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# Pseudo almost periodic solutions for a class of differential equation with delays depending on state

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**Abstract:** In this paper, the exponential dichotomy, and Tikhonov and Banach fixed point theorems are used to study the existence and uniqueness of pseudo almost periodic solutions of a class of iterative functional differential equations of the form

$$x'(t) = \sum_{n=1}^k \sum_{l=1}^{\infty} C_{l,n}(t)(x^{[n]}(t))^l + G(t),$$

where  $x^{[n]}(t)$  denotes  $n$ th iterate of  $x(t)$ .

**Keywords:** Iterative differential equation, pseudo almost periodic solutions, fixed point theorem

**MSC:** 39B12, 39B82

## 1 Introduction

Delay differential equation of the form

$$x'(t) = f(t, x(t), x(t - \tau_1(t)), \dots, x(t - \tau_k(t)))$$

has been discussed in [1, 2]. In particular, the delay functions  $\tau_j(t)$ ,  $j = 0, 1, \dots, k$  depend not only on unknown function, but also state, the delay functions  $\tau_j(t, x(t))$ ,  $j = 0, 1, \dots, k$  have been studied in many literatures. In [3], Cooke pointed out that it is highly desirable to establish the existence and stability properties of periodic solutions for equations of the form

$$x'(t) + ax(t - h(t, x(t))) = F(t),$$

in which the lag  $h(t, x(t))$  implicitly involves  $x(t)$ . Si and Wang [4] discussed the smooth solutions of equation of

$$x'(t) = \lambda_1 x(t) + \lambda_2 x^{[2]}(t) + \dots + \lambda_n x^{[n]}(t) + f(t),$$

where  $x^{[1]}(t) = x(t)$ ,  $x^{[2]}(t) = x(x(t))$ ,  $x^{[3]}(t) = x(x(x(t)))$ ,  $\dots$ , i.e.,  $x^{[i]}(t)$  denotes  $i$ th iterate of  $x(t)$ ,  $i = 1, 2, \dots, n$ . Later, by Schröder transformation, Liu and Si [5] considered the analytic solutions of the form

$$x'(t) = \sum_{n=0}^k \sum_{l=1}^{\infty} C_{l,n}(t)(x^{[n]}(t))^l + G(t),$$

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where  $x^{[n]}(t)$  denotes  $n$ th iterate of  $x(t)$ ,  $n = 0, 1, 2, \dots, k$ . Recently, In [6], Zhao and Fečkan studied the periodic solution of

$$x'(t) = \sum_{n=1}^k \sum_{l=1}^{\infty} C_{l,n}(t)(x^{[n]}(t))^l + G(t), \quad (1.1)$$

where  $x^{[n]}(t)$  denotes  $n$ th iterate of  $x(t)$ ,  $n = 1, 2, \dots, k$ . For some various properties of solutions for several iterative functional differential equations, we refer the interested reader to [7]-[11].

On the other hand, the existence of pseudo almost periodic solutions is among the most attractive topics in qualitative theory of differential equations due to their applications, especially in biology, economics and physics [12]-[16]. As pointed out by Ait Das and Ezzinbi [13], it is an interesting thing to study the pseudo almost periodic systems with delays. It is obviously that iterative functional differential equations are special type state-dependent delay differential equations. In [17], Liu pointed that the properties of the almost periodic functions do not always hold in the set of pseudo almost periodic functions and given an example: when  $x(t)$  is a pseudo almost periodic function,  $x(x(t))$  may not be a pseudo almost periodic function. To the best of our knowledge, there are few results about pseudo almost periodic solutions for iterative functional differential equations except [17], [18] and [19].

In the present work, we propose an existence result for pseudo almost periodic solutions of Eq (1.1) by using Tikhonov fixed theorem. Uniqueness of the solution is achieved by Banach fixed point theorem.

This paper is organized as follows. In Section 2 we give some notes and establish the main existence result. In Section 3, we show that (1.1) has a unique pseudo almost periodic solution under some suitable conditions. Furthermore, we prove the stability depend on  $C_{l,n}$  and  $G$ . In Section 4, we present an example to illustrate the theory. Related problems are also studied in [20].

