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On the existence and multiplicity of solutions to fractional Lane-Emden elliptic systems involving measures

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Abstract: We study positive solutions to the fractional Lane-Emden system

$$\begin{cases} (-\Delta)^s u = v^p + \mu & \text{in } \Omega \\ (-\Delta)^s v = u^q + \nu & \text{in } \Omega \\ u = v = 0 & \text{in } \Omega^c = \mathbb{R}^N \setminus \Omega, \end{cases} \tag{S}$$

where Ω is a C^2 bounded domains in $\mathbb{R}^N$, $s \in (0,1)$, $N > 2s$, $p > 0$, $q > 0$ and $\mu,\ \nu$ are positive measures in Ω. We prove the existence of the minimal positive solution of (S) under a smallness condition on the total mass of μ and ν. Furthermore, if $p, q \in (1, \frac{N+s}{N-s})$ and $0 \le \mu,\ \nu \in L^r(\Omega)$, for some $r > \frac{N}{2s}$, we show the existence of at least two positive solutions of (S). The novelty lies at the construction of the second solution, which is based on a highly nontrivial adaptation of Linking theorem. We also discuss the regularity of the solutions.

Keywords: nonlocal, system, existence, multiplicity, linking theorem, measure data, source terms, positive solution

MSC: Primary 35R11, 35J57, 35J50, 35B09, 35R06

1 Introduction and main results

In this article we consider elliptic system of the type

$$\begin{cases} (-\Delta)^s u = v^p + \mu & \text{in } \Omega \\ (-\Delta)^s v = u^q + \nu & \text{in } \Omega \\ u,\ v \ge 0 & \text{in } \Omega \\ u = v = 0 & \text{in } \Omega^c = \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where Ω is a C^2 bounded domain in $\mathbb{R}^N$, $s \in (0,1)$, $N > 2s$, $p > 0$, $q > 0$ and $\mu,\ \nu$ are positive Radon measures in Ω. Here $(-\Delta)^s$ denotes the fractional Laplace operator defined as follows

$$(-\Delta)^s u(x) = \lim_{\varepsilon \to 0} (-\Delta)^s_\varepsilon u(x),$$

where

$$(-\Delta)^s_\varepsilon u(x) := a_{N,s} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy, \tag{1.2}$$

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and $a_{N,s} = \frac{2^{2s} s\Gamma(N/2+s)}{\pi^{N/2}\Gamma(1-s)}$.

When $s = 1$, $(-\Delta)^s$ coincides the classical laplacian $-\Delta$ and the Lane-Emden system

$$\begin{cases} -\Delta u = v^p + \mu & \text{in } \Omega, \\ -\Delta v = u^q + \nu & \text{in } \Omega, \end{cases} \tag{1.3}$$

has been studied extensively in the literature (see [4, 13, 17, 20, 21, 23, 28, 37, 38] and the references therein). Bidaut-Véron and Yarur [4] provided various necessary and sufficient conditions in terms of estimates on the Green kernel for the existence of solutions of (1.3). When $\mu = \nu = 0$, the structure of solution of (1.3) has been better understood according to the relation between p, q and N. More precisely, if $\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{N-2}{N}$ then (1.3) admits some positive (radial, bounded) classical solutions in $\mathbb{R}^N$ (see Serrin and Zou [38]). On the other hand, the so-called Lane-Emden conjecture states that if

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N} \tag{1.4}$$

then there is no nontrivial classical solution of (1.3) in $\mathbb{R}^N$. The conjecture is known to be true for radial solutions in all dimensions (see Mitidieri [23]). In the nonradial case, partial results have been achieved. Nonexistence was proved by Figueiredo and Felmer in [17] and by Reichel and Zou in [37] for (p, q) in certain subregions of (1.4). Recently, Bidaut-Véron, Nguyen and Véron [5] provided necessary and sufficient conditions for the existence expressed in terms of Riesz or Bessel capacities for quasilinear system of equations.

For nonlocal case, i.e. $s \in (0, 1)$, Quaas and Xia [31] showed the existence of at least one positive viscosity solution for the system of the type

$$\begin{cases} (-\Delta)^s u = v^p & \text{in } \Omega, \\ (-\Delta)^s v = u^q & \text{in } \Omega, \\ u = v = 0 & \text{in } \Omega^c. \end{cases} \tag{1.5}$$

It has been proved by Quaas and Xia [29] that under some conditions on the exponents p and q, system (1.5) does not admit any positive bounded viscosity solution. We also refer [14, 30] for further results in this directions.

Nonlocal equations with measure data have been investigated in [2, 9, 10, 12, 27] and the references therein. More precisely, fractional elliptic equations with interior measure data were studied by Chen and Véron [12] and by Chen and Quaas [10], while the equations with measure boundary data were carried out by Nguyen and Véron [27] (for absorption nonlinearity) and by Bhakta and Nguyen [2] (for source nonlinearity).

1.1 Main results

Before stating the main results, we introduce necessary notations.

For $\phi \geq 0$, denote by $\mathfrak{M}(\Omega, \phi)$ the space of Radon measures τ on Ω satisfying $\int_\Omega \phi\, d|\tau| < \infty$ and by $\mathfrak{M}^+(\Omega, \phi)$ the positive cone of $\mathfrak{M}(\Omega, \phi)$. For $\kappa > 0$, denote by $L^\kappa(\Omega, \phi)$ the space of measurable functions w such that $\int_\Omega |w|^\kappa \phi dx < \infty$. We denote $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. When $\phi = \delta^s$, we can define the space $\mathfrak{M}(\Omega, \delta^s)$ and $L^\kappa(\Omega, \delta^s)$. Let $G_s = G_s^\Omega$ be the Green kernel of $(-\Delta)^s$ in Ω. We denote the associated Green operator $\mathbb{G}_s$ as follows:

$$\mathbb{G}_s[\tau](x) := \int_\Omega G_s(x, y) d\tau(y), \quad \tau \in \mathfrak{M}(\Omega, \delta^s).$$

Important estimates concerning the Green kernel are presented Section 2.

Definition 1.1. (Weak solution) Let $\mu,\ \nu \in \mathfrak{M}^+(\Omega, \delta^s)$. We say that (u, v) is a weak solution of (1.1) if $u,\ v \in L^1(\Omega)$, $v^p,\ u^q \in L^1(\Omega, \delta^s)$ and

$$\begin{cases} \int\limits_\Omega u(-\Delta)^s \xi dx = \int\limits_\Omega v^p \xi dx + \int\limits_\Omega \xi\, d\mu, \\ \int\limits_\Omega v(-\Delta)^s \xi dx = \int\limits_\Omega u^q \xi dx + \int\limits_\Omega \xi\, d\nu, \end{cases} \qquad \forall \xi \in \mathbb{X}_s(\Omega), \tag{1.6}$$

where $\mathbb{X}_s(\Omega) \subset C(\mathbb{R}^N)$ denotes the space of test functions ξ satisfying

(i) $\operatorname{supp}(\xi) \subset \bar{\Omega}$,

(ii) $(-\Delta)^s \xi(x)$ exists for all $x \in \Omega$ and $|(-\Delta)^s \xi(x)| \leq C$ for some $C > 0$,

(iii) there exists $\varphi \in L^1(\Omega, \delta^s)$ and $\varepsilon_0 > 0$ such that $|(-\Delta)^s_\varepsilon \xi| \leq \varphi$ a.e. in Ω, for all $\varepsilon \in (0, \varepsilon_0]$.

Remark 1.2. We observe that, by [27, Proposition A], (u, v) is a weak solution of (1.1) if and only if

$$u = \mathbb{G}_s[v^p] + \mathbb{G}_s[\mu] \quad \text{and} \quad v = \mathbb{G}_s[u^q] + \mathbb{G}_s[\nu]. \tag{1.7}$$

Define

$$N_s := \frac{N + s}{N - s}. \tag{1.8}$$

Our first result is the existence of the minimal weak solutions of (1.1).

Theorem 1.3. (Minimal solution) *Let $p, q > 0$ with $p \leq q$, $pq \neq 1$ and $q\frac{p+1}{q+1} < N_s$. Assume $\mu, \nu \in \mathfrak{M}^+(\Omega, \delta^s)$ and $\mathbb{G}_s[\mu] \in L^q(\Omega, \delta^s)$. Then system* (1.1) *admits a positive weak solution $(\underline{u}_\mu, \underline{v}_\nu)$ for $\|\mu\|_{\mathfrak{M}(\Omega,\delta^s)}$ and $\|\nu\|_{\mathfrak{M}(\Omega,\delta^s)}$ small if $pq > 1$ and for any $\mu, \nu \in \mathfrak{M}^+(\Omega, \delta^s)$ if $pq < 1$. This solution satisfies*

$$\underline{u}_\mu \geq \mathbb{G}_s[\mu], \quad \underline{v}_\nu \geq \mathbb{G}_s[\nu] \quad \text{a.e. in } \Omega.$$

Moreover, it is the minimal positive weak solution of (1.1) *in the sense that if (u, v) is a positive weak solution of* (1.1) *then $\underline{u}_\mu \leq u$ and $\underline{v}_\nu \leq v$ a.e. in Ω.*

In addition, if $q < N_s$ then there exists a positive constant $K = K(N, s, p, q, \|\mu\|_{\mathfrak{M}(\Omega,\delta^s)}, \|\nu\|_{\mathfrak{M}(\Omega,\delta^s)})$ such that $K \to 0$ as $(\|\mu\|_{\mathfrak{M}(\Omega,\delta^s)}, \|\nu\|_{\mathfrak{M}(\Omega,\delta^s)}) \to (0, 0)$ and

$$\max\{\underline{u}_\mu, \underline{v}_\nu\} \leq K\mathbb{G}_s[\tilde{\mu} + \tilde{\nu}] \quad \text{a.e. in } \Omega, \tag{1.9}$$

where $\tilde{\mu} = \frac{\mu}{\|\mu\|_{\mathfrak{M}(\Omega,\delta^s)}}$ and $\tilde{\nu} = \frac{\nu}{\|\nu\|_{\mathfrak{M}(\Omega,\delta^s)}}$.

The existence of the second solution is stated in the following theorem.

Theorem 1.4. (Second solution) *Assume $0 \leq \mu,\ \nu \in L^r(\Omega)$ for some $r > \frac{N}{2s}$ and $1 < p \leq q < N_s$, where N_s is defined in* (1.8). *There exists $t^* > 0$ such that if*

$$\max\{\|\mu\|_{L^r(\Omega)}, \|\nu\|_{L^r(\Omega)}\} < t^*$$

then system (1.1) *admits at least two positive weak solutions (u_μ, v_ν) and $(\underline{u}_\mu, \underline{v}_\nu)$ with $u_\mu \gneqq \underline{u}_\mu$ and $v_\nu \gneqq \underline{v}_\nu$, where $(\underline{u}_\mu, \underline{v}_\nu)$ is the minimal solution constructed in Theorem 1.3.*

If, in addition, $\mu, \nu \in L^r(\Omega) \cap L^\infty_{loc}(\Omega)$ then $u_\mu > \underline{u}_\mu$ and $v_\nu > \underline{v}_\nu$ in Ω.

1.2 Methodology and novelty

It is worth mentioning that the existence results in Theorem 1.3 and Theorem 1.4 rely on completely different methods. More precisely, the proof of Theorem 1.3 is in spirit of Bidaut-Véron and Yarur [4], based on a delicate construction of a supersolution. The main ingredient is a series of estimates concerning the Green kernel (see

Lemmas 2.9, 2.10, 2.11 and 2.12). Theorem 1.4 is obtained by using a variational approach. Because of the interplay of the two components u and v, the analysis of the associated energy functional (see (5.9)) becomes complicated and consequently the Mountain-pass theorem is inapplicable. Therefore, we employ the Linking theorem instead. We would like to point out that though Linking theorem is a classical theorem in variational methods (due to Rabinowitz [32]) which had been used previously in many papers, see for e.g. Bonheure, dos Santos and Tavares [7], Figueiredo et al. [18, 19], Lam and Lu [22] in the local cases (also see Ferrero and Saccon [16] where Linking theorem was used for single equation with measure data in the local case) and Servadei and Valdinoci [39, 40], Molica Bisci, Radulescu and Servadei [24], Mosconi, Perera, Squassinaa and Yang [25] in the nonlocal cases, it cannot be applied in a straightforward way in our framework due to the structure of the system (1.1). In order to construct the second solution of (1.1), we reduce (1.1) to a new system which has variational structure. Moreover, we require the data μ and ν to be sufficiently regular, namely $\mu, \nu \in L^r(\Omega)$ for some $r > \frac{N}{2s}$. This enables us to deduce the boundedness of the minimal solution constructed in Theorem 1.3, which in turn allows to establish the geometry of the Linking theorem. As a consequence, we are able to prove the existence of a variational solution which is in fact a weak solution due to a result of Abatangelo [1] and greater than the minimal solution. The idea to construct second solution for nonlocal system of equations is relatively new and different from the path of the papers mentioned above. The proof also involves the minimal solution that was constructed in Theorem 1.3.

Organization of the paper. In Section 2, we collect some known estimates on the Green kernel from different papers and prove important estimates regarding the Green operator (see Lemmas 2.10, 2.11, 2.12). These estimates are the main ingredient in the proof of Theorem 1.3 which is presented in Section 3. In Section 4, we discuss a priori estimates, as well as regularity properties, of weak solutions. Section 5 deals with the proof of Theorem 1.4 which is based on the Linking theorem.

Notations. Throughout the present paper, we denote by $c, c', c_1, c_2, C, \dots$ positive constants that may vary from line to line. If necessary, the dependence of these constants will be made precise.

2 Estimates on Green kernel

We denote by G_s the Green kernel of $(-\Delta)^s$ in Ω respectively. More precisely, for every $y \in \Omega$,

$$\begin{cases} (-\Delta)^s G_s(.,y) = \delta_y & \text{in } \Omega \\ G_s(.,y) = 0 & \text{in } \Omega^c, \end{cases} \tag{2.1}$$

where δ_y is the Dirac mass at y.

