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Periodic traveling fronts for partially degenerate reaction-diffusion systems with bistable and time-periodic nonlinearity

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Abstract: This paper is concerned with the periodic traveling fronts for partially degenerate reaction-diffusion systems with bistable and time-periodic nonlinearity. We first determine the signs of wave speeds for two monostable periodic traveling fronts of the system. Then, we prove the existence of periodic traveling fronts connecting two stable periodic solutions. An estimate of the wave speed is also obtained. Further, we prove the monotonicity, uniqueness (up to a translation), Liapunov stability and exponentially asymptotical stability of the smooth bistable periodic traveling fronts.

Keywords: Time-periodic reaction-diffusion systems; Periodic traveling fronts; Bistable nonlinearity; Partially degenerate systems

MSC: 35K57; 37L60; 92D25

1 Introduction

This paper is concerned with the periodic traveling wave solutions of the following n-dimensional time-periodic reaction-diffusion system:

$$u_t = D\Delta u + f(u, t), \quad x \in \mathbb{R}, \; t > 0, \tag{1.1}$$

where $n \in \mathbb{N}$ with $n \geq 2$, $u = (u_1, \cdots, u_n)^{\mathrm{T}}$ and $D = \mathrm{diag}(d_1, \cdots, d_n)$ with $d_i \geq 0,\; i = 1, \cdots n$. The nonlinearity $f(\cdot, \cdot) = (f_1(\cdot, \cdot), \cdots, f_n(\cdot, \cdot))^{\mathrm{T}} \in C^{2,1}(\mathbb{R}^{n+1}, \mathbb{R}^n)$ is T-periodic in t for some period $T > 0$, i.e. $f(\cdot, t + T) = f(\cdot, t)$ for all $t \in \mathbb{R}$. It is clear that (1.1) is a time-periodic generalization of the following reaction-diffusion system:

$$u_t = D\Delta u + f(u), \quad x \in \mathbb{R}, \; t > 0. \tag{1.2}$$

In the past decades, the issue on the traveling wave solutions of system (1.2) with monostable or bistable nonlinearity have been well addressed, see [1–4] for monotone systems; [5] for non-monotone systems; also the references cited therein.

In recent years, stimulating by a great deal of examples of biological and physical systems with time-periodic parameters, periodic traveling wave solutions have also been widely investigated in time-periodic reaction-diffusion systems, see e.g. [6–17]. More precisely, Alikakos et al. [6] studied the existence, uniqueness and stability of bistable periodic traveling waves of (1.1) with $n = 1$. More recently, Fang and Zhao [7] developed the theory of bistable traveling fronts for monotone evolution systems with some compact conditions. As an application, their theory has been used to establish the existence of bistable traveling waves for

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the time periodic reaction-diffusion system (1.1) when $d_i > 0$ for all $i = 1, \cdots, n$ (i.e. the non-degenerate case). In addition, Bao and Wang [8] applied Fang and Zhao's theory to study the existence of bistable periodic traveling fronts for a time-periodic Lotka-Volterra competition system with positive diffusion coefficients. Zhao and Ruan [16, 17] studied the existence, uniqueness and stability of periodic traveling waves for monostable time-periodic reaction-diffusion systems. We also refer to Bao et al. [9], Wang [13], and Wang and Wu [14], Sheng et al. [18] for periodic traveling curved fronts of time-periodic diffusion systems.

On the other hand, in population biology and epidemiology, evolution is quite often described by reaction-diffusion systems with some but not all diffusion coefficients are zeros. Such systems are called partially degenerate systems. Two typical and important examples are the following periodic epidemic system [10]:

$$\begin{cases} u_t = du_{xx} - a_{11}u(x,t) + a_{12}v(x,t), \\ v_t = -a_{22}v(x,t) + g(u(x,t),t), \end{cases} \tag{1.3}$$

where $d > 0$, $g(\cdot, t+T) = g(\cdot, t)$ for some $T > 0$, and the periodic reaction-diffusion population model with a quiescent state [19]:

$$\begin{cases} u_t = d_1u_{xx} + f(u(x,t),t) - y_1(t)u(x,t) + y_2(t)v(x,t), \\ v_t = y_1(t)u(x,t) - y_2(t)v(x,t), \end{cases} \tag{1.4}$$

where $d_1 > 0$, $y_i(t+T) = y_i(t)$, $i = 1, 2$ and $f(\cdot, t+T) = f(\cdot, t)$ for some $T > 0$. For more details on models (1.3) and (1.4), we refer to [10, 19] and the references cited therein. Motivated by the models (1.3) and (1.4), the purpose of this article is to study the bistable periodic traveling wave solutions of the general partially degenerate system (1.1).

Throughout this paper, we always use the usual notations for the standard ordering in $\mathbb{R}^n$, and impose the following basic assumptions:

(C1) **[Partially degenerate system]** There exists $i_0 \in \{1, \cdots, n\}$ such that $d_i > 0$ for $i = 1, \cdots, i_0$ and $d_i = 0$ for $i = i_0 + 1, \cdots, n$.

(C2) **[Bistability]** The Poincaré map $P(\alpha) := v(T; \alpha)$, where $v(t; \alpha)$ is the solution to

$$v'(t) = f(v,t), \ t \in \mathbb{R}, \ v(0;\alpha) = \alpha \in \mathbb{R}^n, \tag{1.5}$$

has exactly three fixed points w^-, $\bar{w}$ and w^+ satisfying $w^- < \bar{w} < w^+$,

$$r^\pm := r(\mathrm{D}P(w^\pm)) < 1 \text{ and } \bar{r} := r(\mathrm{D}P(\bar{w})) > 1.$$

Here, $r(L)$ denotes the spectral radius of an operator L.

Let's set $w^\pm(t) = v(t; w^\pm)$, $\bar{w}(t) = v(t; \bar{w})$, and $\Omega_\pm$ by the domain of attraction of $w^\pm(t)$, respectively. According to (C2), we may assume that $w^\pm(t) \pm 2\delta_0\mathbf{1} \in \Omega_\pm$, where $\mathbf{1} := (1, \cdots, 1) \in \mathbb{R}^n$ and $\delta_0 > 0$ is small enough. Moreover, we can assume that

$$w^-(t) = \mathbf{0} := (0, \cdots, 0) \text{ and } w^+(t) = \mathbf{1}. \tag{1.6}$$

In fact, let $z = (z_1, \cdots, z_n)^{\mathrm{T}}$ with $z_i := \dfrac{u_i - w_i^-(t)}{w_i^+(t) - w_i^-(t)}$, $i = 1, \cdots, n$. From (1.1), z_i satisfies

$$(z_i)_t = d_i\Delta z_i + g_i(z,t),$$

where $w(t) := \mathrm{diag}(w_1^+(t) - w_1^-(t), \cdots, w_n^+(t) - w_n^-(t))$ and

$$g_i(z,t) := \frac{f_i(w(t)z + w^-(t), t) - f_i(w^-(t), t) - [f_i(w^+(t), t) - f_i(w^-(t), t)]z_i}{w_i^+(t) - w_i^-(t)}, \ i = 1, \cdots, n.$$

It is clear that $g(\cdot, t+T) = g(\cdot, t)$, $g(\mathbf{0}, t) = g(\mathbf{1}, t) = \mathbf{0}$. In this situation, $w^-(t)$ and $w^+(t)$ reduce to $\mathbf{0}$ and $\mathbf{1}$. Therefore, without loss of generality, we always assume (1.6) in the rest of this paper. Moreover, we have $\mathbf{0} \ll \bar{w}(t) \ll \mathbf{1}$ for $t > 0$ (c.f. the non-autonomous version of [20, Theorem 4.1.1]) provided that the function $f(\cdot, \cdot)$ satisfies the following monotone and irreducible conditions:

(C3) **[Monotonicity]** $\dfrac{\partial f_i(u,t)}{\partial u_j} \geq 0$ for all $u \in I_0 := [-\delta_0\mathbf{1}, \mathbf{1} + \delta_0\mathbf{1}]$, $t \geq 0$ and $1 \leq i \neq j \leq n$.

(C4) **[Irreducibility]** $Df(u,t)$ is irreducible for all $u \in I_0$ and $t \geq 0$.
Before to state the main results, we first introduce the definition of periodic traveling wave solution of (1.1) (see e.g. [6, 7, 10, 16]).

Definition 1.1. (1) *A solution $u(x,t)$ of* (1.1) *is called a periodic traveling wave solution connecting* **0** *and* **1** *if there exist a function $U(\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}^n$ and a constant $c \in \mathbb{R}$ such that*

(i) *$u(x,t) = U(\xi,t)$, $\xi = x + ct$, and $U(\xi,t)$ is periodic in t, i.e. $U(\cdot,t+T) = U(\cdot,t)$, $\forall t \in \mathbb{R}$.*

(ii) *$U(-\infty,t) := \lim_{\xi\to-\infty} U(\xi,t) = \mathbf{0}$ and $U(+\infty,t) := \lim_{\xi\to+\infty} U(\xi,t) = \mathbf{1}$ uniformly in $t \in [0,T]$.*

Moreover, if $U(\xi,\cdot)$ is non-decreasing in $\xi \in \mathbb{R}$, then we call it a periodic traveling front.

(2) *A periodic traveling wave connecting* **0** *and* **1** *is smooth if $U(x,t) \in C^{2,1}(\mathbb{R}^2,\mathbb{R}^n)$, $U_x(\cdot,\cdot)$ and $U_{xx}(\cdot,\cdot)$ are bounded.*

In this article, we first determine the signs of the wave speeds of the monostable periodic traveling fronts of (1.1) connecting **0** and $\bar{w}(t)$, and $\bar{w}(t)$ and **1** (see Lemma 3.1). Then, based on these results, we use a generalized comparison principle for weak sub- and supersolutions (which may not be continuous) and the "vanishing viscosity" method (c.f. [1, 12]) to prove the existence of the bistable periodic traveling front of (1.1) connecting **0** and **1**. More precisely, for any $\epsilon \in (0,1]$, we construct an auxiliary non-degenerate and time-periodic reaction-diffusion system with diffusion coefficients $D^\epsilon = \mathrm{diag}(d_1^\epsilon, \cdots, d_n^\epsilon)$, where $d_i^\epsilon = d_i > 0$ for $i = 1, \cdots, i_0$ and $d_i^\epsilon = \epsilon$ for $i = i_0, \cdots, n$ (see (4.1)). Following the results in [7, Theorem10], for any $\epsilon \in (0,1]$, the auxiliary non-degenerate system admits a bistable periodic traveling front $U^\epsilon(x + c_\epsilon t, t)$ connecting **0** and **1**. The existence of the bistable periodic traveling front of (1.1) connecting **0** and **1** is then obtained by letting $\epsilon \to 0$. To find a convergent subsequence of $\{U^\epsilon(x + c_\epsilon t, t)\}_{\epsilon\in(0,1]}$, we need to prove the boundedness of $\{c^\epsilon\}_{\epsilon\in(0,1]}$ and the compactness of $\{U^\epsilon(x,t) : \epsilon \in (0,1]\}$. The former part can be proved by constructing a pair of explicit sub- and supersolutions (see Lemma 4.2). The major difficulty is the verification of compactness of $\{U^\epsilon(x,t) : \epsilon \in (0,1]\}$. Since $d_i^\epsilon = \epsilon \in (0,1]$ when $d_i = 0$, $U_x^\epsilon(x,t)$ and $U_{xx}^\epsilon(x,t)$ may not be uniformly bounded for all $\epsilon \in (0,1]$. To overcome this difficulty, we shall show that $\{U^\epsilon(x,t) : \epsilon \in (0,1]\}$ is pre-compact in $L^1_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^n)$ (see Lemma 4.4).

The existence result of bistable periodic traveling fronts is stated as follow.

Theorem 1.2. **[Existence]** *Assume* (C1)-(C4). *Then* (1.1) *admits a periodic traveling front $U(x+ct,t)$ connecting* **0** *and* **1** *with speed $|c| \leq C_0$, where $C_0 = C_0(d_1,\cdots,d_{i_0},f)$ is a constant given by* (4.9).

It should be pointed out that the existence part in Theorem 1.2 can also be established by applying the theory developed by Fang and Zhao [7] for the monotone semiflow with weak compactness. The reasons why we use the vanishing viscosity method are stated in the sequel.

(1) Some preliminary lemmas have their own interests. For example, Lemma 3.1 provided some information on the signs of the wave speeds of the monostable periodic traveling fronts connecting **0** and $\bar{w}(t)$, and $\bar{w}(t)$ and **1**, respectively.

(2) As a by-product, we can obtain an estimate for the wave speed of the bistable periodic traveling front although its sign cannot be determined.

(3) Our result demonstrates a way of applying the vanishing viscosity method to study the time-periodic reaction-diffusion systems which are partially degenerate.

In addition to the existence of the bistable periodic traveling front of (1.1), it is natural to ask whether such traveling fronts can determine the long term behavior of the corresponding Cauchy type problem of (1.1). From [3, 6], we know that that the bistable traveling fronts of the reaction-diffusion system (1.1) are stable when $f(u,t) = f(u)$ or $n = 1$, which naturally determine the long time dynamical behavior of the corresponding initial value problem. This motivates us to consider the stability of the bistable periodic traveling fronts of system (1.1).

In this work, we shall generalize the squeezing technique to prove the asymptotic stability of the smooth bistable periodic traveling fronts for system (1.1). This technique was introduced in Chen [21] to prove the global asymptotic stability of bistable traveling fronts for nonlocal evolution equations and has been used to study the asymptotic stability of traveling fronts for various evolution systems, see e.g. [3, 22–26]. Since our problem is a general time-periodic system, this generalization is nontrivial and needs some new techniques. For example, the constructions of the sub- and supersolutions are different to those in [3, 21, 22] (see Lemma 6.1). In fact, the sub- and supersolutions contain a non-monotone (w.r.t. x) and time periodic function $p(x,t)$ (see (5.5)). To overcome this difficulty, the shift parameters ξ_1 and ξ_2 in two component parts of the sub- and supersolution might be different. Moreover, we need to establish a strong comparison principle (see Lemma 2.5) under the following strongly irreducible assumption:
(C4)$'$ **[Strong irreducibility]** The matrix $\left(\min_{u\in[-\delta_0\mathbf{1},\mathbf{1}+2\delta_0\mathbf{1}],t\geq 0}\partial_j f_i(u,t)\right)$ is irreducible.
We now state the result on the monotonicity and stability of the periodic traveling fronts.

Theorem 1.3. **[Monotonicity]** *Assume* (C1)-(C3) *and* (C4)$'$. *Let $U(x+ct,t)$ be a smooth periodic traveling wave of* (1.1) *connecting* $\mathbf{0}$ *and* $\mathbf{1}$. *Then,* $U_x(\cdot,\cdot)\gg\mathbf{0}$.