## 2 Existence result

In this section, the existence of pseudo almost periodic solutions of equation (1.1) will be studied. Throughout this paper, it will be assumed that

(H)  $C_{1,1}(t) : \mathbb{R} \rightarrow (-\infty, 0)$  is an almost periodic function,  $C_{l,n}(t) : \mathbb{R} \rightarrow \mathbb{R}$  are pseudo almost periodic functions, and

$$C_{1,1}^+ = \sup_{t \in \mathbb{R}} C_{1,1}(t) < 0, \quad C_{l,n}^+ = \sup_{t \in \mathbb{R}} |C_{l,n}(t)|, \quad G^+ = \sup_{t \in \mathbb{R}} |G(t)|,$$

where  $l = 1, 2, \dots; n = 2, \dots, k$ .

For convenience, we will use  $C(\mathbb{R}, \mathbb{R})$  to denote the set of all continuous functions from  $\mathbb{R}$  to  $\mathbb{R}$  endowed with the usual metric  $d(f, g) = \sum_{m=1}^{\infty} 2^{-m} \frac{\|f-g\|_m}{1+\|f-g\|_m}$  for  $\|f-g\|_m = \max_{t \in [-m, m]} |f(t) - g(t)|$ , so the topology on  $C(\mathbb{R}, \mathbb{R})$  is the uniform convergence on each compact intervals of  $\mathbb{R}$ . We also consider the set  $BC(\mathbb{R}, \mathbb{R})$  of all bounded and continuous functions from  $\mathbb{R}$  to  $\mathbb{R}$  with the norm  $\|f\| = \sup_{t \in \mathbb{R}} |f(t)|$ , so the topology on  $BC(\mathbb{R}, \mathbb{R})$  is the uniform convergence on  $\mathbb{R}$ . Denote by  $AP(\mathbb{R}, \mathbb{R})$  the set of all almost periodic functions from  $\mathbb{R}$  to  $\mathbb{R}$ . Define the set

$$PAP_0(\mathbb{R}, \mathbb{R}) = \left\{ g \in BC(\mathbb{R}, \mathbb{R}) \mid \lim_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^T |g(t)| dt = 0 \right\}.$$

A function  $\varphi \in BC(\mathbb{R}, \mathbb{R})$  is called pseudo almost periodic if it can be expressed as  $\varphi = h + g$ , where  $h \in AP(\mathbb{R}, \mathbb{R})$  and  $g \in PAP_0(\mathbb{R}, \mathbb{R})$ . The collection of such functions will be denoted by  $PAP(\mathbb{R}, \mathbb{R})$ . In particular,  $(PAP(\mathbb{R}, \mathbb{R}), \|\cdot\|)$  is a Banach space [21]. For  $M, L > 0$ , define

$$B_C(M, L) = \left\{ \varphi \in C(\mathbb{R}, \mathbb{R}) \mid |\varphi(t)| \leq M, |\varphi(t_2) - \varphi(t_1)| \leq L|t_2 - t_1|, \right. \\ \left. \text{for all } t, t_1, t_2 \in \mathbb{R} \right\},$$

$$B_S(M, L) = S(\mathbb{R}, \mathbb{R}) \cap B_C(M, L), \quad S \in \{PAP, AP, PAP_0\}.$$

From [17], it is easy to see that  $B_S(M, L)$ ,  $S \in \{PAP, AP, PAP_0, C\}$  are closed convex and bounded subsets of  $BC(\mathbb{R}, \mathbb{R})$  and  $C(\mathbb{R}, \mathbb{R})$ , respectively. Furthermore, by the Arzelà-Ascoli theorem, the subset  $B_C(M, L)$  is compact in  $C(\mathbb{R}, \mathbb{R})$ .

