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Periodic solutions for second order differential equations with indefinite singularities

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Abstract: In this paper, the problem of periodic solutions is studied for second order differential equations with indefinite singularities

$$x''(t) + f(x(t))x'(t) + \varphi(t)x^m(t) - \frac{\alpha(t)}{x^\mu(t)} + \frac{\beta(t)}{x^\gamma(t)} = 0,$$

where $f \in C((0, +\infty), \mathbb{R})$ may have a singularity at the origin, the signs of φ and α are allowed to change, m is a non-negative constant, μ and γ are positive constants. The approach is based on a continuation theorem of Manásevich and Mawhin with techniques of a priori estimates.

Keywords: Second order differential equation; Continuation theorem; Periodic solution; Indefinite singularity

MSC 2010: 34B16; 34B18; 34C25

1 Introduction

In the past years, the problem of existence of periodic solutions to second order differential equations with definite singularities, either attractive type or repulsive type, was extensively studied by many researchers [1]-[14]. In [15], Hakl and Torres investigated the problem of periodic solutions to the equation

$$x''(t) = \frac{g(t)}{x^\mu(t)} - \frac{h(t)}{x^\gamma(t)} + r(t), \tag{1.1}$$

where g, h, r are T−periodic functions with $g, h, r \in L([0, T], \mathbb{R})$. Chu and et. al in [16] studied the problem of twist periodic solutions to (1.1) for the case of $r(t) \equiv 0$. We notice that in [15] and [16], the functions of $g(t)$ and $h(t)$ are required to be $g(t) \geq 0$ and $h(t) \geq 0$ a.e. $t \in [0, T]$. Recently, the periodic problem for second order differential equations with indefinite singularities has attract much attention from researchers (see [12] and [17]-[20]). For example, in [18], the authors considered the existence of positive periodic solutions to the equation like

$$x''(t) = \frac{\alpha(t)}{x^\mu}, \tag{1.2}$$

where the sign of weight function $\alpha(t)$ can change on $[0, T]$, and $\mu \geq 1$. The equations like (1.2) with indefinite singularities can be used to model some important problems appearing in many physical contexts (see [21] and the references therein).

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In this paper, we consider the problem of periodic solutions to the equation

$$x''(t) + f(x(t))x'(t) + \varphi(t)x^m(t) - \frac{\alpha(t)}{x^\mu(t)} + \frac{\beta(t)}{x^y(t)} = 0, \tag{1.3}$$

where $f \in C((0, +\infty), \mathbb{R})$, φ, α and β are T−periodic and in $L([0, T], \mathbb{R})$, m, μ and y are constants with $m \geq 0$ and $\mu \geq y > 0$. By using a continuation theorem of Manásevich and Mawhin, some new results on the existence of periodic solutions to (1.3) are obtained. In (1.3), the signs of $\alpha(t)$, $\beta(t)$ and $\varphi(t)$ are all allowed to change. This means that the singularity associated to restoring force $-\frac{\alpha(t)}{x^\mu} + \frac{\beta(t)}{x^y}$ at $x = 0$ is indefinite type(see [18, 19]). The periodic problem for equation (1.3) has been investigated in recent paper [20]. However, in [20], $\beta(t) \equiv 0$, and the functions of $\alpha(t)$ and $\varphi(t)$ are required to be either $\alpha(t) \geq 0$ a.e. $t \in [0, T]$, $\bar{\alpha} > 0$ and $\bar{\varphi} > 0$, or $\varphi(t) \geq 0$ a.e. $t \in [0, T]$, $\bar{\alpha} > 0$ and $\bar{\varphi} > 0$. The significance of present paper relies in the following aspects. Firstly, the coefficient function $f(x)$ associated to friction term $f(x)x'$ may have a singularity at $x = 0$. Secondly, compared with the case where the signs of functions $\varphi(t)$, $\alpha(t)$ and $\beta(t)$ are in definite, the work of obtaining the estimates of periodic solutions to (1.3) is more difficult. In order to overcome this difficulty, we propose a function $F(x) = \int_1^x f(s)ds$. By analyzing some properties of $F(x)$ at $x = 0$ and $x = +\infty$, we investigate the mechanism under which how the singularity associated to $f(x)$ at $x = 0$ influences the priori estimates of periodic solutions. Moreover, the constant μ in (1.3) is allowed to be in (0, 1). For this case, even if $\alpha(t)$ and $\beta(t)$ are constant functions, the singular restoring force $-\frac{\alpha(t)}{x^\mu} + \frac{\beta(t)}{x^y}$ has a weak singularity at $x = 0$. Finally, by using a theorem in present paper, a new result on the existence of periodic solutions is obtained for Rayleigh-Plesset equation

$$\rho\left(RR'' + \frac{3}{2}(R')^2\right) = [P_v - P_\infty(t)] + P_{g_0}\left(\frac{R_0}{R}\right)^{3k} - \frac{2S}{R} - \frac{4\nu R'}{R} \tag{1.4}$$

in the case of $k \in (\frac{1}{3}, +\infty)$. Equation (1.4) is used in physics of fluids to model the oscillations of the radius $R(t)$ of a spherical bubble immersed in a fluid under the influence of a periodic acoustic field P_∞(see [21]-[25]), and $P_\infty \in L([0, T], \mathbb{R})$. The physical meaning of the rest of the parameters in (1.4) can also be seen in Section 3 of [5]. By using the methods of upper and lower solutions, the authors in [5] obtained the following result:

Theorem 1.1. *Suppose that $P_v > \overline{P_\infty}$ and*

$$\frac{5}{2\rho}(P_v - P_\infty(t)) \leq \left(\frac{6k-2}{5}\right)\left[\frac{(\frac{2}{5})^{\frac{2}{5}}(5S)^{\frac{6k}{5}}}{(\frac{6k}{5})^{\frac{6k}{5}}\left(\frac{5P_{g_0}R_0^{3k}}{2\rho}\right)^{\frac{6k-2}{5}}}\right]^{\frac{5}{6k-2}}, \ for\ t \in [0, T]. \tag{1.5}$$

Then there exists at least one positive T−periodic solution to the Rayleigh-Plesset equation (1.4) *in the case of $k \in (\frac{1}{3}, +\infty)$.*

