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Anisotropic problems with unbalanced growth

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Abstract: The main purpose of this paper is to study a general class of (p, q)-type eigenvalues problems with lack of compactness. The reaction is a convex-concave nonlinearity described by power-type terms. Our main result establishes a complete description of all situations that can occur. We prove the existence of a critical positive value λ^* such that the following properties hold: (i) the problem does not have any entire solution in the case of low perturbations (that is, if $0 < \lambda < \lambda^*$); (ii) there is at least one solution if $\lambda = \lambda^*$; and (iii) the problem has at least two entire solutions in the case of high perturbations (that is, if $\lambda > \lambda^*$). The proof combines variational methods, analytic tools, and monotonicity arguments.

Keywords: Nonlinear elliptic equation; nonhomogeneous differential operator; variable exponent; weak solution; mountain pass

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1 Introduction

The initial motivation of this paper goes back to the paper by Alama and Tarantello [1], where there are studied combined effects of convex and concave terms for nonlinear elliptic equations with Dirichlet boundary condition. Alama and Tarantello were concerned with the following semilinear elliptic problem

$$\begin{cases} -\Delta u - \lambda u = k(x)u^q - h(x)u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where $1 < p < q$, λ is a real parameter, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, and the potentials $h, k \in L^1(\Omega)$ are nonnegative. Let λ_1 be the first eigenvalue of the Laplace operator in $H_0^1(\Omega)$. The main result in [1] establishes that for all $\lambda \in \mathbb{R}$ in a neighbourhood of λ_1, problem (1.1) has nontrivial weak solutions under natural growth hypotheses on h and k. More precisely, Alama and Tarantello proved existence, nonexistence and multiplicity properties depending on λ and according to the integrability properties of the ratio k^{p-1}/h^{q-1}.

Nonlinear boundary value problems with reaction described by convex–concave nonlinear terms have been also studied in the pioneering paper by Ambrosetti, Brezis and Cerami [2]. The authors considered the

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following semilinear elliptic problem with Dirichlet boundary condition

$$\begin{cases} -\Delta u = \lambda u^{q-1} + u^{p-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, λ is a positive parameter, and $1 < q < 2 < p < 2^\star$ ($2^\star = 2N/(N-2)$ if $N \geq 3$, $2^\star = +\infty$ if $N = 1, 2$). The authors proved the existence of a critical value $\lambda_0 > 0$ such that problem (1.2) has at least two solutions for all $\lambda \in (0, \lambda_0)$, one solution for $\lambda = \lambda_0$, and no solution exists for all $\lambda > \lambda_0$.

For related contributions, we refer to Bartsch and Willem [8], Filippucci, Pucci and Rădulescu [13], Pucci and Rădulescu [23], Rădulescu and Repovš [25], etc.

Motivated by the above mentioned papers, we are concerned with existence and multiplicity properties of solutions in a different abstract setting and with lack of compactness. The feature of the present paper is that we consider a nonlinear Dirichlet problem driven by a general nonhomogenous differential operator, which was introduced by Barile and Figueiredo [5]. This operator generalizes several standard operators, including the p-Laplacian, the (p, q)-Laplace operator, the generalized mean curvature operator, etc. In particular, the associated energy can be a double phase functional.

The study of non-autonomous functionals characterized by the fact that the energy density changes its ellipticity and growth properties according to the point has been initiated by Marcellini [16–18]. Recently, important contributions are due to Mingione *et al.* [6, 12]. These papers are in relationship with the works of Zhikov [29, 30], which describe the behavior of phenomena arising in nonlinear elasticity. In fact, Zhikov intended to provide models for strongly anisotropic materials in the context of homogenisation. In particular, he considered the following model functional

$$\mathcal{P}_{p,q}(u) := \int_\Omega (|Du|^p + a(x)|Du|^q)dx, \quad 0 \leq a(x) \leq L,\ 1 < p < q, \tag{1.3}$$

where the modulating coefficient $a(x)$ dictates the geometry of the composite made of two differential materials, with hardening exponents p and q, respectively.

Another significant model example of a functional with (p, q)–growth studied by Mingione *et al.* is given by

$$u \mapsto \int_\Omega |Du|^p \log(1 + |Du|)dx, \quad p \geq 1,$$

which is a logarithmic perturbation of the p-Dirichlet energy.

Intensive research work has been devoted to nonlinear PDEs with double phase energy; see, e.g., [4, 7, 11, 19, 20, 24, 26, 28].

Some of the main abstract methods used in this paper can be found in Ambrosetti and Rabinowitz [3], Brezis and Nirenberg [10], Pucci and Rădulescu [22], and the monograph by Papageorgiou, Rădulescu and Repovš [21].