On the other hand, the subset  $B_C(M, L)$  is not precompact in  $BC(\mathbb{R}, \mathbb{R})$ , since a sequence  $\left\{M \operatorname{sech} \frac{L(t+n)}{M}\right\}_{n=1}^{\infty} \in B_C(M, L)$  has no a convergent subsequence in  $BC(\mathbb{R}, \mathbb{R})$ . As in [19], we have following Theorem.

**Theorem 2.1.** *The subsets  $B_S(M, L)$ ,  $S \in \{PAP, AP, PAP_0, C\}$  are not precompact in  $BC(\mathbb{R}, \mathbb{R})$ . They are precompact in  $C(\mathbb{R}, \mathbb{R})$ . Furthermore,  $B_{AP}(M, L)$  and  $B_{PAP_0}(M, L)$  are just precompact in  $C(\mathbb{R}, \mathbb{R})$ , while  $B_C(M, L)$  is compact in  $C(\mathbb{R}, \mathbb{R})$ .*

So (non-)compactness of  $B_{PAP}(M, L)$  in  $C(\mathbb{R}, \mathbb{R})$  is still open.

Finally, we recall Tikhonov fixed point theorem.

**Theorem 2.2.** (Tikhonov) *Let  $\Omega$  be a non-empty compact convex subset of a locally convex topological vector space  $X$ . Then any continuous function  $A : \Omega \rightarrow \Omega$  has a fixed point.*

Now we apply Theorem 2.2. Let  $X$  be either  $PAP(\mathbb{R}, \mathbb{R})$  or  $BC(\mathbb{R}, \mathbb{R})$  but fixed. If  $\varphi \in X$ , from Corollary 5.4 in [21], it follows  $\varphi^{[2]} \in X$ . Consider an auxiliary equation

$$x'(t) = C_{1,1}(t)x(t) + \sum_{n=2}^k C_{1,n}(t)\varphi^{[n]}(t) + \sum_{n=1}^k \sum_{l=2}^{\infty} C_{l,n}(t)(\varphi^{[n]}(t))^l + G(t). \quad (2.2)$$

where  $G \in PAP(\mathbb{R}, \mathbb{R})$  and  $C_{1,1}(t) < 0$ . It is easy to see that the linear equation

$$x'(t) = C_{1,1}(t)x(t)$$

admits an exponential dichotomy on  $\mathbb{R}$ , by Theorem 2.3 in [15], we know that (2.2) has exactly one solution

$$x_{\varphi}(t) = \int_{-\infty}^t e^{\int_s^t C_{1,1}(u)du} \left( \sum_{n=2}^k C_{1,n}(s)\varphi^{[n]}(s) + \sum_{n=1}^k \sum_{l=2}^{\infty} C_{l,n}(s)(\varphi^{[n]}(s))^l + G(s) \right) ds$$

in  $X$ .

Let  $A : B_C(M, L) \rightarrow BC(\mathbb{R}, \mathbb{R})$  be defined by

$$(A\varphi)(t) = \int_{-\infty}^t e^{\int_s^t C_{1,1}(u)du} \left( \sum_{n=2}^k C_{1,n}(s)\varphi^{[n]}(s) + \sum_{n=1}^k \sum_{l=2}^{\infty} C_{l,n}(s)(\varphi^{[n]}(s))^l + G(s) \right) ds, \quad t \in \mathbb{R}. \quad (2.3)$$

Note  $A : B_{PAP}(M, L) \rightarrow PAP(\mathbb{R}, \mathbb{R})$ .

**Lemma 2.1.** *For any  $\varphi, \psi \in B_C(M, L)$ ,  $t_1, t_2 \in \mathbb{R}$ , the following inequality hold,*

$$\|\varphi^{[n]} - \psi^{[n]}\| \leq \sum_{j=0}^{n-1} L^j \|\varphi - \psi\|, \quad n = 1, 2, \dots \quad (2.4)$$

*Proof.* It can be obtained by direct calculation by the definition of  $B_C(M, L)$ . □