Theorem 1.4. *Assume* (C1)-(C3) *and* (C4)$'$. *Let $U(x+ct,t)$ be a smooth periodic traveling wave of* (1.1) *connecting* $\mathbf{0}$ *and* $\mathbf{1}$. *Then the following statements hold:*

(i) **[Uniform stability]** *For any $\epsilon>0$, there exists $\delta>0$ such that for any $\phi\in L^\infty(\mathbb{R})$ with $\|\phi(\cdot)-U(\cdot,0)\|_{L^\infty(\mathbb{R})}<\delta$, there holds*
$$\|u(\cdot,t;\phi)-U(\cdot+ct,t)\|_{L^\infty(\mathbb{R})}<\epsilon \text{ for all } t>0,$$
where $u(x,t;\phi)$ is the unique solution of (1.1) *with $u(\cdot,0;\phi)=\phi(\cdot)$.*

(ii) **[Asymptotic stability]** *For any $\phi\in L^\infty(\mathbb{R})$ with $\phi(\cdot)\in[-\delta_0\mathbf{1},\mathbf{1}+\delta_0\mathbf{1}]$ and*
$$\liminf_{x\to+\infty}\phi(x)\gg\bar{w} \text{ and } \limsup_{x\to-\infty}\phi(x)\ll\bar{w}, \tag{1.7}$$
where $\bar{w}=\bar{w}(0)$, there exists $\xi_\star\in\mathbb{R}$ such that
$$\|u(\cdot,t;\phi)-U(\cdot+ct+\xi_\star,t)\|_{L^\infty(\mathbb{R})}\to 0 \text{ exponentially as } t\to\infty.$$

As a direct result of part (ii) of Theorem 1.4, we obtain the uniqueness of the smooth periodic traveling waves.

Theorem 1.5. **[Uniqueness]** *Assume* (C1)-(C3) *and* (C4)$'$. *Let $U^{(i)}(x+ct,t)(i=1,2)$ be two periodic traveling waves of* (1.1) *connecting* $\mathbf{0}$ *and* $\mathbf{1}$ *with one of them being smooth. Then, $c_1=c_2$ and there exists $\xi^\star\in\mathbb{R}$ such that $U^{(1)}(\xi,t)=U^{(2)}(\xi+\xi^\star,t)$, $\forall\xi,t\in\mathbb{R}$.*

Remark 1.6. (1) If $d_i>0$, $i=1,\cdots,n$, according to the result in [7, Theorem 10], system (1.1) admits a periodic traveling front $V(x+ct,t)$ connecting $\mathbf{0}$ and $\mathbf{1}$. Moreover, $V(x,t)$ is continuous in x. By the parabolic theory, it is easy to see that $V(x+ct,t)$ is a smooth periodic traveling front. Therefore, Theorems 1.3-1.5 imply that such a periodic traveling front is strictly monotone, uniqueness and stable. However, when system (1.1) is partially degenerate, even in the time independent case, it may admit non-smooth traveling waves (see e.g. [1]). Whether a non-smooth traveling wave is monotone, uniqueness and asymptotically stable remains open. We will study these problems somewhere else.

(2) We also mention that our results can be used to study some biological and epidemiological models described by partially degenerate reaction-diffusion systems with periodic coefficients, such as models (1.3) and (1.4). Furthermore, the techniques in this paper can be extended to the following mixed dispersal evolution system with bistable nonlinearity (c.f. [22, 27]):
$$u_t=D\Delta u+B\int_{\mathbb{R}}J(x-y)\big[u(y)-u(x)\big]dy+f(u,t),\quad x\in\mathbb{R},\ t>0, \tag{1.8}$$
where $n\in\mathbb{N}$, $u=(u_1,\cdots,u_n)^{\mathrm{T}}$, $D=\operatorname{diag}(d_1,\cdots,d_n)$, and $B=\operatorname{diag}(b_1,\cdots,b_n)$ with $d_i,\ b_i\geq 0$, $i=1,\cdots,n$ and $(d_1+b_1,\cdots,d_n+b_n)\neq\mathbf{0}$.

The rest of this paper is organized as follows. In Section 2, we establish the well-posedness of Cauchy problem for (1.1) and some key comparison principles for later use. Section 3 is devoted to the study of signs of wave speeds for two monostable periodic traveling fronts connecting $\mathbf{0}$ and $\bar{w}(t)$, and $\bar{w}(t)$ and $\mathbf{1}$, respectively. In Section 4, we prove the existence of the bistable periodic traveling fronts. An estimate of the wave speed is also established. Section 5 focus on the monotonicity of the smooth periodic traveling waves. Finally, we prove the stability and uniqueness of the smooth periodic traveling fronts in Section 6.

2 Well-posedness and comparison principles

In this section, we give the well-posedness to the Cauchy problem of (1.1) and establish some key comparison principles, especially a strong comparison principle under the assumption (C4)′. First, given any $\tau \geq 0$, we consider the initial value problem of (1.1):

$$\begin{cases} u_t = D\Delta u + f(u,t), \quad x \in \mathbb{R}, \ t > \tau, \\ u(x,\tau) = \phi(x) \in L^\infty(\mathbb{R}). \end{cases} \tag{2.1}$$

Let's set

$$\gamma := \max_{1\leq i\leq n} \sup_{u\in[-\delta_0\mathbf{1},\mathbf{1}+\delta_0\mathbf{1}],\ t\geq 0} \left|\frac{\partial}{\partial u_i} f_i(u,t)\right| \text{ and } Q(u)(x,t) := f(u(x,t),t) + \gamma u(x,t),$$

and define a family of mappings $T_i(t) : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ by $T_i(0) = I$ and

$$T_i(t)[\phi](x) = \begin{cases} e^{-\gamma t} \displaystyle\int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi d_i t}} e^{-\frac{y^2}{4d_i t}} \phi(x-y)dy, & \text{if } i \in \{1,\cdots,i_0\}, \\ e^{-\gamma t}\phi(x), & \text{if } i \in \{i_0+1,\cdots,n\}, \end{cases}$$

$\forall t > 0$, $x \in \mathbb{R}$ and $\phi(\cdot) \in L^\infty(\mathbb{R})$. Furthermore, we denote $T(t) := \operatorname{diag}(T_1(t),\cdots,T_n(t))$. It is clear that (2.1) can be transformed to the following integral form:

$$u(x,t;\tau,\phi) = T(t-\tau)[\phi](x) + \int_\tau^t T(t-s)[Q(u)(\cdot,s)](x)ds, \ x \in \mathbb{R}, \ t > \tau. \tag{2.2}$$

A solution of (2.2) is called a mild solution of (2.1). We have the following well-posedness result.

Lemma 2.1. *Assume* (C1)-(C4). *For any $\phi(\cdot) \in L^\infty(\mathbb{R})$ with $\phi(\cdot) \in [-\delta_0\mathbf{1}, \mathbf{1}+\delta_0\mathbf{1}]$,* (2.1) *admits a unique mild solution $u(x,t;\tau,\phi) \in C^1([\tau,\infty), L^\infty(\mathbb{R}))$. Moreover, if $\phi(\cdot)$ is continuous, then $u(x,t;\tau,\phi)$ is a classical solution of* (2.1).

Proof. The proof is similar to [28, Theorem 2.2] by using the Banach's fixed point theorem. We omit it here. □

Next, we introduce the definitions of (weak) supersolution and subsolution of (1.1).

Definition 2.2. (1) *A function $u : \mathbb{R} \times [\tau,\infty) \to [-\delta_0\mathbf{1}, \mathbf{1}+\delta_0\mathbf{1}]$ is called a weak supersolution* (*or a weak subsolution*) *of* (1.1) *on $[\tau,\infty)$ if*

$$u(x,t) \geq (or \leq)\ T(t-\tau)[u(\cdot,\tau)](x) + \int_\tau^t T(t-r)[Q(u)(\cdot,r)](x)dr$$

for any $x \in \mathbb{R}$, $\tau \leq t < \infty$.

(2) *Let $w = (w_1,\cdots,w_m) : \mathbb{R}\times[\tau,\infty) \to [-\delta_0\mathbf{1},\mathbf{1}+\delta_0\mathbf{1}]$ be a continuous function such that $w_i(x,t)$ is C^1 in t for $i = 1,\cdots,n$ and C^2 in x for $i = 1,\cdots,i_0$. If $w(x,t)$ satisfies*

$$w_t \geq (or \leq) D\Delta w + f(w,t), \quad \forall x \in \mathbb{R}, \ t > \tau,$$

then it is called a supersolution (*or subsolution*) *of* (1.1) *on $[\tau,+\infty)$.*

According to Definition 2.2, it is clear that a supersolution (or subsolution) is a weak supersolution (or subsolution). Furthermore, the weak super- and subsolutions are not necessarily continuous. The following generalized comparison principle for weak sub- and supersolutions will play a crucial role in proving the existence of the bistable periodic traveling fronts.

Lemma 2.3. *Assume* (C1)-(C4). *Let* $u^{\pm}(x,t)$ *be a pair of weak super- and subsolution of* (1.1) *on* $[\tau,+\infty)$. *If* $u^{-}(x,\tau)\le u^{+}(x,\tau)$, $\forall x\in\mathbb{R}$, *then* $u^{-}(x,t)\le u^{+}(x,t)$, $\forall x\in\mathbb{R}$, $t>\tau$.

Proof. This lemma is a non-autonomous version of [1, Lemma 2.1] and can be proved by using a similar argument as in [1, Lemma 2.1]. Here, we omit its proof. □

In addition, the following result follows from Lemma 2.3 and Definition 2.2 directly.

Lemma 2.4. *Assume* (C1)-(C4). *Let* $u^{\pm}(x,t)$ *be a pair of super- and sub-solution of* (1.1) *on* $[\tau,+\infty)$ *with* $u^{-}(x,\tau)\le u^{+}(x,\tau)$, $\forall x\in\mathbb{R}$. *Then* $u^{-}(x,t)\le u^{+}(x,t)$, $\forall x\in\mathbb{R}$, $t>\tau$.

To prove the asymptotic stability of the periodic traveling waves by applying the squeezing technique, we further establish the following strong comparison principle for the periodic system (1.1) under the assumption (C4)$'$.

Lemma 2.5. **[Strong comparison principle]** *Assume* (C1)-(C3) *and* (C4)$'$. *Let* $u^{\pm}(x,t) = (u_1^{\pm}(x,t),\cdots,u_n^{\pm}(x,t))$ *be a pair of super- and subsolution of* (1.1) *on* $[\tau,+\infty)$ *with* $u^{-}(x,\tau)\le u^{+}(x,\tau)$, $\forall x\in\mathbb{R}$.

(i) *There exists a positive and non-increasing function* $\Theta(\cdot)$ *defined on* $[0,+\infty)$ *such that*

$$u_l^{+}(x,\tau+1)-u_l^{-}(x,\tau+1)\ge\Theta(|x|)\sum_{j=1}^{n}\int_0^1[u_j^{+}(y,\tau)-u_j^{-}(y,\tau)]dy, \tag{2.3}$$

for any $l\in\{1,\cdots,n\}$ *and* $x\in\mathbb{R}$.

(ii) *If* $u^{-}(\cdot,\tau)\ne u^{+}(\cdot,\tau)$, *then* $u^{-}(x,t)\ll u^{+}(x,t)$, $\forall x\in\mathbb{R}$, $t>\tau$.

Proof. (i) Let $w_i(x,t):=u_i^{+}(x,t)-u_i^{-}(x,t)$ for $i=1,\cdots,n$, $x\in\mathbb{R}$ and $t\ge\tau$. By Lemma 2.4, we see that $0\le w_i(x,t)\le 1+2\delta_0$ for $i=1,\cdots,n$, $x\in\mathbb{R}$ and $t\ge\tau$. Moreover, by (C4)$'$ and Definition 2.2, we have

$$\begin{aligned}(w_i)_t =&d_i(w_i)_{xx}+f_i(u^{+}(x,t),t)-f_i(u^{-}(x,t),t)\\ =&d_i(w_i)_{xx}-Mw_i+[M+\frac{\partial f_i(\eta(x,t),t)}{\partial u_i}]w_i+\sum_{1\le j\ne i\le n}\frac{\partial f_i(\eta(x,t),t)}{\partial u_j}w_j\\ \ge&d_i(w_i)_{xx}-Mw_i+\sum_{1\le j\ne i\le n}\frac{\partial f_i(\eta(x,t),t)}{\partial u_j}w_j \qquad (2.4)\\ \ge&d_i(w_i)_{xx}-Mw_i,\ \forall x\in\mathbb{R},\ t>\tau,\ i=1,\cdots,n, \qquad (2.5)\end{aligned}$$

where $\eta(x,t):=\vartheta u^{+}(x,t)+(1-\vartheta)u^{-}(x,t)$ with $\vartheta\in(0,1)$ and

$$M:=1+\max_{1\le i\le n}\sup_{u\in[-\delta_0\mathbf{1},\mathbf{1}+2\delta_0\mathbf{1}],\ t\ge\tau}\left|\partial f_i(u,t)/\partial u_i\right|.$$

Recall that $d_i>0$ for $i=1,\cdots,i_0$ and $d_i=0$ for $i=i_0+1,\cdots,n$. Let $d_M:=\max\{d_1,\cdots,d_{i_0}\}$ and $d_m:=\min\{d_1,\cdots,d_{i_0}\}$. By (2.5), for any $i\in\{1,\cdots,i_0\}$, we have

$$w_i(x,t)\ge e^{-M(t-\tau)}\int_{-\infty}^{+\infty}\frac{e^{-\frac{(x-y)^2}{4d_i(t-\tau)}}}{\sqrt{4\pi d_i(t-\tau)}}w_i(y,\tau)dy$$

$$\geq e^{-M(t-\tau)} \frac{e^{-\frac{(|x|+1)^2}{4d_i(t-\tau)}}}{\sqrt{4\pi d_i(t-\tau)}} \int_0^1 w_i(y,\tau)dy \geq \Gamma(|x|, t-\tau)e^{-M(t-\tau)} \int_0^1 w_i(y,\tau)dy, \tag{2.6}$$

for $x \in \mathbb{R},\ t > \tau$, where $\Gamma(|x|, s) := \exp\{-\frac{(|x|+1)^2}{4d_m s}\}/\sqrt{4\pi d_M s},\ s > 0$. On the other hand, for any $i \in \{i_0 + 1, \cdots, n\}$, it follows from (2.5) that

$$w_i(x,t) \geq e^{-M(t-\tau)} w_i(x,\tau),\ \forall x \in \mathbb{R},\ t > \tau. \tag{2.7}$$

Now, we consider the following two cases.

Case 1. $l \in \{1, \cdots, i_0\}$. In this case, it follows from (2.6) that

$$w_l(x,t) \geq \Gamma(|x|, t-\tau)e^{-M(t-\tau)} \int_0^1 w_l(y,\tau)dy,\ \forall x \in \mathbb{R},\ t > \tau. \tag{2.8}$$

Given any $j \in \{1, \cdots, n\}\setminus\{l\}$. Since the matrix $(m_{i,k}) := \left(\min_{u\in[-\delta_0\mathbf{1}, \mathbf{1}+2\delta_0\mathbf{1}], t\geq 0} \partial_k f_i(u,t)\right)$ is irreducible, there exists a distinct sequence $i_1, i_2, \cdots, i_r$ with $i_1 = l$ and $i_r = j$ such that $m_{i_s i_{s+1}} > 0, \forall s = 1, \cdots, r-1$. We first prove the following claim.