2 Preliminaries

Throughout this paper, let $C_T = \{x \in C(\mathbb{R}, \mathbb{R}) : x(t + T) = x(t) \text{ for all } t \in \mathbb{R}\}$ with the norm defined by $|x|_\infty = \max_{t\in[0,T]} |x(t)|$, and $C_T^1 = \{x \in C^1(\mathbb{R}, \mathbb{R}) : x(t + T) = x(t) \text{ for all } t \in \mathbb{R}\}$ with the norm defined by $||x||_{C_T^1} = \max\{|x|_\infty, |x'|_\infty\}$. For any T−periodic function $y(t)$ with $y \in L([0, T], \mathbb{R})$, $y_+(t)$ and $y_-(t)$ is denoted by $\max\{y(t), 0\}$ and $-\min\{y(t), 0\}$, respectively, and $\bar{y} = \frac{1}{T}\int_0^T y(s)ds$. Clearly, $y(t) = y_+(t) - y_-(t)$ for all $t \in \mathbb{R}$, and $\bar{y} = \overline{y_+} - \overline{y_-}$.

Lemma 2.1 ([20, 26]). *Assume that there exist positive constants M_0, M_1 and M_2 with $0 < M_0 < M_1$, such that the following conditions hold.*
1.For any $\lambda \in (0, 1]$, each possible positive T−periodic solution x to the equation

$$u''(t) + \lambda f(u(t))u'(t) + \lambda\varphi(t)u^m(t) - \frac{\lambda\alpha(t)}{x^\mu(t)} + \frac{\lambda\beta(t)}{x^y(t)} = 0$$

satisfies the inequalities $M_0 < x(t) < M_1$ *and* $|x'(t)| < M_2$, for all $t \in [0, T]$.
2.Each possible solution c to the equation

$$\frac{\bar{\alpha}}{c^{\mu}} - \frac{\bar{\beta}}{c^{y}} - \overline{\varphi}c^{m} = 0$$

satisfies the inequality $M_0 < c < M_1$.
3.The inequality

$$\Big(\frac{\bar{\alpha}}{M_0^{\mu}} - \frac{\bar{\beta}}{M_0^{y}} - \overline{\varphi}M_0^{m}\Big)\Big(\frac{\bar{\alpha}}{M_1^{\mu}} - \frac{\bar{\beta}}{M_1^{y}} - \overline{\varphi}M_1^{m}\Big) < 0$$

holds.
Then, equation (1.3) *has at least one T–periodic solution u such that* $M_0 < u(t) < M_1$ *for all* $t \in [0, T]$.

Remark 2.1 If $\bar{\varphi} > 0$ and $\bar{\alpha} > 0$, then we have from the condition of $m \geq 0$ and $\mu > y > 0$ that there are two constants D_1 and D_2 with $0 < D_1 < D_2$ such that

$$\frac{\bar{\alpha}}{x^{\mu}} - \frac{\bar{\beta}}{x^{y}} - \bar{\varphi}x^{m} > 0, \text{ for all } x \in (0, D_1)$$

and

$$\frac{\bar{\alpha}}{x^{\mu}} - \frac{\bar{\beta}}{x^{y}} - \bar{\varphi}x^{m} < 0, \text{ for all } x \in (D_2, \infty).$$

Lemma 2.2 ([5]). *Let* $u \in [0, \omega] \to \mathbb{R}$ *be an arbitrary absolutely continuous function with* $u(0) = u(\omega)$.*Then the inequality*

$$(\max_{t\in[0,\omega]} u(t) - \min_{t\in[0,\omega]} u(t))^2 \leq \frac{\omega}{4}\int_4^{\omega} |u'(s)|^2 ds$$

holds.

Now, we embed equation (1.3) into the following equations family with a parameter $\lambda \in (0, 1]$

$$x'' + \lambda f(x)x' + \lambda\varphi(t)x^{m} - \frac{\lambda\alpha(t)}{x^{\mu}} = 0, \lambda \in (0, 1]. \tag{2.1}$$

Let

$$D = \Big\{x \in C_T^1 : x'' + \lambda f(x)x' + \lambda\varphi(t)x^{m} - \frac{\lambda\alpha(t)}{x^{\mu}} + \frac{\lambda\beta(t)}{u^{y}(t)} = 0, \\ \lambda \in (0, 1]; x(t) > 0, \forall t \in [0, T]\Big\}, \tag{2.2}$$

$$F(x) = \int_1^x f(s)ds \tag{2.3}$$

and

$$G(x) = \int_1^x s^{\mu}f(s)ds. \tag{2.4}$$

Lemma 2.3. *Assume* $\bar{\alpha} > 0$ *and* $\bar{\varphi} > 0$, *then for each* $u \in D$, *there are constants* $\tau_1, \tau_2 \in [0, T]$ *such that*

$$u(\tau_1) \leq \max\Big\{1, \big(\frac{\overline{\alpha_+} + \overline{\beta_-}}{\bar{\varphi}}\big)^{\frac{1}{m+y}}\Big\} := A_0 \tag{2.5}$$

and

$$u(\tau_2) \geq A_1, \tag{2.6}$$

where

$$A_1 = \begin{cases} \min\Big\{\big(\frac{\bar{\alpha}}{2\overline{\varphi_+}}\big)^{\frac{1}{\mu+m}}, \big(\frac{\bar{\alpha}}{2\overline{\beta_+}}\big)^{\frac{1}{\mu-y}}\Big\}, & \overline{\beta_+} > 0 \\ \big(\frac{\bar{\alpha}}{\overline{\varphi_+}}\big)^{\frac{1}{\mu+m}}, & \overline{\beta_+} = 0, \end{cases} \tag{2.7}$$

m, μ and y are determined in equation (1.3)

Proof. Let $u \in D$, then

$$u''(t) + \lambda f(u(t))u'(t) + \lambda\varphi(t)u^m(t) - \frac{\lambda\alpha(t)}{u^\mu(t)} + \frac{\lambda\beta(t)}{u^\gamma(t)} = 0, \tag{2.8}$$

which together with the fact of $u(t) > 0$ for all $t \in [0, T]$ gives

$$\frac{u''}{u^m} + \frac{\lambda f(u)u'}{u^m} + \lambda\varphi(t) - \frac{\lambda\alpha(t)}{u^{m+\mu}} + \frac{\lambda\beta(t)}{u^{\gamma+m}(t)} = 0.$$