**Lemma 2.2.** *Operator  $A : B_C(M, L) \rightarrow C(\mathbb{R}, \mathbb{R})$  is continuous.*

*Proof.* Let  $\varphi_j \rightarrow \varphi_0$  as  $j \rightarrow \infty$  for  $\varphi_j \in B_C(M, L)$ ,  $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$  uniformly on any compact interval  $[-m, m]$ ,  $m \in \mathbb{N}$  of  $\mathbb{R}$ . Set

$$h_j(s) = \sum_{n=2}^k C_{1,n}(s)\varphi_j^{[n]}(s) + \sum_{n=1}^k \sum_{l=2}^{\infty} C_{l,n}(s)(\varphi_j^{[n]}(s))^l + G(s), \quad j \in \mathbb{N}_0.$$

Then  $h_j \rightarrow h_0$  uniformly on  $[-m, m]$ . Next we have

$$\left| e^{\int_s^{-m} C_{1,1}(u) du} h_j(s) \right| \leq \left( M \sum_{n=2}^k C_{1,n}^+ + \sum_{n=1}^k \sum_{l=2}^{\infty} C_{l,n}^+ M^l + G^+ \right) e^{-C_{1,1}^+(s+m)}, \quad s \in [-\infty, -m].$$

Since

$$\int_{-\infty}^{-m} e^{-C_{1,1}^+(s+m)} ds = -\frac{1}{C_{1,1}^+} < \infty,$$

we can apply the Lebesgue dominated convergence theorem to obtain  $(A\varphi_j)(-m) \rightarrow (A\varphi_0)(-m)$ . From (2.2),

$$x_j'(t) - x_0'(t) = C_{1,1}(t)(x_j(t) - x_0(t)) + h_j(t) - h_0(t) \quad (2.5)$$

for  $x_j(t) = (A\varphi_j)(t)$ ,  $k \in \mathbb{N}_0$ . Integrating the both sides of (2.5) from  $-m$  to  $t$ , we have

$$x_j(t) - x_0(t) = x_j(-m) - x_0(-m) + \int_{-m}^t C_{1,1}(s)(x_j(s) - x_0(s)) ds + \int_{-m}^t (h_j(s) - h_0(s)) ds$$

and

$$|x_j(t) - x_0(t)| \leq |x_j(-m) - x_0(-m)| + 2m\|h_j - h_0\|_m - C_{1,1}^+ \int_{-m}^t |x_j(s) - x_0(s)| ds$$

for any  $t \in [-m, m]$ . Then Gronwall's inequality implies

$$\|x_j - x_0\|_m \leq (|x_j(-m) - x_0(-m)| + 2m\|h_j - h_0\|_m) e^{-2C_{1,1}^+ m},$$

which means

$$\|A\varphi_j - A\varphi_0\|_m \leq (|A\varphi_j(-m) - A\varphi_0(-m)| + 2m\|h_j - h_0\|_m) e^{-2C_{1,1}^+ m}.$$

Hence  $A\varphi_j(t) \rightarrow A\varphi_0(t)$  uniformly on  $t \in [-m, m]$ . Since  $m \in \mathbb{N}$  is arbitrarily, we get  $A\varphi_j \rightarrow A\varphi_0$  in  $C(\mathbb{R}, \mathbb{R})$ , i.e.,  $A : B_C(M, L) \rightarrow C(\mathbb{R}, \mathbb{R})$  is continuous. This proves the continuity of  $A$ .  $\square$

Of course, then  $A : B_{PAP}(M, L) \rightarrow C(\mathbb{R}, \mathbb{R})$  is continuous as well.

