac{c_2}{2} \int\limits_{|\nabla\psi|>1/t} t^p |\nabla\psi|^p dx + \frac{c_2}{2} \int\limits_{|\nabla\psi|\leq 1/t} t^q |\nabla\psi|^q dx \\ & + \frac{c_2}{2} \int\limits_{|\psi|>1/t} t^p |\psi|^p dx + \frac{c_2}{2} \int\limits_{|\psi|\leq 1/t} t^q |\psi|^q dx \\ & - \frac{t^r}{r} \int\limits_{\mathbb{R}^N} a(x)|\psi|^r dx - \lambda t \int\limits_{\mathbb{R}^N} f\psi dx \\ \leq\ & \frac{c_2}{2} \left[t^p \int\limits_{|\nabla\psi|>1/t} |\nabla\psi|^p dx + t^q \int\limits_{|\nabla\psi|\leq 1/t} |\nabla\psi|^q dx \right] \\ & + \frac{c_2}{2} \left[t^p \int\limits_{|\psi|>1/t} |\psi|^p dx + t^q \int\limits_{|\psi|\leq 1/t} |\psi|^q dx \right] - c_3 t^r - c_4 t,\end{aligned}$$

where c_3 and c_4 are positive numbers.

By assumption (2.3) we conclude that $\mathcal{J}(t\psi) < 0$ for t large enough. □

The following result establishes a low perturbation property, which remains valid only for small perturbations from the origin of the parameter λ. Roughly speaking, this result shows that the mountain pass geometry does not change in the case of 'low' perturbations.

Lemma 2. *There exists positive numbers a and ρ such that $\mathcal{J}(u) \geq a$ for all $u \in \mathcal{X}$ with $\|u\|_{p,q} = \rho$.*

Proof. By hypothesis (φ_2) we have for all $u \in \mathcal{X}$

$$\begin{aligned}\mathcal{J}(u) \geq\ & \frac{c_1}{2} \int\limits_{|\nabla u|>1} |\nabla u|^p dx + \frac{c_1}{2} \int\limits_{|\nabla u|\leq 1} |\nabla u|^q dx \\ & + \frac{c_1}{2} \int\limits_{|u|>1} |u|^p dx + \frac{1_2}{2} \int\limits_{|u|\leq 1} |u|^q dx \\ & - \frac{1}{r} \int\limits_{\mathbb{R}^N} a(x)|u|^r dx - |\lambda| \int\limits_{\mathbb{R}^N} |fu| dx.\end{aligned}$$

Thus, there exists $c_5 > 0$ such that for all $u \in \mathcal{X}$

$$\mathcal{J}(u) \geq c_5 \|u\|_{p,q}^q - \frac{\|a\|_\infty}{r} \|u\|_r^r - |\lambda| \int\limits_{\mathbb{R}^N} |fu| dx. \tag{3.5}$$

By [1, Theorem 2.1], there exists $c_6 > 0$ such that for all $u \in \mathcal{X}$

$$\|u\|_r \leq c_6 \|u\|_{p,q}. \tag{3.6}$$

Using (2.4) and (3.6), relation (3.5) yields for all $u \in \mathcal{X}$

$$\begin{aligned}\mathcal{J}(u) & \geq c_5 \|u\|_{p,q}^q - \frac{c_6\|a\|_\infty}{r} \|u\|_{p,q}^r - c_6 |\lambda| \|f\|_{L^{p'} \cap L^{q'}} \|u\|^\star \\ & \geq c_5 \|u\|_{p,q}^q - \frac{c_6\|a\|_\infty}{r} \|u\|_{p,q}^r - c_7 \,|\lambda|\, \|u\|_{p,q}.\end{aligned} \tag{3.7}$$

Next, using assumption (2.3), we find $a_0 > 0$ and $\rho > 0$ such that $\mathcal{J}(u) \geq a_0$ for all $u \in \mathcal{X}$ with $\|u\|_{p,q} = \rho$. We fix some small $\rho > 0$ with this property. Taking $\lambda \in \mathbb{R}$ such that $|\lambda| \leq a_0/(2c_7\rho)$ and using (3.7), we conclude that

$$\mathcal{J}(u) \geq a := a_0/2 > 0 \text{ for all } u \in \mathcal{X} \text{ with } \|u\|_{p,q} = \rho.$$

The proof of Lemma 2 is now complete. □

Set

$$\mathcal{F} := \{y \in C([0,1]; \mathcal{X});\ y(0) = 0,\ y(1) = \zeta\},$$

where ζ is as in Lemma 1.