Integrating the above equality over the interval $[0, T]$, we obtain

$$\int_0^T \frac{u''}{u^m}dt + \lambda\int_0^T \varphi(t)dt = \lambda\int_0^T \frac{\alpha(t)}{u^{m+\mu}}dt - \lambda\int_0^T \frac{\beta(t)}{u^{\gamma+m}(t)}dt,$$

i.e.,

$$\int_0^T \frac{u''}{u^m}dt + \lambda T\bar{\varphi} = \lambda\int_0^T \frac{\alpha(t)}{u^{m+\mu}}dt - \lambda\int_0^T \frac{\beta(t)}{u^{\gamma+m}(t)}dt. \tag{2.9}$$

Since

$$\int_0^T \frac{u''}{u^m}dt \geq 0, \tag{2.10}$$

it follows from (2.9) that

$$T\bar{\varphi} \leq \int_0^T \frac{\alpha(t)}{u^{m+\mu}}dt - \int_0^T \frac{\beta(t)}{u^{\gamma+m}(t)}dt \leq \int_0^T \frac{\alpha_+(t)}{u^{m+\mu}(t)}dt + \int_0^T \frac{\beta_-(t)}{u^{\gamma+m}(t)}dt. \tag{2.11}$$

From (2.11), we can conclude that there is a point $\xi \in [0, T]$ such that

$$u(\xi) \leq \max\left\{1, (\frac{\overline{\alpha_+} + \overline{\beta_-}}{\bar{\varphi}})^{\frac{1}{m+\gamma}}\right\}. \tag{2.12}$$

If (2.12) does not hold, then

$$u(t) > \max\left\{1, (\frac{\overline{\alpha_+} + \overline{\beta_-}}{\bar{\varphi}})^{\frac{1}{m+\gamma}}\right\} \text{ for all } t \in [0, T],$$

which implies that

$$u(t) > 1 \text{ for all } t \in [0, T] \tag{2.13}$$

and

$$u(t) > (\frac{\overline{\alpha_+} + \overline{\beta_-}}{\bar{\varphi}})^{\frac{1}{m+\gamma}} \text{ for all } t \in [0, T]. \tag{2.14}$$

From (2.11), (2.13) and $\mu > \gamma$, and by using mean value theorem of integrals, we have that there exists a point $\eta \in [0, T]$ such that

$$T\bar{\varphi} \leq \frac{T\overline{\alpha_+} + T\overline{\beta_-}}{u^{m+\gamma}(\eta)},$$

i.e.,

$$u(\eta) \leq \left(\frac{\overline{\alpha_+} + \overline{\beta_-}}{\bar{\varphi}}\right)^{\frac{1}{m+\gamma}}, \tag{2.15}$$

which contradicts to (2.14). This contradiction verifies (2.12), and so (2.5) holds. Meanwhile, we can assert that (2.6) is true. In fact, multiplying two sides of (2.8) with $u^\mu(t)$ and integrating it over the interval $[0, T]$, we obtain that

$$-\int_0^T u^\mu(t)u''(t)dt - \lambda\int_0^T \varphi(s)u^{\mu+m}ds - \lambda\int_0^T \beta(t)u^{\mu-\gamma}(t)dt + \lambda T\bar{\alpha} = 0,$$

which together with

$$-\int_0^T u^{\mu}(t)u''(t)dt = \int_0^T u^{\mu-1}(s)|u'(s)|^2 dt \geq 0$$

yields

$$T\bar{\alpha} \leq \int_0^T \varphi(s)u^{\mu+m}ds + \int_0^T \beta(t)u^{\mu-\gamma}(t)dt \leq \int_0^T \varphi_+(s)u^{\mu+m}(s)ds + \int_0^T \beta_+(t)u^{\mu-\gamma}(t)dt.$$

Thus, there is a point $\eta \in [0, T]$ such that

$$u(\eta) \geq A_1,$$

where A_1 is defined by (2.7). This implies that (2.6) holds. □

Lemma 2.4. *Assume $\bar{\alpha} > 0$ and $\bar{\varphi} > 0$, and suppose that the following assumptions*

$$C_0 := \sup_{x\in[A_1,+\infty)} \left[F(x) + T\overline{\varphi_-}x^m\right] < +\infty, \tag{2.16}$$

and

$$\lim_{s\to 0^+} \left(F(s) - \frac{T\overline{\alpha_-}}{s^{\mu}} - \frac{T\overline{\beta_+}}{s^{\gamma}}\right) > C_0 \tag{2.17}$$

hold, where $F(x)$ is defined by (2.3).

Then there exists a constant $\gamma_0 > 0$, such that

$$\min_{t\in[0,T]} u(t) \geq \gamma_0, \text{ uniformly for } u \in D.$$

Proof. For each $u \in D$, we have

$$u''(t) + \lambda f(u(t))u'(t) + \lambda\varphi(t)u^m(t) - \frac{\lambda\alpha(t)}{u^{\mu}(t)} + \frac{\lambda\beta(t)}{u^{\gamma}(t)} = 0, \quad \lambda \in (0, 1]. \tag{2.18}$$

Since $u(t)$ is a T–periodic function, there are two points $t_1, t_2 \in \mathbb{R}$ such that $0 < t_2 - t_1 \leq T$ with $u(t_1) = \max_{t\in[0,T]} u(t)$ and $u(t_2) = \min_{t\in[0,T]} u(t)$. Assumptions of $\bar{\alpha} > 0$ and $\bar{\varphi} > 0$ implies that (2.6) in Lemma 2.3 holds. This gives

$$A_1 \leq u(t_1) < +\infty,$$

to which by applying assumption (2.16) yields

$$F(u(t_1)) + T\overline{\varphi_-}u^m(t_1) \leq \sup_{A_1\leq s<+\infty} \left[F(s) + T\overline{\varphi_-}s^m\right] := C_0 < +\infty. \tag{2.19}$$

By integrating (2.18) over the interval $[t_1, t_2]$, we get

$$\begin{aligned}\int_{t_1}^{t_2} f(u(t))u'(t)dt \\ = \int_{t_1}^{t_2} \frac{\alpha(t)}{u^{\mu}(t)}dt - \int_{t_1}^{t_2} \frac{\beta(t)}{u^{\gamma}(t)}dt - \int_{t_1}^{t_2} \varphi(t)u^m(t)dt.\end{aligned} \tag{2.20}$$