**Theorem 2.3.** Suppose (H) holds, then Eq. (1.1) has a solution

$$\varphi \in \overline{B_{PAP}(M, L)} \subset C(\mathbb{R}, \mathbb{R})$$

for the constants  $M$  and  $L$  satisfy

$$\sum_{n=1}^k \sum_{l=1}^{\infty} C_{l,n}^+ M^l + G^+ \leq 0, \quad (2.6)$$

$$2(M \sum_{n=2}^k C_{1,n}^+ + \sum_{n=1}^k \sum_{l=2}^{\infty} C_{l,n}^+ M^l + G^+) \leq L. \quad (2.7)$$

For any  $\varphi \in B_C(M, L)$ ,  $t, t_1, t_2 \in \mathbb{R}$ , from (2.6) we have

$$\begin{aligned} |(A\varphi)(t)| &\leq \sum_{n=2}^k C_{1,n}^+ \left| \int_{-\infty}^t e^{\int_s^t C_{1,1}(u) du} \varphi^{[n]}(s) ds \right| + \sum_{n=1}^k \sum_{l=2}^{\infty} C_{l,n}^+ \left| \int_{-\infty}^t e^{\int_s^t C_{1,1}(u) du} (\varphi^{[n]}(s))^l ds \right| \\ &\quad + \left| \int_{-\infty}^t e^{\int_s^t C_{1,1}(u) du} G(s) ds \right| \end{aligned}$$

$$\leq -\frac{M}{C_{1,1}^+} \sum_{n=2}^k C_{1,n}^+ - \frac{1}{C_{1,1}^+} \sum_{n=1}^k \sum_{l=2}^{\infty} C_{l,n}^+ M^l - \frac{G^+}{C_{1,1}^+} \\ \leq M.$$

Without loss of generality, assume  $t_2 \geq t_1$ , using (2.7)

$$\begin{aligned} |(A\varphi)(t_2) - (A\varphi)(t_1)| &\leq \left| \int_{-\infty}^{t_2} e^{\int_s^{t_2} C_{1,1}(u)du} \left( \sum_{n=2}^k C_{1,n}(s) \varphi^{[n]}(s) + \sum_{n=1}^k \sum_{l=2}^{\infty} C_{l,n}(s) (\varphi^{[n]}(s))^l \right. \right. \\ &\quad \left. \left. + G(s) \right) ds - \int_{-\infty}^{t_1} e^{\int_s^{t_1} C_{1,1}(u)du} \left( \sum_{n=2}^k C_{1,n}(s) \varphi^{[n]}(s) \right. \right. \\ &\quad \left. \left. + \sum_{n=1}^k \sum_{l=2}^{\infty} C_{l,n}(s) (\varphi^{[n]}(s))^l + G(s) \right) ds \right| \\ &\leq \sum_{n=2}^k \left[ \left| \int_{-\infty}^{t_1} e^{\int_s^{t_1} C_{1,1}(u)du} \left( e^{\int_{t_1}^{t_2} C_{1,1}(u)du} - 1 \right) C_{1,n}(s) \varphi^{[n]}(s) ds \right| \right. \\ &\quad \left. + \left| \int_{t_1}^{t_2} e^{\int_s^{t_2} C_{1,1}(u)du} C_{1,n}(s) \varphi^{[n]}(s) ds \right| \right] \\ &\quad + \sum_{n=1}^k \sum_{l=2}^{\infty} \left[ \left| \int_{-\infty}^{t_1} e^{\int_s^{t_1} C_{1,1}(u)du} \left( e^{\int_{t_1}^{t_2} C_{1,1}(u)du} - 1 \right) C_{l,n}(s) (\varphi^{[n]}(s))^l ds \right| \right. \\ &\quad \left. + \left| \int_{t_1}^{t_2} e^{\int_s^{t_2} C_{1,1}(u)du} C_{l,n}(s) (\varphi^{[n]}(s))^l ds \right| \right] \\ &\quad + \left[ \left| \int_{-\infty}^{t_1} e^{\int_s^{t_1} C_{1,1}(u)du} \left( e^{\int_{t_1}^{t_2} C_{1,1}(u)du} - 1 \right) G(s) ds \right| \right. \\ &\quad \left. + \left| \int_{t_1}^{t_2} e^{\int_s^{t_2} C_{1,1}(u)du} G(s) ds \right| \right] \\ &\leq 2M \sum_{n=2}^k C_{1,n}^+ |t_2 - t_1| + 2 \sum_{n=1}^k \sum_{l=2}^{\infty} C_{l,n}^+ M^l |t_2 - t_1| + 2G^+ |t_2 - t_1| \\ &\leq L |t_2 - t_1|. \end{aligned}$$