Following are some useful estimates on the Green kernel.

Lemma 2.1. ([11, Corollary 1.3]) *There exists a positive constant $c = c(N, s, \Omega)$ such that*

$$\begin{aligned} c^{-1} \min\{|x-y|^{2s-N}, \delta(x)^s\delta(y)^s|x-y|^{-N}\} &\le G_s(x,y) \\ &\le c\min\{|x-y|^{2s-N}, \delta(x)^s\delta(y)^s|x-y|^{-N}\}, \quad \forall x \neq y,\ x,\ y \in \Omega. \end{aligned} \tag{2.2}$$

Lemma 2.2. ([11, Theorem 1.1]) *There exists a positive constant $C = C(N, s, \Omega)$ such that, for every $(x,y) \in \Omega \times \Omega$, $x \neq y$,*

$$G_s(x,y) \le C\delta(y)^s|x-y|^{s-N}, \tag{2.3}$$

$$G_s(x,y) \le C\frac{\delta(y)^s}{\delta(x)^s}|x-y|^{2s-N}. \tag{2.4}$$

Remark 2.3. Let $\theta \in (0,1]$, then there exists a positive constant $c = c(N, s, \theta, \Omega)$ such that $\mathbb{G}_s[\lambda]^\theta \ge c\delta^s$ a.e. in Ω for every measure $\lambda \in \mathfrak{M}^+(\Omega, \delta^s)$ such that $\|\lambda\|_{\mathfrak{M}(\Omega,\delta^s)} = 1$. Indeed, since $|x-y| < d_\Omega := \operatorname{diam}(\Omega)$, applying (2.2) we have

$$G_s(x,y) > c^{-1}\min\{d_\Omega^{2s-N}, d_\Omega^{-N}\delta(x)^s\delta(y)^s\}.$$

Further, $\max\{\delta(x), \delta(y)\} < d_\Omega$ implies $d_\Omega^{-N}\delta(x)^s\delta(y)^s < d_\Omega^{2s-N}$. Consequently, $G_s(x,y) > c^{-1}d_\Omega^{-N}\delta(x)^s\delta(y)^s$. It follows that $\mathbb{G}_s[\lambda] \geq c\delta^s$ a.e. in Ω. Therefore

$$\mathbb{G}_s[\lambda]^\theta \geq c^\theta\delta^{\theta s} \geq c_1\delta^s \quad \text{a.e. in } \Omega.$$

Remark 2.4. Let $\theta \in (0,1]$, then there exists a constant $c = c(N,s,\theta,\Omega)$ such that $\mathbb{G}_s[\lambda]^\theta \geq c\mathbb{G}_s[1]$ in Ω for every measure $\lambda \in \mathfrak{M}^+(\Omega,\delta^s)$ such that $\|\lambda\|_{\mathfrak{M}(\Omega,\delta^s)} = 1$. Indeed, by [12, (2.18)], we have $c_1^{-1}\delta^s \leq \mathbb{G}_s[1] \leq c_1\delta^s$ in Ω for some constant $c_1 = c_1(N,s,\Omega)$. This, combined with Remark 2.3, leads to the desired estimate.

We recall below the definition of Marcinkiewicz space (or weak L^p space). See the paper of Benilan, Brezis and Crandall [6] or the hanbook of Véron [43] for more details.

Definition 2.5. (Marcinkiewicz space) Let $\Omega \subset \mathbb{R}^N$ be a domain and λ be a positive Borel measure in Ω. For $\kappa > 1$, $\kappa' = \frac{\kappa}{1-\kappa}$ and $u \in L^1_{loc}(\Omega,\lambda)$, we set

$$\|u\|_{M^\kappa(\Omega,\lambda)} := \inf\left\{c \in [0,\infty] : \int_E |u|\,d\lambda \leq c\left(\int_E d\lambda\right)^{\frac{1}{\kappa'}}, \quad \forall E \subset \text{Borel set}\right\}$$

and

$$M^\kappa(\Omega,\lambda) := \left\{u \in L^1_{loc}(\Omega,\lambda) : \|u\|_{M^\kappa(\Omega,\lambda)} < \infty\right\}.$$

$M^\kappa(\Omega,\lambda)$ is called the Marcinkiewicz space with exponent κ (or weak L^κ space) with quasi-norm $\|.\|_{M^\kappa(\Omega,\lambda)}$.

The next lemma establishes a relation between Lebesgue space norm and Marcinkiewicz quasi-norm.

Lemma 2.6. ([6, Lemma A.2(ii)]) *Assume* $1 \leq q < \kappa < \infty$ *and* $u \in L^1_{loc}(\Omega,\lambda)$. *Then there exists* $C(q,\kappa) > 0$ *such that for any Borel subset E of* Ω

$$\int_E |u|^q\,d\lambda \leq C(q,\kappa)\|u\|^q_{M^\kappa(\Omega,\lambda)}\left(\int_E d\lambda\right)^{1-\frac{q}{\kappa}}.$$

We set

$$k_{\alpha,y} := \begin{cases} \dfrac{N+\alpha}{N-2s+y} & \text{if } \alpha < \dfrac{Ny}{N-2s} \\ \dfrac{N}{N-2s} & \text{otherwise.} \end{cases} \tag{2.5}$$

Estimates of Green operator are presented below.

Lemma 2.7. ([12, Proposition 2.2]) *Let* $\alpha, y \in [0,s]$ *and* $k_{\alpha,y}$ *be as in* (2.5). *There exists* $c = c(N,s,\alpha,y,\Omega) > 0$ *such that*

$$\|\mathbb{G}_s[\lambda]\|_{M^{k_{\alpha,y}}(\Omega,\delta^\alpha)} \leq c\|\lambda\|_{\mathfrak{M}(\Omega,\delta^y)} \quad \forall \lambda \in \mathfrak{M}(\Omega,\delta^y). \tag{2.6}$$

Lemma 2.8. ([35, Proposition 1.4]) *(i) If* $t > \frac{N}{2s}$ *then there exists* $c = c(N,s,t,\Omega)$ *such that*

$$\|\mathbb{G}_s[\lambda]\|_{C^{\min\{s,2s-\frac{N}{t}\}}(\Omega)} \leq c\|\lambda\|_{L^t(\Omega)} \quad \forall \lambda \in L^t(\Omega). \tag{2.7}$$

(ii) If $1 < t < \frac{N}{2s}$ *then there exists a constant* $c = c(N,s,t)$ *such that*

$$\|\mathbb{G}_s[\lambda]\|_{L^{\frac{Nt}{N-2ts}}(\Omega)} \leq c\|\lambda\|_{L^t(\Omega)} \quad \forall \lambda \in L^t(\Omega). \tag{2.8}$$

The following estimate was prove by Chen and Song [11, Theorem 1.6]).

Lemma 2.9. (3G-estimate) *There exists a positive constant* $C = C(\Omega,s)$ *such that*

$$\frac{G_s(x,y)G_s(y,z)}{G_s(x,z)} \leq C\frac{|x-z|^{N-2s}}{|x-y|^{N-2s}|y-z|^{N-2s}}, \quad \forall (x,y,z) \in \Omega\times\Omega\times\Omega. \tag{2.9}$$

Next, in the spirit of [4], we will prove some important estimates concerning the Green operator which will be used in Section 3.

Lemma 2.10. *Let* $0 < p < N_s$ *and* $\lambda \in \mathfrak{M}^+(\Omega, \delta^s)$ *such that* $\|\lambda\|_{\mathfrak{M}(\Omega,\delta^s)} = 1$. *Then there exists a constant* $C = C(N, s, p, \Omega) > 0$ *such that*

$$\mathbb{G}_s[\mathbb{G}_s[\lambda]^p] \le C\mathbb{G}_s[\lambda] \quad \text{a.e. in } \Omega. \tag{2.10}$$

Proof. First, we consider the case $p > 1$. From Lemma 2.7 and the embedding $M^{N_s}(\Omega, \delta^s) \subset L^p(\Omega, \delta^s)$, we deduce that $\mathbb{G}_s[\lambda] \in L^p(\Omega, \delta^s)$. We write

$$\mathbb{G}_s[\lambda](y) = \int_\Omega G_s(y, z)d\lambda(z) = \int_\Omega \frac{G_s(y, z)}{\delta(z)^s}\delta(z)^s d\lambda(z).$$

Therefore, using Hölder inequality, we obtain

$$\mathbb{G}_s[\lambda](y)^p \le \int_\Omega \left(\frac{G_s(y, z)}{\delta(z)^s}\right)^p \delta(z)^s d\lambda(z),$$

as $\|\lambda\|_{\mathfrak{M}(\Omega,\delta^s)} = 1$. Consequently,

$$\mathbb{G}_s[\mathbb{G}_s[\lambda]^p](x) \le \int_\Omega\int_\Omega G_s(x, y)G_s(y, z)^p\delta(z)^{s(1-p)}d\lambda(z)dy. \tag{2.11}$$

Now applying Lemma 2.9 and (2.3) to the right-hand side of the above expression, we obtain

$$\begin{aligned}
&\int_\Omega\int_\Omega G_s(x, y)G_s(y, z)^p\delta(z)^{s(1-p)}d\lambda(z)dy\\
&= \int_\Omega\int_\Omega G_s(x, y)G_s(y, z)\Big(\frac{G_s(y, z)}{\delta(z)^s}\Big)^{p-1}d\lambda(z)dy\\
&\le C_1\int_\Omega G_s(x, z)\int_\Omega \frac{|x-z|^{N-2s}}{|x-y|^{N-2s}|y-z|^{N-2s}}|y-z|^{-(N-s)(p-1)}dyd\lambda(z)\\
&\le C_2\int_\Omega G_s(x, z)\int_\Omega \big[|y-z|^{2s-N-(N-s)(p-1)} + |x-y|^{-(N-2s)}|y-z|^{-(N-s)(p-1)}\big]dyd\lambda(z)\\
&\le C_3\int_\Omega G_s(x, z)\Big[\int_{\Omega\cap\{|x-y|\ge|y-z|\}} |y-z|^{2s-N-(N-s)(p-1)}dy\\
&\qquad + \int_{\Omega\cap\{|x-y|\le|y-z|\}} |x-y|^{2s-N-(N-s)(p-1)})dy\Big]d\lambda(z)\\
&\le C_4\int_\Omega G_s(x, z)d\lambda(z),
\end{aligned} \tag{2.12}$$

where $C_i = C_i(N, s, p, \Omega)$ $(i = 1, 2, 3, 4)$. Here in the second estimate, we have used the inequality $|x - z| \le |x - y| + |y - z|$ and in the last estimate we have used the fact that $p < N_s$. Hence combining (2.11) and (2.12), we derive that

$$\mathbb{G}_s[\mathbb{G}_s[\lambda]^p](x) \le C\int_\Omega G_s(x, z)d\lambda(z) = C\mathbb{G}_s[\lambda](x)$$

where $C = C(N, s, p, \Omega)$. Note that the above argument is still valid for the case $p = 1$.

Next we consider the case $0 < p < 1$. Then we have

$$\mathbb{G}_s[\lambda]^p \le C(p)(1 + \mathbb{G}_s[\lambda]) \quad \text{a.e. in } \Omega.$$

This yields

$$\mathbb{G}_s[\mathbb{G}_s[\lambda]^p] \le C(p)(\mathbb{G}_s[1] + \mathbb{G}_s[\mathbb{G}_s[\lambda]]) \quad \text{a.e. in } \Omega.$$

By applying the case $p = 1$, we have $\mathbb{G}_s[\mathbb{G}_s[\lambda]] \le C\mathbb{G}_s[\lambda]$ with $C = C(N, p, s, \Omega)$. Therefore, combining the above results along with Remark 2.4 with $\theta = 1$, we derive (2.10). □

Lemma 2.11. *Let $0 < p < N_s$, $\lambda \in \mathfrak{M}^+(\Omega, \delta^s)$ with $\|\lambda\|_{\mathfrak{M}(\Omega,\delta^s)} = 1$. Let θ be such that*

$$\max(0, p - N_s + 1) < \theta \le 1. \tag{2.13}$$

Then there exists a positive constant $C = C(N, s, p, \theta, \Omega)$ such that

$$\mathbb{G}_s[\mathbb{G}_s[\lambda]^p] \le C\mathbb{G}_s[\lambda]^\theta \quad \textit{a.e. in } \Omega. \tag{2.14}$$

Proof. First we assume that $p > 1$. In view of the proof of Lemma 2.10, we have

$$\begin{aligned}\mathbb{G}_s[\mathbb{G}_s[\lambda]^p](x) &\le C \int_\Omega \int_\Omega G_s(x,y) G_s(y,z)^p \delta(z)^{s(1-p)} d\lambda(z) dy \\ &= C \int_\Omega \int_\Omega G_s(x,y)^{1-\theta} G_s(x,y)^\theta G_s(y,z)^\theta \left(\frac{G_s(y,z)}{\delta(z)^s}\right)^{p-\theta} \delta(z)^{s(1-\theta)} d\lambda(z) dy.\end{aligned} \tag{2.15}$$

By (2.9) and the inequality $|x-z| \le |x-y| + |y-z|$, we have

$$G_s(x,y)^\theta G_s(y,z)^\theta \le C G_s(x,z)^\theta (|x-y|^{(2s-N)\theta} + |y-z|^{(2s-N)\theta}). \tag{2.16}$$

Combining (2.15) and (2.16) yields

$$\mathbb{G}_s[\mathbb{G}_s[\lambda]^p](x) \le C \int_\Omega G_s(x,z)^\theta \delta(z)^{s(1-\theta)} \int_\Omega I_{x,z}(y) dy d\lambda(z) \tag{2.17}$$

where

$$I_{x,z}(y) = \int_\Omega G_s(x,y)^{1-\theta} \left(\frac{G_s(y,z)}{\delta(z)^s}\right)^{p-\theta} (|x-y|^{(2s-N)\theta} + |y-z|^{(2s-N)\theta}) dy.$$

Applying (2.2) and (2.3), we obtain

$$\begin{aligned}I_{x,z}(y) &\le C \int_\Omega |x-y|^{(2s-N)(1-\theta)} |y-z|^{(s-N)(p-\theta)} (|x-y|^{(2s-N)\theta} + |y-z|^{(2s-N)\theta}) dy \\ &\le C \int_{\{y\in\Omega : |x-y|\le|y-z|\}} |x-y|^{2s-N+(s-N)(p-\theta)} dy \\ &\quad + C \int_{\{y\in\Omega : |x-y|\ge|y-z|\}} |y-z|^{2s-N+(s-N)(p-\theta)} dy \\ &\le C.\end{aligned} \tag{2.18}$$

Here in the last inequlaity we have used the fact that $\theta > p - N_s + 1$. Thus

$$\begin{aligned}\mathbb{G}_s[\mathbb{G}_s[\lambda]^p](x) &\le C \int_\Omega \left(\frac{G_s(x,z)}{\delta(z)^s}\right)^\theta \delta(z)^s d\lambda(z) \\ &\le C \left(\int_\Omega G_s(x,z) d\lambda(z)\right)^\theta,\end{aligned} \tag{2.19}$$

where in the last line we have used Hölder inequality with exponent $\frac{1}{\theta}$. Note that the above approach is still valid for the case $p = 1$.