It follows from (2.19), (2.20) and the condition $\bar{\varphi} > 0$ that

$$\begin{aligned}F(u(t_2)) &= F(u(t_1)) - \int_{t_1}^{t_2} \varphi(t)u^m(t)dt + \int_{t_1}^{t_2} \frac{\alpha(t)}{u^{\mu}(t)}dt - \int_{t_1}^{t_2} \frac{\beta(t)}{u^{\gamma}(t)}dt \\ &\leq F(u(t_1)) + \int_0^T \varphi_-(t)u^m(t)dt + \int_0^T \frac{\alpha_+(t)}{u^{\mu}(t)}dt + \int_0^T \frac{\beta_-(t)}{u^{\gamma}(t)}dt.\end{aligned} \tag{2.21}$$

On the other hand, by integrating (2.18) on $[0, T]$, we get

$$\int_0^T \frac{\alpha(t)}{u^\mu(t)} dt = \int_0^T \frac{\beta(t)}{u^y(t)} dt + \int_0^T \varphi(t) u^m(t) dt,$$

i.e.,

$$\int_0^T \frac{\alpha_+(t)}{u^\mu(t)} dt + \int_0^T \frac{\beta_-(t)}{u^y(t)} dt + \int_0^T \varphi_-(t) u^m(t) dt$$
$$= \int_0^T \frac{\alpha_-(t)}{u^\mu(t)} dt + \int_0^T \frac{\beta_+(t)}{u^y(t)} dt + \int_0^T \varphi_+(t) u^m(t) dt.$$

Substituting it into (2.21), we have from (2.19) that

$$\begin{aligned} &F(u(t_2)) \\ \le &F(u(t_1)) + \int_0^T \frac{\alpha_-(t)}{u^\mu(t)} dt + \int_0^T \frac{\beta_+(t)}{u^y(t)} dt + \int_0^T \varphi_+(t) u^m(t) dt \\ \le &F(u(t_1)) + T\overline{\varphi_+} u^m(t_1) + \frac{T\overline{\alpha_-}}{u^\mu(t_2)} + \frac{T\overline{\beta_+}}{u^y(t_2)} \\ \le &C_0 + \frac{T\overline{\alpha_-}}{u^\mu(t_2)} + \frac{T\overline{\beta_+}}{u^y(t_2)}, \end{aligned}$$

and so

$$F(u(t_2)) - \frac{T\overline{\alpha_-}}{u^\mu(t_2)} - \frac{T\overline{\beta_+}}{u^y(t_2)} \le C_0. \tag{2.22}$$

Assumption (2.17) ensures that there exists a constant $y_0 > 0$ such that

$$F(s) - \frac{T\overline{\alpha_-}}{s^\mu} - \frac{T\overline{\beta_+}}{s^y} > C_0, \ \text{ for all } s \in (0, y_0). \tag{2.23}$$

Combining (2.23) with (2.22), we can get

$$\min_{t \in [0,T]} u(t) = u(t_2) \ge y_0. \tag{2.24}$$

□

Lemma 2.5. *Assume $\bar{\alpha} > 0$ and $\bar{\varphi} > 0$. Suppose further that the following assumptions*

$$B_0 := \inf_{x \in [A_1, +\infty)} \left[F(x) - T\overline{\varphi_+} x^m \right] > -\infty \tag{2.25}$$

and

$$\lim_{s \to 0^+} \left(F(s) + \frac{T\overline{\alpha_-}}{s^\mu} + \frac{T\overline{\beta_+}}{s^y} \right) < B_0 \tag{2.26}$$

hold. Then there exists a constant $y_1 > 0$, such that

$$\min_{t \in [0,T]} u(t) \ge y_1, \ \text{ uniformly for } u \in D.$$

Proof. Suppose that $u \in D$, then u satisfies (2.18). Let t_1 and t_2 be defined as same as the ones in the proof of Lemma 2.4. From (2.6) in Lemma 2.3, we see that

$$A_1 \le u(t_1) < +\infty,$$

which together with assumption (2.25), yields that

$$F(u(t_1)) - T\overline{\varphi_+}u^m(t_1) \geq \inf_{A_1 \leq s < +\infty} \left[F(s) - T\overline{\varphi_+}s^m\right] := B_0. \tag{2.27}$$

By integrating (2.18) over the interval $[t_1, t_2]$, we get

$$\begin{aligned} &F(u(t_2)) \\ &= F(u(t_1)) + \int_{t_1}^{t_2} \frac{\alpha(t)}{u^\mu(t)}dt - \int_{t_1}^{t_2} \frac{\beta(t)}{u^\gamma(t)}dt - \int_{t_1}^{t_2} \varphi(t)u^m(t)dt \\ &\geq F(u(t_1)) - \int_0^T \frac{\alpha_-(t)}{u^\mu(t)}dt - \int_0^T \frac{\beta_+(t)}{u^\gamma(t)}dt - \int_0^T \varphi_+(t)u^m(t)dt \\ &\geq F(u(t_1)) - \frac{T\overline{\alpha_-}}{u^\mu(t_2)} - \frac{T\overline{\beta_+}}{u^\gamma(t_2)} - T\overline{\varphi_+}u^m(t_1), \end{aligned}$$

i.e.,

$$F(u(t_2)) + \frac{T\overline{\alpha_-}}{u^\mu(t_2)} - \frac{T\overline{\beta_+}}{u^\gamma(t_2)} \geq F(u(t_1)) - T\overline{\varphi_+}u^m(t_1).$$

From (2.27), we obtain

$$F(u(t_2)) + \frac{T\overline{\alpha_-}}{u^\mu(t_2)} + \frac{T\overline{\beta_+}}{u^\gamma(t_2)} \geq B_0. \tag{2.28}$$

Assumption (2.26) implies that there exists a constant $y_1 > 0$ such that

$$F(s) + \frac{T\overline{\alpha_-}}{s^\mu} + \frac{T\overline{\beta_+}}{s^\gamma} < B_0, \text{ for all } s \in (0, y_1),$$

which together with (2.28) gives

$$\min_{t\in[0,T]} u(t) = u(t_2) \geq y_1.$$

□

3 Main Results

Theorem 3.1. *Assume $\bar{\varphi} > 0$ and $\bar{\alpha} > 0$. Suppose further*

$$\lim_{s\to 0^+} \left(F(s) - \frac{T\overline{\alpha_+}}{s^\mu} - \frac{T\overline{\beta_-}}{s^\gamma}\right) = +\infty \tag{3.1}$$

and

$$\lim_{x\to +\infty} \left(F(x) + T\overline{\varphi_+}x^m\right) = -\infty. \tag{3.2}$$