Claim: There exist positive functions $\theta_{i_1,i_s}(\cdot,\cdot)$ $s = 1, \cdots, r$ such that each $\theta_{i_1,i_s}(x,\cdot)$ is non-increasing in $x \in [0, \infty)$ and

$$w_{i_1}(x,t) \geq \theta_{i_1,i_s}(|x|, t-\tau) \int_0^1 w_{i_s}(z,\tau)dz,\ s = 1, \cdots, r. \tag{2.9}$$

It is clear that (2.9) holds for $s = 1$ with $\theta_{i_1,i_1}(|x|, s) = \Gamma(|x|, s)e^{-Ms}$. We now prove that (2.9) holds for $s = 2$. Using (2.4), (2.6) and (2.7), we have

$$\begin{aligned}
& w_{i_1}(x,t) \\
\geq & \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-y)^2}{4d_{i_1}(t-\tau)}} e^{-M(t-\tau)}}{\sqrt{4\pi d_{i_1}(t-\tau)}} w_{i_1}(y,\tau)dy + m_{i_1 i_2} \int_\tau^t \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-y)^2}{4d_{i_1}(t-r)}} e^{-M(t-r)}}{\sqrt{4\pi d_{i_1}(t-r)}} w_{i_2}(y,r)dydr \\
\geq & m_{i_1 i_2} \int_\tau^t \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-y)^2}{4d_{i_1}(t-r)}} e^{-M(t-r)}}{\sqrt{4\pi d_{i_1}(t-r)}} w_{i_2}(y,r)dydr \\
\geq & \begin{cases} e^{-M(t-\tau)} m_{i_1 i_2} \displaystyle\int_\tau^t \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-y)^2}{4d_{i_1}(t-r)}}}{\sqrt{4\pi d_{i_1}(t-r)}} \Gamma(|y|, r-\tau) \int_0^1 w_{i_2}(z,\tau)dzdydr, & \text{if } i_2 \in \{1, \cdots, i_0\} \\ e^{-M(t-\tau)} m_{i_1 i_2} \displaystyle\int_\tau^t \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-y)^2}{4d_{i_1}(t-r)}}}{\sqrt{4\pi d_{i_1}(t-r)}} w_{i_2}(y,\tau)dydr, & \text{if } i_2 \in \{i_0+1, \cdots, n\} \end{cases} \\
\geq & \begin{cases} e^{-M(t-\tau)} m_{i_1 i_2} \displaystyle\int_\tau^t \int_0^1 \Gamma(|x|, t-r)\Gamma(|y|, r-\tau)dydr \int_0^1 w_{i_2}(z,\tau)dz, & \text{if } i_2 \in \{1, \cdots, i_0\} \\ e^{-M(t-\tau)} m_{i_1 i_2} \displaystyle\int_\tau^t \Gamma(|x|, t-r)dr \int_0^1 w_{i_2}(z,\tau)dz, & \text{if } i_2 \in \{i_0+1, \cdots, n\} \end{cases} \\
\geq & \theta_{i_1,i_2}(|x|, t-\tau) \int_0^1 w_{i_2}(z,\tau)dz,
\end{aligned}$$

(2.10), (2.11)

for $x \in \mathbb{R},\ t > \tau$, where

$$\theta_{i_1,i_2}(|x|, s) = e^{-Ms} m_{i_1 i_2} \min\left\{ \int_0^s \int_0^1 \Gamma(|x|, s-r)\Gamma(|y|, r)dydr,\ \int_0^s \Gamma(|x|, s-r)dr \right\}.$$

Next, we prove (2.9) for $s = 3$ by considering the following two subcases.

Subcase 1. $i_2 \in \{1, \cdots, i_0\}$. Since $m_{i_2 i_3} > 0$, by the same method as above, we can show that there is a positive function $\hat{\theta}_{i_2,i_3}(\cdot,\cdot)$ such that $\hat{\theta}_{i_2,i_3}(x,\cdot)$ is non-increasing in $x \in [0,\infty)$ and

$$w_{i_2}(x,t) \geq \hat{\theta}_{i_2,i_3}(|x|, t-\tau) \int_0^1 w_{i_3}(y,\tau)dy,\ \forall x \in \mathbb{R},\ t > \tau. \tag{2.12}$$

It then follows from (2.10) and (2.12) that

$$\begin{aligned} w_{i_1}(x,t) \geq& m_{i_1 i_2} \int_\tau^t \int_0^1 \Gamma(|x|, t-r)e^{-M(t-r)} w_{i_2}(y,r)dydr \\ \geq& m_{i_1 i_2} \int_\tau^t \int_0^1 \Gamma(|x|, t-r)e^{-M(t-r)} \hat{\theta}_{i_2,i_3}(|y|, r-\tau)dydr \int_0^1 w_{i_3}(z,\tau)dz \\ =& m_{i_1 i_2} \int_0^{t-\tau} \int_0^1 \Gamma(|x|, t-\tau-r)e^{-M(t-\tau-r)} \hat{\theta}_{i_2,i_3}(|y|, r)dydr \int_0^1 w_{i_3}(z,\tau)dz \\ =:& \theta^{(1)}_{i_1,i_3}(|x|, t-\tau) \int_0^1 w_{i_3}(z,\tau)dz. \end{aligned} \tag{2.13}$$

Subcase 2. $i_2 \in \{i_0+1, \cdots, n\}$. From (2.4), we obtain

$$w_{i_2}(x,t) \geq m_{i_2 i_3} \int_\tau^t e^{-M(t-s)} w_{i_3}(x,s)ds. \tag{2.14}$$

If $i_3 \in \{1, \cdots, i_0\}$, by (2.6), we get

$$w_{i_2}(x,t) \geq e^{-M(t-\tau)} m_{i_2 i_3} \int_\tau^t \Gamma(|x|, s-\tau)ds \int_0^1 w_{i_3}(y,\tau)dy := \check{\theta}_{i_2,i_3}(|x|, t-\tau) \int_0^1 w_{i_3}(y,\tau)dy.$$

Hence,

$$\begin{aligned} w_{i_1}(x,t) \geq& m_{i_1 i_2} \int_\tau^t \int_0^1 \Gamma(|x|, t-r)e^{-M(t-r)} w_{i_2}(y,r)dydr \\ \geq& m_{i_1 i_2} \int_\tau^t \int_0^1 \Gamma(|x|, t-r)e^{-M(t-r)} \check{\theta}_{i_2,i_3}(|y|, r-\tau)dydr \int_0^1 w_{i_3}(z,\tau)dz \\ =& m_{i_1 i_2} \int_0^{t-\tau} \int_0^1 \Gamma(|x|, t-\tau-r)e^{-M(t-\tau-r)} \check{\theta}_{i_2,i_3}(|y|, r)dydr \int_0^1 w_{i_3}(z,\tau)dz \\ =:& \theta^{(2)}_{i_1,i_3}(|x|, t-\tau) \int_0^1 w_{i_3}(z,\tau)dz,\ x \in \mathbb{R},\ t > \tau. \end{aligned} \tag{2.15}$$

If $i_3 \in \{i_0+1, \cdots, n\}$, using (2.10), (2.14) and (2.7), we have

$$
\begin{aligned}
w_{i_1}(x,t) \geq & m_{i_1 i_2} \int_\tau^t \int_0^1 \Gamma(|x|, t-r) e^{-M(t-r)} w_{i_2}(y,r) dy dr \\
\geq & m_{i_1 i_2} m_{i_2 i_3} \int_\tau^t \int_0^1 \Gamma(|x|, t-r) e^{-M(t-r)} \int_\tau^r e^{-M(r-s)} w_{i_3}(y,s) ds dy dr \\
\geq & m_{i_1 i_2} m_{i_2 i_3} \int_\tau^t \int_0^1 \Gamma(|x|, t-r) \int_\tau^r e^{-M(t-s)} e^{-M(s-\tau)} w_{i_3}(y,\tau) ds dy dr \\
= & m_{i_1 i_2} m_{i_2 i_3} e^{-M(t-\tau)} \int_0^{t-\tau} r\Gamma(|x|, t-\tau-r) dr \int_0^1 w_{i_3}(y,\tau) dy \\
=: & \theta^{(3)}_{i_1,i_3}(|x|, t-\tau) \int_0^1 w_{i_3}(y,\tau) dy.
\end{aligned}
\tag{2.16}
$$

Let $\theta_{i_1,i_3}(|x|, s) := \min\{\theta^{(1)}_{i_1,i_3}(|x|, s), \theta^{(2)}_{i_1,i_3}(|x|, s), \theta^{(3)}_{i_1,i_3}(|x|, s)\}$, one can see that (2.9) holds for $s = 3$. The cases for (2.9) with $s = 4, \cdots, r$ can be proved similarly. Hence the claim follows.

By the claim, we obtain

$$
w_{i_1}(x,t) \geq \theta_{i_1,i_r}(|x|, t-\tau) \int_0^1 w_{i_r}(z,\tau) dz, \ \forall x \in \mathbb{R}, \ t > \tau.
$$

Thus, combining with (2.8), we deduce that for any $j = 1, \cdots, n$,

$$
w_l(x,t) \geq \theta_{l,j}(|x|, t-\tau) \int_0^1 w_j(z,\tau) dz, \ \forall x \in \mathbb{R}, \ t > \tau.
$$

Therefore, we conclude that

$$
\begin{aligned}
w_l(x,t) \geq & \left(\frac{1}{n} \min_{j,l=1,\cdots,n} \theta_{l,j}(|x|, t-\tau)\right) \sum_{j=1}^n \int_0^1 w_j(z,\tau) dz \\
=: & \Theta^{(1)}(|x|, t-\tau) \sum_{j=1}^n \int_0^1 w_j(z,\tau) dz, \ \forall x \in \mathbb{R}, \ t > \tau.
\end{aligned}
\tag{2.17}
$$

It is clear that is $\Theta^{(1)}(x,\cdot)$ non-increasing in $x \in [0,\infty)$.

Case 2. $l \in \{i_0+1, \cdots, n\}$. Since $i_0 \geq 1$, we see that $l \neq 1$. Using the irreducibility of the matrix $(m_{i,k}) = \left(\min_{u \in [-\delta_0 \mathbf{1}, \mathbf{1}+2\delta_0 \mathbf{1}], t \geq 0} \partial_k f_i(u,t)\right)$ again, there exists a distinct sequence $i_1, i_2, \cdots, i_r$ with $i_1 = l$ and $i_r = 1$ such that $m_{i_s i_{s+1}} > 0$, $\forall s = 1, \cdots, r-1$. Noting that

$$
w_1(x,t) \geq \Theta^{(1)}(|x|, t-\tau) \sum_{j=1}^n \int_0^1 w_j(z,\tau) dz, \ \forall x \in \mathbb{R}, \ t > \tau,
\tag{2.18}
$$

we can easily obtain that there exists a positive function $\Theta^{(2)}(\cdot,\cdot)$ such that $\Theta^{(2)}(x,\cdot)$ is non-increasing in $x \in [0,\infty)$ and

$$
w_l(x,t) \geq \Theta^{(2)}(|x|, t-\tau) \sum_{j=1}^n \int_0^1 w_j(z,\tau) dz, \ \forall x \in \mathbb{R}, \ t > \tau.
\tag{2.19}
$$

Combining the above two cases, the assertion (i) holds by setting

$$\Theta(|x|) := \min\{\Theta^{(1)}(|x|, 1), \Theta^{(2)}(|x|, 1)\}.$$

(ii) Clearly, the assertion follows from (2.17) and (2.19). This completes the proof. □

3 Wave speeds of monostable periodic traveling fronts

In this section, we shall determine the signs of wave speeds for two monostable periodic traveling fronts connecting $\mathbf{0}$ and $\bar{w}(t)$, and $\bar{w}(t)$ and $\mathbf{1}$, respectively. In fact, we have the following result.

Lemma 3.1. *Assume* (C1)-(C4). *Then, the following statements hold.*
(1) *If $U(x+ct, t)$ is a non-decreasing periodic traveling wave of* (1.1) *with $U(-\infty, t) = \bar{w}(t)$ and $U(+\infty, t) = \mathbf{1}$, then $c > 0$.*
(2) *If $U(x+ct, t)$ is a non-decreasing periodic traveling wave of* (1.1) *with $U(-\infty, t) = \mathbf{0}$ and $U(+\infty, t) = \bar{w}(t)$, then $c < 0$.*

The proof of Lemma 3.1 is based on the results of spreading speeds (see [10]) for monostable systems. To consider the general monostable case, we make the following assumption:
(C2)′ **[Monostability]** The Poincaré map $P(\alpha) := v(T; \alpha)$ of (1.5) has exactly two fixed points $\mathbf{0}$ and $\mathbf{1}$ satisfying $r_0 := r(DP(\mathbf{0})) > 1$.
From the assumption (C2)′, system (1.5) has two periodic solutions $v(t; \mathbf{0})$ and $v(t; \mathbf{1})$. As mentioned in the introduction, we can assume that $\mathbf{0} = v(t; \mathbf{0})$ and $\mathbf{1} = v(t; \mathbf{1})$, $\forall t \in \mathbb{R}$. Applying the theory of spreading speed for monotone periodic semiflow developed by Liang et al. [10], one can show that (1.1) has a spreading speed under the assumptions (C1), (C2)′, (C3) and (C4). The proof is similar to that of [10, Section 3]. For the sake of reader's convenience, we sketch it in the sequel.

Let $\mathcal{C}$ be the space of all bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$. For any $\phi^1, \phi^2 \in \mathcal{C}$, we write $\phi^1 \le \phi^2$ ($\phi^1 \ll \phi^2$) if $\phi^1(x) \le \phi^2(x)$ ($\phi^1(x) \ll \phi^2(x)$), $\forall x \in \mathbb{R}$ and $\phi^1 < \phi^2$ if $\phi^1 \le \phi^2$ and $\phi^1 \ne \phi^2$. We further equip $\mathcal{C}$ with the compact open topology, that is, a sequence ϕ_n converges to ϕ in $\mathcal{C}$ if and only if $\phi_n(x)$ converges to $\phi(x)$ in $\mathbb{R}^n$ uniformly for x in any bounded subset of $\mathbb{R}$. The following norm on $\mathcal{C}$ can induce such topology:

$$\|\varphi\|_{\mathcal{C}} := \sum_{l=1}^{\infty} \frac{\max_{x\in\mathbb{R}, |x|\le l} \|\varphi(x)\|}{2^l}, \quad \forall \varphi \in \mathcal{C}.$$

Clearly, the topology generated by $\|\cdot\|_{\mathcal{C}}$ and the compact open topology on $\mathcal{C}$ are equivalent on any uniformly bounded subset of $\mathcal{C}$. We also denote $\mathcal{C}_{\mathbf{1}} := \{\phi \in \mathcal{C} : \mathbf{0} \le \phi(\cdot) \le \mathbf{1}\}$.