If $0 < p < 1$ then

$$\mathbb{G}_s[\mathbb{G}_s[\lambda]^p] \le C(\mathbb{G}_s[1] + \mathbb{G}_s[\mathbb{G}_s[\lambda]]) \le C(\mathbb{G}_s[1] + \mathbb{G}_s[\lambda]^\theta).$$

Then (2.14) follows by a similar argument as in the proof of Lemma 2.10 by using Remark 2.4 with $\theta \le 1$. □

In the sequel, without loss of generality, we may assume that

$$0 < p \le q. \tag{2.20}$$

Hence, if $pq \ge 1$ then

$$p \le q\frac{p+1}{q+1} \le p\frac{q+1}{p+1} \le q. \tag{2.21}$$

Put

$$t_s := q\,(p - N_s + 1)\,.$$

Notice that if $q\frac{p+1}{q+1} < N_s$ then $t_s < q\frac{p+1}{q+1} < N_s$.

Lemma 2.12. *Let $p, q > 0$, $p \le q$ and $\lambda \in \mathfrak{M}^+(\Omega, \delta^s)$ with $\|\lambda\|_{\mathfrak{M}(\Omega,\delta^s)} = 1$. Assume $q\frac{p+1}{q+1} < N_s$. Then for any $t \in (\max(0, t_s), q]$, there exists a positive constant $c = c(N, p, q, s, t)$ such that*

$$\mathbb{G}_s[\mathbb{G}_s[\lambda]^p]^q \le c\mathbb{G}_s[\lambda]^t \quad \text{a.e. in } \Omega. \tag{2.22}$$

In particular,

$$\mathbb{G}_s[\mathbb{G}_s[\lambda]^p]^q \le C\mathbb{G}_s[\lambda]^{q\frac{p+1}{q+1}} \quad \text{a.e. in } \Omega, \tag{2.23}$$

$$\mathbb{G}_s[\mathbb{G}_s[\mathbb{G}_s[\lambda]^p]^q] \le C\mathbb{G}_s[\lambda] \quad \text{a.e. in } \Omega, \tag{2.24}$$

where $C = C(N, p, q, s)$.

Proof. Since $p \le q$ from (2.20), it follows that $p \le q\frac{p+1}{q+1}$. Therefore, from the assumption, it follows $p < N_s$. Hence $\max(0, p - N_s + 1) < 1$. Let $t \in (\max(0, t_s), q]$ then $\max(0, p - N_s + 1) < \frac{t}{q} \le 1$. Therefore, applying Lemma 2.11 with θ replaced by $\frac{t}{q}$, we obtain

$$\mathbb{G}_s[\mathbb{G}_s[\lambda]^p] \le c\mathbb{G}_s[\lambda]^{\frac{t}{q}},$$

which implies (2.22).

Since $t_s < q\frac{p+1}{q+1} \le q$, taking $t = q\frac{p+1}{q+1}$ in (2.22) yields (2.23). Next, since $q\frac{p+1}{q+1} < N_s$, combining (2.23) along with Lemma 2.10, we have

$$\mathbb{G}_s[\mathbb{G}_s[\mathbb{G}_s[\lambda]^p]^q] \le C\mathbb{G}_s[\mathbb{G}_s[\lambda]^{q\frac{p+1}{q+1}}] \le C\mathbb{G}_s[\lambda] \quad \text{a.e. in } \Omega.$$

This completes the proof. □

3 Construction of the minimal solution

This section is devoted to the construction of the minimal solution of (1.1). We borrow some idea from [4].

Lemma 3.1. *Assume $p, q > 0$ and $\mu, \nu \in \mathfrak{M}^+(\Omega, \delta^s)$. Assume in addition that there exist functions $V \in L^p(\Omega, \delta^s)$ and $U \in L^q(\Omega, \delta^s)$ such that*

$$\begin{aligned} U &\ge \mathbb{G}_s[V^p] + \mathbb{G}_s[\mu] \quad \text{a.e. in } \Omega,\\ V &\ge \mathbb{G}_s[U^q] + \mathbb{G}_s[\nu] \quad \text{a.e. in } \Omega. \end{aligned} \tag{3.1}$$

Then there exists a positive minimal weak solution $(\underline{u}_\mu, \underline{v}_\nu)$ of (1.1) satisfying

$$\mathbb{G}_s[\mu] \le \underline{u}_\mu \le U, \quad \mathbb{G}_s[\nu] \le \underline{v}_\nu \le V \quad \text{a.e. in } \Omega. \tag{3.2}$$

Proof. Put $u_0 := \mathbb{G}_s[\mu]$, $v_0 := \mathbb{G}_s[\nu]$ and for $n \geq 1$, define

$$\begin{aligned} u_n &:= \mathbb{G}_s[v_{n-1}^p] + \mathbb{G}_s[\mu], \\ v_n &:= \mathbb{G}_s[u_{n-1}^q] + \mathbb{G}_s[\nu]. \end{aligned} \tag{3.3}$$

Clearly $u_0 \leq U$ and $v_0 \leq V$. Therefore,

$$u_1 = \mathbb{G}_s[v_0^p] + \mathbb{G}_s[\mu] \leq \mathbb{G}_s[V^p] + \mathbb{G}_s[\mu] \leq U.$$

Similarly, $v_1 \leq \mathbb{G}_s[U^q] + \mathbb{G}_s[\nu] \leq V$. By induction, it follows that $u_n \leq U$ and $v_n \leq V$ for every $n \geq 1$. Also, it is easy to see that $\{u_n\}$ and $\{v_n\}$ are increasing sequences. Hence $u_n \uparrow \underline{u}_\mu \leq U \in L^q(\Omega, \delta^s)$ and $v_n \uparrow \underline{v}_\nu \leq V \in L^p(\Omega, \delta^s)$. Therefore $\mathbb{G}_s[v_n^p] \uparrow \mathbb{G}_s[\underline{v}_\nu^p]$ and $\mathbb{G}_s[u_n^q] \uparrow \mathbb{G}_s[\underline{u}_\mu^q]$ a.e. in Ω. Letting $n \to \infty$ in (3.3), we deduce that

$$\underline{u}_\mu = \mathbb{G}_s[\underline{v}_\nu^p] + \mathbb{G}_s[\mu] \quad \text{and} \quad \underline{v}_\nu = \mathbb{G}_s[\underline{u}_\mu^q] + \mathbb{G}_s[\nu].$$

This means that $(\underline{u}_\mu, \underline{v}_\nu)$ is a weak solution of (1.1).

Next, let (u, v) be any positive weak solution (1.1). Then

$$u = \mathbb{G}_s[v^p] + \mathbb{G}_s[\mu] \geq u_0, \quad v = \mathbb{G}_s[u^q] + \mathbb{G}_s[\nu] \geq v_0.$$

Thus

$$u \geq \mathbb{G}_s[v_0^p] + \mathbb{G}_s[\mu] \geq u_1, \quad v \geq \mathbb{G}_s[u_0^q] + \mathbb{G}_s[\nu] \geq v_1.$$

By induction it follows that $u \geq u_n$ and $v \geq v_n$ for all $n \geq 1$. Hence $u \geq \underline{u}_\mu$ and $v \geq \underline{v}_\nu$. This completes the lemma. □

Remark 3.2. In stead of studing system (1.1), in the sequel, we will work on the following system

$$\begin{cases} (-\Delta)^s u = v^p + \rho\mu & \text{in } \Omega, \\ (-\Delta)^s v = u^q + \tau\nu & \text{in } \Omega, \\ u = v = 0 & \text{in } \Omega^c, \end{cases} \tag{3.4}$$

where ρ, τ are positive parameters and $\mu, \nu \in \mathfrak{M}^+(\Omega, \delta^s)$ such that $\|\mu\|_{\mathfrak{M}(\Omega,\delta^s)} = \|\nu\|_{\mathfrak{M}(\Omega,\delta^s)} = 1$. The advantage is when dealing with system (3.4), we can easily apply Lemma 2.11 and Lemma 2.12 for $\mathbb{G}_s[\mu]$ and $\mathbb{G}_s[\nu]$ and require only the smallness of the parameters ρ and τ, which improves considerably the exposition.

We recall that in the sequel, we assume that $0 < p \leq q$ and hence if $pq \geq 1$ then (2.21) holds.

Theorem 3.3. *Let $p, q, \rho, \tau > 0$ and $\mu, \nu \in \mathfrak{M}^+(\Omega, \delta^s)$ such that $\|\mu\|_{\mathfrak{M}(\Omega,\delta^s)} = \|\nu\|_{\mathfrak{M}(\Omega,\delta^s)} = 1$. Assume $p \leq q$, $pq \neq 1$, $q\frac{p+1}{q+1} < N_s$ and $\mathbb{G}_s[\mu] \in L^q(\Omega, \delta^s)$. Then system (3.4) admits the minimal weak solution $(\underline{u}_{\rho\mu}, \underline{v}_{\tau\nu})$ for ρ and τ small if $pq > 1$, for any $\rho > 0$ and $\tau > 0$ if $pq < 1$.*

In addition, if $p \leq q < N_s$, then there exists a constant $K = K(N, s, p, q, \Omega, \rho, \tau)$ such that $K \to 0$ as $(\rho, \tau) \to (0, 0)$ and

$$\max\{\underline{u}_{\rho\mu}, \underline{v}_{\tau\nu}\} \leq K\mathbb{G}_s[\mu + \nu] \quad \text{a.e. in } \Omega. \tag{3.5}$$

Proof. Fix numbers $\vartheta_i > 0$, $(i = 1, 2)$ and set

$$\Psi := \mathbb{G}_s[\vartheta_1\mu]^q + \vartheta_2\nu. \tag{3.6}$$

For $\kappa \in (0, 1]$, put

$$\rho = \kappa^{\frac{1}{q}}\vartheta_1, \quad \tau = \kappa\vartheta_2 \tag{3.7}$$

and consider system (3.4) with ρ and τ as in (3.7). From the assumptions, $\mathbb{G}_s[\mu]^q \in L^1(\Omega, \delta^s)$ and $\nu \in \mathfrak{M}^+(\Omega, \delta^s)$, it follows that $\Psi \in \mathfrak{M}^+(\Omega, \delta^s)$. Also we note that $p \leq q$ and $q\frac{p+1}{q+1} < N_s$ imply $p < N_s$. Set

$$V := A\mathbb{G}_s[\kappa\Psi] \quad \text{and} \quad U := \mathbb{G}_s[V^p] + \mathbb{G}_s[\rho\mu]$$

where $A > 0$ will be determined later on. Then,

$$\begin{aligned} U^q &\le c(\mathbb{G}_s[V^p]^q + \mathbb{G}_s[\rho\mu]^q) \\ &= c\left\{A^{pq}\kappa^{pq}\mathbb{G}_s[\mathbb{G}_s[\Psi]^p]^q + \mathbb{G}_s[\rho\mu]^q\right\} \end{aligned}$$

where $c = c(p, q)$. It follows that

$$\begin{aligned} \mathbb{G}_s[U^q] + \mathbb{G}_s[\tau\nu] &\le c(A^{pq}\kappa^{pq}\mathbb{G}_s[\mathbb{G}_s[\mathbb{G}_s[\Psi]^p]^q] + \mathbb{G}_s[\mathbb{G}_s[\rho\mu]^q]) + \mathbb{G}_s[\tau\nu] \\ &= c(A^{pq}\kappa^{pq}\mathbb{G}_s[\mathbb{G}_s[\mathbb{G}_s[\Psi]^p]^q] + \kappa\mathbb{G}_s[\mathbb{G}_s[\vartheta_1\mu]^q]) + \kappa\mathbb{G}_s[\vartheta_2\nu] \\ &\le c(A^{pq}\kappa^{pq}\mathbb{G}_s[\mathbb{G}_s[\mathbb{G}_s[\Psi]^p]^q] + \kappa\mathbb{G}_s[\Psi]). \end{aligned} \tag{3.8}$$

Since $q\frac{p+1}{q+1} < N_s$, applying Lemma 2.12, we have

$$\mathbb{G}_s[\mathbb{G}_s[\mathbb{G}_s[\Psi]^p]^q] \le C\|\Psi\|^{pq-1}_{\mathfrak{M}(\Omega,\delta^s)}\mathbb{G}_s[\Psi] \tag{3.9}$$

where $C = C(N, s, p, q, \Omega)$. Combining the definition of Ψ in (3.6), Lemma 2.12, Lemma 2.7 and the assumption that $\|\mu\|_{\mathfrak{M}(\Omega,\delta^s)} = \|\nu\|_{\mathfrak{M}(\Omega,\delta^s)} = 1$, we can estimate $C^{-1} \le \|\Psi\|_{\mathfrak{M}(\Omega,\delta^s)} \le C$ for some positive constant C independent of A and κ. This, combined with (3.8) and (3.9) implies

$$\mathbb{G}_s[U^q] + \mathbb{G}_s[\tau\nu] \le C(A^{pq}\kappa^{pq} + \kappa)\mathbb{G}_s[\Psi]. \tag{3.10}$$

for some positive constant C independent of A and κ.