Then equation(1.3) *has at least one positive T–periodic solution.*

Proof. Firstly, we will prove that there exist two constants $y_2 > 0$ and $y_3 > 0$ with $y_3 > y_2$, such that

$$\min_{t\in[0,T]} u(t) \geq y_2, \text{ uniformly for } u \in D \tag{3.3}$$

and

$$\max_{t\in[0,T]} u(t) \leq y_3, \text{ uniformly for } u \in D. \tag{3.4}$$

For each $u \in D$, u satisfies (2.18). Let t_1 and t_2 be defined as same as the ones in the proof of Lemma 2.4, that is $0 < t_2 - t_1 \leq T$, $u(t_1) = \max_{t\in[0,T]} u(t)$ and $u(t_2) = \min_{t\in[0,T]} u(t)$. Take $t_3 = t_1 + T$, then $u(t_3) = \max_{t\in[0,T]} u(t)$

and $0 \le t_3 - t_2 < T$. It is easy to see that (2.16) can be deduced from (3.2), and (3.1) verifies (2.17). By using Lemma 2.4, as well as (2.5) in Lemma 2.3, we know that

$$A_0 \ge \min_{t\in[0,T]} u(t) = u(t_2) \ge y_0. \tag{3.5}$$

Integrating (2.18) over the interval $[t_2, t_3]$, we have

$$\begin{aligned}
&F(u(t_3))\\
=&F(u(t_2)) + \int_{t_2}^{t_3} \frac{\alpha(t)}{u^\mu(t)}dt - \int_{t_2}^{t_3} \frac{\beta(t)}{u^\gamma(t)}dt - \int_{t_2}^{t_3} \varphi(t)u^m(t)dt\\
\ge&F(u(t_2)) - \int_0^T \frac{\alpha_-(t)}{u^\mu(t)}dt - \int_0^T \frac{\beta_+(t)}{u^\gamma(t)}dt - \int_0^T \varphi_+(t)u^m(t)dt\\
\ge&F(u(t_2)) - \frac{T\overline{\alpha_-}}{u^\mu(t_2)} - \frac{T\overline{\beta_+}}{u^\gamma(t_2)} - T\overline{\varphi_+}u^m(t_3),
\end{aligned}$$

i.e.,

$$F(u(t_3)) + T\overline{\varphi_+}u^m(t_3) \ge F(u(t_2)) - \frac{T\overline{\alpha_-}}{u^\mu(t_2)} - \frac{T\overline{\beta_+}}{u^\gamma(t_2)},$$

which together with (3.5) gives

$$F(u(t_3)) + T\overline{\varphi_+}u^m(t_3) \ge \inf_{x\in[y_0,A_0]} \left[F(x) - \frac{T\overline{\alpha_-}}{x^\mu} - \frac{T\overline{\beta_+}}{x^\gamma}\right] := \rho_0 > -\infty.$$

It follows from (3.2) that there is a constant $y_3 > 0$ such that

$$\max_{t\in[0,T]} u(t) = u(t_3) \le y_3. \tag{3.6}$$

From (3.5) and (3.6), we see that the conclusions of (3.3) and (3.4) hold. Next, we will show that there exists a positive constant ρ_1 such that

$$\max_{t\in[0,T]} |u'(t)| \le \rho, \text{ uniformly for } u \in D. \tag{3.7}$$

In fact, if $u \in D$, then

$$u''(t) + \lambda f(u(t))u'(t) + \lambda\varphi(t)u^m(t) - \frac{\lambda\alpha(t)}{u^\mu(t)} + \frac{\lambda\beta(t)}{u^\gamma(t)} = 0, \lambda \in (0,1]. \tag{3.8}$$

Let u attain its maximum over $[0, T]$ at $t_1 \in [0, T]$, then $u'(t_1) = 0$ and we deduce from (3.8) that

$$u'(t) = \lambda \int_{t_1}^{t} [-f(u(s))u'(s) - \varphi(s)u^m(s) + \frac{\alpha(s)}{u^\mu(s)} - \frac{\beta(s)}{u^\gamma(s)}]ds$$

for all $t \in [t_1, t_1 + T]$. Thus,

$$\begin{aligned}
|u'(t)| \le&\lambda|F(u(t)) - F(u(t_1))| + \lambda \int_{t_1}^{t_1+T} \left|\frac{\alpha(s)}{u^\mu(s)}\right|ds + \lambda \int_{t_1}^{t_1+T} \left|\frac{\beta(s)}{u^\gamma(s)}\right|ds\\
&+ \int_{t_1}^{t_1+T} |\varphi(s)|u^m(s)ds\\
\le&2 \max_{y_2\le u\le y_3} |F(u)| + \int_0^T \left|\frac{\alpha(t)}{u^\mu(t)}\right|dt + \int_0^T \left|\frac{\beta(t)}{u^\gamma(t)}\right|dt + \int_0^T |\varphi(s)|u^m(s)ds.
\end{aligned}$$

It follows from (3.4) and (3.3) that

$$\begin{aligned}|u'(t)| \le& 2 \max_{y_0\le u\le y_3} |F(u)| + \frac{T\overline{|\alpha|}}{y_1^\mu} + \frac{T\overline{|\beta|}}{y_1^\gamma} + Ty_3^m\overline{|\varphi|} \\ :=& \rho, \text{ for all } t \in [0, T],\end{aligned}$$

and so

$$\max_{t\in[0,T]} |u'(t)| \le \rho, \text{ uniformly for } u \in D,$$

which implies that (3.7) holds. Let $m_0 = \min\{D_1, y_2\}$ and $m_1 = \max\{y_3, D_2\}$ be two constants, where D_1 and D_2 are constants determined in Remark 2.1, then from (3.4), (3.3) and (3.7), we see that each possible positive T–periodic solution u to equation(2.1) satisfies

$$m_0 < u(t) < m_1, |u'(t)| < \rho_1 \text{ for all } t \in [0, T]$$

This implies that condition 1 of Lemma 2.1 is satisfied. Also, we can deduce from Remark 2.1 that

$$\frac{\bar\alpha}{x^\mu} - \frac{\bar\beta}{x^\gamma} - \bar\varphi x^m > 0, \text{ for } x \in (0, m_0]$$

and

$$\frac{\bar\alpha}{x^\mu} - \frac{\bar\beta}{x^\gamma} - \bar\varphi x^m < 0, \text{ for } x \in [m_1, +\infty),$$

which gives that condition 2 of Lemma 2.1 holds. Furthermore, we have

$$\left(\frac{\bar\alpha}{m_0^\mu} - \frac{\bar\beta}{m_0^\gamma} - \bar\varphi m_0^m\right)\left(\frac{\bar\alpha}{m_1^\mu} - \frac{\bar\beta}{m_1^\gamma} - \bar\varphi m_1^m\right) < 0.$$

So condition 3 of Lemma 2.1 holds, too. By using Lemma 2.1, we see that equation(1.3) has at least one positive T–periodic solution.