Recall that a family of operators $\{Q_t\}_{t\ge 0}$ is said to be a T-periodic semiflow on a metric space $(\mathcal{X}, \bar{d})$ provided that the following properties hold:
(i) $Q_0[\varphi] = \varphi$, $\forall \varphi \in \mathcal{X}$; (ii) $Q_t[Q_T[\varphi]] = Q_{t+T}[\varphi]$, $\forall t \ge 0$ and $\forall \varphi \in \mathcal{X}$;
(iii) $Q(t, \varphi) := Q_t[\varphi]$ is continuous in (t, φ) on $[0, \infty) \times \mathcal{X}$.
The map Q_T is called the Poincaré map associated with this periodic semiflow. Let $\{Q_t\}_{t\ge 0}$ be the semiflow on $\mathcal{C}_{\mathbf{1}}$ associated with system (1.1), i.e.

$$Q_t[\varphi](x) = u(x, t; \varphi), \ \forall \varphi \in \mathcal{C}_{\mathbf{1}}, \ x \in \mathbb{R}.$$

Note that for any $(t_0, \varphi_0) \in \mathbb{R}_+ \times \mathcal{C}_{\mathbf{1}}$, we have

$$\|Q_t[\varphi] - Q_{t_0}[\varphi_0]\|_{\mathcal{C}} \le \|Q_t[\varphi] - Q_t[\varphi_0]\|_{\mathcal{C}} + \|Q_t[\varphi_0] - Q_{t_0}[\varphi_0]\|_{\mathcal{C}}.$$

By a similar argument as in [29, Theorem 8.5.2], $Q_t[\varphi]$ is continuous at (t_0, φ_0) with respect to the compact open topology. Moreover, by [30, Corallary 5], we have the following result.

Lemma 3.2. *Assume* (C1), (C2)′, (C3) *and* (C4). *Then* $\{Q_t\}_{t\geq 0}$ *is a monotone periodic semiflow on* $\mathcal{C}_{\mathbf{1}}$.

Let $\mathcal{R}[\varphi](x) := \varphi(-x)$ be the reflection operator. Given $h \in \mathbb{R}$, we define the translation operator $\mathcal{T}_h[\cdot]$ by $\mathcal{T}_h[\varphi](x) = \mathcal{T}_h[\varphi](x-h)$. In order to apply [10, Theorem 2.1], we need verify that the map $Q := Q_T : \mathcal{C}_{\mathbf{1}} \to \mathcal{C}_{\mathbf{1}}$ satisfies the following assumptions:
(A1) $Q[\mathcal{R}[u]] = \mathcal{R}[Q[u]]$, $\mathcal{T}_h[Q[u]] = Q[\mathcal{T}_h[u]]$ for $h \in \mathbb{R}$;
(A2) $Q : \mathcal{C}_{\mathbf{1}} \to \mathcal{C}_{\mathbf{1}}$ is continuous with respect to the compact open topology;
(A3) $\{Q[u](x) : u \in \mathcal{C}_{\mathbf{1}}, x \in \mathbb{R}\}$ is a bounded subset of $\mathbb{R}^n$;
(A4) $Q : \mathcal{C}_{\mathbf{1}} \to \mathcal{C}_{\mathbf{1}}$ is monotone in the sense that $Q[\varphi] \geq Q[\psi]$ whenever $\varphi \geq \psi$ in $\mathcal{C}_{\mathbf{1}}$;
(A5) $Q : [\mathbf{0}, \mathbf{1}] \to [\mathbf{0}, \mathbf{1}]$ admits exactly two fixed points $\mathbf{0}$ and $\mathbf{1}$, and for any $\epsilon > 0$, there is a $\alpha \in [\mathbf{0}, \mathbf{1}]$ with $\|\alpha\| < \epsilon$ such that $Q[\alpha] \gg \alpha$.

Lemma 3.3. *Assume* (C1), (C2)′, (C3) *and* (C4). *Then, the Poincaré map* $Q = Q_T$ *satisfies* (A1)–(A5), *and* Q_t *satisfies* (A1) *for any* $t > 0$.

Proof. It is easy to see that $Q = Q_T$ satisfies (A1)–(A4), and Q_t satisfies (A1) for any $t > 0$. Let $\widehat{Q}_t$ be the restriction of Q_t to $[\mathbf{0}, \mathbf{1}]$. Then, $\widehat{Q}_t$ is the periodic semiflow on $[\mathbf{0}, \mathbf{1}]$ generated by the periodic cooperative and irreducible system (1.5). By the nonautonomous version of [20, Theorem 4.1.1], it follows that $\widehat{Q}_t$ is strongly monotone on $[\mathbf{0}, \mathbf{1}]$ for all $t > 0$. Thus, the Dancer-Hess connecting orbit lemma (see e.g. [31]) yields that the map $\widehat{Q}_T$ admits a strongly monotone full orbit connecting $\mathbf{0}$ to $\mathbf{1}$. Therefore, (A5) holds for Q_T. This completes the proof. □

By [10, Theorem A], it then follows that the Poincaré map Q_T has an asymptotic speed of spread (spreading speed for short) $c_\star > 0$. Moreover, [10, Theorem 2.1] implies that $c_\star/T$ is the spreading speed for solutions of (1.1), that is, the following result holds.

Theorem 3.4. *Assume* (C1), (C2)′, (C3) *and* (C4). *Let* $c_\star$ *be the spreading speed of* Q_T. *Then the following statements hold true:*
(1) *For any* $c > c_\star/T$, *if* $\varphi \in \mathcal{C}_{\mathbf{1}}$ *with* $\mathbf{0} \leq \varphi \ll \mathbf{1}$, *and* $\varphi(x) = \mathbf{0}$ *for* x *outside a bounded interval, then* $\lim_{t\to\infty, |x|\geq ct} u(x, t; \varphi) = \mathbf{0}$.
(2) *For any* $c < c_\star/T$ *and* $\sigma \in [\mathbf{0}, \mathbf{1}]$ *with* $\sigma \gg \mathbf{0}$, *there exists an integer* $r_\sigma > 0$ *such that if* $\varphi \in \mathcal{C}_{\mathbf{1}}$ *with* $\varphi(x) \gg \sigma$ *for* x *on an interval of length* $2r_\sigma$, *then* $\lim_{t\to\infty, |x|\leq ct} \|u(x, t; \varphi) - \mathbf{1}\| = 0$.

Next, we investigate the lower bound of $c_\star$ by considering the linearized system of (1.1) at the zero solution:

$$u_t = D\Delta u + \mathrm{D}f(\mathbf{0}, t)u. \tag{3.1}$$

Let's write $u(x, t) = e^{-\mu x}v(t)$ for some $\mu > 0$. It is obvious that $v(t)$ satisfies the system

$$v'(t) = [D\mu^2 + \mathrm{D}f(\mathbf{0}, t)]v. \tag{3.2}$$

Let $\lambda(\mu)$ be the principal Floquet multiplier of the linear periodic cooperative and irreducible system (3.2) and $\Phi(\mu) := [\ln \lambda(\mu)]/\mu$. Then, the following result holds.

Lemma 3.5. *Assume* (C1), (C2)′, (C3) *and* (C4). *Then,* $c_\star \geq \bar{c} = \inf_{\mu>0} \Phi(\mu) > 0$.

Proof. By (C2)′, we know that $\lambda(0) = r_0 > 1$, which implies that $\Phi(0+) = +\infty$. Using the same argument as in [10, Section 3], one can easily show that $\Phi(+\infty) = +\infty$. It then follows from [32, Lemma 3.8] that $\Phi(\mu)$ attains its minimum at some finite value and $\inf_{\mu>0} \Phi(\mu) > 0$.

Take $L := \max_{z\in[\mathbf{0},\mathbf{1}], t\in[0,T], i=1,\cdots,n} |\partial_i f_i(z, t)|$ and any $\epsilon \in (0, 1)$. From the proof of [35, Lemma 4.1], for any $t \in [0, T]$, there exists $\eta(t) \gg \mathbf{0}$ such that

$$f(z, t) \geq (1-\epsilon)\mathrm{D}f(\mathbf{0}, t)z - \epsilon L z, \quad \forall z \in [\mathbf{0}, \eta(t)].$$

In view of the continuity of $f(\cdot, t)$ and $\mathrm{D}f(\mathbf{0}, t)$ with respect to $t \in [0, T]$, we can choose $\eta(t)$ appropriately such that it is continuous on $[0, T]$. Take $\eta = \min_{t\in[0,T]} \eta(t) \gg \mathbf{0}$. Then,

$$f(z, t) \geq (1-\epsilon)\mathrm{D}f(\mathbf{0}, t)z - \epsilon Lz, \quad \forall t \in [0, T],\ z \in [\mathbf{0}, \eta].$$

Due to the periodicity of $f(\cdot, t)$ with respect to t, we have

$$f(z, t) \geq (1-\epsilon)\mathrm{D}f(\mathbf{0}, t)z - \epsilon Lz =: f^{\epsilon}(z, t), \quad \forall t \geq 0,\ z \in [\mathbf{0}, \eta].$$

By the continuous dependence of solutions on initial conditions, there is a sufficient small $\delta \gg \mathbf{0}$ in $\mathbb{R}^n$ such that the solution $z(t;\delta)$ of (1.5) with $z(0;\delta) = \delta$ satisfies $z(t;\delta) \leq \eta$ for any $t \in [0, T]$. It follows from the comparison principle that

$$Q_t[\phi](x) = u(x, t; \phi) \leq z(t;\delta) \leq \eta \text{ for any } \phi \in \mathcal{C}_{\delta},\ x \in \mathbb{R} \text{ and } t \in [0, T].$$

Let $M_t^{\epsilon}[\cdot]$ be the solution map associated with the linear periodic system

$$v_t = D\Delta v + f^{\epsilon}(v, t). \tag{3.3}$$

Then $Q_t[\phi] \geq M_t^{\epsilon}[\phi]$ for any $\phi \in \mathcal{C}_{\delta}$ and $t \in [0, T]$. Note that $\partial_j f_i^{\epsilon}(\cdot,\cdot) = (1-\epsilon)\partial_j f_i(\cdot,\cdot)$, $\forall j \neq i$. Let $\Phi^{\epsilon}(\mu) = [\ln \lambda^{\epsilon}(\mu)]/\mu$, where $\lambda^{\epsilon}(\mu)$ be the principal Floquet multiplier of the linear periodic cooperative and irreducible system

$$w_i' = d_i\mu^2 w_i + (1-\epsilon)\sum_{j=1}^{n} \partial_j f_i(\mathbf{0}, t)w_j - \epsilon L w_i,\ i = 1, \cdots, n. \tag{3.4}$$

It then follows from [10, Theorem B(2)] that $c_{\star} \geq \inf_{\mu>0} \Phi^{\epsilon}(\mu)$. Letting $\epsilon \to 0$, we conclude that $c_{\star} \geq \bar{c} = \inf_{\mu>0} \Phi(\mu)$. This completes the proof. □

According to Theorem 3.4 and Lemma 3.5, we are ready to give the proof of Lemma 3.1.

Proof of Lemma 3.1.

We only prove the assertion (1), since the assertion (2) can be discussed similarly. Let's denote $v = (v_1, \cdots, v_n)$ with $v_i = (u_i - \bar{w}_i(t))/(1-\bar{w}_i(t))$ for $i = 1, \cdots, n$. From (1.1), v_i satisfies

$$(v_i)_t = D\Delta v_i + g_i(v, t), \tag{3.5}$$

where $w(t) := \mathrm{diag}(1-\bar{w}_1(t), \cdots, 1-\bar{w}_n(t))$ and

$$g_i(z, t) := \frac{f_i(w(t)z + \bar{w}(t), t) - f_i(\bar{w}(t), t) - [f_i(1, t) - f_i(\bar{w}(t), t)]z_i}{1-\bar{w}_i(t)},\ i = 1, \cdots, n.$$

It is clear that $g(\cdot, t+T) = g(\cdot, t)$ and $g(\mathbf{0}, t) = g(\mathbf{1}, t) = \mathbf{0}$, $\forall t \in \mathbb{R}$. Since

$$\partial_j g_i(z, t) = (1-\bar{w}_i(t))^{-1}\partial_j f_i\big(w(t)z + \bar{w}(t), t\big)(1-\bar{w}_j(t)), \forall i \neq j,$$

it's easy to verify that (C2)′, (C3) and (C4) hold for system (3.5). Then it follows from Theorem 3.4 and Lemma 3.5 that system (3.5) admits a spreading speed $c_{\star}^{g}/T$ which has a positive lower bound $\bar{c}_g/T > 0$. Moreover, the function $\hat{U}(x+ct, t) = (\hat{U}_1(x+ct, t), \cdots, \hat{U}_n(x+ct, t))$ with

$$\hat{U}_i(x+ct, t) = \frac{U_i(x+ct, t) - \bar{w}_i(t)}{1-\bar{w}_i(t)}, i = 1, \cdots, n$$

is a non-decreasing periodic traveling wave of (3.5) satisfying $\hat{U}(-\infty, t) = \mathbf{0}$ and $\hat{U}(+\infty, t) = \mathbf{1}$.

We claim that $c \geq \bar{c}_g/T$. In fact, if the claim is false, then we can choose $c_1 \in (c, \bar{c}_g/T)$. In view of $\hat{U}(x, t)$ is a non-decreasing in x and $\hat{U}(+\infty, t) = \mathbf{1}$, we can choose $\phi \in \mathcal{C}_{\mathbf{1}}$ with $\phi(\cdot) \leq \hat{U}(\cdot, 0)$ and $\phi(+\infty) = \frac{1}{2}\mathbf{1}$. Thus, by Theorem 3.4, we have $\lim_{t\to\infty} \|u(-c_1 t, t; \phi) - \mathbf{1}\| = 0$. On the other hand, from Lemma 2.3, we have $u(x, t; \phi) \leq \hat{U}(x+ct, t)$ which implies that

$$\lim_{t\to\infty} u(-c_1 t, t; \phi) \leq \liminf_{t\to\infty} \hat{U}((c-c_1)t, t) = \mathbf{0}.$$

This contradiction shows that $c \geq \bar{c}_g/T > 0$. This completes the proof.

4 Existence of periodic traveling fronts

Based on the result of Lemma 3.1, we are ready to prove the existence of bistable periodic traveling front by using the vanishing viscosity method. Moreover, an estimate of the wave speed is also established. Throughout this section, we always assume the conditions (C1)-(C4).