We define

$$c := \inf_{y\in\mathcal{F}} \max_{t\in[0,1]} \mathcal{J}(y(t)). \tag{3.8}$$

We observe that $c \geq 0$.

By Theorem 2, there exists a sequence $(u_n) \subset \mathcal{X}$ such that

$$\mathcal{J}(u_n) \to c \text{ and } \mathcal{J}'(u_n) \to 0 \text{ as } n \to \infty. \tag{3.9}$$

Lemma 3. *The sequence* $(u_n) \subset \mathcal{X}$ *satisfying* (3.9) *is bounded.*

Proof. We have

$$\begin{aligned}\mathcal{J}(u_n) - \frac{1}{r}\mathcal{J}'(u_n)u_n \quad &= \int\limits_{\mathbb{R}^N} \left(\frac{1}{2}\varphi(|\nabla u_n|^2) - \frac{1}{r}\varphi'(|\nabla u_n|^2)|\nabla u_n|^2\right) dx \\ &+ \int\limits_{\mathbb{R}^N} \left(\frac{1}{2}\varphi(u_n^2) - \frac{1}{r}\varphi'(u_n^2)u_n^2\right) dx \\ &-\lambda\left(1 - \frac{1}{r}\right)\int\limits_{\mathbb{R}^N} f(x)u_n dx.\end{aligned} \tag{3.10}$$

By (2.4) we obtain for all n and some positive constant c_8

$$\begin{aligned}\mathcal{J}(u_n) - \frac{1}{r}\mathcal{J}'(u_n)u_n \quad &\geq \int\limits_{\mathbb{R}^N} \left(\frac{1}{2}\varphi(|\nabla u_n|^2) - \frac{1}{r}\varphi'(|\nabla u_n|^2)|\nabla u_n|^2\right) dx \\ &+ \int\limits_{\mathbb{R}^N} \left(\frac{1}{2}\varphi(u_n^2) - \frac{1}{r}\varphi'(u_n^2)u_n^2\right) dx \\ &-c_8\,|\lambda|\,\|u_n\|_{p,q}.\end{aligned} \tag{3.11}$$

Assumption (φ_4) implies that

$$\frac{1}{2}\,\varphi(t) - \frac{1}{r}\,t\varphi'(t) \geq \frac{1-\mu}{2}\,\varphi(t) \text{ for all } t \geq 0,$$

where $\mu \in (0,1)$. Returning to (3.11) we deduce that for all n

$$\begin{aligned}\mathcal{J}(u_n) - \frac{1}{r}\mathcal{J}'(u_n)u_n \quad &\geq \frac{1-\mu}{2}\int\limits_{\mathbb{R}^N} \left[\varphi(|\nabla u_n|^2) + \varphi(u_n^2)\right] dx \\ &-c_8\,|\lambda|\,\|u_n\|_{p,q} \\ &\geq c_{10}\min\{\|u_n\|_{p,q}^p, \|u_n\|_{p,q}^q\} - c_8\,|\lambda|\,\|u_n\|_{p,q},\end{aligned} \tag{3.12}$$

for some $c_{10} > 0$.

On the other hand, since $\{u_n\}$ satisfies (3.9), we have

$$\mathcal{J}(u_n) - \frac{1}{r}\mathcal{J}'(u_n)u_n = c + O(1) + o(\|u_n\|_{p,q}) \text{ as } n \to \infty. \tag{3.13}$$

Combining relations (3.12) and (3.13) we conclude that the sequence $(u_n) \subset \mathcal{X}$ is bounded. □

4 Proof of the main result

In this section we give the proof of Theorem 1.

By Lemma 3 and [3, Corollary 2.11] we can assume that, up to a subsequence,

$$u_n \rightharpoonup U \text{ in } \mathcal{X}. \tag{4.14}$$

Lemma 4. *The function U given in* (4.14) *is a solution of problem* (2.2).

Proof. Let $\eta \in C_c^\infty(\mathbb{R}^N)$ and denote $\omega := \operatorname{supp}(\eta)$.