This shows that  $A\varphi \in B_C(M, L)$ . Hence  $A : B_C(M, L) \rightarrow B_C(M, L)$  and also  $A : B_{PAP}(M, L) \rightarrow B_{PAP}(M, L)$ . By Lemma 2.2 we get  $A : \overline{B_{PAP}(M, L)} \rightarrow \overline{B_{PAP}(M, L)}$ . So all conditions of Tikhonov fixed theorem are satisfied for  $A$  with  $\Omega = \overline{B_{PAP}(M, L)}$  and  $X = C(\mathbb{R}, \mathbb{R})$ . Thus there exists a fixed point  $\varphi$  in  $\overline{B_{PAP}(M, L)}$  of  $A$ . This is equivalent to say that  $\varphi$  is a solution of (1.1) in  $\overline{B_{PAP}(M, L)}$ . This completes the proof.  $\square$

### 3 Uniqueness and stability

In this section, uniqueness and stability of (1.1) will be proved.

In addition to the assumption of Theorem 2.3, suppose that

$$\sum_{n=2}^k \sum_{j=0}^{n-1} L^j C_{1,n}^+ + \sum_{n=1}^k \sum_{l=2}^{\infty} \sum_{j=0}^{n-1} L^j C_{l,n}^+ l M^{l-1} < -C_{1,1}^+, \quad (3.8)$$

then Eq. (1.1) has unique solution in  $B_{PAP}(M, L)$ .

For operator  $A$  from  $B_{PAP}(M, L)$  into  $B_{PAP}(M, L)$ , where we consider  $B_{PAP}(M, L)$  in  $BC(\mathbb{R}, \mathbb{R})$ . For  $\varphi, \psi \in B_{PAP}(M, L)$ , by (2.3) and (2.4)

$$\begin{aligned} |\varphi(t) - \psi(t)| &= |(A\varphi)(t) - (A\psi)(t)| \\ &\leq \sum_{n=2}^k \left| \int_{-\infty}^t e^{\int_s^t C_{1,1}(u)du} C_{1,n}(s) (\varphi^{[n]}(s) - \psi^{[n]}(s)) ds \right| \\ &\quad + \sum_{n=1}^k \sum_{l=2}^{\infty} \left| \int_{-\infty}^t e^{\int_s^t C_{1,1}(u)du} C_{l,n}(s) ((\varphi^{[n]}(s))^l - (\psi^{[n]}(s))^l) ds \right| \\ &\leq -\frac{1}{C_{1,1}^+} \left( \sum_{n=2}^k \sum_{j=0}^{n-1} L^j C_{1,n}^+ + \sum_{n=1}^k \sum_{l=2}^{\infty} \sum_{j=0}^{n-1} L^j C_{l,n}^+ l M^{l-1} \right) \|\varphi - \psi\| \\ &= \Gamma \|\varphi - \psi\|, \end{aligned}$$

where  $\Gamma = -\frac{1}{C_{1,1}^+} \left( \sum_{n=2}^k \sum_{j=0}^{n-1} L^j C_{1,n}^+ + \sum_{n=1}^k \sum_{l=2}^{\infty} \sum_{j=0}^{n-1} L^j C_{l,n}^+ l M^{l-1} \right)$ .

Thus

$$\|\varphi - \psi\| \leq \Gamma \|\varphi - \psi\|.$$

By (3.8), we know  $\Gamma < 1$  and the fixed point  $\varphi$  must be unique.  $\square$

The unique solution obtained in Theorem 3.1 depends continuously on the given functions  $C_{l,n}(t)$  and  $G(t)$ , for  $l = 1, 2, \dots, \infty; n = 1, \dots, k$ .