For any $\epsilon \in (0, 1]$, we consider the following non-degenerate reaction-diffusion system

$$u_t = D^{\epsilon}\Delta u + f(u, t), \quad x \in \mathbb{R}, \ t > 0, \tag{4.1}$$

where $D^{\epsilon} = \mathrm{diag}(d_1^{\epsilon}, \cdots, d_n^{\epsilon})$ with

$$d_i^{\epsilon} := \begin{cases} d_i, & \text{if } d_i > 0, \\ \epsilon, & \text{if } d_i = 0. \end{cases} \tag{4.2}$$

Following [7, Theorem 10], we have the following existence result.

Lemma 4.1. *For any $\epsilon > 0$, system* (4.1) *admits a periodic traveling front $U^{\epsilon}(x + c_{\epsilon}t, t)$ connecting* $\mathbf{0}$ *and* $\mathbf{1}$.

To find a convergent subsequence of $\{U^{\epsilon}(x + c_{\epsilon}t, t)\}_{\epsilon\in(0,1]}$, we need to study the boundedness of $\{c^{\epsilon}\}_{\epsilon\in(0,1]}$ and the compactness of $\{U^{\epsilon}(x, t) : \epsilon \in (0, 1]\}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^n)$.

First, we construct a pair of sub- and supersolutions to show the boundedness of $\{c^{\epsilon}\}_{\epsilon\in(0,1]}$. Let $\mu^{\pm} := \frac{1}{T}\ln r^{\pm} < 0$, where r^- and r^+ are the spectral radius of $\mathrm{D}P(\mathbf{0})$ and $\mathrm{D}P(\mathbf{1})$, respectively. From [33, Lemma 2.1], there exist positive and T-periodic functions $s_{\pm}(t) = ((s_{\pm})_1(t), \cdots, (s_{\pm})_n(t))$ with $s_{\pm}(\cdot) \le \mathbf{1}$ such that $v^{\pm}(t) = e^{\mu^{\pm}t}s_{\pm}(t)$ are solutions of the T-periodic linear system:

$$v'(t) = \mathrm{D}f(w^{\pm}(t), t)v(t)$$

with the initial value $v^{\pm}(0) \gg \mathbf{0}$, where $w^-(t) \equiv \mathbf{0}$ and $w^+(t) \equiv \mathbf{1}$. It is clear that $s_{\pm}(t)$ satisfies

$$s'_{\pm}(t) = \mathrm{D}f(w^{\pm}(t), t)s_{\pm}(t) - \mu^{\pm}s_{\pm}(t). \tag{4.3}$$

Let δ_0 be the constant defined in Section 1, and

$$\zeta(s) := \frac{1}{2}(1 + \tanh\frac{s}{2}), \ s \in \mathbb{R}. \tag{4.4}$$

Obviously, $0 < \zeta(\cdot) < 1$, $\zeta'(s) = \zeta(s)(1 - \zeta(s))$ and $\zeta''(s) = \zeta(s)(1 - \zeta(s))(1 - 2\zeta(s))$. Hence, $\zeta'(\cdot) > 0$ and $|\zeta''(\cdot)| \le 1$. The sub- and supersolutions of (4.1) are constructed as follows.

Lemma 4.2. *There exist $\kappa_0 \in (0, \delta_0)$, $\sigma_0 > 0$ and $C_0 > 0$, which are independent of $\epsilon \in (0, 1]$, such that for any $\kappa \in [\kappa_0/2, \kappa_0]$, $\sigma \in [\sigma_0/2, \sigma_0]$ and $C \ge C_0$, the functions*

$$\begin{aligned} u^-(x - Ct, t; \sigma, \kappa) &:= (\mathbf{1} + \kappa s_-(t) - \kappa s_+(t))\zeta(\sigma(x - Ct)) - \kappa s_-(t), \\ u^+(x + Ct, t; \sigma, \kappa) &:= (\mathbf{1} + \kappa s_+(t) - \kappa s_-(t))\zeta(\sigma(x + Ct)) + \kappa s_-(t) \end{aligned}$$

are sub- and supersolutions of (4.1) *on* $[0, \infty)$, *respectively.*

Proof. Since $0 < s_{\pm}(\cdot) \le \mathbf{1}$ and $\zeta(\cdot) \in [0, 1]$, we see that for any $\kappa \in (0, \delta_0)$, $\sigma > 0$ and $C > 0$,

$$-\delta_0\mathbf{1} \le u^{\pm}(x \pm Ct, t; \sigma, \kappa) \le \mathbf{1} + \delta_0\mathbf{1}, \ \forall x \in \mathbb{R}, \ t \ge 0.$$

We only prove $u^-(x - Ct, t; \sigma, \kappa)$ is a subsolution of (4.1), since the proof for the supersolution is similar. For convenience, we denote $u^-(x - Ct, t; \sigma, \kappa)$ by $u^-(x - Ct, t)$. As mentioned in the introduction, we may assume that $w^-(t) = \mathbf{0}$ and $w^+(t) = \mathbf{1}$ are two periodic solutions of (1.5). It follows that $f(\mathbf{0}, t) = f(\mathbf{1}, t) = \mathbf{0}$, $\forall t \ge 0$. Since $\mu^- < 0$ and

$$f(-\kappa s_-(t), t) = -\mathrm{D}f(\mathbf{0}, t)\kappa s_-(t) - o(\kappa)s_-(t) \text{ as } \kappa \to 0,$$

there exists $\kappa_1 > 0$ such that for any $\kappa \in (0, \kappa_1]$,

$$f(-\kappa s_-(t), t) \geq -\mathrm{D}f(\mathbf{0}, t)\kappa s_-(t) + \frac{\mu^-}{2}\kappa s_-(t) = -\kappa s_-'(t) - \frac{\mu^-}{2}\kappa s_-(t), \ \forall t \geq 0.$$

Thus, for any $\kappa \in (0, \kappa_1]$, we have

$$f(-\kappa s_-(t), t) + \kappa s_-'(t) \geq -\frac{\mu^-}{2}\kappa s_-(t) \geq -\frac{\mu^-}{2}\kappa \min_{t\in[0,T]} s_-(t) \gg \mathbf{0}, \ \forall t \geq 0. \tag{4.5}$$

Similarly, there exists $\kappa_2 > 0$ such that for any $\kappa \in (0, \kappa_2]$,

$$f(\mathbf{1} - \kappa s_+(t), t) + \kappa s_+'(t) \geq -\frac{\mu^+}{2}\kappa s_+(t) \geq -\frac{\mu^+}{2}\kappa \min_{t\in[0,T]} s_+(t) \gg \mathbf{0}, \ \forall t \geq 0. \tag{4.6}$$

Choose $\kappa_0 \in (0, \min\{\kappa_1, \kappa_2, \delta_0\})$ such that $\mathbf{1} + \kappa s_-(t) - \kappa s_+(t) \gg \mathbf{0}$ for any $\kappa \in (0, \kappa_0]$ and $t \geq 0$. Note that $s_\pm(t)$ and $s_\pm'(t)$ are bounded on $[0, \infty)$. By (4.5) and (4.6), there exist $\vartheta_0 \in (0, 1)$ and $\sigma_0 > 0$ such that for any $\kappa \in [\kappa_0/2, \kappa_0]$, $\sigma \in [\sigma_0/2, \sigma_0]$ and $\vartheta \in [0, \vartheta_0]$, there holds

$$\begin{aligned} &f\big((\mathbf{1} + \kappa s_-(t) - \kappa s_+(t))\vartheta - \kappa s_-(t), t\big) + \kappa s_-'(t) - \kappa(s_-'(t) - s_+'(t))\vartheta \\ \gg &D^1(\mathbf{1} + \kappa s_-(t) - \kappa s_+(t))\sigma^2, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &f\big((\mathbf{1} + \kappa s_-(t) - \kappa s_+(t))(1-\vartheta) - \kappa s_-(t), t\big) + \kappa s_-'(t) - \kappa(s_-'(t) - s_+'(t))(1-\vartheta) \\ \gg &D^1(\mathbf{1} + \kappa s_-(t) - \kappa s_+(t))\sigma^2, \end{aligned} \tag{4.8}$$

where $D^1 := \mathrm{diag}(d_1, \cdots, d_{i_0}, 1, \cdots, 1)$. Let's set

$$\begin{aligned} |f|_{\max} :=& \max_{u\in[-\delta_0\mathbf{1}, \mathbf{1}+\delta_0\mathbf{1}], t\geq 0, i=1,\cdots,n} \{|f_i(u,t)|\}, \\ m_\zeta :=& \min_{\zeta(s)\in[\vartheta_0, 1-\vartheta_0]} \zeta'(s), \quad s_{\min} := \min_{t\geq 0, i=1,\cdots,n} \{(s_-)_i(t) - (s_+)_i(t)\}, \\ C_0 :=& \Big\{|f|_{\max} + \bar{d}_M\sigma_0^2\big[1 + \kappa_0\big(\max_{t\geq 0}\{\|s_-(t)\| + \|s_+(t)\|\}\big)\big] \\ &+ 2\kappa_0 \max_{t\geq 0}\|s_-'(t)\| + \kappa_0 \max_{t\geq 0}\|s_+'(t)\|\Big\}\Big/\Big[\frac{\sigma_0}{2}(1 + \frac{\kappa_0}{2}s_{\min})m_\zeta\Big] > 0, \end{aligned} \tag{4.9}$$

where $\bar{d}_M = \max\{d_1, \cdots, d_{i_0}, 1\}$. Note that $1 + \kappa s_{\min} > 0$ for all $\kappa \in (0, \kappa_0]$. Clearly, for any $C \geq C_0$, $\sigma \in [\sigma_0/2, \sigma_0]$ and $\kappa \in [\kappa_0/2, \kappa_0]$, there holds

$$\begin{aligned} C\sigma(1 + \kappa s_{\min})m_\zeta \geq& |f|_{\max} + \bar{d}_M\sigma_0^2\big[1 + \kappa_0\big(\max_{t\geq 0}\{\|s_-(t)\| + \|s_+(t)\|\}\big)\big] \\ &+ 2\kappa \max_{t\geq 0}\|s_-'(t)\| + \kappa \max_{t\geq 0}\|s_+'(t)\|. \end{aligned} \tag{4.10}$$

To prove $u^-(x - Ct, t)$ is a subsolution of (4.1), it suffices to show that

$$u_t^-(\xi, t) - Cu_\xi^-(\xi, t) - D^\epsilon u_{\xi\xi}^-(\xi, t) - f(u^-(\xi, t), t) \leq 0,$$

where $\xi := x - Ct$. Moreover, direct computations show that

$$\begin{aligned} &u_t^-(\xi, t) - Cu_\xi^-(\xi, t) - D^\epsilon u_{\xi\xi}^-(\xi, t) - f(u^-(\xi, t), t) \\ =&\kappa(s_-'(t) - s_+'(t))\zeta(\sigma\xi) - \kappa s_-'(t) - \sigma C(\mathbf{1} + \kappa s_-(t) - \kappa s_+(t))\zeta'(\xi) - \\ &D^\epsilon\sigma^2(\mathbf{1} + \kappa s_-(t) - \kappa s_+(t))\zeta''(\xi) - f\big((\mathbf{1} + \kappa s_-(t) - \kappa s_+(t))\zeta(\sigma\xi) - \kappa s_-(t), t\big). \end{aligned} \tag{4.11}$$

In the sequel, we consider the following two cases:

(i) $\zeta(\sigma\xi) \in [0, \vartheta_0] \cup [1 - \vartheta_0, 1]$ and (ii) $\zeta(\sigma\xi) \in [\vartheta_0, 1 - \vartheta_0]$.

For case (i), it follows from (4.7), (4.8) and (4.11) that

$$\begin{aligned}&u_t^-(\xi,t)-Cu_\xi^-(\xi,t)-D^\epsilon u_{\xi\xi}^-(\xi,t)-f(u^-(\xi,t),t)\\ \le&\kappa(s_-'(t)-s_+'(t))\zeta(\sigma\xi)-\kappa s_-'(t)+D^1\sigma^2(\mathbf{1}+\kappa s_-(t)-\kappa s_+(t))-\\ &f\big((\mathbf{1}+\kappa s_-(t)-\kappa s_+(t))\zeta(\sigma\xi)-\kappa s_-(t),t\big)\le 0.\end{aligned}\tag{4.12}$$

For case (ii), it follows from (4.10) and (4.11) that

$$\begin{aligned}&(u_i^-)_t(\xi,t)-C(u_i^-)_\xi(\xi,t)-D^\epsilon(u_i^-)_{\xi\xi}(\xi,t)-f_i(u^-(\xi,t),t)\\ =&\kappa[(s_-)_i'(t)-(s_+)_i'(t)]\zeta(\sigma\xi)-\kappa(s_-)_i'(t)-\sigma C[1+\kappa(s_-)_i(t)-\kappa(s_+)_i(t)]\zeta'(\xi)-\\ &d_i^\epsilon\sigma^2(1+\kappa(s_-)_i(t)-\kappa(s_+)_i(t))\zeta''(\xi)-f_i((\mathbf{1}+\kappa s_-(t)-\kappa s_+(t))\zeta(\sigma\xi)-\kappa s_-(t),t)\\ \le&2\kappa\max_{t\ge0}\|s_-'(t)\|+\kappa\max_{t\ge0}\|s_+'(t)\|-C\sigma(1+\kappa s_{\min})m_\zeta+\\ &\bar d_M\sigma_0^2\big[1+\kappa\big(\max_{t\ge0}\{\|s_-(t)\|+\|s_+(t)\|\}\big)\big]+|f|_{\max}\le 0,\ \forall i=1,\cdots,n.\end{aligned}\tag{4.13}$$

Combining the above two cases, we conclude that $u^-(x,t)$ is a subsolution of (4.1) on $[0,\infty)$. This completes the proof. □

Based on the above lemma, we can show that c^ϵ is bounded for any $\epsilon\in(0,1]$.