We will choose A and κ such that

$$\mathbb{G}_s[U^q] + \mathbb{G}_s[\tau\nu] \le V. \tag{3.11}$$

For that, it is sufficient to choose A and κ such that

$$C(A^{pq}\kappa^{pq} + \kappa)\mathbb{G}_s[\Psi] \le V.$$

This holds if

$$C(A^{pq}\kappa^{pq-1} + 1) \le A. \tag{3.12}$$

If $pq > 1$ then we can choose $A > 0$ large enough and then choose $\kappa > 0$ small enough (depending on A) such that (3.12) holds. If $pq < 1$ then for any $\kappa > 0$ there exists A large enough such that (3.12) holds. For such A and $\kappa > 0$, we obtain (3.11). Consequently, (U, V) satisfies (3.1). By Lemma 3.1, there exists a weak solution $(\underline{u}_{\rho\mu}, \underline{v}_{\tau\nu})$ of (3.4) for $\rho > 0$ and $\tau > 0$ small if $pq > 1$, for any $\rho > 0$ and $\tau > 0$ if $pq < 1$. Moreover, $(\underline{u}_{\rho\mu}, \underline{v}_{\tau\nu})$ satisfies (3.2).

Next, assuming in addition that $p \le q < N_s$, we will demonstrate (3.5). From the definition of V and Lemma 2.11, we see that

$$\begin{aligned} V &= A\kappa\mathbb{G}_s[\Psi] = A\kappa(\mathbb{G}_s[\mathbb{G}_s[\vartheta_1\mu]^q] + \mathbb{G}_s[\vartheta_2\nu]) \\ &\le cA\kappa\mathbb{G}_s[\mu + \nu]. \end{aligned} \tag{3.13}$$

It follows that

$$V^p \le C\kappa^p(\mathbb{G}_s[\mu]^p + \mathbb{G}_s[\nu]^p).$$

Therefore,

$$\begin{aligned} U &\le C\kappa^p(\mathbb{G}_s[\mathbb{G}_s[\mu]^p] + \mathbb{G}_s[\mathbb{G}_s[\nu]^p]) + \kappa^{\frac{1}{q}}\mathbb{G}_s[\vartheta_1\mu] \\ &\le C\kappa^p\mathbb{G}_s[\mu + \nu] + \kappa^{\frac{1}{q}}\vartheta_1\mathbb{G}_s[\mu] \\ &\le C\max\{\kappa^p, \kappa^{\frac{1}{q}}\}\mathbb{G}_s[\mu + \nu]. \end{aligned} \tag{3.14}$$

Combining (3.14) and (3.13) along with (3.2) leads to (3.5). □

Remark 3.4. By Lemma 2.8(i), we see that if $\mu, \nu \in L^r(\Omega)$ with $r > \frac{N}{2s}$ then $\mathbb{G}_s[\mu + \nu] \in L^\infty(\Omega)$. From Theorem 3.3, if ρ and τ are small then system (3.4) admits a minimal solution $(\underline{u}_{\rho\mu}, \underline{v}_{\tau\nu})$ which satisfies (3.5). It follows that $\underline{u}_{\rho\mu}, \underline{v}_{\tau\nu} \in L^\infty(\Omega)$. Moreover, $(\underline{u}_{\rho\mu}, \underline{v}_{\tau\nu}) \to (0, 0)$ a.e. in Ω as $(\rho, \tau) \to (0, 0)$.

Proof of Theorem 1.3 completed: Combining Theorem 3.3 along with Remark 3.2, the proof of the theorem follows.

4 A priori estimates and regularity

In this section, we provide a priori estimates, as well as regularity properties, of weak solutionsof (1.1).

4.1 A priori estimates

Lemma 4.1. *Assume $p > 1$, $q > 1$ and $\mu, \nu \in \mathfrak{M}^+(\Omega, \delta^s)$. If (u, v) is a weak solution of* (1.1) *then there is a positive constant $c = c(N, s, p, q, \Omega)$ such that*

$$\begin{aligned} \|u\|_{L^1(\Omega)} + \|v\|_{L^p(\Omega,\delta^s)} &\le c(1 + \|\mu\|_{\mathfrak{M}(\Omega,\delta^s)}), \\ \|v\|_{L^1(\Omega)} + \|u\|_{L^q(\Omega,\delta^s)} &\le c(1 + \|\nu\|_{\mathfrak{M}(\Omega,\delta^s)}). \end{aligned} \tag{4.1}$$

Proof. We prove this lemma in the spirit of [2, Lemma 4.1]. Let (λ_1, φ_1) be the first eigenvalue and corresponding positive eigenfunction of $(-\Delta)^s$ in $X_0(\Omega)$ (see the definition of X_0 in (5.3)). By [12, Lemma 2.1(ii)], $\varphi_1 \in \mathbb{X}_s(\Omega)$, and hence by taking $\zeta = \varphi_1$ in (1.6), we obtain

$$\begin{aligned} \lambda_1 \int_\Omega u\varphi_1 \, dx &= \int_\Omega v^p \varphi_1 \, dx + \lambda_1 \int_\Omega \varphi_1 \, d\mu, \\ \lambda_1 \int_\Omega v\varphi_1 \, dx &= \int_\Omega u^q \varphi_1 \, dx + \lambda_1 \int_\Omega \varphi_1 \, d\nu. \end{aligned} \tag{4.2}$$

By Young's inequality, we get

$$\begin{aligned} \int_\Omega v\varphi_1 \, dx &\le (2\lambda_1)^{-1} \int_\Omega v^p \varphi_1 \, dx + (2\lambda_1)^{\frac{1}{p-1}} \int_\Omega \varphi_1 \, dx, \\ \int_\Omega u\varphi_1 \, dx &\le (2\lambda_1)^{-1} \int_\Omega u^q \varphi_1 \, dx + (2\lambda_1)^{\frac{1}{q-1}} \int_\Omega \varphi_1 \, dx. \end{aligned} \tag{4.3}$$

Substituting (4.3) in (4.2) we have

$$\begin{aligned} 2\int_\Omega v^p \varphi_1 \, dx + 2\lambda_1 \int_\Omega \varphi_1 \, d\mu &\le \int_\Omega u^q \varphi_1 \, dx + (2\lambda_1)^{\frac{q}{q-1}} \int_\Omega \varphi_1 \, dx, \\ 2\int_\Omega u^q \varphi_1 \, dx + 2\lambda_1 \int_\Omega \varphi_1 \, d\nu &\le \int_\Omega v^p \varphi_1 \, dx + (2\lambda_1)^{\frac{p}{p-1}} \int_\Omega \varphi_1 \, dx. \end{aligned} \tag{4.4}$$

Therefore,

$$\frac{3}{2}\int_\Omega v^p \varphi_1 \, dx + \lambda_1 \int_\Omega \varphi_1 \, d(2\mu + \nu) \le \left(\frac{(2\lambda_1)^{\frac{p}{p-1}}}{2} + (2\lambda_1)^{\frac{q}{q-1}} \right) \int_\Omega \varphi_1 \, dx.$$

Since the second term on the left-hand side of above expression is nonnegative, taking into account that $c^{-1}\delta^s < \varphi_1 < c\delta^s$ in Ω, we have

$$\|v\|^p_{L^p(\Omega,\delta^s)} \le C(\lambda_1) \int_\Omega \delta^s \, dx \le c'. \tag{4.5}$$

Similarly

$$\|u\|^q_{L^q(\Omega,\delta^s)} \le C'. \tag{4.6}$$

Next, combining (1.7), Lemma 2.6 and Lemma 2.7 with $y = s$, $\alpha = 0$ we obtain

$$\begin{aligned} \|u\|_{L^1(\Omega)} &\le \|\mathbb{G}_s[v^p]\|_{L^1(\Omega)} + \|\mathbb{G}_s[\mu]\|_{L^1(\Omega)} \\ &\le C\big(\|\mathbb{G}_s[v^p]\|_{M^{N_s}(\Omega)} + \|\mathbb{G}_s[\mu]\|_{M^{\frac{N}{N-s}}(\Omega)}\big) \\ &\le C\big(\|v\|^p_{L^p(\Omega,\delta^s)} + \|\mu\|_{\mathfrak{M}(\Omega,\delta^s)}\big). \end{aligned} \tag{4.7}$$

Hence first expression of (4.1) holds by combining (4.5) and (4.7). Similarly, the second expression of (4.1) follows. □

4.2 Regularity

Theorem 4.2. *Let $p, q \in (1, N_s)$. (i) Assume $\mu, \nu \in L^r(\Omega)$ for some $r > \frac{N}{2s}$. If (u, v) is a nonnegative weak solution of* (1.1) *then $u, v \in L^\infty_{loc}(\Omega)$.*

(ii) Assume $\mu, \nu \in L^r(\Omega) \cap L^\infty_{loc}(\Omega)$. If (u, v) is a nonnegative weak solution of (1.1) *then $u, v \in C^\alpha_{loc}(\Omega)$ for some $\alpha \in (0, 2s)$.*

Proof. (i) We first assume that $\mu, \nu \in L^r(\Omega)$ for some $r > \frac{N}{2s}$. Let (u, v) be a nonnegative weak solution of (1.1). Then $u \in L^1(\Omega) \cap L^q(\Omega, \delta^s)$, $v \in L^1(\Omega) \cap L^p(\Omega, \delta^s)$ and (u, v) satisfies (1.7). Let $x_0 \in \Omega$ and $r > 0$ such that $B(x_0, 2r) \subset\subset \Omega$. For any $j \in \mathbb{N}$, set $B_j := B(x_0, 2^{-j} r)$. For any $j \in \mathbb{N}$, we can write

$$\begin{aligned} u &= \mathbb{G}_s[\chi_{\Omega \setminus B_j} v^p] + \mathbb{G}_s[\chi_{B_j} v^p] + \mathbb{G}_s[\mu], \\ v &= \mathbb{G}_s[\chi_{\Omega \setminus B_j} u^q] + \mathbb{G}_s[\chi_{B_j} u^q] + \mathbb{G}_s[\nu]. \end{aligned} \tag{4.8}$$

Observe that, for $x \in B_{j+1}$, by (2.2),

$$\begin{aligned} \mathbb{G}_s[\chi_{\Omega \setminus B_j} v^p](x) &= \int_{\Omega \setminus B_j} v(y)^p G_s(x, y)\,dy \le C\delta(x)^s \int_{\Omega \setminus B_j} v(y)^p \delta(y)^s |x - y|^{-N}\,dy \\ &\le C 2^{(j+1)N} r^{-N} \|v\|^p_{L^p(\Omega, \delta^s)} < \infty. \end{aligned}$$

Therefore,

$$\mathbb{G}_s[\chi_{\Omega \setminus B_j} v^p] \in L^\infty(B_{j+1}) \quad \forall j \in \mathbb{N}. \tag{4.9}$$

Similarly,

$$\mathbb{G}_s[\chi_{\Omega \setminus B_j} u^q] \in L^\infty(B_{j+1}) \quad \forall j \in \mathbb{N}. \tag{4.10}$$

Since $\mu, \nu \in L^r(\Omega)$ for $r > \frac{N}{2s}$, by Lemma 2.8 (i), we deduce

$$\mathbb{G}_s[\mu],\ \mathbb{G}_s[\nu] \in L^\infty(\Omega). \tag{4.11}$$

Further, as $u \in L^q(\Omega, \delta^s)$ and $v \in L^p(\Omega, \delta^s)$, we have $\chi_{B_0} u \in L^q(\Omega)$ and $\chi_{B_0} v \in L^p(\Omega)$ and therefore, applying Lemma 2.7 and Lemma 2.6 we have

$$\|\mathbb{G}_s[\chi_{B_0} u^q]\|_{L^\theta(\Omega)} \le c \|\mathbb{G}_s[\chi_{B_0} u^q]\|_{M^{\frac{N}{N-2s}}(\Omega)} \le c' \|\chi_{B_0} u\|^q_{L^q(\Omega)},$$

for every $1 < \theta < \frac{N}{N-2s}$. This in turn implies $\mathbb{G}_s[\chi_{B_0} u^q]$, $\mathbb{G}_s[\chi_{B_0} v^p] \in L^\theta(B_0)$ for every $1 < \theta < \frac{N}{N-2s}$. This and (4.8) – (4.11) yield $u, v \in L^\theta(B_2)$ for every $1 < \theta < \frac{N}{N-2s}$. Set,

$$\ell_0 := \frac{1}{2}(1 + \frac{N_s}{p}), \quad \tilde{\ell}_0 := \frac{1}{2}(1 + \frac{N_s}{q}).$$

Then $1 < p\ell_0,\ q\tilde{\ell}_0 < N_s < \frac{N}{N-2s}$ and hence $u \in L^{q\tilde{\ell}_0}(B_2)$ and $v \in L^{p\ell_0}(B_2)$. Without loss of generality, we can assume $\ell_0,\ \tilde{\ell}_0 \neq \frac{N}{2s}$. If $\ell_0,\ \tilde{\ell}_0 > \frac{N}{2s}$, then by Lemma 2.8 (i), $\mathbb{G}_s[\chi_{B_2} v^p]$, $\mathbb{G}_s[\chi_{B_2} u^q] \in L^\infty(B_2)$. This and (4.8) – (4.11) imply $u,\ v \in L^\infty(B_4)$. If $\ell_0 < \frac{N}{2s}$ or $\tilde{\ell}_0 < \frac{N}{2s}$, then by Lemma 2.8 (ii) we obtain $\mathbb{G}_s[\chi_{B_2} v^p] \in L^{p\ell_1}(B_2)$ or $\mathbb{G}_s[\chi_{B_2} u^q] \in L^{q\tilde{\ell}_1}(B_2)$ respectively, where

$$\ell_1 := \frac{1}{p} \frac{N\ell_0}{N - 2\ell_0 s}, \quad \tilde{\ell}_1 := \frac{1}{q} \frac{N\tilde{\ell}_0}{N - 2\tilde{\ell}_0 s}.$$

Then from (4.8) – (4.11), $v \in L^{p\ell_1}(B_4)$ or $u \in L^{q\tilde{\ell}_1}(B_4)$. We have

$$\frac{\ell_1}{\ell_0} = \frac{1}{p} \frac{N}{N - 2\ell_0 s} > \frac{1}{p} \frac{N}{N - 2s} > \ell_0.$$

This implies that $\ell_1 > \ell_0^2 > \ell_0 > 1$. Similarly, $\tilde{\ell}_1 > \tilde{\ell}_0^2 > \tilde{\ell}_0 > 1$. Now if ℓ_1 or $\tilde{\ell}_1 \neq \frac{N}{2s}$, then continuing the bootstrap method as in the proof of [2, Theorem 1.6]), we can conclude that $u,\ v \in L^\infty(B_{2(k+1)})$. Consequently, $u,\ v \in L^\infty_{loc}(\Omega)$.