Notation

Throughout this paper, we denote by $W^{1,p}(\mathbb{R}^N)$ the Sobolev space endowed with the norm

$$\|u\|_{W^{1,p}} = \left(\int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p)\, dx\right)^{1/p}.$$

We denote by $L^s_m(\mathbb{R}^N)$, $1 \leq s < \infty$ the weighted Lebesgue space

$$L^s_m(\mathbb{R}^N) = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^N) \,:\, \int_{\mathbb{R}^N} m(x)|u|^s dx < \infty \right\},$$

where $m(x)$ is a positive continuous function on $\mathbb{R}^N$. This weighted function space is endowed with the norm

$$\|u\|_{m,s} = \left(\int_{\mathbb{R}^N} m(x)|u|^s dx\right)^{1/s}.$$

If $m(x) \equiv 1$ in $\mathbb{R}^N$, the norm is denoted by $\|\cdot\|_s$.

2 The main result

Barile and Figueiredo [5] studied nonlinear elliptic problems driven by the potential $a : [0,\infty) \to (0,\infty)$, which is a continuously differentiable function satisfying the following hypotheses:

(a1) there exist positive constants c_i $(i = 0, 1, 2, 3)$ and real numbers $1 < p \le q$ such that

$$c_0 + c_1 t^{(q-p)/p} \le a(t) \le c_2 + c_3 t^{(q-p)/p} \quad \text{for all } t \ge 0;$$

(a2) there exists $\alpha \ge q/p$ such that

$$\frac{1}{\alpha} a(t)t \le A(t) = \int_0^t a(s)ds \quad \text{for all } t \ge 0;$$

(a3) the mapping $[0,\infty) \ni t \mapsto a(t)t^{(p-2)/p}$ is increasing.

In this paper, we are concerned with the study of the following nonlinear eigenvalue problem.

$$\begin{cases} -\mathrm{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) + a(|u|^p)|u|^{p-2}u = \lambda\,|u|^{r-2}u - m(x)|u|^{s-2}u, & x \in \mathbb{R}^N \\ u \ge 0, & x \in \mathbb{R}^N \\ u \not\equiv 0, \end{cases} \tag{2.4}$$

where λ is a positive parameter and

$$1 < p \le q < r < s < p^\star. \tag{2.5}$$

We assume that $m : \mathbb{R}^N \to (0,\infty)$ is a continuous function satisfying the following (normalized) growth condition

$$\int_{\mathbb{R}^N} m(x)^{r/(r-s)} dx = 1. \tag{2.6}$$

As usual, we have denoted by $p^\star$ the critical Sobolev exponent, that is, $p^\star = Np/(N-p)$ if $p < N$ and $p^\star = +\infty$ if $p \ge N$.

Let $E \subset W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ be the function space defined by

$$E = \left\{ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N) \;:\; \int_{\mathbb{R}^N} m(x)|u|^s dx < \infty \right\}.$$

Then E is a Banach space endowed with the norm

$$\|u\|_E = \|u\|_{W^{1,p}} + \|u\|_{W^{1,q}} + \|u\|_{m,s}.$$

We point out that by hypothesis (2.5) and Sobolev embeddings, the space E is continuously embedded into $L^r(\mathbb{R}^N)$.

We say that $u \in E \setminus \{0\}$ is a solution of problem (2.4) if $u(x) \geq 0$ a.e. in $\mathbb{R}^N$ and

$$\begin{aligned}\int_{\mathbb{R}^N} a(|\nabla u|^p)|\nabla u|^{p-2} \quad \nabla u \nabla v dx + \int_{\mathbb{R}^N} a(|u|^p)|u|^{p-2} uv dx = \\ \lambda \int_{\mathbb{R}^N} |u|^{r-2} uv dx - \int_{\mathbb{R}^N} m(x)|u|^{s-2} uv dx,\end{aligned} \tag{2.7}$$

for all $v \in E$.

We state in what follows the main result of this paper. Roughly speaking, this results establishes that problem (2.4) does not have any solutions in the case of small perturbations. However, this problem admits at least two solutions in the case of high perturbations. In both cases, the term “perturbation" should be understood in relationship with the values of the positive parameter λ that is associated to the power-type reaction $|u|^{r-2}u$ in problem (2.4).

Theorem 1. *Assume that hypotheses (2.5) and (a1)–(a3) are fulfilled. Then there exists $\lambda^* > 0$ such that the following properties hold.*

(i) If $0 < \lambda < \lambda^*$, then problem (2.4) does not have any solution.

(ii) If $\lambda = \lambda^*$, then problem (2.4) has at least one solution.

(iii) If $\lambda > \lambda^*$, then problem (2.4) has at least two solutions.

According to Barile and Figueiredo [5], the following operators are suitable to the hypotheses of Theorem 1.

(i) If $a \equiv 1$, then $\operatorname{div}(a(|\nabla u|)|\nabla u|^{p-2}\nabla u) = \Delta_p u$.

(ii) If $a(t) = 1 + t^{(q-p)/p}$, then $\operatorname{div}(a(|\nabla u|)|\nabla u|^{p-2}\nabla u) = \Delta_p u + \Delta_q u$.