Lemma 4.3. *$|c_\epsilon|\le C_0$ for any $\epsilon\in(0,1]$, where $C_0=C_0(d_1,\cdots,d_{i_0},f)$ is given by (4.9), which is independent of ϵ.*

Proof. From Lemma 4.2, we see that there exist $\kappa_0>0$, $\sigma_0>0$ and $C_0>0$ with

$$(\mathbf{1}+\kappa_0 s_-(0)-\kappa_0 s_+(0))\zeta(0)-\kappa_0 s_-(0)\ge\frac{1}{2}\zeta(0)\mathbf{1}\gg\mathbf{0},\tag{4.14}$$

which are independent of $\epsilon\in(0,1]$, such that $u^-(x-C_0t,t;\sigma_0,\kappa_0)$ and $u^+(x+C_0t,t;\sigma_0,\kappa_0)$ are sub- and supersolutions of system (4.1) on $[0,\infty)$. Note that

$$u^-(-\infty,0;\sigma_0,\kappa_0)=-\kappa_0 s_-(0)\ll\mathbf{0}=U^\epsilon(-\infty,0)$$
and
$$u^-(x,0;\sigma_0,\kappa_0)\le\mathbf{1}-\kappa_0 s_+(0)\ll\mathbf{1}=U^\epsilon(+\infty,0),\ \forall x\in\mathbb{R}.$$

Then, for any $\epsilon\in(0,1]$, there exists $\eta_\epsilon\in\mathbb{R}$ such that $U^\epsilon(\cdot+\eta_\epsilon,0)\ge u^-(\cdot,0;\sigma_0,\kappa_0)$. By Lemma 2.4, we obtain

$$U^\epsilon(x+c^\epsilon t+\eta_\epsilon,t)\ge u^-(x-C_0t,t;\sigma_0,\kappa_0),\forall x\in\mathbb{R},\,t\ge0,$$

which implies that

$$U^\epsilon(x+(c^\epsilon+C_0)t+\eta_\epsilon,t)\ge u^-(x,t;\sigma_0,\kappa_0),\ \forall x\in\mathbb{R},\ t\ge0.\tag{4.15}$$

Letting $x=0$ and $t=nT$, $n\in\mathbb{N}$, in (4.15), we obtain

$$U^\epsilon((c^\epsilon+C_0)nT+\eta_\epsilon,nT)\ge u^-(0,nT;\sigma_0,\kappa_0)=(\mathbf{1}+\kappa_0 s_-(nT)-\kappa_0 s_+(nT))\zeta(0)-\kappa_0 s_-(nT).$$

By (4.14) and the periodicity of $U^\epsilon(\cdot,t)$ and $s_\pm(t)$, it follows that

$$U^\epsilon((c^\epsilon+C_0)nT+\eta_\epsilon,0)\ge(\mathbf{1}+\kappa_0 s_-(0)-\kappa_0 s_+(0))\zeta(0)-\kappa_0 s_-(0)\ge\frac{1}{2}\zeta(0)\mathbf{1}.$$

If $c^\epsilon+C_0<0$, letting $n\to\infty$ in above inequality, we get

$$U^\epsilon(-\infty,0)=\mathbf{0}\ge\frac{1}{2}\zeta(0)\mathbf{1}\gg\mathbf{0}.$$

This contradiction implies that $c^\epsilon\ge -C_0$ for any $\epsilon\in(0,1]$. Similarly, we can prove that $c^\epsilon\le C_0$ for any $\epsilon\in(0,1]$. This completes the proof. □

Next, we study the compactness of $\{U^\epsilon(x,t):\epsilon\in(0,1]\}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^n)$.

Lemma 4.4. *$\{U^\epsilon(x,t):\epsilon\in(0,1]\}$ is pre-compact in $L^1_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^n)$.*

Proof. The idea of the proof follows from [34, Lemma 5] and [12, Lemma 2.5]. Clearly, it suffices to prove that $\{U_i^\epsilon(x,t):\epsilon\in(0,1]\}$ is pre-compact in $L^1_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R})$, $\forall i=1,\cdots,n$.

Given any $i\in\{1,\cdots,n\}$. Since $0\le U_i^\epsilon(\cdot,\cdot)\le 1$ and $U_i^\epsilon(x,t)$ is monotone in x, we know that $\{U_i^\epsilon(\cdot,t):\epsilon\in(0,1],t\in\mathbb{R}\}$ is pre-compact in $L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$. Thus, for any $r>0$, there exists a function $z_r(\cdot)$ which is continuous, and non-decreasing, such that $z_r(0)=0$ and

$$\int\limits_{|x|\le r}|U_i^\epsilon(x+\triangle x,t)-U_i^\epsilon(x,t)|dx\le z_r(|\triangle x|),\forall\epsilon\in(0,1]\text{ and }t\in\mathbb{R}.\tag{4.16}$$

We may assume that $z_{r_1}(\cdot)\le z_{r_2}(\cdot)$ for $r_1\le r_2$. Now we prove the following claim.

Claim: *For any $r>0$, there exists a function $\bar z_r(\cdot)$ which is continuous and non-decreasing, such that $\bar z_r(0)=0$ and*

$$\int\limits_{|x|\le r}|U_i^\epsilon(x,t+\triangle t)-U_i^\epsilon(x,t)|dx\le \bar z_r(|\triangle t|),\forall\epsilon\in(0,1]\text{ and }t\in\mathbb{R}.\tag{4.17}$$

Given $r>0$. For any $h\in(0,r)$, we define

$$\beta(x):=\begin{cases}\mathrm{sign}(U_i^\epsilon(x,t+\triangle t)-U_i^\epsilon(x,t)), & \text{if } |x|\le r-h,\\ 0, & \text{if } |x|>r-h,\end{cases}$$

$$\rho(x)=\beta^h(x):=\int\limits_{-\infty}^{+\infty}\frac{1}{h}\delta\Big(\frac{x-y}{h}\Big)\beta(y)dy,\ x\in\mathbb{R},$$

where $\delta(\cdot)$ is an infinitely differentiable function on $\mathbb{R}$ with

$$\delta(\cdot)\ge 0,\ \delta(s)=0\text{ for }|s|\ge 1\text{ and }\int\limits_{-\infty}^{+\infty}\delta(s)ds=1.$$

Clearly, $\mathrm{supp}\rho\subseteq\{x\in\mathbb{R}\,\big|\,|x|\le r\}$, $|\rho(\cdot)|\le 1$, $|\rho'(\cdot)|\le C_1/h$, and $|\rho''(\cdot)|\le C_1/h^2$, where $C_1>0$ is a constant independent of ϵ and h. Note that $U^\epsilon(x,t)$ satisfies

$$(U_i^\epsilon)_t+c^\epsilon(U_i^\epsilon)_x=d_i^\epsilon(U_i^\epsilon)_{xx}+f_i(U^\epsilon,t),\ i=1,\cdots,n.$$

For any $h\in(0,\min\{1,r/2\}]$, we have

$$\begin{aligned}\int\limits_{|x|\le r}\rho(x)[U_i^\epsilon(x,t+\triangle t)-U_i^\epsilon(x,t)]dx&=\int\limits_{|x|\le r}\int\limits_t^{t+\triangle t}\rho(x)(U_i^\epsilon)_t(x,t)dtdx\\&=\int\limits_{|x|\le r}\int\limits_t^{t+\triangle t}\rho(x)[-c^\epsilon(U_i^\epsilon)_x+d_i^\epsilon(U_i^\epsilon)_{xx}+f_i(U^\epsilon,t)]dtdx\\&=\int\limits_t^{t+\triangle t}\int\limits_{|x|\le r}\big[c^\epsilon\rho'(x)U_i^\epsilon+d_i^\epsilon\rho''(x)U_i^\epsilon+f_i(U^\epsilon,t)\rho(x)\big]dxdt\\&\le\Big|\int\limits_t^{t+\triangle t}\int\limits_{|x|\le r}\big[C_0|\rho'(x)|+d_M|\rho''(x)|+L_f|\rho(x)|\big]dxdt\Big|\\&\le C_2|\triangle t|\max_{|x|\le r}\big[|\rho'(x)|+|\rho''(x)|+|\rho(x)|\big]\end{aligned}$$

$$\leq C_2|\triangle t|\Big[\frac{C_1}{h}+\frac{C_1}{h^2}+1\Big]\leq |\triangle t|\frac{C_2(2C_1+1)}{h^2},$$

where $\bar{d}_M := \max\{1, d_1, \cdots, d_{i_0}\}$,

$$L_f := \max_{u\in[\mathbf{0},\mathbf{1}],t\in[0,T],i=1,\cdots,n}|f_i(u,t)| \text{ and } C_2 := \max\{C_0, \bar{d}_M, L_f\}.$$

According to [12, Lemma 2.1] (see also [34]), for any $h \in (0, \min\{1, r/2\}]$, we obtain

$$\begin{aligned}\int\limits_{|x|\leq r}|U_i^\epsilon(x,t+\triangle t)-U_i^\epsilon(x,t)|dx &\leq \int\limits_{|x|\leq r-h}|U_i^\epsilon(x,t+\triangle t)-U_i^\epsilon(x,t)|dx + C_3 h\\ &\leq \int\limits_{|x|\leq r-h}\rho(x)[U_i^\epsilon(x,t+\triangle t)-U_i^\epsilon(x,t)]dx + C_4 z_{r-h}(h) + C_3 h\\ &\leq \int\limits_{|x|\leq r}\rho(x)[U_i^\epsilon(x,t+\triangle t)-U_i^\epsilon(x,t)]dx + C_4 z_r(h) + C_3 h\\ &\leq C_5\Big[\frac{|\triangle t|}{h^2}+z_r(h)+h\Big],\end{aligned}$$

where C_3 and C_4 are constants independent of h, ϵ, t and $|\triangle t|$, and $C_5 := \max\{C_2(2C_1+1), C_3, C_4\}$. Then it follows that

$$\int\limits_{|x|\leq r}|U_i^\epsilon(x,t+\triangle t)-U_i^\epsilon(x,t)|dx \leq C_5 \min_{h\in(0,\min\{1,\frac{r}{2}\}]}\Big[\frac{|\triangle t|}{h^2}+z_r(h)+h\Big] =: \bar{z}_r(|\triangle t|).$$

This proves the claim. By (4.16) and (4.17), we conclude that $\{U_i^\epsilon(x,t) : \epsilon \in (0,1]\}$ is pre-compact in $L^1_{\text{loc}}(\mathbb{R}^2,\mathbb{R})$. This completes the proof. □

We now give the proof of Theorem 1.2.

Proof of Theorem 1.2.

By the boundedness of $\{c^\epsilon\}_{\epsilon\in(0,1]}$, there exists a subsequence $\{c^{\epsilon_j}\}_{j\geq 1}$ such that $\epsilon_j \to 0$ and c^{ϵ_j} converges to some $c_0 \in \mathbb{R}$. For convenience, we denote c^{ϵ_j} and $D^{\epsilon_j} = \text{diag}(d_1^{\epsilon_j}, \cdots, d_n^{\epsilon_j})$ by c^j and $D^j = \text{diag}(d_1^j, \cdots, d_n^j)$, respectively. Let $U^j(\cdot,\cdot)$ be the corresponding wave profile with wave speed c^j. Since $U_1^j(-\infty,0) = 0$, $U_1^j(+\infty,0) = 1$ and $0 < \bar{w}_1(0) < 1$, we can choose ξ^j and η^j such that

$$U_1^j(\xi^j,0)=\frac{\bar{w}_1(0)}{2} \text{ and } U_1^j(\eta^j,0)=\frac{1+\bar{w}_1(0)}{2}.$$

Define $V^j(\cdot,t) = U^j(\cdot+\xi^j,t)$ and $W^j(\cdot,t) = U^j(\cdot+\eta^j,t)$. Then,

$$V_1^j(0,0)=\frac{\bar{w}_1(0)}{2} \text{ and } W_1^j(0,0)=\frac{1+\bar{w}_1(0)}{2}. \tag{4.18}$$

By Lemma 4.4, there exist a function $V(\cdot,\cdot) = (V_1(\cdot,\cdot),\cdots,V_n(\cdot,\cdot))^{\mathrm{T}} \in L^1_{\text{loc}}(\mathbb{R}^2,\mathbb{R}^n)$ and a subsequence of $\{V^j(\cdot,\cdot)\}$, still denoted by $\{V^j(\cdot,\cdot)\}$, such that

$$V^j(x,t)\to V(x,t) \text{ for a.e. } (x,t)\in\mathbb{R}^2.$$

Thus, there is a set $D_0 \subseteq \mathbb{R}$ with $m(\mathbb{R}\setminus D_0) = 0$ such that for any $t \in D_0$,

$$V^j(x,t)\to V(x,t) \text{ for a.e. } x\in\mathbb{R}. \tag{4.19}$$

By the monotonicity of $V^j(x,t)$ in x and Helly's theorem (see [36, P165]), we may assume that $0 \in D_0$ and for each $t \in \mathbb{R}\setminus D_0$, there is $\{j_l\} \subseteq \{j\}$ such that

$$V^{j_l}(x,t)\to V(x,t) \text{ for a.e. } x\in\mathbb{R}. \tag{4.20}$$

In view of the boundedness of $V^j(\cdot,\cdot)$, we may assume that $V^{j_l}(0,0)\to V(0,0)$ and hence $V_1(0,0)=\bar{w}_1(0)/2$. Moreover, we may assume that $V(x,t)$ is non-decreasing in x. Let $u^j(x,t):=V^j(x+c_0t,t)$ and $\bar{V}(x,t)=V(x+c_0t,t)$. One can verify that

$$u^j_t+(c^j-c_0)u^j_x=D^ju^j_{xx}+f(u^j(x,t),t). \tag{4.21}$$

By (4.19) and (4.20), for given $0<t<\infty$, there exists $\{j_l\}\subseteq\{j\}$ such that for any $s\in(D_0\cap[0,t])\cup\{0,t\}$,

$$u^{j_l}(x,s)\to\bar{V}(x,s) \text{ for a.e. } x\in\mathbb{R}. \tag{4.22}$$

Recall that $d_i>0$ for $i=1,\cdots,i_0$ and $d_i=0$ for $i=i_0+1,\cdots,n$. For $i=1,\cdots,i_0$, we define two families of mappings $T^j_i(t),\ T_i(t):L^\infty(\mathbb{R})\to L^\infty(\mathbb{R})$ by $T^j_i(0)=T_i(0)=I$, and

$$T^j_i(t)[\phi](x):=\frac{1}{\sqrt{4\pi d_it}}\int_{-\infty}^{+\infty}\exp\Big\{-\frac{(x-(c^j-c_0)t-y)^2}{4d_it}\Big\}\phi(y)dy,$$

$$T_i(t)[\phi](x):=\frac{1}{\sqrt{4\pi d_it}}\int_{-\infty}^{+\infty}\exp\Big\{-\frac{(x-y)^2}{4d_it}\Big\}\phi(y)dy,$$

for any $\phi\in L^\infty(\mathbb{R})$, $t>0$, $x\in\mathbb{R}$. If $i\in\{1,\cdots,i_0\}$, by (4.21), we have

$$u^{j_l}_i(x,t)=T^{j_l}_i(t)[u^{j_l}_i(\cdot,0)](x)+\int_0^t T^{j_l}_i(t-s)[f_i(u^{j_l}_i(\cdot,s),s)](x)ds,$$

which implies that

$$\bar{V}_i(x,t)=T_i(t)[\bar{V}_i(\cdot,0)](x)+\int_0^t T_i(t-s)[f_i(\bar{V}_i(\cdot,s),s)](x)ds,\ \text{ for a.e. } x\in\mathbb{R}. \tag{4.23}$$