(ii) If $\mu, \nu \in L^r(\Omega) \cap L^\infty_{loc}(\Omega)$ then by part (i), we have $v^p + \mu,\ u^q + \nu \in L^\infty_{loc}(\Omega)$. Further, as $u, v \in L^1(\Omega)$, applying Schauder estimate [34], we have $u, v \in C^\alpha_{loc}(\Omega)$, for some $\alpha \in (0, 2s)$.

□

5 Construction of a second solution

In this section we assume $1 < p \leq q < N_s$. Then it follows that $q\frac{p+1}{q+1} < N_s$. Using Linking theorem, we will construct a second weak solution of (3.4) when $\mu, \nu \in L^r(\Omega)$, for $r > \frac{N}{2s}$ with $\|\mu\|_{L^r(\Omega)} = \|\nu\|_{L^r(\Omega)} = 1$.

By Theorem 3.3, if $\rho > 0$ and $\tau > 0$ are small then there exists the minimal positive week solution, denoted by $(\underline{u}_{\rho\mu}, \underline{v}_{\tau\nu})$, of (3.4). We would like to apply Linking theorem to find a variational weak solution of

$$\begin{cases} (-\Delta)^s u = (\underline{v}_{\tau\nu} + v^+)^p - \underline{v}_{\tau\nu}^p & \text{in } \Omega \\ (-\Delta)^s v = (\underline{u}_{\rho\mu} + u^+)^q - \underline{u}_{\rho\mu}^q & \text{in } \Omega \\ \quad u = 0 = v & \text{in } \Omega^c, \end{cases} \tag{5.1}$$

where $u^+ := \max(u, 0)$ and $u^- := -\min(0, u)$.

From Remark 3.4, we observe that there exists a constant $M > 0$ such that

$$\max\{\underline{u}_{\rho\mu}, \underline{v}_{\tau\nu}\} < M \quad \text{in } \Omega. \tag{5.2}$$

Define

$$X_0 := \{w \in H^s(\mathbb{R}^N) : w = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega\}, \tag{5.3}$$

where $H^s(\mathbb{R}^N)$ is the standard fractional Sobolev space on $\mathbb{R}^N$. It is well-known that

$$\|w\|_{X_0} := \left(\int_Q \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} dxdy \right)^{\frac{1}{2}}, \tag{5.4}$$

where $Q = \mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)$, is a norm on X_0 and $(X_0, ||.||_{X_0})$ is a Hilbert space, with the inner product

$$\langle \phi, \psi \rangle_{X_0} := \int_Q \frac{(\phi(x) - \phi(y))(\psi(x) - \psi(y))}{|x - y|^{N+2s}} dxdy.$$

Put

$$2_s^\star := \frac{2N}{N - 2s}.$$

It is easy to check that (see [36])

$$\int_\Omega \psi(-\Delta)^s \phi \, dx = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} \phi (-\Delta)^{\frac{s}{2}} \psi \, dx \quad \forall \phi, \psi \in X_0.$$

It is also well known that the embedding $X_0 \hookrightarrow L^r(\mathbb{R}^N)$ is compact, for any $r \in [1, 2_s^\star)$ and $X_0 \hookrightarrow L^{2_s^\star}(\mathbb{R}^N)$ is continuous.

Definition 5.1. We say that a solution (u, v) of (1.1) is stable (resp. semistable) if

$$\begin{cases} \|\phi\|_{X_0}^2 > \text{(resp. } \geq) \; p \displaystyle\int_\Omega v^{p-1} \phi^2 dx, \\ \|\phi\|_{X_0}^2 > \text{(resp. } \geq) \; q \displaystyle\int_\Omega u^{q-1} \phi^2 dx, \end{cases} \quad \forall \phi \in X_0 \setminus \{0\}. \tag{5.5}$$

Proposition 5.2. *Assume $p, q \in (1, N_s)$ and μ, ν are positive functions in $L^r(\Omega)$ for some $r > \frac{N}{2s}$ such that $\|\mu\|_{L^r(\Omega)} = \|\nu\|_{L^r(\Omega)} = 1$. For $\rho > 0$ and $\tau > 0$ small, let $(\underline{u}_{\rho\mu}, \underline{v}_{\tau\nu})$ be the minimal solution of (3.4) obtained in Theorem 3.3. There exists $t_0 > 0$ such that if $\max\{\rho, \tau\} < t_0$ then $(\underline{u}_{\rho\mu}, \underline{v}_{\tau\nu})$ is stable. Moreover, there exists a positive constant $C = C(N, s, p, q, t_0)$ such that*

$$\begin{cases} \|\phi\|_{X_0}^2 - p \displaystyle\int_\Omega \underline{v}_{\tau\nu}^{p-1} \phi^2 dx \geq C\|\phi\|_{X_0}^2, \\ \|\phi\|_{X_0}^2 - q \displaystyle\int_\Omega \underline{u}_{\rho\mu}^{q-1} \phi^2 dx \geq C\|\phi\|_{X_0}^2, \end{cases} \quad \forall \phi \in X_0 \setminus \{0\}. \tag{5.6}$$

Proof. **Step 1:** We show that there exists $t_0 > 0$ such that $(\underline{u}_{\rho\mu}, \underline{v}_{\tau\nu})$ is stable provided $\max\{\rho, \tau\} < t_0$.

Indeed, from Remark 3.4, it follows that for any $\phi \in X_0 \setminus \{0\}$, there exists $t_0 > 0$ small such that if $\max\{\rho, \tau\} < t_0$, there hold

$$\int_\Omega \underline{v}_{\tau\nu}^{p-1}\phi^2\,dx \le \|\underline{v}_{\tau\nu}\|_{L^\infty(\Omega)}^{p-1}\int_\Omega \phi^2 dx \le \frac{1}{p}\|\phi\|_{X_0}^2,$$
$$\int_\Omega \underline{u}_{\rho\mu}^{q-1}\phi^2\,dx \le \|\underline{u}_{\rho\mu}\|_{L^\infty(\Omega)}^{q-1}\int_\Omega \phi^2 dx \le \frac{1}{q}\|\phi\|_{X_0}^2.$$

This completes Step 1.

Step 2: We prove (5.6). Assume $(\rho, \tau) \in (0, t_0) \times (0, t_0)$ and put

$$\rho' = \frac{\rho + t_0}{2}, \quad \tau' = \frac{\tau + t_0}{2}.$$

Set

$$\alpha = \max\left\{\left(\frac{\rho}{\rho'}\right)^{\frac{1}{q}}, \left(\frac{\tau}{\tau'}\right)^{\frac{1}{p}}\right\} < 1.$$

Let $(\underline{u}_{\rho'\mu}, \underline{v}_{\tau'\nu})$ and $(\underline{u}_{\rho\mu}, \underline{v}_{\tau\nu})$ be the solutions of (3.4) with data $(\rho'\mu, \tau'\nu)$ and $(\rho\mu, \tau\nu)$ respectively. Since $p, q > 1$ and $\alpha < 1$, it is easy to see that

$$\alpha\underline{u}_{\rho'\mu} = \mathbb{G}_s[\alpha\underline{v}_{\tau'\nu}^p] + \mathbb{G}_s[\alpha\rho'\mu] \ge \mathbb{G}_s[(\alpha\underline{v}_{\tau'\nu})^p] + \mathbb{G}_s[\rho\mu],$$
$$\alpha\underline{v}_{\tau'\nu} = \mathbb{G}_s[\alpha\underline{u}_{\rho'\mu}^q] + \mathbb{G}_s[\alpha\tau'\nu] \ge \mathbb{G}_s[(\alpha\underline{u}_{\rho'\mu})^q] + \mathbb{G}_s[\tau\nu].$$

Consequently, in view of the proof of Lemma 3.1, we deduce $\alpha\underline{u}_{\rho'\mu} \ge \underline{u}_{\rho\mu}$ and $\alpha\underline{v}_{\tau'\nu} \ge \underline{v}_{\tau\nu}$. Furthermore, since $(\rho', \tau') \in (0, t_0) \times (0, t_0)$, by Step 1, we assert that $(\underline{u}_{\rho'\mu}, \underline{v}_{\tau'\nu})$ is stable. Therefore,

$$\begin{aligned} 0 < \|\phi\|_{X_0}^2 - p\int_\Omega \underline{v}_{\tau'\nu}^{p-1}\phi^2 dx &\le \|\phi\|_{X_0}^2 - p\alpha^{1-p}\int_\Omega \underline{v}_{\tau\nu}^{p-1}\phi^2 dx \\ &= \alpha^{1-p}\Big(\alpha^{p-1}\|\phi\|_{X_0}^2 - p\int_\Omega \underline{v}_{\tau\nu}^{p-1}\phi^2 dx\Big). \end{aligned} \tag{5.7}$$

Hence,

$$\begin{aligned} \|\phi\|_{X_0}^2 - p\int_\Omega \underline{v}_{\tau\nu}^{p-1}\phi^2 dx &= (1-\alpha^{p-1})\|\phi\|_{X_0}^2 + \alpha^{p-1}\|\phi\|_{X_0}^2 - p\int_\Omega \underline{v}_{\tau\nu}^{p-1}\phi^2 dx \\ &> (1-\alpha^{p-1})\|\phi\|_{X_0}^2. \end{aligned} \tag{5.8}$$

Similarly, one can prove

$$\|\phi\|_{X_0}^2 - q\int_\Omega \underline{u}_{\rho\mu}^{q-1}\phi^2 dx > (1-\alpha^{q-1})\|\phi\|_{X_0}^2.$$

Hence (5.6) holds with $C = \min\{1-\alpha^{p-1}, 1-\alpha^{q-1}\}$. □

The norm of an element $z = (u, v) \in X_0 \times X_0$ is defined by

$$\|z\|_{X_0\times X_0} := \|(u, v)\|_{X_0\times X_0} = \left(\|u\|_{X_0}^2 + \|v\|_{X_0}^2\right)^{\frac{1}{2}}.$$

Definition 5.3. Let $(X, \|.\|_X)$ be a real Banach space with its dual $(X^\star, \|.\|_{X^\star})$ and $I \in C^1(X, \mathbb{R})$. For $c \in \mathbb{R}$, we say that I satisfies Cerami condition at level c (in short, $(C)_c$) if for any sequence $\{w_n\} \subset X$ with

$$I(w_n) \to c, \quad \|I'(w_n)\|_{X^\star}(1 + \|w_n\|_X) \to 0,$$

there is a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $\{w_{n_k}\}$ converges strongly in X.

We say that $\{w_n\} \subset X$ is a Palais-Smale sequence of I at level c if

$$I(w_n) \to c, \quad \|I'(w_n)\|_{X^\star} \to 0.$$

The energy functional associated to (5.1) is

$$\begin{aligned} I(u,v) := & \int\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{N+2s}}\,dxdy - \int\limits_{\Omega} H(\underline{v}_{\tau\nu}, v)dx \\ & - \int\limits_{\Omega} \tilde{H}(\underline{u}_{\rho\mu}, u)dx \quad \forall\,(u,v)\in X_0\times X_0, \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} H(r,t) &:= \frac{1}{p+1}\Big[(r+t^+)^{p+1} - r^{p+1} - (p+1)r^p t^+\Big], \\ \tilde{H}(r,t) &:= \frac{1}{q+1}\Big[(r+t^+)^{q+1} - r^{q+1} - (q+1)r^q t^+\Big], \quad r\ge 0. \end{aligned} \tag{5.10}$$

Therefore,

$$\begin{aligned} I'(u,v)(\phi,\psi) \quad = & \int\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\big(\phi(x)-\phi(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{N+2s}}\,dxdy \\ & + \int\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\big(u(x)-u(y)\big)\big(\psi(x)-\psi(y)\big)}{|x-y|^{N+2s}}\,dxdy \\ & - \int\limits_{\Omega} h(\underline{v}_{\tau\nu}, v)\psi dx - \int\limits_{\Omega} \tilde{h}(\underline{u}_{\rho\mu}, u)\phi dx, \end{aligned}$$

where

$$h(r,t) := (r+t^+)^p - r^p \quad \text{and} \quad \tilde{h}(r,t) := (r+t^+)^q - r^q, \quad r\ge 0.$$

It is easy to see that if $z=(u,v)$ is a critical point of I then (u,v) solves (5.1). We will find these critical points using Linking Theorem in the spirit of [19].

Lemma 5.4. *(i) There hold*

$$\frac{1}{p+1}t^{p+1} < H(r,t), \qquad \frac{1}{q+1}t^{q+1} < \tilde{H}(r,t) \quad \textit{for} \quad r,\ t>0. \tag{5.11}$$

(ii) Given any $M>0$, there exist $\theta>2$ and $T>0$ such that

$$H(r,t) \le \frac{1}{\theta}h(r,t)t, \quad \tilde{H}(r,t) \le \frac{1}{\theta}\tilde{h}(r,t)t, \quad \textit{for} \quad 0\le r\le M,\ t\ge T, \tag{5.12}$$

where T depend on M, p, q, θ.