(iii) If $a(t) = 1 + (1+t)^{-p/(p-2)}$, then

$$\operatorname{div}(a(|\nabla u|)|\nabla u|^{p-2}\nabla u) = \Delta_p u + \operatorname{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{(1+|\nabla u|^p)^{(p-2)/p}}\right).$$

(iv) If $a(t) = 1 + t^{(q-p)/p} + (1+t)^{-p/(p-2)}$, then

$$\operatorname{div}(a(|\nabla u|)|\nabla u|^{p-2}\nabla u) = \Delta_p u + \Delta_q u + \operatorname{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{(1+|\nabla u|^p)^{(p-2)/p}}\right).$$

The energy functional associated to problem (2.4) is $\mathcal{J}_\lambda : E \to \mathbb{R}$ defined by

$$\mathcal{J}_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}^N} A(|\nabla u|^p)dx + \frac{1}{p}\int_{\mathbb{R}^N} A(|u|^p)dx - \frac{\lambda}{r}\int_{\mathbb{R}^N} |u|^r dx + \frac{1}{s}\int_{\mathbb{R}^N} m(x)|u|^s dx.$$

Recall that $A(t) = \int_0^t a(s)ds$.

Next, by hypothesis (a1), we have

$$a(t)t^{(p-1)/p} \leq c_2 t^{(p-1)/p} + c_3 t^{(q-1)/p} \quad \text{for all } t \geq 0.$$

This subcritical growth condition implies that $\mathcal{J}_\lambda$ is well defined. Moreover, by standard arguments, the functional $\mathcal{J}_\lambda$ is of class C^1 and its Gâteaux directional derivative is given by

$$\begin{aligned}\langle \mathcal{J}'_\lambda(u), v\rangle &= \int_{\mathbb{R}^N} a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \nabla v dx + \int_{\mathbb{R}^N} a(|u|^p)|u|^{p-2} uv dx \\ &-\lambda \int_{\mathbb{R}^N} |u|^{r-2} uv dx + \int_{\mathbb{R}^N} m(x)|u|^{s-2} uv dx \quad \text{for all } u, v \in E.\end{aligned}$$

This shows that the nonnegative nontrivial critical points of $\mathcal{J}_\lambda$ correspond to the solutions of problem (2.4).

Let us assume that u is a solution of problem (2.4). Then the corresponding $\lambda \in \mathbb{R}$ is an "eigenvalue" associated to the "eigenfunction" u. This terminology is in accordance with the related notions introduced by Fučik, Nečas, Souček and Souček [14, p. 117] in the context of *nonlinear* operators. Indeed, if we define the nonlinear operators

$$Su = \frac{1}{p}\int_{\mathbb{R}^N} A(|\nabla u|^p)dx + \frac{1}{p}\int_{\mathbb{R}^N} A(|u|^p)dx - \frac{1}{s}\int_{\mathbb{R}^N} m(x)|u|^s dx$$

and

$$Tu = \frac{1}{r}\int_{\mathbb{R}^N} |u|^r dx$$

then λ is an eigenvalue for the pair (S, T) (in the sense of [14]) if and only if there is a corresponding eigenfunction u which is a solution of problem (2.4) as described in (2.7).

The strategy to prove Theorem 1 is the following.

(a) We first establish that there is $\lambda^* > 0$ such that problem (2.4) has no solution for all $\lambda < \lambda^*$ (case of "low perturbations"). This implies that solutions could exist only in the case of "high perturbations", namely if λ is large enough. The proof of this assertion yields an energy lower bound of solutions in term of λ, which is useful to conclude that problem (2.4) has a non-trivial solution if $\lambda = \lambda^*$.

(b) Next, we show that there exists $\lambda^{**} > 0$ such that problem (2.4) has at least two solutions for all $\lambda > \lambda^{**}$. Finally, combining the properties of λ^* and λ^{**} we conclude that $\lambda^* = \lambda^{**}$.

The proof of Theorem 1 uses some ideas developed in Alama and Tarantello [1], Filippucci, Pucci and Rădulescu [13], and Pucci and Rădulescu [23].

3 Preliminary results

We start with a basic property of the energy functional $\mathfrak{J}_\lambda$.

Lemma 1. *The functional $\mathfrak{J}_\lambda$ is coercive.*

Proof. We first observe that using hypotheses (a1) and (a2) we have for all $u \in E$

$$A(|\nabla u|^p) \geq \frac{1}{\alpha} a(|\nabla u|^p)|\nabla u|^p \geq \frac{c_0|\nabla u|^p + c_1|\nabla u|^q}{\alpha}$$

and

$$A(|u|^p) \geq \frac{1}{\alpha} a(|u|^p)|u|^p \geq \frac{c_0|u|^p + c_1|u|^q}{\alpha}.$$

Therefore

$$\frac{1}{p}\int_{\mathbb{R}^N} A(|\nabla u|^p)dx + \frac{1}{p}\int_{\mathbb{R}^N} A(|u|^p)dx \geq \frac{c_0}{\alpha}\|u\|^p_{W^{1,p}} + \frac{c_1}{\alpha}\|u\|^q_{W^{1,q}}. \tag{3.8}$$

Next, for fixed $a, b \in \mathbb{R}$ and $0 < c < d$, we consider the mapping

$$[0, +\infty) \ni t \mapsto \varphi(t) := at^c - bt^d.$$

By straightforward computation we deduce that

$$\varphi(t) \leq C_1 a \left(\frac{a}{b}\right)^{c/(c-d)} \quad \text{for all } t \geq 0,$$

where C_1 depends only on c, d.