By Lemma 3 and [3, Proposition 2.17(ii)] we can assume, going eventually to a subsequence, that

$$u_n \to U \text{ in } L^r(\omega). \tag{4.15}$$

Define the functionals $\mathcal{J}_1, \mathcal{J}_2 : \mathcal{X} \to \mathbb{R}$ by

$$\mathcal{J}_1(v) = \frac{1}{2} \int_\omega \varphi(|\nabla v|^2)dx + \frac{1}{2} \int_\omega \varphi(v^2)dx$$

and

$$\mathcal{J}_2(v) = \frac{1}{r} \int_\omega a(x)|v|^r dx + \lambda \int_\omega f(x)v dx.$$

Thus, by relation (3.9), we have

$$\mathcal{J}_1'(u_n)(\eta) - \mathcal{J}_2'(u_n)(\eta) \to 0 \quad \text{as } n \to \infty. \tag{4.16}$$

Next, by (4.15) and since the function

$$\mathcal{X} \ni u \mapsto F(x, u) \in L^1(\omega)$$

is compact, we obtain

$$\mathcal{J}_2(u_n) \to \mathcal{J}_2(U) \text{ as } n \to \infty$$

and

$$\mathcal{J}_2'(u_n)(\eta) \to \mathcal{J}_2'(U)(\eta) \text{ as } n \to \infty.$$

Using these relations in conjunction with (4.16) we have

$$\mathcal{J}_1'(u_n)(\eta) \to \mathcal{J}_2'(U)(\eta) \text{ as } n \to \infty. \tag{4.17}$$

Next, by condition (φ_5), we deduce that A is a nonlinear convex mapping. It follows that

$$\mathcal{J}_1(u_n) - \mathcal{J}_1(U) \le \mathcal{J}_1'(u_n)(u_n - U) \text{ for all } n \ge 1. \tag{4.18}$$

Combining now (4.14) and (4.17), relation (4.18) implies that

$$\mathcal{J}_1(U) \ge \limsup_{n\to\infty} \mathcal{J}_1(u_n).$$

Since $\mathcal{J}_1$ is convex and continuous, then it is lower semicontinuous. It follows that

$$\mathcal{J}_1(U) \le \liminf_{n\to\infty} \mathcal{J}_1(u_n).$$

Thus, the last two relations yield

$$A(u_n) \to A(U) \quad \text{as } n \to \infty.$$

Using the same ideas as in [2, p. 210], we obtain as $n \to \infty$

$$\nabla u_n \to \nabla U \text{ in } L^p(\omega) + L^q(\omega),$$

$$\int_\omega a(x)|u_n|^r dx \to \int_\omega a(x)|U|^r dx$$

and

$$\int_\omega f(x)u_n dx \to \int_\omega f(x)U dx.$$

It follows that

$$\int_\omega \varphi'(|\nabla U|^2)\nabla U\nabla\eta dx + \int_\omega \varphi'(U^2)U\eta dx = \int_\omega a(x)|U|^{r-2}U\eta dx + \lambda\int_\omega f(x)\eta dx. \tag{4.19}$$

By density arguments (using the definition of $\mathcal{X}$), we deduce that relation (4.19) holds for all $\eta \in \mathcal{X}$, that is, U is a solution of (2.2). The proof of Lemma 4 is now complete. □

We conclude the proof of Theorem 1 by establishing that $U \neq 0$.

By relation (3.9) we deduce that there is a positive integer N such that for all $n \geq N$ we have

$$\begin{aligned}\frac{c}{2} &\leq \mathcal{J}(u_n) - \frac{1}{2}\mathcal{J}'(u_n)(u_n)\\ &= \frac{1}{2}\int_{\mathbb{R}^N}\left[\varphi(|\nabla u_n|^2) - \varphi'(|\nabla u_n|^2)|\nabla u_n|^2\right] dx\\ &+\frac{1}{2}\int_{\mathbb{R}^N}\left(\varphi(u_n^2) - \varphi'(u_n^2)u_n^2\right) dx\\ &+\left(\frac{1}{2}-\frac{1}{r}\right)\int_{\mathbb{R}^N} a(x)|u_n|^r dx - \frac{\lambda}{2}\int_{\mathbb{R}^N} f(x)u_n dx.\end{aligned} \tag{4.20}$$