□

Theorem 3.2. *Assume $\bar\varphi > 0$ and $\bar\alpha > 0$. Suppose further*

$$\lim_{s\to 0^+}\left(F(s) + \frac{T\overline{\alpha_+}}{s^\mu} + \frac{T\overline{\beta_-}}{s^\gamma}\right) = -\infty \tag{3.9}$$

and

$$\lim_{x\to+\infty}\left(F(x) - T\overline{\varphi_+}x^m\right) = +\infty. \tag{3.10}$$

Then equation(1.3) *has at leat one positive T–periodic solution.*

Proof. From the proof of Theorem 3.1, we see that it suffices for us to verify (3.4) and (3.3). In order to do it, let $u \in D$, u satisfies (2.18). Let t_2 and t_3 be defined as same as the ones in the proof of Theorem 3.1, that is $0 \le t_3 - t_2 < T$, $u(t_3) = \max_{t\in[0,T]} u(t)$ and $u(t_2) = \min_{t\in[0,T]} u(t)$. It is easy to see that (2.25) can be deduced from (3.10), and (3.9) verifies (2.26). By using Lemma 2.5 and (2.5) in Lemma 2.4, we know that

$$A_0 \ge \min_{t\in[0,T]} u(t) = u(t_2) \ge y_1. \tag{3.11}$$

Integrating (2.18) over the interval $[t_2, t_3]$, we have

$$\begin{aligned}&F(u(t_3))\\ =& F(u(t_2)) + \int_{t_2}^{t_3} \frac{\alpha(t)}{u^\mu(t)}dt - \int_{t_2}^{t_3} \frac{\beta(t)}{u^\gamma(t)}dt - \int_{t_2}^{t_3} \varphi(t)u^m(t)dt \\ \le& F(u(t_2)) + \int_0^T \frac{\alpha_+(t)}{u^\mu(t)}dt + \int_0^T \frac{\beta_-(t)}{u^\gamma(t)}dt + \int_0^T \varphi_-(t)u^m(t)dt \\ \le& F(u(t_2)) + \frac{T\overline{\alpha_+}}{u^\mu(t_2)} + \frac{T\overline{\beta_-}}{u^\gamma(t_2)} + T\overline{\varphi_-}u^m(t_3),\end{aligned}$$

i.e.,

$$F(u(t_3)) - T\overline{\varphi_-}u^m(t_3) \le F(u(t_2)) + \frac{T\overline{\alpha_+}}{u^\mu(t_2)} + \frac{T\overline{\beta_-}}{u^\gamma(t_2)}, \tag{3.12}$$

which together with (3.11) and gives

$$F(u(t_3)) - T\overline{\varphi_-}u^m(t_3) \le \sup_{x\in[y_0,A_0]} \left[F(x) + \frac{T\overline{\alpha_+}}{x^\mu} + \frac{T\overline{\beta_-}}{x^\gamma}\right] := C_0 < +\infty.$$

It follows from (3.10) and the condition $\bar{\varphi} > 0$ that there is a constant $\rho_2 > 0$ such that

$$\max_{t\in[0,T]} u(t) = u(t_3) \le \rho_2. \tag{3.13}$$

From (3.11) and (3.13), we see that the conclusions of (3.3) and (3.4) hold. □

Theorem 3.3. *Assume $\bar{\varphi} > 0$, $\bar{\alpha} > 0$ and*

$$\lim_{x\to+\infty} \left[G(x) - T\overline{\beta_+}x^{\mu-\gamma} - T\overline{\varphi_+}x^{\mu+m}\right] = +\infty. \tag{3.14}$$

If

$$\lim_{x\to 0^+} G(x) < C_0 - T\overline{\alpha_-}, \tag{3.15}$$

then equation(1.3) *has at least one positive T–periodic solution, where $G(x)$ is defined by* (2.4) *and*

$$C_0 := \inf_{x\in[A_1,+\infty)} \left[G(x) - T\overline{\beta_+}x^{\mu-\gamma} - T\overline{\varphi_+}x^{\mu+m}\right]. \tag{3.16}$$

.

Remark 3.1. Assumption (3.14) implies that $C_0 \in (-\infty, +\infty)$ is a constant.

Proof. Similar to the proof of Theorem 3.2, we need only prove the estimates of (3.4) and (3.3). Let $u \in D$, then u satisfies (2.18). Let t_1 and t_2 be as same as the ones in the proof of Lemma 2.4. Multiplying (2.18) with $u^\mu(t)$ and integrating it on $[t_4, t_2]$, we get

$$\begin{aligned}&\int_{t_1}^{t_2} u''(t)u^\mu(t)dt + \lambda\int_{t_1}^{t_2} u^\mu(t)f(u(t))u'(t)dt\\ &= \lambda\int_{t_4}^{t_2}\alpha(t)dt - \lambda\int_{t_4}^{t_2}\varphi(t)u^{m+\mu}(t)dt - \lambda\int_{t_4}^{t_2}\beta(t)u^{\mu-\gamma}(t)dt.\end{aligned} \tag{3.17}$$

Since $\int_{t_1}^{t_2} u''(t)u^\mu(t)dt \le 0$, it follows from (3.17) and (3.16) that

$$\begin{aligned}G(u(t_2)) &\ge G(u(t_1)) + \int_{t_1}^{t_2}\alpha(t)dt - \int_{t_1}^{t_2}\varphi_+(t)u^{m+\mu}(t)dt - \int_{t_1}^{t_2}\beta_+(t)u^{\mu-\gamma}(t)dt\\ &\ge G(u(t_1)) - T\overline{\alpha_-} - T\overline{\varphi_+}u^{m+\mu}(t_1) - T\overline{\beta_+}u^{\mu-\gamma}(t_1).\end{aligned} \tag{3.18}$$