Applying this inequality for $a = \lambda/r$, $b = m(x)/s$, $c = r$, $d = s$, we find

$$\frac{\lambda}{r}|u(x)|^r - \frac{1}{s}m(x)|u(x)|^s \leq C_2\lambda^{s/(s-r)}(m(x))^{r/(r-s)} \quad \text{for all } x \in \Omega.$$

By integration and using the normalized assumption (2.6) we obtain

$$\frac{\lambda}{r}\,\|u\|_r^r - \frac{1}{s}\,\|u\|_{m,s}^s \le C_3(\lambda). \tag{3.9}$$

Combining relations (3.8) and (3.9), we conclude that $\mathcal{J}_\lambda$ is coercive. □

Next, with the same arguments as in Barile and Figueiredo [5, p. 460] and the proof of Lemma 2 in Pucci and Rădulescu [23], we can establish that the energy functional $\mathcal{J}_\lambda : E \to \mathbb{R}$ is weakly lower semicontinuous.

3.1 Case of low perturbations

In this subsection we prove that solutions of problem (2.4) cannot exist if the positive parameter λ is small enough. This corresponds to the case of "low perturbations".

Assume that $u \in E$ is an eigenfunction of problem (2.4) corresponding to the eigenvalue $\lambda > 0$. Choosing $v = u$ in relation (2.7) we obtain

$$\int_{\mathbb{R}^N} a(|\nabla u|^p)|\nabla u|^p\,dx + \int_{\mathbb{R}^N} a(|u|^p)|u|^p\,dx + \int_{\mathbb{R}^N} m(x)|u|^s\,dx = \lambda \int_{\mathbb{R}^N} |u|^r\,dx,$$

hence

$$\int_{\mathbb{R}^N} a(|\nabla u|^p)|\nabla u|^p\,dx + \int_{\mathbb{R}^N} a(|u|^p)|u|^p\,dx + \|u\|_{m,s}^s = \lambda\,\|u\|_r^r. \tag{3.10}$$

Next, by the Young inequality and using hypothesis (2.5),

$$\lambda|u|^r \le \frac{r}{s}\left(m(x)^{r/s}|u|^r\right)^{s/r} + \frac{s-r}{s}\left(\frac{\lambda}{m(x)^{r/s}}\right)^{s/(s-r)}.$$

Integrating this inequality over Ω we obtain

$$\lambda\,\|u\|_r^r \le \frac{r}{s}\,\|u\|_{m,s}^s + \frac{s-r}{s}\,\lambda^{s/(s-r)} \int_{\mathbb{R}^N} m(x)^{r/(r-s)}\,dx.$$

Thus, by the normalized growth condition (2.6),

$$\lambda\,\|u\|_r^r \le \frac{r}{s}\,\|u\|_{m,s}^s + \frac{s-r}{s}\,\lambda^{s/(s-r)}. \tag{3.11}$$

Combining relations (3.10) and (3.11) we find

$$\int_{\mathbb{R}^N} a(|\nabla u|^p)|\nabla u|^p\,dx + \int_{\mathbb{R}^N} a(|u|^p)|u|^p\,dx \le \frac{r-s}{s}\,\|u\|_{m,s}^s + \frac{s-r}{s}\,\lambda^{s/(s-r)}.$$

Using now hypothesis (2.5) we obtain

$$\int_{\mathbb{R}^N} a(|\nabla u|^p)|\nabla u|^p\,dx + \int_{\mathbb{R}^N} a(|u|^p)|u|^p\,dx \le \frac{s-r}{s}\,\lambda^{s/(s-r)}. \tag{3.12}$$

Next, by (a1),

$$c_0 t^p + c_1 t^q \le a(t^p)t^p \quad \text{for all } t \ge 0.$$

Applying this inequality in relation (3.12) it follows that

$$c_0 \int_{\mathbb{R}^N} |\nabla u|^p\,dx + c_1 \int_{\mathbb{R}^N} |\nabla u|^q\,dx + c_0 \int_{\mathbb{R}^N} |u|^p\,dx + c_1 \int_{\mathbb{R}^N} |u|^q\,dx \le \frac{s-r}{s}\,\lambda^{s/(s-r)}.$$

Therefore

$$c_0\,\|u\|_{W^{1,p}}^p + c_1\,\|u\|_{W^{1,q}}^q \le \frac{s-r}{s}\,\lambda^{s/(s-r)}.$$

By hypothesis (2.5) we have $p < r < p^\star$ and $q < r < p^\star < q^\star$. Thus, by the Sobolev embedding theorem, the spaces $W^{1,p}(\mathbb{R}^N)$ and $W^{1,q}(\mathbb{R}^N)$ are continuously embedded into $L^r(\mathbb{R}^N)$. It follows that there exists a positive constant C_0 such that

$$\|u\|_r^p \le C_0 \|u\|_{W^{1,p}}^p \quad \text{for all } u \in W^{1,p}(\mathbb{R}^N) \tag{3.13}$$

and

$$\|u\|_r^q \le C_0 \|u\|_{W^{1,q}}^q \quad \text{for all } u \in W^{1,q}(\mathbb{R}^N). \tag{3.14}$$