Since the mapping $t \mapsto \varphi(t^2)$ is convex (by hypothesis (φ_5)) we have

$$\varphi(t^2) \leq \varphi'(t^2)t^2,$$

hence

$$\varphi(|\nabla u_n|^2) - \varphi'(|\nabla u_n|^2)|\nabla u_n|^2 \leq 0 \tag{4.21}$$

and

$$\varphi(u_n^2) - \varphi'(u_n^2)u_n^2 \leq 0. \tag{4.22}$$

Let c be the real number defined in (3.8). By definition, we have $c \geq 0$. We claim that

$$c > 0. \tag{4.23}$$

Indeed, if not, then for all $\varepsilon > 0$ there is $y \in \mathcal{F}$ such that

$$0 \leq \max_{t\in[0,1]} \mathcal{J}(y(t)) < \varepsilon.$$

Let a be the real positive number given by Lemma 2 and take $0 < \varepsilon < a$. Fix $y_0 \in \mathcal{F}$ joining the origin and ζ given by Lemma 1 with R big enough, hence $y_0(0) = 0$ and $y_0(1) = \zeta$. It follows that $y_0(0) = 0$ and $\|y_0(1)\| > R$. By continuity, there exists $t_0 \in (0, 1)$ such that $\|y_0(t_0)\| = R$, hence

$$\|\mathcal{J}(y_0(t_0))\| = a > \varepsilon.$$

This contradiction shows that our claim (4.23) is true.

We first assume that $r > 2$. Returning to (4.20) and using (4.22) and (4.23), it follows that for all n big enough we have

$$0 < \frac{c}{2} \leq \left(\frac{1}{2}-\frac{1}{r}\right)\int_{\mathbb{R}^N} a(x)|u_n|^r dx - \frac{\lambda}{2}\int_{\mathbb{R}^N} f(x)u_n dx. \tag{4.24}$$

Arguing by contradiction, we assume that $U = 0$. We first assume that $r \leq 2$. Then by the Hölder-type inequality (2.4) and (4.24) we obtain

$$0 < \frac{c}{2} \leq \frac{|\lambda|}{2}\,\|f\|_{L^{p'}\cap L^{q'}}\,\|u_n\|^\star.$$

If either $\lambda = 0$ or (up to a subsequence) $\|u_n\|^* \to 0$ as $n \to \infty$, then we have a contradiction. If not, we can take $|\lambda|$ small enough (as requested in Theorem 1) in order to find a contradiction. So, $U \neq 0$.

We now consider the case $r > 2$. Using again relation (4.24) we obtain for all n sufficiently large

$$0 < \frac{c}{2} \leq c_{11} \int_{\mathbb{R}^N} a(x)|u_n|^r dx + c_{12}\,|\lambda|\,\|u_n\|^*, \tag{4.25}$$

where c_{11} and c_{12} are positive constants. We argue again by contradiction and assume that $U = 0$. Then with an argument as above we obtain that for all n large enough we have

$$0 < \frac{c}{4} \leq c_{11} \int_{\mathbb{R}^N} a(x)|u_n|^r dx.$$

Using now (4.15) and the assumption $U = 0$ we obtain a contradiction.

We can give a direct argument in order to show that $U \neq 0$. Indeed, by (4.25), it follows that (u_n) does not converge strongly to 0 in $L^r(\mathbb{R}^N)$. From now on, with the same argument as in Gazzola and Rădulescu [14, pp. 55-56], we obtain that $U \neq 0$.

We conclude that U is a nontrivial solution of problem (2.2) and the proof is complete. □

Comments. (i) Related arguments can be applied in order to show that the same low-perturbation result remains true if hypotheses (a) is replaced by conditions
(a_1) $a \in L^\infty_{loc}(\mathbb{R}^N \setminus \{0\})$ and $\operatorname{essinf}_{x\in\mathbb{R}^N} a(x) > 0$;
(a_2) $\lim_{x\to 0} a(x) = \lim_{|x|\to\infty} a(x) = +\infty$.

(ii) We also point out that the same arguments can be developed in order to analyze the "critical frequency case", which corresponds to potentials $a(x)$ satisfying $\liminf_{|x|\to\infty} a(x) = 0$; see Byeon and Wang [10].

(iii) We do not have any results in the case of "high perturbation", that is, if $|\lambda|$ is big enough. It seems that the same arguments based on the mountain pass theorem cannot be applied.

(iv) We conjecture that with arguments as those developed in Rădulescu and Smets [25] we can prove that, in fact, problem (2.2) has at least two nontrivial solutions, provided that $|\lambda|$ is small enough.

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