If $i\in\{i_0+1,\cdots,n\}$, by the dominated convergence theorem and Fubini theorem for Lebesgue integrals, for any $s(\cdot)\in C^\infty(\mathbb{R})$ with compact support, we have

$$\begin{aligned}\int_{\mathbb{R}}\bar{V}_i(x,t)s(x)dx&=\lim_{l\to\infty}\int_{\mathbb{R}}u^{j_l}_i(x,t)s(x)dx\\
&=\lim_{l\to\infty}\int_{\mathbb{R}}\Big\{u^{j_l}_i(x,0)s(x)dx+\\
&\qquad\int_0^t\big[-(c^{j_l}-c_0)(u^{j_l}_i)_x(x,s)+d^{j_l}_i(u^{j_l}_i)_{xx}(x,s)+f_i(u^{j_l}(x,s),s)\big]ds\Big\}s(x)dx\\
&=\lim_{l\to\infty}\int_{\mathbb{R}}u^{j_l}_i(x,0)s(x)dx+\lim_{l\to\infty}\int_0^t\int_{\mathbb{R}}f_i(u^{j_l}(x,s),s)s(x)dxds+\\
&\qquad\lim_{l\to\infty}\int_0^t\int_{\mathbb{R}}[(c^{j_l}-c_0)u^{j_l}_i(x,s)s'(x)+d^{j_l}_iu^{j_l}_i(x,s)s''(x)]dxds\\
&=\int_{\mathbb{R}}\bar{V}_i(x,0)s(x)dx+\int_0^t\int_{\mathbb{R}}f_i(\bar{V}(x,s),s)s(x)dxds\\
&=\int_{\mathbb{R}}\bar{V}_i(x,0)s(x)dx+\int_{\mathbb{R}}\int_0^t f_i(\bar{V}(x,s),s)s(x)dxds.\end{aligned}$$

Hence, for $i \in \{i_0 + 1, \cdots, n\}$,

$$\bar{V}_i(x,t) = \bar{V}_i(x,0) + \int_0^t f_i(\bar{V}(x,s),s)ds, \text{ for a.e. } x \in \mathbb{R}. \tag{4.24}$$

Note that (4.23)-(4.24) just hold for a.e. $x \in \mathbb{R}$. Let's define $V^-(x,t) := \lim_{h\to 0^-} \bar{V}(x+h,t) = \lim_{h\to 0^-} V(x+c_0t+h,t) =: \phi(x+c_0t,t)$ for all $x \in \mathbb{R}$, then it follows from (4.23)-(4.24) and the monotonicity of $\bar{V}(x,t)$ in x that

$$V_i^-(x,t) = T_i(t)[V_i^-(\cdot,0)](x) + \int_0^t T_i(t-s)[f_i(V_i^-(\cdot,s),s)](x)ds, \ \forall i \in \{1,\cdots,i_0\}, \tag{4.25}$$

$$V_i^-(x,t) = V_i^-(x,0) + \int_0^t f_i(V^-(x,s),s)ds, \ \forall i \in \{i_0+1,\cdots,n\}, \tag{4.26}$$

for any $x \in \mathbb{R}$, $t > 0$.

Similarly, there exist a function $W(\cdot,\cdot) = (W_1(\cdot,\cdot),\cdots,W_n(\cdot,\cdot))^{\mathrm{T}} \in L^1_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^n)$ and a subsequence of $\{W^j(\cdot,\cdot)\}$, still denoted by $\{W^j(\cdot,\cdot)\}$, such that

$$W^j(x,t) \to W(x,t) \text{ for a.e. } (x,t) \in \mathbb{R}^2.$$

Define $\bar{W}(x,t) = W(x+c_0t,t)$ and

$$W^+(x,t) = \lim_{h\to 0^+} \bar{W}(x+h,t) = \lim_{h\to 0^+} W(x+c_0t+h,t) =: \psi(x+c_0t,t).$$

Using the similar argument as above, we can obtain

$$W_i^+(x,t) = T_i(t)[W_i^+(\cdot,0)](x) + \int_0^t T_i(t-s)[f_i(W_i^+(\cdot,s),s)](x)ds, \ \forall i \in \{1,\cdots,i_0\},$$

$$W_i^+(x,t) = W_i^+(x,0) + \int_0^t f_i(W^+(x,s),s)ds, \ \forall i \in \{i_0+1,\cdots,n\},$$

for any $x \in \mathbb{R}$, $t > 0$. It is clear that $V^-(\cdot,t+T) = V^-(\cdot,t)$ and $W^+(\cdot,t+T) = W^+(\cdot,t)$.

We now consider the boundary behaviors of $\phi(\cdot,t)$ and $\psi(\cdot,t)$ at $\pm\infty$. Clearly, $\phi(\pm\infty,t)$ both exists and are periodic solutions of the system:

$$u(t) = u(0) + \int_0^t f(u(s),s)ds,$$

and hence they satisfy (1.5). Since $\bar{V}(x,t)$ is non-decreasing in x, it follows that

$$\phi_1(0,0) = V_1^-(0,0) = \lim_{h\to 0^-} \bar{V}_1(h,0) \le \bar{V}_1(0,0) = V_1(0,0) = \bar{w}_1(0)/2.$$

By our assumptions, system (1.5) has exactly three periodic solutions $\mathbf{0}$, $\bar{w}(t)$ and $\mathbf{1}$ with $\mathbf{0} \ll \bar{w}(t) \ll \mathbf{1}$, $\forall t \in \mathbb{R}$. Thus, $\phi(-\infty,t) = \mathbf{0}$ and $\phi(+\infty,t) = \bar{w}(t)$ or $\mathbf{1}$. Similarly, we obtain $\psi(+\infty,t) = \mathbf{1}$ and $\psi(-\infty,t) = \mathbf{0}$ or $\bar{w}(t)$. By Lemma 3.1, we see that $\phi(+\infty,t) = \bar{w}(t)$ and $\psi(-\infty,t) = \bar{w}(t)$ cannot hold simultaneously because $V^-(x,t) = \phi(x+c_0t,t)$ and $W^+(x,t) = \psi(x+c_0t,t)$ are non-decreasing periodic traveling waves of (1.1) with the same wave speed c_0. That is, either $V^-(x,t) = \phi(x+c_0t,t)$ or $W^+(x,t) = \psi(x+c_0t,t)$ is a periodic traveling front of (1.1) connecting $\mathbf{0}$ and $\mathbf{1}$ with speed c_0. Since $|c_\epsilon| \le C_0$ for any $\epsilon \in (0,1]$, we conclude that $|c_0| \le C_0$. This completes the proof of Theorem 1.2.

5 Monotonicity of periodic traveling fronts

This section is devoted to the monotonicity of the periodic traveling waves. Throughout this section, we assume that (C1)-(C3) and (C4)′ hold and $U(x+ct,t)$ is a smooth periodic traveling wave of (1.1) connecting $\mathbf{0}$ and $\mathbf{1}$.

We first consider the following linear initial value problem:

$$\begin{cases} v_t + cv_x = D\Delta v + \mathrm{D}f(U,t)v, \ x \in \mathbb{R}, t > 0, \\ v(x,0) = \varphi(x) \in L^\infty(\mathbb{R}). \end{cases} \tag{5.1}$$

Three important lemmas are established in the sequel.

Lemma 5.1. *Let $v(x,t;\varphi)$ be a solution of system* (5.1) *with $v(\cdot,0;\varphi)=\varphi(\cdot)$. If $\varphi(\cdot) > \mathbf{0}$, then $v(x,t;\varphi) \gg \mathbf{0}$, $\forall x \in \mathbb{R}$, $t \geq 0$.*

Proof. The proof is similar to that of Lemma 2.5. We omit it here. □

Lemma 5.2. *Given any $L>0$ and $c \in \mathbb{R}$. Let $\bar{u}(x,t)$, $\underline{u}(x,t) : \mathbb{R}\times[\tau,\infty) \to [-\delta_0\mathbf{1}, \mathbf{1}+\delta_0\mathbf{1}]$ be two continuous functions which satisfy*

$$\text{(i) } \bar{u}_t + c\bar{u}_x \geq D\Delta\bar{u} + f(\bar{u},t) \text{ and } \underline{u}_t + c\underline{u}_x \leq D\Delta\underline{u} + f(\underline{u},t), \ \forall x \in \mathbb{R}\setminus[-L,L], \ t>0;$$
$$\text{(ii) } \bar{u}(x,0) \geq \underline{u}(x,0), \ \forall x \in \mathbb{R} \text{ and } \bar{u}(x,t) \geq \underline{u}(x,t), \ \forall x \in [-L,L], \ t \geq 0.$$

Then $\bar{u}(x,t) \geq \underline{u}(x,t)$ for all $x \in \mathbb{R}$, $t \geq 0$.

Proof. Let $v(x,t) = (v_1(x,t),\cdots,v_n(x,t)) := \bar{u}(x,t) - \underline{u}(x,t)$, $\forall x \in \mathbb{R}$, $t \geq 0$. Then $v(x,0) \geq 0$, $-\mathbf{1} - 2\delta_0\mathbf{1} \leq v(x,t) \leq \mathbf{1} + 2\delta_0\mathbf{1}$ for $x \in \mathbb{R}$, $t \geq 0$, and

$$(v_i)_t + c(v_i)_x - d_i(v_i)_{xx} \geq f_i(\bar{u},t) - f_i(\underline{u},t), \ i = 1,\cdots,n. \tag{5.2}$$

Choosing $K>0$ large enough such that $\frac{7}{4}K - |c| - \frac{3}{2}L_f - \max\{d_1,\cdots,d_{i_0}\} > 0$, where

$$L_f := \max_{1\leq i\leq n} \sup_{u\in[-\delta_0\mathbf{1},\mathbf{1}+\delta_0\mathbf{1}],t\geq 0} |\partial f_i(u,t)/\partial u_i|.$$

Suppose the assertion is false, then there exist $i \in \{1,\cdots,n\}$, $\varpi > 0$ and $t_0 > 0$ such that

$$v_i(x,t) > -\varpi e^{2Kt} \text{ for } x \in \mathbb{R}, t \in [0,t_0), \quad \inf_{x\in\mathbb{R}} v_i(x,t_0) = -\varpi e^{2Kt_0}$$

and $v_j(x,t) \geq 0$ for $x \in \mathbb{R}$, $t \in [0,t_0]$, $j \neq i$. Thus, there exists a bounded set $S \subset \mathbb{R}$ with positive Lebesgue measure such that $v_i(x,t_0) \leq -\frac{15}{16}\varpi e^{2Kt_0}$ for $x \in S$.

Let $\zeta_0(x)$ be a smooth function satisfying

$$\min_{x\in\mathbb{R}} \zeta_0(x) = 1, \ \zeta_0(x) = 1 \text{ for all } x \in S, \ \sup_{x\in\mathbb{R}} \zeta_0(x) = \zeta_0(\pm\infty) = 3,$$
$$|\zeta_0'(\cdot)| \leq 1 \text{ and } |\zeta_0''(\cdot)| \leq 1$$

For any $\alpha \in [0,1]$, we define the function

$$z(x,t;\alpha) := -\varpi(\frac{3}{4} + \alpha\zeta_0(x))e^{2Kt}, \ x \in \mathbb{R}, t \in [0,t_0].$$

It is clear that $\frac{\partial}{\partial\alpha}z(x,t;\alpha) < 0$, $\forall \alpha \in [0,1]$, $x \in \mathbb{R}$, $t \in [0,t_0]$,

$$z(x,t;\frac{1}{4}) = -\varpi(\frac{3}{4} + \frac{1}{4}\zeta_0(x))e^{2Kt} \leq -\varpi e^{2Kt} \leq v_i(x,t) \text{ for } x \in \mathbb{R}, t \in [0,t_0];$$
$$z(x,t_0;\frac{1}{8}) = -\varpi(\frac{3}{4} + \frac{1}{8}\zeta_0(x))e^{2Kt_0} = -\frac{7}{8}\varpi e^{2Kt_0} > v_i(x,t_0) \text{ for } x \in S.$$

Thus,

$$\alpha_\star := \inf\{\alpha \in (\frac{1}{8}, \frac{1}{4}] \big| v_i(x,t) \geq z(x,t;\alpha) \text{ for } x \in \mathbb{R}, t \in [0,t_0]\}.$$

is well defined. Obviously, $v_i(x,t) \geq z(x,t;\alpha_\star)$ for $x \in \mathbb{R}$, $t \in [0,t_0]$.

Since $v_i(x,0) \geq 0 > z(x,0;\alpha_\star)$ for $x \in \mathbb{R}$;

$$z(\pm\infty, t; \alpha_\star) \leq -\frac{9}{8}\varpi e^{2Kt} < v_i(x,t) \text{ for } x \in \mathbb{R}, t \in [0,t_0];$$

$$v_i(x,t) > z(x,t;\alpha_\star) \text{ for } x \in [-L,L], t \in [0,t_0],$$

we deduce that the function $w(x,t) := v_i(x,t) - z(x,t;\alpha_\star)$ attains its infimum 0 at $(x_1,t_1) \in \mathbb{R}\backslash[-L,L]\times(0,t_0]$. Therefore, $w(x_1,t_1) = 0$, $w_t(x_1,t_1) \leq 0$, $w_x(x_1,t_1) = 0$ and $w_{xx}(x_1,t_1) \geq 0$. From (5.2) and the fact that $v_j(x,t) \geq 0$ for $x \in \mathbb{R}$, $t \in [0,t_0]$, $j \neq i$, we have

$$\begin{aligned}
0 \geq & w_t(x_1,t_1) + cw_x(x_1,t_1) - d_i w_{xx}(x_1,t_1) \\
\geq & f_i(\bar{u}(x_1,t_1),t_1) - f_i(\underline{u}(x_1,t_1),t_1) - [z_t + cz_x - d_i z_{xx}]|_{(x_1,t_1)} \\
= & \sum_{j=1}^{n} \partial_j f_i(\eta(x_1,t_1),t_1)v_j(x_1,t_1) + \varpi[2K(\frac{3}{4} + \alpha_\star\zeta_0(x))e^{2Kt_1} + c\alpha_\star\zeta_0'(x)e^{2Kt_1} - d_i\alpha_\star\zeta_0''(x)e^{2Kt_1}] \\
\geq & \partial_i f_i(\eta(x_1,t_1),t_1)v_i(x_1,t_1) + \big[\frac{7}{4}K - |c| - d_i\big]\varpi e^{2Kt_1} \\
= & -\partial_i f_i(\eta(x_1,t_1),t_1)\varpi(\frac{3}{4} + \alpha_\star\zeta_0(x_1))e^{2Kt_1} + \big[\frac{7}{4}K - |c| - d_i\big]\varpi e^{2Kt_1} \\
\geq & \big[\frac{7}{4}K - |c| - \frac{3}{2}L_f - \max\{d_1,\cdots,d_{i_0}\}\big]\varpi e^{2Kt_1} > 0,
\end{aligned}$$

where $\eta(x_1,t_1) = \theta\bar{u}(x_1,t_1) + (1-\theta)\underline{u}(x_1,t_1)$ with $\theta \in (0,1)$. This contradiction implies that $\bar{u}(x,t) \geq \underline{u}(x,t)$ for all $x \in \mathbb{R}$ and $t \geq 0$. □