Assuming that u is a solution of problem (2.4), relation (3.10) yields

$$\int_{\mathbb{R}^N} a(|\nabla u|^p)|\nabla u|^p dx + \int_{\mathbb{R}^N} a(|u|^p)|u|^p dx \le \lambda \|u\|_r^r. \tag{3.15}$$

Using now hypothesis (a1) and relation (3.15), we obtain

$$c_0\|u\|_{W^{1,p}}^p + c_1\|u\|_{W^{1,q}}^q \le \lambda \|u\|_r^r.$$

Thus, by (3.13) and (3.14), there exists a positive constant C_4 not depending on the solution u such that

$$\begin{aligned}\|u\|_r^p + \|u\|_r^q &\le C_4 \left(\int_{\mathbb{R}^N} a(|\nabla u|^p)|\nabla u|^p dx + \int_{\mathbb{R}^N} a(|u|^p)|u|^p dx\right)\\ &\le C_4\lambda\|u\|_r^r.\end{aligned}$$

This relation implies that

$$\|u\|_r \ge \max\{(C_4\lambda)^{1/(p-r)}, (C_4\lambda)^{1/(q-r)}\}.$$

Using this estimate in conjunction with hypothesis (a1) and relations (3.12), (3.13) and (3.14), we obtain that there exists $\Lambda > 0$ such that $\lambda > \Lambda$. In particular, solutions do not exist if $\lambda \le \Lambda$.

We set

$$\lambda^\star := \sup\{\Lambda > 0 \,:\, \text{problem (2.4) does not have a solution}\}. \tag{3.16}$$

The definition of $\lambda^\star$ implies that problem (2.4) does not have any solution for all $\lambda \in (0, \lambda^\star)$.

4 Proof of Theorem 1

We first prove the existence of two solutions, provided that $\lambda > 0$ is large enough. The first solution is obtained by the direct method of the calculus of variations and the corresponding energy is negative. The second solution of problem (2.4) is obtained by applying the mountain pass theorem without the Palais-Smale condition. The energy of this solution is positive.

Since $\mathcal{J}_\lambda$ is coercive and lower semicontinuous, then it has a global minimizer $u_0 \in E$, see Lemmas 1, 2 and Theorem 1.2 in Struwe [27]. It follows that u_0 is a critical point of $\mathcal{J}_\lambda$. In order to show that u_0 is a solution of problem (2.4) it remains to prove that $u_0 \neq 0$ and u_0 is nonnegative, provided that λ is sufficiently large.

We first establish that the solution u_0 is nontrivial. For this purpose we show that $\mathcal{J}_\lambda(u_0) < 0$.

Consider the following constrained minimization problem

$$m_0 := \inf_{u\in E}\left\{\frac{r}{p}\int_{\mathbb{R}^N}[A(|\nabla u|^p)dx + A(|u|^p)]dx + \frac{r}{s}\int_{\mathbb{R}^N} m(x)|u|^s dx;\ \int_{\mathbb{R}^N}|u|^r dx = 1\right\}.$$

We observe that $m_0 > 0$. Indeed, by Hölder's inequality and hypothesis (2.6), for all $u \in E$ with $\|u\|_r = 1$

$$\begin{aligned}1 = \int_{\mathbb{R}^N}|u|^r dx &\le \left(\int_{\mathbb{R}^N} m(x)^{r/(r-s)}\right)^{(s-r)/r}\left(\int_{\mathbb{R}^N} m(x)|u|^s dx\right)^{r/s}\\ &= \left(\int_{\mathbb{R}^N} m(x)|u|^s dx\right)^{r/s},\end{aligned}$$

hence $m_0 \geq r/s > 0$.

Fix $\lambda > m_0$. Then there exists $v \in E$ with $\int_{\mathbb{R}^N} |v|^r dx = 1$ such that

$$\lambda > \frac{r}{p} \int_{\mathbb{R}^N} [A(|\nabla v|^p) + A(|v|^p)]dx + \frac{r}{s} \int_{\mathbb{R}^N} m(x)|v|^s dx.$$

Therefore

$$\begin{aligned} \mathcal{J}_\lambda(v) &= \frac{1}{p} \int_{\mathbb{R}^N} A(|\nabla v|^p)dx + \frac{1}{p} \int_{\mathbb{R}^N} A(|v|^p)dx - \frac{\lambda}{r} \int_{\mathbb{R}^N} |v|^r dx \\ &+ \frac{1}{s} \int_{\mathbb{R}^N} m(x)|v|^s dx < 0. \end{aligned}$$

This shows that $\inf_{u\in E} \mathcal{J}_\lambda(u) < 0$ for λ large enough, say for $\lambda > \lambda_\star$. In this case, problem (2.4) admits a nontrivial solution u_1, which is a global minimizer of $\mathcal{J}_\lambda$. Moreover, since $\mathcal{J}_\lambda(|u_0|) \leq \mathcal{J}_\lambda(u_0)$, we can also assume that $u_0 \geq 0$ in Ω.