By using (2.6) and (3.16), we get

$$\begin{aligned}G(u(t_2)) &\ge \inf_{x\in[A_1,+\infty)}\left[G(x) - T\overline{\varphi_+}x^{m+\mu} - T\overline{\beta_+}x^{\mu-\gamma}\right] - T\overline{\alpha_-}\\ &= C_0 - T\overline{\alpha_-},\end{aligned}$$

which together with (3.15) verifies (3.4). Thus, according to (2.5), we arrive at

$$y_2 \le u(t_2) \le A_0. \tag{3.19}$$

On the other hand, by (3.18) again, we obtain from (3.19) that

$$\begin{aligned} & G(u(t_1)) - T\overline{\varphi_+}u^{m+\mu}(t_1) - T\overline{\beta_+}u^{\mu-\gamma}(t_1) \\ & \leq G(u(t_2)) + T\overline{\alpha_-} \\ & \leq \max_{y_2 \leq x \leq A_0} G(x) + T\overline{\alpha_-}. \end{aligned} \tag{3.20}$$

By assumption (3.14), we see that there is a constant $y_3 > y_2$ such that

$$\begin{aligned} & G(s) - T\overline{\varphi_+}s^{m+\mu} - T\overline{\beta_+}s^{\mu-\gamma} \\ > & \max_{y_2 \leq x \leq A_0} G(x) + T\overline{\alpha_-}, \text{ for all } s \in (y_3, +\infty). \end{aligned} \tag{3.21}$$

Therefore, (3.20) and (3.21) imply

$$\max_{t \in [0,T]} u(t) = u(t_4) \leq y_3. \tag{3.22}$$

From (3.19) and (3.22), we see that the (3.4) and (3.3) hold. □

Theorem 3.4. *Assume $\bar{\varphi} > 0$, $\bar{\alpha} > 0$ and*

$$\lim_{x \to +\infty} \left[G(x) + T\overline{\beta_+}x^{\mu-\gamma} + T\overline{\varphi_+}x^{\mu+m}\right] = -\infty. \tag{3.23}$$

If

$$\lim_{x \to 0^+} G(x) > C_1 + T\overline{\alpha_-}, \tag{3.24}$$

then equation(1.3) *has at least one positive T–periodic solution, where*

$$C_1 := \sup_{x \in [A_1, +\infty)} \left[G(x) + T\overline{\beta_+}x^{\mu-\gamma} + T\overline{\varphi_+}x^{\mu+m}\right]. \tag{3.25}$$

Remark 3.2. From assumption (3.23), one can find that $C_1 \in (-\infty, +\infty)$ is a constant.

Proof. Let $u \in D$. If set $t_4 = t_2 - T$, then $0 < t_1 - t_4 \leq T$ and $u(t_4) = \min_{t \in [0,T]} u(t)$. Multiplying (2.18) with $u^\mu(t)$ and integrating it on $[t_4, t_1]$, we get

$$\begin{aligned} & \int_{t_4}^{t_1} u''(t)u^\mu(t)dt + \lambda \int_{t_4}^{t_1} u^\mu(t)f(u(t))u'(t)dt \\ & = \lambda \int_{t_4}^{t_1} \alpha(t)dt - \lambda \int_{t_4}^{t_1} \varphi(t)u^{m+\mu}(t)dt - \lambda \int_{t_4}^{t_1} \beta(t)u^{\mu-\gamma}(t)dt. \end{aligned} \tag{3.26}$$

By using $\int_{t_4}^{t_1} u''(t)u^\mu(t)dt \leq 0$, it follows from (3.26) that

$$\begin{aligned} G(u(t_1)) & \geq G(u(t_4)) + \int_{t_4}^{t_1} \alpha(t)dt - \int_{t_4}^{t_1} \varphi_+(t)u^{m+\mu}(t)dt - \int_{t_4}^{t_1} \beta_+(t)u^{\mu-\gamma}(t)dt \\ & \geq G(u(t_4)) - T\overline{\alpha_-} - T\overline{\varphi_+}u^{m+\mu}(t_1) - T\overline{\beta_+}u^{\mu-\gamma}(t_1), \end{aligned} \tag{3.27}$$

which together with (3.25) gives

$$\begin{aligned} & G(u(t_4)) \\ & \leq G(u(t_1)) + T\overline{\varphi_+}u^{m+\mu}(t_1) + T\overline{\beta_+}u^{\mu-\gamma}(t_1) + T\overline{\alpha_-} \\ & \leq \sup_{x \in [A_1, +\infty)} \left[G(x) + T\overline{\beta_+}x^{\mu-\gamma} + T\overline{\varphi_+}x^{\mu+m}\right] + T\overline{\alpha_-} \\ & \leq C_1 + T\overline{\alpha_-}. \end{aligned}$$

In view of (3.24), we get that there is a constant $y_4 > 0$ such that

$$\min_{t\in[0,T]} u(t) > y_4.$$

This together with (2.5) yields

$$y_4 \le u(t_4) \le A_0.$$

It follows from (3.27) that

$$\begin{aligned} &G(u(t_1)) + T\overline{\varphi_+}u^{m+\mu}(t_1) + T\overline{\beta_+}u^{\mu-y}(t_1) + T\overline{\alpha_-} \\ \ge &G(u(t_4)) - T\overline{\alpha_-} \\ \ge &\min_{y_4\le x\le A_0} G(x) - T\overline{\alpha_-}, \end{aligned}$$

to which by applying (3.23), we get that there is a constant $y_5 > y_4$ such that

$$\max_{t\in[0,T]} u(t) = u(t_1) < y_5.$$

□

Example 3.1: Consider the following equation

$$x''(t) - (x^m + \frac{1}{x^2})x'(t) + (1 + 2\sin t)x^m(t) - \frac{1 - 2\cos t}{x^{\frac{2}{3}}(t)} + \frac{\sin t}{x^{\frac{1}{2}}(t)} = 0, \tag{3.28}$$

where $m \in [0, +\infty)$ is a constant.