*Proof.* Under the assumptions of Theorem 3.1, if any functions  $C_{l,n}(t)$ ,  $\tilde{C}_{l,n}(t)$  and  $G(t)$ ,  $\tilde{G}(t)$  are given, and satisfy (2.6) and (2.7) for  $l = 1, \dots, \infty; n = 1, \dots, k$ . Then there are two unique corresponding functions  $x(t)$  and  $\tilde{x}(t)$  in  $B_{PAP}(M, L)$  such that

$$\begin{aligned} x(t) &= (Ax)(t) \\ &= \int_{-\infty}^t e^{\int_s^t C_{1,1}(u)du} \left( \sum_{n=2}^k C_{1,n}(s) x^{[n]}(s) + \sum_{n=1}^k \sum_{l=2}^{\infty} C_{l,n}(s) (x^{[n]}(s))^l + G(s) \right) ds \end{aligned}$$

and

$$\begin{aligned} \tilde{x}(t) &= (\tilde{A}\tilde{x})(t) \\ &= \int_{-\infty}^t e^{\int_s^t \tilde{C}_{1,1}(u)du} \left( \sum_{n=2}^k \tilde{C}_{1,n}(s) \tilde{x}^{[n]}(s) + \sum_{n=1}^k \sum_{l=2}^{\infty} \tilde{C}_{l,n}(s) (\tilde{x}^{[n]}(s))^l + \tilde{G}(s) \right) ds. \end{aligned}$$

Then

$$\|x - \tilde{x}\| = \|Ax - \tilde{A}\tilde{x}\| \leq \|Ax - A\tilde{x}\| + \|A\tilde{x} - \tilde{A}\tilde{x}\|,$$

i.e.,

$$\|x - \tilde{x}\| \leq \frac{\|A\tilde{x} - \tilde{A}\tilde{x}\|}{1 - \Gamma}. \quad (3.9)$$

Note

$$\begin{aligned} |(A\tilde{x})(t) - (\tilde{A}\tilde{x})(t)| &\leq \left| \int_{-\infty}^t e^{\int_s^t C_{1,1}(u)du} \left( \sum_{n=2}^k C_{1,n}(s) \tilde{x}^{[n]}(s) + \sum_{n=1}^k \sum_{l=2}^{\infty} C_{l,n}(s) (\tilde{x}^{[n]}(s))^l + G(s) \right) ds \right. \\ &\quad \left. - \int_{-\infty}^t e^{\int_s^t \tilde{C}_{1,1}(u)du} \left( \sum_{n=2}^k \tilde{C}_{1,n}(s) \tilde{x}^{[n]}(s) + \sum_{n=1}^k \sum_{l=2}^{\infty} \tilde{C}_{l,n}(s) (\tilde{x}^{[n]}(s))^l + \tilde{G}(s) \right) ds \right| \end{aligned}$$

$$\begin{aligned}
&\leq \left| \int_{-\infty}^t \left( e^{\int_s^t C_{1,1}(u)du} - e^{\int_s^t \tilde{C}_{1,1}(u)du} \right) \left( \sum_{n=2}^k C_{1,n}(s) \tilde{x}^{[n]}(s) + \sum_{n=1}^k \sum_{l=2}^{\infty} C_{l,n}(s) (\tilde{x}^{[n]}(s))^l \right. \right. \\
&\quad \left. \left. + G(s) \right) ds \right| + \sum_{n=2}^k \left| \int_{-\infty}^t e^{\int_s^t \tilde{C}_{1,1}(u)du} \left( C_{1,n}(s) - \tilde{C}_{1,n}(s) \right) \tilde{x}^{[n]}(s) ds \right| \\
&\quad + \sum_{n=1}^k \sum_{l=2}^{\infty} \left| \int_{-\infty}^t e^{\int_s^t \tilde{C}_{1,1}(u)du} \left( C_{l,n}(s) - \tilde{C}_{l,n}(s) \right) (\tilde{x}^{[n]}(s))^l ds \right| \\
&\quad + \left| \int_{-\infty}^t e^{\int_s^t \tilde{C}_{1,1}(u)du} \left( G(s) - \tilde{G}(s) \right) ds \right| \\
&\leq \frac{1}{C_{1,1}^+ \tilde{C}_{1,1}^+} \left( M \sum_{n=2}^k C_{1,n}^+ + \sum_{n=1}^k \sum_{l=2}^{\infty} C_{l,n}^+ M^l + G^+ \right) \|C_{1,1} - \tilde{C}_{1,1}\| \\
&\quad - \frac{M}{\tilde{C}_{1,1}^+} \sum_{n=2}^k \|C_{1,n} - \tilde{C}_{1,n}\| - \frac{1}{\tilde{C}_{1,1}^+} \sum_{n=1}^k \sum_{l=2}^{\infty} M^l \|C_{l,n} - \tilde{C}_{l,n}\| \\
&\quad - \frac{1}{\tilde{C}_{1,1}^+} \|G - \tilde{G}\| \\
&\leq \Delta \left( \sum_{l=1}^{\infty} \sum_{n=1}^k \|C_{l,n} - \tilde{C}_{l,n}\| + \|G - \tilde{G}\| \right), \tag{3.10}
\end{aligned}$$