Our next purpose is to establish the existence of a second solution $u_1 \geq 0$ of problem (2.4) for all $\lambda > \lambda_\star$. This will be done by using the mountain pass theorem without the Palais-Smale condition of Brezis and Nirenberg [10].

Fix $\lambda > \lambda_\star$. Since we are looking for nonnegative solutions, it is natural to consider the truncation

$$h(x,t) = \begin{cases} 0 & \text{if } t < 0 \\ \lambda t^{r-1} - m(x)t^{s-1} & \text{if } 0 \leq t \leq u_0(x) \\ \lambda u_0^{r-1} - m(x)u_0^{s-1} & \text{if } t > u_0(x). \end{cases}$$

Set $H(x,t) = \int_0^t h(x,s)ds$ and consider the C^1-functional $\mathcal{H} : E \to \mathbb{R}$ defined by

$$\mathcal{H}(u) = \frac{1}{p} \int_{\mathbb{R}^N} A(|\nabla u|^p)dx + \frac{1}{p} \int_{\mathbb{R}^N} A(|u|^p)dx - \int_{\mathbb{R}^N} H(x,u)dx.$$

A simple argument shows that $\mathcal{H}$ is coercive.

The following auxiliary result establishes an interesting location property of the critical points of $\mathcal{H}$ with respect to the solution u_0.

Lemma 2. *If u is an arbitrary critical point of $\mathcal{H}$, then $u \leq u_0$.*

Proof. Fix $u \in E$ an arbitrary critical point of $\mathcal{H}$, hence $\mathcal{H}'(u) = 0$. Since u_0 solves problem (2.4), then $\mathcal{J}_\lambda'(u_0) = 0$. We have

$$\begin{aligned} 0 &= \langle \mathcal{H}'(u) - \mathcal{J}_\lambda'(u_0), (u-u_0)^+ \rangle \\ &= \int_{\mathbb{R}^N} (a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u - a(|\nabla u_0|^p)|\nabla u_0|^{p-2}\nabla u_0)\nabla(u-u_0)^+ dx \\ &+ \int_{\mathbb{R}^N} (a(|u|^p)|u|^{p-2}u - a(|u_0|^p)|u_0|^{p-2}u_0)(u-u_0)^+ dx \\ &- \int_{\mathbb{R}^N} (h(x,u) - \lambda u_0^{r-1} + m(x)u_0^{s-1})(u-u_0)^+ dx. \end{aligned}$$

Taking into account the definition of h, the last integral in the above expression vanishes. It follows that

$$\begin{aligned}
0 = &\langle \mathcal{H}'(u) - \mathcal{J}'_\lambda(u_0), (u-u_0)^+\rangle = \\
&\int\limits_{[u>u_0]} (a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u - a(|\nabla u_0|^p)|\nabla u_0|^{p-2}\nabla u_0)\nabla(u-u_0)^+ dx + \\
&\int\limits_{[u>u_0]} (a(|u|^p)|u|^{p-2}u - a(|u_0|^p)|u_0|^{p-2}u_0)(u-u_0)^+ dx = \\
&\int\limits_{[u>u_0]} (a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u - a(|\nabla u_0|^p)|\nabla u_0|^{p-2}\nabla u_0)\nabla(u-u_0) dx + \\
&\int\limits_{[u>u_0]} (a(|u|^p)|u|^{p-2}u - a(|u_0|^p)|u_0|^{p-2}u_0)(u-u_0) dx.
\end{aligned} \tag{4.17}$$

Since $a(t) \geq c_0$ for all $t \geq 0$ (by hypothesis (a1)) and using the monotonicity assumption (a3), we deduce that there exists $C_5 > 0$ such that for all $x, y \in \mathbb{R}^n$ and all $n \geq 1$

$$(a(|x|^p)|x|^{p-2}x - a(|y|^p)|y|^{p-2}y)\cdot(x-y) \geq C_5|x-y|^p.$$

Combining this inequality with relation (4.17) we obtain

$$\begin{aligned}
0 &= \langle \mathcal{H}'(u) - \mathcal{J}'_\lambda(u_0), (u-u_0)^+\rangle \\
&\geq C_5 \int\limits_{[u>u_0]} (|\nabla(u-u_0)|^p + |u-u_0|^p) dx \\
&\geq 0.
\end{aligned}$$

We conclude that $u \leq u_0$. □

We prove in what follows that $\mathcal{H}$ satisfies the geometric hypotheses of the mountain pass theorem. The existence of a “valley" is guaranteed by the fact that $\mathcal{H}(u_0) = \mathcal{J}_\lambda(u_0) < 0$. The following result establishes the existence of a “mountain" between the origin and u_0.