(iii) Let $0<\kappa<p+1$, then there exists a constant $C=C(p,q,\kappa)>0$ such that

$$H(r,t),\ \tilde{H}(r,t) \ge t^\kappa - C \quad \textit{for} \quad r,\ t>0. \tag{5.13}$$

Proof. (i) Estimate (5.11) was proved in [26, Lemma C.2(ii)].

(ii) First let us choose $\theta_1\in(2,p+1)$ arbitrarily and fix it. Next, we define

$$y(r,t) := h(r,t)t - \theta_1 H(r,t).$$

From the definition of $h(r,t)$ and $H(r,t)$, a straight forward computation yields that

$$y(r,t) = t^2\Big[p\big(1-\frac{\theta_1}{2}\big)r^{p-1} + \frac{p(p-1)}{2}\big(1-\frac{\theta_1}{3}\big)r^{p-2}t + \cdots + \big(1-\frac{\theta_1}{p+1}\big)t^{p-1}\Big].$$

Therefore, there exits $0<T=T(p,M,\theta_1)$ such that $y(r,t)>0$ for $t\ge T$, $r\le M$. Similarly we can prove the other inequality by choosing $\theta_2\in(2,q+1)$. Then by take $\theta=\min\{\theta_1,\theta_2\}$, we obtain (5.12).

(iii) Since $\kappa < p + 1 \le q + 1$, applying Young's inequality, we have

$$t^\kappa \le \frac{1}{p+1} t^{p+1} + c_1 \quad \text{and} \quad t^\kappa \le \frac{1}{q+1} t^{q+1} + c_2$$

where $c_1 = c_1(\kappa, p)$ and $c_2 = c_2(\kappa, q)$. Taking $C = \max\{c_1, c_2\}$, it follows

$$\frac{1}{p+1} t^{p+1}, \ \frac{1}{q+1} t^{q+1} \ge t^\kappa - C.$$

Combining this with (i), (5.13) follows. □

Remark 5.5. Combining Lemma 5.4 along with the fact that $H(r, t) = 0$, for $t \le 0$, it holds

$$H(r, t) \ge 0, \quad \tilde{H}(r, t) \ge 0, \quad \forall\, t \in \mathbb{R}, \forall r \ge 0. \tag{5.14}$$

We also observe that ([26, Lemma C.2(iii)]) for any $\varepsilon > 0$, there exists $c_\varepsilon > 0$, such that

$$H(r, t) - \frac{p}{2} r^{p-1} t^2 \le \varepsilon r^{p-1} t^2 + c_\varepsilon t^{p+1}, \quad r,\ t \ge 0. \tag{5.15}$$

Notation: For the rest of this section, we denote by $\|\cdot\|$, the norm in X_0, by $\|(\cdot,\cdot)\|$ the norm in $X_0 \times X_0$ and by $\langle \cdot, \cdot \rangle$ the inner product in X_0.

Next, we prove that I has the geometry of the Linking theorem.

5.1 Geometry of the Linking Theorem

We define,

$$E^+ := \{(u, u) \,:\, u \in X_0\} \quad \text{and} \quad E^- := \{(u, -u) \,:\, u \in X_0\}.$$

Lemma 5.6. *There exist $\varrho,\ \sigma > 0$ such that $I(u, v) \ge \sigma$ for all $(u, v) \in S := \partial B_\varrho \cap E^+$.*

Proof. From the definition of $I(u, u)$, we have

$$\begin{aligned} I(u,u) &= \frac{1}{2}\Big(\|u\|^2 - p\int_\Omega \underline{v}_{\tau\nu}^{p-1} u^2\,dx\Big) + \frac{1}{2}\Big(\|u\|^2 - q\int_\Omega \underline{u}_{\rho\mu}^{q-1} u^2\,dx\Big) \\ &\quad -\Big(\int_\Omega H(\underline{v}_{\tau\nu}, u)\,dx - \frac{p}{2}\int_\Omega \underline{v}_{\tau\nu}^{p-1} u^2\,dx\Big) - \Big(\int_\Omega \tilde{H}(\underline{u}_{\rho\mu}, u)\,dx - \frac{q}{2}\int_\Omega \underline{u}_{\rho\mu}^{q-1} u^2\,dx\Big). \end{aligned}$$

Applying (5.15) and (5.6) to the above line and using (5.2) and Sobolev inequality, we obtain

$$\begin{aligned} I(u,u) &\ge C\|u\|^2 - \varepsilon\int_\Omega \underline{v}_{\tau\nu}^{p-1} u^2\,dx - C_\varepsilon\int_\Omega u^{p+1}\,dx - \varepsilon\int_\Omega \underline{u}_{\rho\mu}^{q-1} u^2\,dx - C_\varepsilon\int_\Omega u^{q+1}\,dx \\ &\ge (C - M^{p-1}S^{-1}\varepsilon - M^{q-1}S^{-1}\varepsilon)\|u\|^2 - C\|u\|^{p+1} - C\|u\|^{q+1}, \end{aligned}$$

where S is the Sobolev constant. Now, choosing $\varepsilon > 0$ and $\varrho > 0$ small enough, we find one $\sigma > 0$ such that $I(u, u) \ge \sigma$ when $\|u\| = \varrho$, as $p,\ q > 1$. This proves the lemma. □

Let $\psi_0 \in X_0$ be a fixed nonnegative function with $\|\psi_0\| = 1$ and

$$Q_{\psi_0} := \{r(\psi_0, \psi_0) + w \,:\, w \in E^-, \|w\| \le R_0,\ 0 \le r \le R_1\}.$$

Lemma 5.7. *There exist constants $R_0,\ R_1 > 0$, which depend on ψ_0, such that $I(u, v) \le 0$ for all $(u, v) \in \partial Q_{\psi_0}$.*

Proof. We note that boundary ∂Q_{ψ_0} of the set Q_{ψ_0} is taken in the space $\mathbb{R}(\psi_0, \psi_0) \oplus E^-$ and consists of three parts. We estimate I on these parts as below.

Case 1: $z \in \partial Q_{\psi_0} \cap E^-$ and of the form $z = (u, -u) \in E^-$. Then, thanks to (5.14), it follows

$$I(z) = -\|u\|^2 - \int_\Omega H(\underline{v}_{\tau\nu}, -u)dx - \int_\Omega \tilde{H}(\underline{u}_{\rho\mu}, u)dx \le 0.$$

Case 2: $z = R_1(\psi_0, \psi_0) + (u, -u) \in \partial Q_{\psi_0}$ with $\|(u, -u)\| \le R_0$. Thus,

$$I(z) = R_1^2\|\psi_0\|^2 - \|u\|^2 - \int_\Omega H(\underline{v}_{\tau\nu}, R_1\psi_0 - u)dx - \int_\Omega \tilde{H}(\underline{u}_{\rho\mu}, R_1\psi_0 + u)dx. \tag{5.16}$$

For $2 < \kappa < p + 1 \le q + 1$, set

$$\xi(t) := \begin{cases} t^\kappa & \text{if } t \ge 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Then, applying (5.14) and (5.13) to (5.16), we get

$$I(z) \quad \le \quad R_1^2 - \int_\Omega \xi(R_1\psi_0 - u)dx - \int_\Omega \xi(R_1\psi_0 + u)dx + C$$

where $C = C(p, q, \kappa)$ is the constant in (5.13).

Now, using convexity of the function ξ, we obtain

$$I(z) \le R_1^2 - 2R_1^\kappa \int_\Omega |\psi_0|^\kappa dx + C.$$

Therefore, since $\kappa > 2$, taking R_1 large enough (depending on ψ_0), it follows that $I(z) \le 0$.

Case 3: $z = r(\psi_0, \psi_0) + (u, -u) \in \partial Q_{\psi_0}$ with $\|(u, -u)\| = R_0$ and $0 \le r \le R_1$.

Then, using (5.14) it follows that

$$I(z) \le r^2\|\psi_0\|^2 - \|u\|^2 \le R_1^2 - \frac{1}{2}R_0^2.$$

Choosing $R_0 \ge \sqrt{2}R_1$, we have $I(z) \le 0$.

Combining case 2 and case 3, in order that the geometry of Linking theorem holds, we choose R_0, R_1 large enough with $R_0 \ge \sqrt{2}R_1$. □

Our next aim is to prove that Cerami sequences are bounded.

Proposition 5.8. *Let $(u_m, v_m) \in X_0 \times X_0$ such that*

(i) $I(u_m, v_m) = c + \delta_m$, where $\delta_m \to 0$ as $m \to \infty$.

(ii) $(1 + \|(u_m, v_m)\|)|I'(u_m, v_m)(\phi, \psi)| \le \varepsilon_m\|(\phi, \psi)\|$ for $\phi, \psi \in X_0 \times X_0$ and $\varepsilon_m \to 0$ as $m \to \infty$. Then,

$$\begin{aligned} \|u_m\| &\le C, & \|v_m\| &\le C \\ \int_\Omega h(\underline{v}_{\tau\nu}, v_m)v_m dx &\le C, & \int_\Omega \tilde{h}(\underline{u}_{\rho\mu}, u_m)u_m dx &\le C \\ \int_\Omega H(\underline{v}_{\tau\nu}, v_m)dx &\le C, & \int_\Omega \tilde{H}(\underline{u}_{\rho\mu}, u_m)dx &\le C. \end{aligned}$$

Proof. Choosing $(\phi, \psi) = (v_m, 0)$ and $(\phi, \psi) = (0, u_m)$ in (ii), we have

$$\begin{aligned} \left| \|v_m\|^2 - \int_\Omega \tilde{h}(\underline{u}_{\rho\mu}, u_m)v_m dx \right| &\le \varepsilon_m\|v_m\|, \\ \left| \|u_m\|^2 - \int_\Omega h(\underline{v}_{\tau\nu}, v_m)u_m dx \right| &\le \varepsilon_m\|u_m\|. \end{aligned} \tag{5.17}$$

Now choosing $(\phi, \psi) = (u_m, 0)$ and $(\phi, \psi) = (0, v_m)$ in (ii), we have

$$\left|\langle u_m, v_m\rangle - \int_\Omega \tilde{h}(\underline{u}_{\rho\mu}, u_m)u_m dx\right| \le \varepsilon_m,$$
$$\left|\langle u_m, v_m\rangle - \int_\Omega h(\underline{v}_{\tau\nu}, v_m)v_m dx\right| \le \varepsilon_m. \tag{5.18}$$

On the other hand, from (i), we obtain

$$\langle u_m, v_m\rangle - \int_\Omega H(\underline{v}_{\tau\nu}, v_m)dx - \int_\Omega \tilde{H}(\underline{u}_{\rho\mu}, u_m)dx = c + \delta_m. \tag{5.19}$$

Combining (5.17) and (5.19) and using (5.12), we get

$$\begin{aligned} 2c + 2\delta_m &= \langle u_m, v_m\rangle - 2\int_\Omega H(\underline{v}_{\tau\nu}, v_m)dx + \langle u_m, v_m\rangle - 2\int_\Omega \tilde{H}(\underline{u}_{\rho\mu}, u_m)dx \\ &\ge -2\varepsilon_m + \int_\Omega \tilde{h}(\underline{u}_{\rho\mu}, u_m)u_m dx + \int_\Omega h(\underline{v}_{\tau\nu}, v_m)v_m dx \\ &\quad -2\int_\Omega \tilde{H}(\underline{u}_{\rho\mu}, u_m)dx - 2\int_\Omega H(\underline{v}_{\tau\nu}, v_m)dx \\ &\ge -2\varepsilon_m + (\theta - 2)\int_{\Omega\cap\{v_m>T\}} H(\underline{v}_{\tau\nu}, v_m)dx + (\theta - 2)\int_{\Omega\cap\{u_m>T\}} \tilde{H}(\underline{u}_{\rho\mu}, u_m)dx + C, \end{aligned}$$

where we have used the fact

$$\int_{\Omega\cap\{v_m\le T\}} H(\underline{v}_{\tau\nu}, v_m)dx < C \quad \text{and} \quad \int_{\Omega\cap\{u_m\le T\}} \tilde{H}(\underline{u}_{\rho\mu}, v_m)dx < C,$$

which follows from the definition of $h(r, t)$, $H(r, t)$ and (3.5). Therefore,

$$\int_\Omega H(\underline{v}_{\tau\nu}, v_m)dx \le C \quad \text{and} \quad \int_\Omega \tilde{H}(\underline{u}_{\rho\mu}, u_m)dx \le C. \tag{5.20}$$

Using (5.12), similarly it can be also shown that

$$\int_\Omega h(\underline{v}_{\tau\nu}, v_m)v_m dx \le C, \quad \int_\Omega \tilde{h}(\underline{u}_{\rho\mu}, u_m)u_m dx \le C. \tag{5.21}$$

Observe that $h(\underline{v}_{\tau\nu}, v_m) = 0$ if $v_m \leq 0$ and $h(\underline{v}_{\tau\nu}, v_m)u_m \leq 0$ if $u_m \leq 0$. Therefore applying Young's inequality, (5.11), (5.20) and the fact that $\underline{u}_{\rho\mu}$ and $\underline{v}_{\tau\nu}$ are bounded (see (5.2)) yields

$$\begin{aligned}
\int_\Omega h(\underline{v}_{\tau\nu}, v_m)u_m dx &\leq \int_{\Omega\cap\{v_m\geq 0,\, u_m\geq 0\}} h(\underline{v}_{\tau\nu}, v_m)u_m dx \\
&\leq \frac{1}{q+1}\int_{\Omega\cap\{v_m\geq 0,\, u_m\geq 0\}} u_m^{q+1} dx + \frac{q}{q+1}\int_{\Omega\cap\{v_m\geq 0,\, u_m\geq 0\}} h(\underline{v}_{\tau\nu}, v_m)^{\frac{q+1}{q}} dx \\
&\leq \int_\Omega \tilde{H}(\underline{u}_{\rho\mu}, u_m)dx + \frac{q}{q+1}\int_{\Omega\cap\{v_m\geq 0,\, u_m\geq 0\}} (\underline{v}_{\tau\nu} + v_m^+)^{\frac{p(q+1)}{q}} dx \\
&\leq C_1 + C(q)\Bigg(\int_\Omega \underline{v}_{\tau\nu}^{\frac{p(q+1)}{q}} dx + \int_{\Omega\cap\{v_m\geq 0,\, u_m\geq 0\}} v_m^{\frac{p(q+1)}{q}} dx\Bigg) \\
&\leq C_1 + C_2 + C_3\Bigg(\int_{\Omega\cap\{v_m\geq 0,\, u_m\geq 0\}} v_m^{p+1} dx\Bigg)^{\frac{(q+1)p}{(p+1)q}} |\Omega|^{\frac{q-p}{(p+1)q}} \\
&\leq C_1 + C_2 + C_4\Bigg(\int_{\Omega\cap\{v_m\geq 0,\, u_m\geq 0\}} H(\underline{v}_{\tau\nu}, v_m)dx\Bigg)^{\frac{(q+1)p}{(p+1)q}} < C.
\end{aligned} \tag{5.22}$$

In above estimate we have also used the fact $p \leq q$ implies $(q+1)p/q \leq p+1$.