where

$$\Delta = -\frac{1}{\tilde{C}_{1,1}^+} \max \left\{ -\frac{1}{C_{1,1}^+} \left( M \sum_{n=2}^k C_{1,n}^+ + \sum_{n=1}^k \sum_{l=2}^{\infty} C_{l,n}^+ M^l + G^+ \right), 1, M, M^2, M^3, \dots \right\}.$$

Using (3.9) and (3.10), we have

$$\|x - \tilde{x}\| \leq \frac{\Delta}{1 - \Gamma} \left( \sum_{l=1}^{\infty} \sum_{n=1}^k \|C_{l,n} - \tilde{C}_{l,n}\| + \|G - \tilde{G}\| \right).$$

This completes the proof.  $\square$

## 4 Example

In this section, an example is provided to illustrate that the assumptions of Theorem 2.3 do not self-contradict.

**Example 4.1.** Now, we will show that the conditions in Theorem 2.3 do not self-contradict. Consider the following equation:

$$\begin{aligned}
x'(t) &= - \sum_{l=1}^{\infty} \frac{|\sin t + \sin \sqrt{3}t| + 8}{100^l} (x(t))^l + \sum_{l=1}^{\infty} \frac{|\cos t + \cos \sqrt{5}t| + 1}{100^l} (x^{[2]}(t))^l \\
&\quad + \frac{1}{100} (\sin t + \sin \sqrt{2}t), \tag{4.11}
\end{aligned}$$

where  $C_{l,1}(t) = -\frac{|\sin t + \sin \sqrt{3}t| + 8}{100^l}$ ,  $C_{l,2}(t) = \frac{|\cos t + \cos \sqrt{5}t| + 1}{100^l}$ , in particular  $C_{1,1}(t) = -\frac{|\sin t + \sin \sqrt{3}t| + 8}{100}$ ,  $G(t) = \frac{1}{100} (\sin t + \sin \sqrt{2}t)$ . Take  $M = 1$ ,  $L = 1$ , a simple calculation yields

$$-\frac{1}{10} \leq C_{1,1}^+ \leq -\frac{2}{25}, \quad C_{s,1}^+ \leq \frac{10}{100^l}, \quad C_{l,2}^+ \leq \frac{3}{100^l}, \quad G^+ \leq \frac{1}{50}, \quad s = 2, 3, \dots, l = 1, 2, \dots$$

$$\sum_{n=1}^2 \sum_{l=1}^{\infty} C_{l,n}^+ M^l + G^+ \leq -\frac{71}{2475} < 0$$

and

$$2(MC_{1,2}^+ + \sum_{n=1}^2 \sum_{l=2}^{\infty} C_{l,n}^+ M^l + G^+) \leq \frac{254}{2475} < 1 = L.$$

then (2.6) and (2.7) are satisfied. By Theorem 2.3, equation (4.11) has a solution in  $B_{PAP}(1, 1)$ .

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