Lemma 3. *There exist positive numbers r and a with $r < \|u_0\|$ such that $\mathcal{H}(u) \geq a$ for all $u \in E$ satisfying $\|u\| = r$.*

Proof. We have

$$\begin{aligned}
\mathcal{H}(u) &= \frac{1}{p}\int\limits_{\mathbb{R}^N} A(|\nabla u|^p)dx + \frac{1}{p}\int\limits_{\mathbb{R}^N} A(|u|^p)dx \\
&- \int\limits_{[u>u_0]} H(x,u)dx - \int\limits_{[0\leq u\leq u_0]} H(x,u)dx \\
&= \frac{1}{p}\int\limits_{\mathbb{R}^N} A(|\nabla u|^p)dx + \frac{1}{p}\int\limits_{\mathbb{R}^N} A(|u|^p)dx \\
&- \frac{\lambda}{r}\int\limits_{[u>u_0]} u_0^r dx - \frac{\lambda}{r}\int\limits_{[0\leq u\leq u_0]} u^r dx \\
&\geq \frac{1}{p}\int\limits_{\mathbb{R}^N} A(|\nabla u|^p)dx + \frac{1}{p}\int\limits_{\mathbb{R}^N} A(|u|^p)dx - \frac{\lambda}{r}\int\limits_{\mathbb{R}^N} u_0^r dx.
\end{aligned} \tag{4.18}$$

On the one hand, by hypotheses (a1) and (a2) we find

$$A(|\nabla u|^p) \geq \frac{1}{\alpha}\, a(|\nabla u|^p)|\nabla u|^p \geq \frac{1}{\alpha}\,(c_0|\nabla u|^p + c_1|\nabla u|^q). \tag{4.19}$$

and

$$A(|u|^p) \geq \frac{1}{\alpha}\, a(|u|^p)|u|^p \geq \frac{1}{\alpha}\,(c_0|u|^p + c_1|u|^q). \tag{4.20}$$

On the other hand, by the Sobolev embedding theorem, there exists $C_6 > 0$ such that

$$\|u\|_r \le C_6 \|u\|_{W^{1,p}} \quad \text{for all } u \in E. \tag{4.21}$$

Combining relations (4.18), (4.19), (4.20), (4.21) and hypothesis (2.5), we deduce that there exist $r < \|u_0\|$ small enough and $a > 0$ such that $\mathcal{H}(u) \ge a$ for all $u \in E$ satisfying $\|u\| = r$. □

Set

$$\mathcal{P} = \{p \in C([0,1], E);\ p(0) = 0 \text{ and } p(1) = u_0\}$$

and

$$c = \inf_{p\in\mathcal{P}} \max_{t\in[0,1]} \mathcal{H}(p(t)) > 0.$$

Applying the mountain pass theorem without the Palais-Smale condition, there exists a sequence (z_n) in E such that

$$\mathcal{H}(z_n) \to c \quad \text{as } n \to \infty \tag{4.22}$$

and

$$\|\mathcal{H}'(z_n)\|_{E^*} \to 0 \quad \text{as } n \to \infty. \tag{4.23}$$

By relation (4.22) and since $\mathcal{H}$ is coercive, we deduce that (z_n) is bounded. So, there exists $u_1 \in E$ such that, up to a subsequence,

$$z_n \rightharpoonup u_1 \quad \text{in } E.$$

Using this information in conjunction with (4.23) we deduce that

$$\mathcal{H}'(z_n)(\varphi) \to \mathcal{H}'(u_1)(\varphi) \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^N).$$

Since $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ and E is continuously embedded into $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$, we deduce that

$$\mathcal{H}'(z_n)(v) \to \mathcal{H}'(u_1)(v) \quad \text{for all } v \in E.$$

By (4.23) we obtain that $\mathcal{H}'(u_1) = 0$, hence u_1 is a solution of problem (2.4). We conclude that problem (2.4) admits at least two solutions for all $\lambda > \lambda_\star$.

Set

$$\lambda^{**} := \inf\{\lambda > 0;\ \text{problem (2.4) has a solution}\}.$$

Recall that

$$\lambda^* := \sup\{\lambda > 0;\ \text{problem (2.4) does not have any solution}\}.$$

Then $0 < \lambda^* \le \lambda^{**} < \infty$.

To complete the proof, we need to argue the following assertions:
(i) problem (2.4) has at least two solutions for all $\lambda > \lambda^{**}$;
(ii) $\lambda^* = \lambda^{**}$ and problem (2.4) has a solution for $\lambda = \lambda^*$.

Assertion (i) follows by standard arguments based on the monotonicity hypothesis (a3). Assertion (ii) is a consequence of the fact that problem (2.4) does not have any solution provided that $\lambda < \lambda^{**}$. In both cases we refer for details to the proof of Theorem 1.1 in [13].

If we replace hypothesis (2.5) with

$$1 < p \le q < s < r < p^*,$$

then the associated energy functional is no longer coercive but has a mountain pass geometry for all $\lambda > 0$. A straightforward argument shows the following properties:
(a) problem (2.4) does not have any solution if $\lambda \le 0$;
(b) problem (2.4) has at least one solution for all $\lambda > 0$.

In this case we can apply the mountain pass theorem of Ambrosetti and Rabinowitz. The mountain pass geometry of the problem is generated by the assumption $1 < p \leq q < s < r < p^*$. We also point out that since $s < r$, then any Palais-Smale sequence is bounded in E. The details of the proof are left to the reader.

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References

[1] S. Alama and G. Tarantello, Elliptic problems with nonlinearities indefinite in sign, *J. Funct. Anal.* **141** (1996), 159-215.