Similarly we can show that $\int_\Omega \tilde{h}(\underline{u}_{\rho\mu}, u_m)v_m dx < C$. Therefore substituting back in (5.17), we obtain $\|u_m\| \leq C$ and $\|v_m\| \leq C$. □

5.2 Finite dimensional problem

Since the functional I is strongly indefinite and defined in infinite dimensional space, no suitable linking theorem is available. We therefore approximate (5.1) with a sequence of finite dimensional problems.

Associated to the eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$ of $((-\Delta)^s, X_0)$, there exits an orthogonal basis $\{\varphi_1, \varphi_2, \cdots\}$ of corresponding eigen functions in X_0 and $\{\varphi_1, \varphi_2, \cdots\}$ is an orthonormal basis for $L^2(\Omega)$. We set

$$\begin{aligned}
E_n^+ &:= \operatorname{span}\{(\varphi_i, \varphi_i) : i = 1, 2, \cdots, n\}, \\
E_n^- &:= \operatorname{span}\{(\varphi_i, -\varphi_i) : i = 1, 2, \cdots, n\}, \\
E_n &:= E_n^+ \oplus E_n^-.
\end{aligned}$$

Let $\psi_0 \in X_0$ be a fixed nonnegative function with $\|\psi_0\| = 1$ and

$$Q_{n,\psi_0} := \{r(\psi_0, \psi_0) + w : w \in E_n^-, \|w\| \leq R_0, 0 \leq r \leq R_1\},$$

where R_0 and R_1 are chosen in Lemma 5.7. Here we recall that these constants depend only on ψ_0, p, q. Next, define

$$H_{n,\psi_0} := \mathbb{R}(\psi_0, \psi_0) \oplus E_n, \quad H^+_{n,\psi_0} := \mathbb{R}(\psi_0, \psi_0) \oplus E_n^+, \quad H^-_{n,\psi_0} := \mathbb{R}(\psi_0, \psi_0) \oplus E_n^-.$$

$$\Gamma_{n,\psi_0} := \{\pi \in C(Q_{n,\psi_0}, H_{n,\psi_0}) : \pi(u, v) = (u, v) \text{ on } \partial Q_{n,\psi_0}\},$$

and

$$c_{n,\psi_0} := \inf_{\pi\in\Gamma_{n,\psi_0}} \max_{(u,v)\in Q_{n,\psi_0}} I\big(\pi(u, v)\big).$$

Using an intersection theorem (see [33, Proposition 5.9]), we have

$$\pi(Q_{n,\psi_0}) \cap (\partial B_\varrho \cap E^+) \neq \emptyset, \quad \forall \pi \in \Gamma_{n,\psi_0}.$$

Thus there exists an $(u, v) \in (Q_{n,\psi_0})$ such that $\pi(u, v) \in \partial B_\varrho \cap E^+$. Combining this with Lemma 5.6, we get $I(\pi(u, v)) \geq \sigma$. This in turn implies $c_{n,\psi_0} \geq \sigma > 0$. Our next goal is to show that c_{n,ψ_0} has an upper bound. For that, we observe that the identity map $\mathrm{Id} : Q_{n,\psi_0} \to H_{n,\psi_0}$ is in Γ_{n,ψ_0}. Thus for an element of the form $z := r(\psi_0, \psi_0) + (u, -u) \in Q_{n,\psi_0}$, we compute

$$\begin{aligned} I(z) &= \langle r\psi_0 + u, r\psi_0 - u\rangle - \int_\Omega H(\underline{v}_{\tau\nu}, r\psi_0 - u)dx - \int_\Omega \tilde{H}(\underline{u}_{\rho\mu}, r\psi_0 + u)dx \\ &= r^2 - \|u\|^2 - \left[\int_\Omega H(\underline{v}_{\tau\nu}, r\psi_0 - u)dx + \int_\Omega \tilde{H}(\underline{u}_{\rho\mu}, r\psi_0 + u)dx\right] \leq R_1^2, \end{aligned}$$

where in the last inequality we have used (5.14). Consequently,

$$\max_{z\in Q_{n,\psi_0}} I(z) \leq R_1^2.$$

Therefore,

$$c_{n,\psi_0} \leq \max_{z\in Q_{n,\psi_0}} I(\mathrm{Id}(z)) = \max_{z\in Q_{n,\psi_0}} I(z) \leq R_1^2.$$

Hence $0 < \sigma \leq c_{c,\psi_0} \leq R_1^2$. We remark here that upper and lower bound do not depend on n. Define,

$$I_{n,\psi_0} := I\big|_{H_{n,\psi_0}}.$$

Thus, in view of Lemmas 5.6 and 5.7, we see that geometry of Linking theorem holds for the functional I_{n,ψ_0}. Hence applying the linking theorem [33, Theorem 5.3] to I_{n,ψ_0}, we obtain a Palais-Smale sequence, which is bounded in view of Proposition 5.8 (also see [19, pg. 1046]). Therefore, using the fact that H_{n,ψ_0} is a finite dimensional space, we obtain the following proposition:

Proposition 5.9. *For every $n \in \mathbb{N}$ and for every $\psi_0 \in X_0$, a fixed nonnegative function with $\|\psi_0\| = 1$, the functional I_{n,ψ_0} has a critical point z_{n,ψ_0} such that*

$$z_{n,\psi_0} \in H_{n,\psi_0}, \quad I'_{n,\psi_0}(z_{n,\psi_0}) = 0, \quad I_{n,\psi_0}(z_{n,\psi_0}) = c_{n,\psi_0} \in [\sigma, R_1^2], \tag{5.23}$$

$$\|z_{n,\psi_0}\| \leq C, \tag{5.24}$$

where C does not depend on n.

5.3 Existence of solution of (5.1).

Step 1: Let $\psi_0 \in X_0$ be a fixed nonnegative function with $\|\psi_0\| = 1$. Then applying Proposition 5.9, we get a sequence $\{z_{n,\psi_0}\}_{n=1}^\infty$ satisfying (5.23) and (5.24). Consequently, there exists $(u_0, v_0) \in X_0 \times X_0$ such that

$$z_{n,\psi_0} := (u_{n,\psi_0}, v_{n,\psi_0}) \rightharpoonup (u_0, v_0) \quad \text{in} \quad X_0 \times X_0, \tag{5.25}$$

$$u_{n,\psi_0} \to u_0, \quad v_{n,\psi_0} \to v_0 \quad \text{in} \quad L^r(\mathbb{R}^N),\ 1 \leq r < 2_s^\star \quad \text{and a.e. in} \quad \Omega. \tag{5.26}$$

Further, applying Proposition 5.8, we conclude

$$\int_\Omega h(\underline{v}_{\tau\nu}, v_{n,\psi_0})v_{n,\psi_0}dx \leq C, \quad \int_\Omega \tilde{h}(\underline{u}_{\rho\mu}, u_{n,\psi_0})u_{n,\psi_0}dx \leq C. \tag{5.27}$$

$$\int_\Omega H(\underline{v}_{\tau\nu}, v_{n,\psi_0})dx \leq C, \quad \int_\Omega \tilde{H}(\underline{u}_{\rho\mu}, u_{n,\psi_0})dx \leq C. \tag{5.28}$$

Next, taking as test functions $(0,\psi)$ and $(\phi,0)$ in (5.23), where $\phi,\ \psi$ are arbitrary functions in $F_n := \mathrm{span}\{\varphi_i : i = 1,2,\cdots,n\}$, we obtain

$$\left\langle \psi, u_{n,\psi_0} \right\rangle = \int_\Omega h(\underline{v}_{\tau\nu}, v_{n,\psi_0})\psi dx \quad \forall \psi \in F_n, \tag{5.29}$$

$$\left\langle \phi, v_{n,\psi_0} \right\rangle = \int_\Omega \tilde{h}(\underline{u}_{\rho\mu}, u_{n,\psi_0})\phi dx \quad \forall \phi \in F_n. \tag{5.30}$$

Now applying (5.2), (5.26) and the fact that $p < N_s < 2_s^\star$, we also have $h(\underline{v}_{\tau\nu}, v_{n,\psi_0})$ and $h(\underline{v}_{\tau\nu}, v_0)$ are L^1 functions. Therefore, using (5.27), (5.26) and an argument similar to the one used in [18, Lemma 2.1], it follows that

$$h(\underline{v}_{\tau\nu}, v_{n,\psi_0}) \to h(\underline{v}_{\tau\nu}, v_0), \quad \tilde{h}(\underline{u}_{\rho\mu}, u_{n,\psi_0}) \to \tilde{h}(\underline{u}_{\rho\mu}, u_0) \quad \text{in} \quad L^1(\Omega).$$

Hence, taking the limit in (5.29) and (5.30) and using the fact that $\cup_{n=1}^\infty F_n$ is dense in X_0, it follows that

$$\langle \psi, u_0 \rangle = \int_\Omega h(\underline{v}_{\tau\nu}, v_0)\psi dx \quad \text{and} \quad \langle \phi, v_0 \rangle = \int_\Omega \tilde{h}(\underline{u}_{\rho\mu}, u_0)\phi dx,$$

for all $\phi, \psi \in X_0$. As a consequence,

$$(-\Delta)^s u_0 = h(\underline{v}_{\tau\nu}, v_0) \quad \text{and} \quad (-\Delta)^s v_0 = \tilde{h}(\underline{u}_{\rho\mu}, u_0), \quad u_0 = v_0 = 0 \quad \text{in } \Omega^c. \tag{5.31}$$

Step 2: In this step we show that u_0 and v_0 are nontrivial and nonnegative.

Suppose not, we assume $u_0 \equiv 0$ in X_0. Plugging back to the equation (5.31), it implies $(-\Delta)^s v_0 = 0$. As a consequence

$$0 = \int_\Omega v_0(-\Delta)^s v_0\, dx = \int_\Omega |(-\Delta)^{\frac{s}{2}} v_0|^2\, dx = \|v_0\|_{X_0}^2.$$

Therefore, $v_0 \equiv 0$ in X_0, that is, $u_{n,\psi_0} \rightharpoonup 0$ and $v_{n,\psi_0} \rightharpoonup 0$ in X_0. Consequently, $u_{n,\psi_0} \to 0$ and $v_{n,\psi_0} \to 0$ in $L^r(\Omega)$ for $1 \le r < 2_s^\star$. Since $p+1,\ q+1 < 2_s^\star$, computing as in (5.22) we obtain

$$\begin{aligned}
\lim_{n\to\infty} \int_\Omega h(\underline{v}_{\tau\nu}, v_{n,\psi_0}) u_{n,\psi_0} dx &\le \lim_{n\to\infty} C\Bigg(\int_{\Omega\cap\{v_{n,\psi_0}\ge 0,\ u_{n,\psi_0}\ge 0\}} u_{n,\psi_0}^{q+1} dx \\
&\qquad + \int_{\Omega\cap\{v_{n,\psi_0}\ge 0,\ u_{n,\psi_0}\ge 0\}} h(\underline{v}_{\tau\nu}, v_{n,\psi_0})^{\frac{q+1}{q}} dx \Bigg) \\
&\le \lim_{n\to\infty} C \int_{\Omega\cap\{v_{n,\psi_0}\ge 0\}} [(\underline{v}_{\tau\nu} + v_{n,\psi_0})^p - \underline{v}_{\tau\nu}^p]^{\frac{q+1}{q}} dx \\
&\le \lim_{n\to\infty} C \int_\Omega \Big[|v_{n,\psi_0}|^p + |\underline{v}_{\tau\nu}|^{p-1} |v_{n,\psi_0}| \Big]^{\frac{q+1}{q}} dx \\
&\le \lim_{n\to\infty} C \Bigg[\int_\Omega |v_{n,\psi_0}|^{\frac{p}{q}(q+1)} dx + \int_\Omega |\underline{v}_{\tau\nu}|^{\frac{1}{q}(p-1)(q+1)} |v_{n,\psi_0}|^{\frac{q+1}{q}} dx \Bigg] \\
&= 0,
\end{aligned}$$

where for the last inequality we have used the fact that $\frac{p}{q}(q+1) \le p+1$ (since $p \le q$), (5.2) and the fact that $\frac{q+1}{q} < 2_s^\star$ (since $\frac{N-2s}{N+2s} < 1 < q$). Similarly it follows that $\int_\Omega \tilde{h}(\underline{u}_{\rho\mu}, u_{n,\psi_0}) v_{n,\psi_0} dx \to 0$. As a consequence, taking $\psi = u_{n,\psi_0}$ in (5.29) and $\phi = v_{n,\psi_0}$ in (5.30) yields $\|u_{n,\psi_0}\| \to 0$ and $\|v_{n,\psi_0}\| \to 0$ respectively. Hence, $u_{n,\psi_0} \to 0$ and $v_{n,\psi_0} \to 0$ strongly in X_0. This in turn, implies $\left\langle u_{n,\psi_0}, v_{n,\psi_0} \right\rangle \to (0,0)$.