Corresponding to (1.3), we have $f(x) = -x^m - \frac{1}{x^2}$, $\mu = \frac{2}{3}$, $y = \frac{1}{2}$, $\varphi(t) = 1 + 2\sin t$, $\alpha(t) = 1 - 2\cos t$, $\beta(t) = \sin t$ and $T = 2\pi$. Clearly, $\bar{\alpha} = 1 > 0$ and $\bar{\varphi} = 1 > 0$. Since $A_1 = \left(\frac{\bar{\alpha}}{\bar{\varphi}}\right)^{\frac{1}{m+\mu}} = 1$ and

$$F(x) = \int_1^x f(s)ds = -\frac{x^{m+1}}{m+1} + \frac{1}{x} - \frac{m}{m+1},$$

we have

$$\lim_{x\to+\infty}\left(F(x) + T\overline{\varphi_+}x^m\right) = -\infty. \tag{3.29}$$

Furthermore, from $\overline{\alpha_+} = \frac{5\pi}{3} - 2$ and $\mu = \frac{2}{3} < 1$, we have

$$\lim_{s\to0^+}\left(F(s) - \frac{T\overline{\alpha_+}}{s^\mu} - \frac{T\overline{\beta_-}}{s^y}\right) = +\infty, \tag{3.30}$$

Thus, by using Theorem 3.1, we have that equation(3.28) has at least one positive 2π–periodic solution.

Now, we study the existence of periodic solutions to equation (1.4). By using the change of variable $R = x^{\frac{2}{5}}$, from [5], we see that equation (1.4) changes to

$$x'' + \frac{4\nu}{x^{\frac{4}{5}}}x' + \frac{5[P_\infty(t) - P_\nu]}{2\rho}x^{\frac{1}{5}} + \frac{5S}{x^{\frac{1}{5}}} - \left(\frac{5P_{g_0}R_0^{3k}}{2\rho}\right)\frac{1}{x^{\frac{6k-1}{5}}} = 0, \tag{3.31}$$

where the parameters of S, ρ, ν, P_{g_0} and R_0 are positive constants, $k \in (\frac{1}{3}, +\infty)$. Corresponding to (1.3), we have $f(x) = \frac{4\nu}{x^{\frac{4}{5}}}$, $\varphi(t) = \frac{5[P_\infty(t)-P_\nu]}{2\rho}$, $\alpha(t) \equiv \frac{5P_{g_0}R_0^{3k}}{2\rho}$, $\beta(t) \equiv 5S$, $\mu = \frac{6k-1}{5}$ and $y = m = \frac{1}{5}$. Thus,

$$G(x) = \int_1^x s^\mu f(s)ds = 4\nu\int_1^x s^{\frac{6k-1}{5}-\frac{4}{5}}ds = \frac{10\nu}{3k}x^{\frac{6k}{5}} - \frac{10\nu}{3k},$$

and from (2.7), we see

$$\begin{aligned} A_1 &= \min\left\{ \left(\frac{\bar{\alpha}}{2\overline{\varphi_+}}\right)^{\frac{1}{\mu+m}}, \left(\frac{\bar{\alpha}}{2\overline{\beta_+}}\right)^{\frac{1}{\mu-\gamma}} \right\} \\ &= \min\left\{ \left(\frac{P_{g_0}R_0^{3k}}{2\overline{[P_\infty(t)-P_v]_+}}\right)^{\frac{1}{\mu+m}}, \left(\frac{P_{g_0}R_0^{3k}}{2S}\right)^{\frac{1}{\mu-\gamma}} \right\}. \end{aligned} \tag{3.32}$$

If $\frac{10\nu}{3k} > \frac{5T}{2\rho}\overline{[P_\infty(t)-P_v]_+}$, then

$$G(x) - T\overline{\beta_+}x^{\mu-\gamma} - T\overline{\varphi_+}x^{\mu+m} = \left(\frac{10\nu}{3k} - \frac{5T}{2\rho}\overline{[P_\infty(t)-P_v]_+}\right)x^{\frac{6k}{5}} - 5TSx^{\frac{6k-2}{5}} - \frac{10\nu}{3k},$$

and so assumption (3.14) holds. Let

$$H(x) = \left(\frac{10\nu}{3k} - \frac{5T}{2\rho}\overline{[P_\infty(t)-P_v]_+}\right)x^{\frac{6k}{5}} - 5TSx^{\frac{6k-2}{5}}.$$

Clearly, if $A_1 > \left(\frac{5TS}{\sigma_1}\right)^{\frac{5}{2}}$, then

$$\begin{aligned} C_0 &:= \inf_{x\in[A_1,+\infty)} \left[G(x) - 2T\overline{\beta_+}x^{\mu-\gamma} - 2T\overline{\varphi_+}x^{\mu+m}\right] \\ &= \inf_{x\in[A_1,+\infty)} H(x) - \frac{10\nu}{3k} \\ &= H(A_1) - \frac{10\nu}{3k} \\ &> -\frac{10\nu}{3k}, \end{aligned} \tag{3.33}$$

where

$$\sigma_1 = \frac{10\nu}{3k} - \frac{5T}{2\rho}\overline{[P_\infty(t)-P_v]_+}. \tag{3.34}$$

Furthermore,

$$\lim_{x\to 0^+} G(x) = -\frac{10\nu}{3k},$$

which together with (3.33) yields

$$\lim_{x\to 0^+} G(x) < C_0 - T\overline{\alpha_-}.$$

This implies that assumption (3.15) holds. Thus, By using Theorem 3.3, we obtain the following result.

Theorem 3.5. *Assume $k \in (\frac{1}{3}, +\infty)$, $\overline{P_\infty} > P_v$. If $\frac{10\nu}{3k} > \frac{5T}{2\rho}\overline{[P_\infty(t)-P_v]_+}$, and*

$$A_1 > \left(\frac{5TS}{\sigma_1}\right)^{\frac{5}{2}},$$

then (3.31) *has at least one positive T−periodic solution, where σ_1 and A_1 is defined by* (3.34) *and* (3.32)*, respectively.*

Remark 3.2: Since the condition $\overline{P_\infty} > P_v$ in Theorem 3.5 is essentially different from the corresponding one of $\overline{P_\infty} < P_v$ in Theorem 1.1, Theorem 1.4 can be regarded as a new result on the existence of periodic solutions to Rayleigh-Plesset equation (1.4).

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