[2] A. Ambrosetti, H. Brezis and G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.* **122** (1994), 519-543.

[3] A. Ambrosetti and P.H. Rabinowitz, Dual variational methods in critical point theory, *J. Funct. Anal.* **14** (1973), 349-381.

[4] A. Bahrouni, V.D. Rădulescu and D.D. Repovš, Double phase transonic flow problems with variable growth: nonlinear patterns and stationary waves, *Nonlinearity* **32** (2019), 2481-2495.

[5] S. Barile and G. Figueiredo, Some classes of eigenvalues problems for generalized p & q-Laplacian type operators on bounded domains, *Nonlinear Analysis* **119** (2015), 457-468.

[6] P. Baroni, M. Colombo and G. Mingione, Harnack inequalities for double phase functionals, *Nonlinear Analysis* **121** (2015), 206–222.

[7] P. Baroni, M. Colombo and G. Mingione, Regularity for general functionals with double phase, *Calc. Var. Partial Differential Equations* **57** (2018), no. 2, Art. 62, 48 pp.

[8] T. Bartsch and M. Willem, On an elliptic equation with concave and convex nonlinearities, *Proc. Amer. Math. Soc.* **123** (1995), 3555-3561.

[9] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2010.

[10] H. Brezis and L. Nirenberg, Remarks on finding critical points, *Comm. Pure Appl. Math.* **44** (1991), no. 8-9, 939-963.

[11] M. Cencelj, V.D. Rădulescu and D.D. Repovš, Double phase problems with variable growth, *Nonlinear Analysis* **177** (2018), 270-287.

[12] M. Colombo and G. Mingione, Regularity for double phase variational problems, *Arch. Ration. Mech. Anal.* **215** (2015), 443-496.

[13] R. Filippucci, P. Pucci and V.D. Rădulescu, Existence and non–existence results for quasilinear elliptic exterior problems with nonlinear boundary conditions, *Comm. in Partial Differential Equations* **33** (2008), 706-717.

[14] S. Fučik, J. Nečas, J. Souček and V. Souček, *Spectral Analysis of Nonlinear Operators*, Lecture Notes in Mathematics, Vol. 346, Springer-Verlag, Berlin-New York, 1973.

[15] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[16] P. Marcellini, On the definition and the lower semicontinuity of certain quasiconvex integrals, *Ann. Inst. H. Poincaré, Anal. Non Linéaire* **3** (1986), 391-409.

[17] P. Marcellini, Regularity and existence of solutions of elliptic equations with p, q–growth conditions, *J. Differential Equations* **90** (1991), 1-30.

[18] P. Marcellini, Everywhere regularity for a class of elliptic systems without growth conditions, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **(4) 23** (1996), no. 1, 1-25.

[19] N.S. Papageorgiou, V.D. Rădulescu and D.D. Repovš, Double-phase problems with reaction of arbitrary growth, *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)* **69** (2018), 69:108.

[20] N.S. Papageorgiou, V.D. Rădulescu and D.D. Repovš, Double-phase problems and a discontinuity property of the spectrum, *Proceedings Amer. Math. Soc.* **147** (2019), 2899-2910.

[21] N.S. Papageorgiou, V.D. Rădulescu and D.D. Repovš, *Nonlinear Analysis – Theory and Methods*, Springer Monographs in Mathematics, Springer, Cham, 2019.

[22] P. Pucci and V.D. Rădulescu, The impact of the mountain pass theory in nonlinear analysis: a mathematical survey, *Boll. Un. Mat. Ital.* B, Ser. IX, **III** (2010), 543-582.

[23] P. Pucci and V.D. Rădulescu, Combined effects in quasilinear elliptic problems with lack of compactness, *Rend. Lincei Mat. Appl.* **22** (2011), 189-205.

[24] V.D. Rădulescu, Isotropic and anisotropic double-phase problems: old and new, *Opuscula Mathematica* **39** (2019), 259-279.

[25] V.D. Rădulescu and D.D. Repovš, Combined effects in nonlinear problems arising in the study of anisotropic continuous media, *Nonlinear Analysis* **75** (2012), no. 3, 1524-1530.

[26] X. Shi, V.D. Rădulescu, D.D. Repovš and Q. Zhang, Multiple solutions of double phase variational problems with variable exponent, *Advances in Calculus of Variations*, in press (doi:10.1515/acv-2018-0003).
[27] M. Struwe, *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Ergebnisse der Mathematik und ihrer Grenzgebiete, A Series of Modern Surveys in Mathematics, Vol. 34, Springer-Verlag, Berlin, xx+302 pp., 2008.
[28] Q. Zhang and V.D. Rădulescu, Double phase anisotropic variational problems and combined effects of reaction and absorption terms, *J. Math. Pures Appl.* **118C** (2018), 159-203.
[29] V.V. Zhikov, Averaging of functionals of the calculus of variations and elasticity, *Math. USSR-Izves.* **29** (1987), 33-66.
[30] V.V. Zhikov, On variational problems and nonlinear elliptic equations with nonstandard growth conditions, *J. Math. Sci.* **173** (2011), 463-570.