Further, combining (5.15) along with (5.14) and (5.2) yields $\int_\Omega H(\underline{v}_{\tau\nu}, v_{n,\psi_0}) dx \to 0$ and $\int_\Omega \tilde{H}(\underline{u}_{\rho\mu}, u_{n,\psi_0}) dx \to 0$. Hence, $c_{n,\psi_0} = I_{n,\psi_0}(u_{n,\psi_0}, v_{n,\psi_0}) \to 0$. This is a contradiction to the fact that $c_{n,\psi_0} \in [\sigma, R_1^2]$. Therefore, $u_0,\ v_0$ are nontrivial.

Since $u_0 \in X_0$, by direct computation it is easy to see that $u_0^+,\ u_0^- \in X_0$. Thus, taking the test function as u_0^- for the first equation in (5.31) yields $0 \leq \langle u_0^+, u_0^- \rangle - \|u_0^-\|^2 \leq -\|u_0^-\|^2$, i.e., $u_0 \geq 0$ a.e.. Similarly, $v_0 \geq 0$. As a result, step 2 follows.

Hence we obtain the existence of a nonnegative nontrivial solution $(u_0, v_0) \in X_0 \times X_0$ of (5.1).

5.4 Proof of Theorem 1.4 completed

In order to construct the second solution of (3.4), we define

$$u_{\rho\mu} = \underline{u}_{\rho\mu} + u_0, \quad v_{\tau\nu} = \underline{v}_{\tau\nu} + v_0, \tag{5.32}$$

where (u_0, v_0) are as defined in Step 1 of Section 5.3. Clearly $u_{\rho\mu} \geq \underline{u}_{\rho\mu}$ and $v_{\tau\nu} \geq \underline{v}_{\tau\nu}$. Moreover, as u and v are nontrivial element in X_0, there exist two positive measure sets $\Omega', \Omega'' \subset \Omega$ such that $u_0 > 0$ in Ω' and , $v_0 > 0$ in Ω''. Thus $u_{\rho\mu} > \underline{u}_{\rho\mu}$ in Ω' and $v_{\tau\nu} > \underline{v}_{\tau\nu}$ in Ω''. Further, as $(u_0, v_0) \in X_0 \times X_0$ is a solution of (5.1), we have

$$\langle u_0, \psi \rangle = \int_\Omega h(\underline{v}_{\tau\nu}, v_0)\psi dx, \quad \langle v_0, \phi \rangle = \int_\Omega \tilde{h}(\underline{u}_{\rho\mu}, u_0)\phi dx, \quad \forall \phi,\ \psi \in X_0. \tag{5.33}$$

Set

$$\mathcal{T}(\Omega) := \{\tilde{\psi} \in C^\infty(\Omega) : \text{ there exists } \psi \in C_0^\infty(\Omega) \text{ such that } \tilde{\psi} = \mathbb{G}_s[\psi]\}.$$

This is a space of test function defined in [1, Page 41]. By [1, Lemma 5.6], $\mathcal{T}(\Omega) \subset X_0$. Therefore, we deduce from (5.33) that

$$\begin{aligned} \int_\Omega u_0(-\Delta)^s\psi\, dx &= \langle u_0, \psi \rangle = \int_\Omega h(\underline{v}_{\tau\nu}, v_0)\psi dx \quad \forall \psi \in \mathcal{T}(\Omega), \\ \int_\Omega v_0(-\Delta)^s\phi\, dx &= \langle v_0, \phi \rangle = \int_\Omega \tilde{h}(\underline{u}_{\rho\mu}, u_0)\phi dx \quad \forall \phi \in \mathcal{T}(\Omega). \end{aligned} \tag{5.34}$$

Moreover, [1, Lemma 5.12 and Lemma 5.13] ensures that $\mathcal{T}(\Omega) \subset \mathbb{X}_s(\Omega)$ and

$$\begin{aligned} \int_\Omega u_0(-\Delta)^s\psi\, dx &= \int_\Omega h(\underline{v}_{\tau\nu}, v_0)\psi dx \quad \forall \psi \in \mathbb{X}_s(\Omega), \\ \int_\Omega v_0(-\Delta)^s\phi\, dx &= \int_\Omega \tilde{h}(\underline{u}_{\rho\mu}, u_0)\phi dx \quad \forall \phi \in \mathbb{X}_s(\Omega). \end{aligned} \tag{5.35}$$

This means that (u_0, v_0) is a weak solution of

$$\begin{cases} (-\Delta)^s u_0 = (\underline{v}_{\tau\nu} + v_0)^p - \underline{v}_{\tau\nu}^p & \text{in } \Omega \\ (-\Delta)^s v_0 = (\underline{u}_{\rho\mu} + u_0)^q - \underline{u}_{\rho\mu}^q & \text{in } \Omega \\ u_0 = v_0 = 0 \quad \text{in } \Omega^c. \end{cases} \tag{5.36}$$

Hence, $(u_{\rho\mu}, v_{\tau\nu})$, as defined in (5.32), is clearly a weak solution of (3.4).

If $\mu, \nu \in L^r(\Omega) \cap L^\infty_{loc}(\Omega)$ then by Theorem 4.2, $u_{\rho\mu}, v_{\tau\nu}, \underline{u}_{\rho\mu}, \underline{v}_{\tau\nu} \in C^\alpha_{loc}(\Omega)$, for some $\alpha \in (0, 2s)$. Therefore, $u_0, v_0 \in C^\alpha_{loc}(\Omega)$. Also since we have $(-\Delta)^s u_0, (-\Delta)^s v_0 \geq 0$ in Ω and $0 \not\equiv u_0, v_0 \geq 0$ in $\mathbb{R}^N$, applying the strong maximum principle [41, Proposition 2.17], we have $u_0, v_0 > 0$ in Ω. Hence from (5.32), we deduce $u_{\rho\mu} > \underline{u}_{\rho\mu}$ and $v_{\tau\nu} > \underline{v}_{\tau\nu}$. In view of Remark 3.2, this completes the proof. □

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References

[1] Abatangelo, N., Large solutions for fractional Laplacian operators, Ph.D. dissertation (2015).

[2] Bhakta, M.; Nguyen, P-T, Boundary value problems with measures for fractional elliptic equations involving source nonlinearities (arXiv:1801.01544).

[3] Bidaut-Véron, M-F.; Vivier, L., An elliptic semilinear equation with source term involving boundary measures: the subcritical case, *Rev. Mat. Iberoamericana* 16 (2000), no. 3, 477–513.

[4] Bidaut-Véron, M-F.; Yarur, C., Semilinear elliptic equations and systems with measure data: existence and a priori estimates, *Adv. Differential Equations* 7 (2002), no. 3, 257–296.

[5] Bidaut-Véron, M-F.; Nguyen, Q-H.; Véron, L., Quasilinear and Hessian Lane-Emden type systems with measure data (arXiv:1705.08136).

[6] Benilan, P.; Brezis, H.; Crandall, M. G., A semilinear equation in $L^1(\mathbb{R}^N)$, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) 2 (1975), no. 4, 523–555.

[7] Bonheure, D.; dos Santos E. M.; Tavares, H., Hamiltonian elliptic systems: a guide to variational frameworks, *Port. Math.* 71 (2014), no. 3-4, 301–395.

[8] Caffarelli, L.; Silvestre, L, The Evans–Krylov theorem for nonlocal fully non linear equations, *Ann. Math.* 174(2) (2011), 1163–1187.

[9] Chen, H.; Felmer, P.; Véron, L., Elliptic equations involving general subcritical source nonlinearity and measures (2014) (arxiv:1409.3067).

[10] Chen, H.; Quaas, A., Classification of isolated singularities of nonnegative solutions to fractional semilinear elliptic equations and the existence results, *J. London Math. Soc.* (2) 97 (2018), 196–221.

[11] Chen, Z.; Song, R., Estimates on Green functions and Poisson kernels for symmetric stable process, *Math. Ann.* 312 (1998), 465–501.

[12] Chen, H; Véron, L, Semilinear fractional elliptic equations involving measures, *J. Differential Equations* 257 (2014), no. 5, 1457–1486.

[13] Cowan, C. Liouville theorems for stable Lane-Emden systems with biharmonic problems, *Nonlinearity* 26 (2013), no. 8, 2357–2371.

[14] Dahmani, Z.; Karami, F.; Kerbal, S., Nonexistence of positive solutions to nonlinear nonlocal elliptic systems, *J. Math. Anal. Appl.* 346(1) (2008), 22–29.

[15] Felmer, P; Quaas, A.; Tan, J., Positive solutions of nonlinear Schrodinger equation with the fractional Laplacian, *Proc. R. Soc. Edinb., Sect. A*, Math. 142 (2012), 1–26.

[16] Ferrero, A.; and Saccon, C., Existence and multiplicity results for semilinear elliptic equations with measure data and jumping nonlinearities, Topol. Methods Nonlinear Anal. Volume 30, Number 1 (2007), 37–65.

[17] de Figueiredo, D. G.; Felmer, P, A Liouville-type theorem for elliptic systems, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) 21 (1994), 387–397.

[18] de Figueiredo, D. G.; Miyagaki, O. H.; Ruf, B., Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range, *Calc. Var. Partial Differential Equations* 3 (1995), no. 2, 139–153.

[19] de Figueiredo, D. G.; do , J. Marcos; Ruf, B., Critical and subcritical elliptic systems in dimension two, *Indiana Univ. Math. J.* 53 (2004), no. 4, 1037–1054.

[20] Gkikas, K. T; Nguyen, P.-T., On the existence of weak solutions of semilinear elliptic equations and systems with Hardy potentials, *J. Differential Equations* 266 (2019), no. 1, 833–875.

[21] García-Huidobro, M.; Manásevich, R.; Mitidieri, E.; Yarur, C. S., Existence and nonexistence of positive singular solutions for a class of semilinear elliptic systems, *Arch. Rational Mech. Anal.* 140 (1997), no. 3, 253–284.

[22] Lam, N.; Lu, G., Elliptic equations and systems with subcritical and critical exponential growth without the Ambrosetti-Rabinowitz condition, *J. Geom. Anal.* 24 (2014), no. 1, 118–143.

[23] Mitidieri, E., Nonexistence of positive solutions of semilinear elliptic systems in $\mathbb{R}^N$, *Differential Integral Equations* 9 (1996), 465–479.

[24] Molica Bisci, G.; Radulescu, V. D.; Servadei, R., Variational Methods for Nonlocal Fractional Problems, Encyclopedia of Mathematics and its Applications, 162. Cambridge University Press, Cambridge, 2016. xvi+383 pp. ISBN: 978-1-107-11194-3.

[25] Mosconi, S. Perera, K.; Squassinaa, M. Yang, Y., The Brezis-Nirenberg problem for the fractionalp-Laplacian, *Calc. Var. Partial Differential Equations* 55 (2016), 55–105.

[26] Naito, Y; Sato, T., Positive solutions for semilinear elliptic equations with singular forcing terms, *J. Differential Equations* 235 (2007), no. 2, 439–483.

[27] Nguyen, P. T; Véron, L, Boundary singularities of solutions to semilinear fractional equations, Adv. Nonlinear Stud. 18 (2018), 237–267.

[28] Poláčik, P.; Quittner, P.; Souplet, P., Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems. *Duke Math.* J. 139 (2007), no. 3, 555–579.

[29] Quaas, A., Xia, A., Liouville type theorems for nonlinear elliptic equations and systems involving fractional Laplacian in

the half space. *Calc. Var. Partial Differential Equations* 52 (2015), no. 3-4, 641–659.
[30] Quaas, A., Xia, A., A Liouville type theorem for Lane-Emden systems involving the fractional Laplacian, *Nonlinearity* 29 (2016), no. 8, 2279–2297.
[31] Quaas, A., Xia, A., Existence results of positive solutions for nonlinear cooperative elliptic systems involving fractional Laplacian, *Commun. Contemp. Math.* 20 (2018), no. 3, 1750032, 22 pp.
[32] Rabinowitz, P. H., Some critical point theorems and applications to semilinear elliptic partial differential equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) 5 (1978), no. 1, 215–223.
[33] Rabinowitz, P. H., Minimax methods in critical point theory with applications to differential equations, *CBMS Regional Conference Series in Mathematics*, 65. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986. viii+100 pp. ISBN: 0-8218-0715-3.
[34] Ros-Oton, X.; Serra, J., The Dirichlet problem for the fractional Laplacian: regularity up to the boundary, *J. Math. Pures Appl.* (9) 101 (2014), no. 3, 275–302.
[35] Ros-Oton, X.; Serra, J., The extremal solution for the fractional Laplacian, *Calc. Var. Partial Differential Equations* 50 (2014), no. 3-4, 723–750.
[36] Ros-Oton, X.; Serra, J., The Pohozaev identity for the fractional Laplacian. *Arch. Ration. Mech. Anal.* 213 (2014), no. 2, 587–628.
[37] Reichel, W.; Zou, H., Non-existence results for semilinear cooperative elliptic systems via moving spheres, *J. Differential Equations* 161 (2000), 219–243.
[38] Serrin, J.; Zou, H., Existence of positive solutions of the Lane-Emden system, *Atti Sem. Mat. Fis. Univ. Modena* 46 (1998), suppl., 369–380.
[39] Servadei, R.; Valdinoci, E, Variational methods for non-local operators of elliptic type, *Discrete Contin. Dyn. Syst.* 33 (2013), no. 5, 2105–2137.
[40] Servadei, R.; Valdinoci, E, Fractional Laplacian equations with critical Sobolev exponent, *Rev. Mat. Complut.* 28 (2015), no. 3, 655–676.
[41] Silvestre, L., Regularity of the obstacle problem for a fractional power of the Laplace operator, *Comm. Pure Appl. Math.* 60 (2007), no. 1, 67–112.
[42] Stein, E.M., Singular Integrals and Differentiability Properties of Functions, Princeton University Press, 1970.
[43] Veron, L., Elliptic equations involving measures, in Stationary Partial Differential Equations, Handbook of Equations vol. I, Elsevier B.V., pp. 593–712 (2004).