Recall that $\mu^\pm = \frac{1}{T}\ln r^\pm < 0$ and $s_\pm(t) = ((s_\pm)_1(t),\cdots,(s_\pm)_n(t))$ are positive and T-periodic functions with $s_\pm(\cdot) \leq \mathbf{1}$ such that $v^\pm(t) = e^{\mu^\pm t}s_\pm(t)$ are solutions of the T-periodic linear system:

$$v'(t) = \mathrm{D}f(w^\pm(t),t)v(t).$$

Let

$$0 < \beta_1 < \frac{1}{2}\min\{-\mu_+,-\mu_-\}, \;\; \bar{p} := \min\{\min_{i=1,\cdots,n,t\geq 0}\{(s_+)_i(t)\}, \min_{i=1,\cdots,n,t\geq 0}\{(s_-)_i(t)\}\} \tag{5.3}$$

and $\zeta_1(x) \in C^2(\mathbb{R},\mathbb{R}_+)$ be a function satisfying

$$\zeta_1(x) = 0 \text{ for } x \leq -2, \zeta_1(x) = 1 \text{ for } x \geq 2, \;\; 0 \leq \zeta_1'(x) \leq 1 \text{ and } |\zeta_1''(x)| \leq 1 \text{ for } x \in \mathbb{R}. \tag{5.4}$$

Then, for $\varepsilon > 0$, we define

$$p(x,t) := \zeta_1(x)s_+(t) + (1-\zeta_1(x))s_-(t) \text{ and } u_\varepsilon(x,t) := U(x,t) + p(x,t)\varepsilon e^{-\beta_1 t}. \tag{5.5}$$

Lemma 5.3. *There exist $\xi_\star \gg 1$ and $\varepsilon_\star > 0$ such that for all* $0 < \varepsilon \leq \varepsilon_\star$,

$$(u_\varepsilon)_t + c(u_\varepsilon)_x \geq D\Delta u_\varepsilon + f(u_\varepsilon,t), \;\; \forall x \in \mathbb{R}\backslash[-\xi_\star,\xi_\star], \;\; t \geq 0.$$

Proof. For simplicity, we denote

$$\mathcal{L}[u_\varepsilon](x,t) := (u_\varepsilon)_t + c(u_\varepsilon)_x - D\Delta u_\varepsilon - f(u_\varepsilon,t).$$

Note that $p(x,t) = s_+(t)$ for $x \geq 2$ and $p(x,t) = s_-(t)$ for $x \leq -2$. Direct computation shows that, for any $|x| \geq 2$ and $t \geq 0$,

$$\varepsilon^{-1}e^{\beta_1 t}\mathcal{L}[u_\varepsilon](x,t) = \varepsilon^{-1}e^{\beta_1 t}\big[U_t + cU_x - D\Delta U - f(U + p\varepsilon e^{-\beta_1 t},t)\big] + p_t - \beta_1 p + cp_x - Dp_{xx}$$

$$=\varepsilon^{-1}e^{\beta_1 t}\left[f(U,t)-f(U+p\varepsilon e^{-\beta_1 t},t)\right]+p_t-\beta_1 p. \tag{5.6}$$

Since $\lim\limits_{x\to+\infty}U(x,t)=\mathbf{1}$ uniformly in t, it follows from (4.3) that

$$\begin{aligned}&\varepsilon^{-1}e^{\beta_1 t}\left[f(U,t)-f(U+p\varepsilon e^{-\beta_1 t},t)\right]+p_t\\ &=-\int_0^1 \mathrm{D}f(U+p\theta\varepsilon e^{-\beta_1 t},t)d\theta p+p_t\longrightarrow -\mathrm{D}f(\mathbf{1},t)s_+(t)+s_+'(t)=-\mu^+s_+(t),\end{aligned} \tag{5.7}$$

as $x\to+\infty$ and $\varepsilon\to 0$ uniformly in $t\geq 0$. Therefore, there exist $\xi_1>2$ and $\varepsilon_1>0$ such that for all $0<\varepsilon<\varepsilon_1$,

$$\varepsilon^{-1}e^{\beta_1 t}\left[f(U,t)-f(U+p\varepsilon e^{-\beta_1 t},t)\right]+p_t>-\frac{\mu^+}{2}s_+(t),\ \forall x>\xi_1,\ t>0.$$

By the above inequality and (5.6), for all $0<\varepsilon<\varepsilon_1$, we have

$$\varepsilon^{-1}e^{\beta_1 t}\mathcal{L}[u_\varepsilon](x,t)\geq[-\frac{\mu^+}{2}-\beta_1]s_+(t)>0,\ \forall x>\xi_1,\ t>0.$$

Similarly, one can show that there exist $\xi_2>2$ and $\varepsilon_2>0$ such that for all $0<\varepsilon<\varepsilon_2$,

$$\varepsilon^{-1}e^{\beta_1 t}\mathcal{L}[u_\varepsilon](x,t)\geq[-\frac{\mu^-}{2}-\beta_1]s_-(t)>0,\ \forall x<-\xi_2,\ t>0.$$

Letting $\varepsilon_\star:=\min\{\varepsilon_1,\varepsilon_2\}$ and $\xi_\star:=\max\{\xi_1,\xi_2\}$, then assertion of the lemma follows. □

Now, we are ready to give the proof of Theorem 1.3.

Proof of Theorem 1.3.

Note that $\lim\limits_{x\to+\infty}U(x,t)=\mathbf{1}$ and $\lim\limits_{x\to-\infty}U(x,t)=\mathbf{0}$ uniformly in t. Form Lemma 5.3, we can choose $\xi_\star$ large enough such that

$$U(x,t)\geq\mathbf{1}-\frac{1}{2}\varepsilon_\star\bar{p}\mathbf{1},\ \forall x\geq\xi_\star,\ t\in\mathbb{R}\text{ and }U(x,t)\leq\frac{1}{2}\varepsilon_\star\bar{p}\mathbf{1},\ \forall x\leq-\xi_\star,\ t\in\mathbb{R}.$$

Then, there exists $z_\star>0$ such that

$$U(x-z,t)\leq\begin{cases}U(x,t), & \text{if } x\in[-\xi_\star,\xi_\star],\ t\in\mathbb{R},\\ U(x,t)+\varepsilon_\star\bar{p}\mathbf{1}, & \text{if } x\in\mathbb{R}\backslash[-\xi_\star,\xi_\star],\ t\in\mathbb{R},\end{cases} \tag{5.8}$$

for all $z\geq z_\star$, where $\xi_\star$ and $\varepsilon_\star$ are given in Lemma 5.3. We first prove that

$$U(x-z,0)\leq U(x,0)\text{ for all }x\in\mathbb{R},\ z\geq z_\star. \tag{5.9}$$

Notice that $w(x,t)=U(x,t)$ satisfies the following system

$$w_t+cw_x=D\Delta w+f(w,t),\ x\in\mathbb{R},t\in\mathbb{R}.$$

By (5.8), for any $z\geq z_\star$, we obtain

$$\begin{aligned}u_{\varepsilon_\star}(x,0)=&U(x,0)+[\zeta_1(x)s_+(0)+(1-\zeta_1(x))s_-(0)]\varepsilon_\star\\ \geq&\begin{cases}U(x,0)+\bar{p}\varepsilon_\star\mathbf{1}\geq U(x-z,0), & \forall x\in\mathbb{R}\backslash[-\xi_\star,\xi_\star],\\ U(x,0)\geq U(x-z,0), & \forall x\in[-\xi_\star,\xi_\star],\end{cases}\end{aligned}$$

which implies that $u_{\varepsilon_\star}(x,0)\geq U(x-z,0)$ for all $x\in\mathbb{R}$. Moreover,

$$u_{\varepsilon_\star}(x,t)\geq U(x,t)\geq U(x-z,t),\ \forall x\in[-\xi_\star,\xi_\star],\ t\geq 0.$$

By Lemmas 5.3 and 5.2, we deduce that $U(x-z,t)\leq u_{\varepsilon_\star}(x,t)$, $\forall x\in\mathbb{R},\ t\geq 0,\ z\geq z_\star$. In particular, $U(x-z,nT)\leq u_{\varepsilon_\star}(x,nT)$, $\forall x\in\mathbb{R}$, $n\in\mathbb{N}$, $z\geq z_\star$. Using the periodicity of $U(\cdot,t)$ and $s_\pm(t)$, we have for any $n\in\mathbb{N}$ and $z\geq z_\star$,

$$U(x-z,0)\leq U(x,0)+[\zeta_1(x)s_+(0)+(1-\zeta_1(x))s_-(0)]\varepsilon_\star e^{-\beta_1 nT},\ \forall x\in\mathbb{R}.$$

Letting $n \to \infty$ in the above inequality, we conclude that (5.9) holds.

Next, we show that (5.9) actually holds for any $z \geq 0$. To this end, we denote

$$z_0 := \inf\{\bar{z} \geq 0 : U(x-z,0) \leq U(x,0),\ \forall z \geq \bar{z},\ x \in \mathbb{R}\}.$$

Clearly, $U(x-z_0,0) \leq U(x,0)$ for $x \in \mathbb{R}$. We claim that $U(x-z_0,0) = U(x,0)$ for $x \in \mathbb{R}$. Suppose the claim is false, by part (ii) of Lemma 2.5, we have

$$U(x-z_0,t) \ll U(x,t) \text{ for } x \in \mathbb{R} \text{ and } t \in \mathbb{R}.$$

Then, there exists $\epsilon > 0$ such that

$$U(x-z,t) \leq \begin{cases} U(x,t), & \text{if } x \in [-\xi_\star, \xi_\star], \\ U(x,t) + \varepsilon_\star \bar{p}\mathbf{1}, & \text{if } x \in \mathbb{R}\backslash[-\xi_\star, \xi_\star], \end{cases} \quad \forall z \geq z_0 - \epsilon.$$

Similarly, one can show that (5.9) holds for any $z \geq z_0 - \epsilon$, which contradicts the choice of z_0. Hence $U(x - z_0, 0) = U(x,0)$ for $x \in \mathbb{R}$. Since $\lim\limits_{x\to+\infty} U(x,0) = \mathbf{1}$ and $\lim\limits_{x\to-\infty} U(x,0) = \mathbf{0}$, we conclude that $z_0 = 0$. Therefore, (5.9) holds for any $z \geq 0$.

By (5.9), we see that $U_x(x,0) \geq \mathbf{0}$ for $x \in \mathbb{R}$. Since $U_x(\cdot,0) \not\equiv \mathbf{0}$ and $v(x,t) := U_x(x,t)$ satisfies the following linear system:

$$v_t + cv_x = D\Delta v + \mathrm{D}f(U,t)v,\ x \in \mathbb{R},\ t \in \mathbb{R}. \tag{5.10}$$

Therefore, it follows from Lemma 5.1 that $U_x(x,t) \gg \mathbf{0}$ for all $x \in \mathbb{R}$ and $t \in \mathbb{R}$. This completes the proof of Theorem 1.3.

6 Stability of periodic traveling fronts

In this section, we prove the uniform and asymptotic stability of the smooth periodic traveling waves under the assumptions (C1)-(C3) and (C4)$'$.

Let $U(x+ct,t)$ be a smooth periodic traveling wave of (1.1) connecting $\mathbf{0}$ and $\mathbf{1}$. By Lemma 1.3, we know that $U_x(\cdot,\cdot) \gg \mathbf{0}$. Given any $\tau \geq 0$. Let β_1, $\bar{p}$, $\zeta_1(x)$ and $p(x,t)$ be the terms defined in (5.3)-(5.5), respectively. We first construct some pairs of sub- and supersolutions.

Lemma 6.1. *There exist $\delta_1 > 0$ and $\sigma_1 > 0$ such that for any $\delta \in (0,\delta_1)$, $\xi_1,\ \xi_2 \in \mathbb{R}$ with $\xi_2 \in [-2+\xi_1, 2+\xi_1]$, the functions*

$$u^{\pm}(x,t) = U\big(x+ct+\xi_1 \pm \sigma_1\delta(1-e^{-\beta_1(t-\tau)}),t\big) \pm \delta p\big(x+ct+\xi_2 \pm \sigma_1\delta(1-e^{-\beta_1(t-\tau)}),t\big)e^{-\beta_1(t-\tau)}$$

are super- and subsolutions of (1.1) *on* $[\tau,+\infty)$.

Proof. For convenience, we denote

$$\xi_i(x,t) := x+ct+\xi_i+\sigma_1\delta(1-e^{-\beta_1(t-\tau)}),\ i=1,2.$$

Clearly, $\xi_1(x,t) = \xi_2(x,t)+\xi_1-\xi_2$. By direct calculations, we have

$$\begin{aligned}\mathcal{H}[u^+](x,t) :=& u_t^+ - D\Delta u^+ - f(u^+,t)\\ =&(c+\sigma_1\delta\beta_1 e^{-\beta_1(t-\tau)})U_x(\xi_1(x,t),t) + U_t(\xi_1(x,t),t) - DU_{xx}(\xi_1(x,t),t) -\\ & f\big(U(\xi_1(x,t),t)+\delta p(\xi_2(x,t),t)e^{-\beta_1(t-\tau)},t\big) + \big[p_t(\xi_2(x,t),t) + (c+\sigma_1\delta\beta_1 e^{-\beta_1(t-\tau)})\\ & p_x(\xi_2(x,t),t)\big]\delta e^{-\beta_1(t-\tau)} - \beta_1 p(\xi_2(x,t),t)e^{-\beta_1(t-\tau)} - Dp_{xx}(\xi_2(x,t),t)\delta e^{-\beta_1(t-\tau)}\end{aligned}$$

$$\begin{aligned}
&=f(U(\xi_1(x,t),t)-f(U(\xi_1(x,t),t)+\delta p(\xi_2(x,t),t)e^{-\beta_1(t-\tau)},t)+\\
&\quad\delta e^{-\beta_1(t-\tau)}\big[\sigma_1\beta_1 U_x(\xi_1(x,t),t)+p_t(\xi_2(x,t),t)-\beta_1 p(\xi_2(x,t),t)+\\
&\quad(c+\sigma_1\delta\beta_1 e^{-\beta_1(t-\tau)})p_x(\xi_2(x,t),t)-Dp_{xx}(\xi_2(x,t),t)\big]\\
&=\delta e^{-\beta_1(t-\tau)}\big[\sigma_1\beta_1 U_x(\xi_1(x,t),t)+K_1(\xi_2(x,t),t)-\beta_1 p(\xi_2(x,t),t)+\\
&\quad(c+\sigma_1\delta\beta_1 e^{-\beta_1(t-\tau)})p_x(\xi_2(x,t),t)-Dp_{xx}(\xi_2(x,t),t)\big],
\end{aligned}\tag{6.1}$$

where

$$K_1(\xi_2(x,t),t):=-\int_0^1 \mathrm{D}f\big(U(\xi_2(x,t)+\xi_1-\xi_2,t)+\delta\theta p(\xi_2(x,t),t)e^{-\beta_1(t-\tau)},t\big)d\theta p(\xi_2(x,t),t)$$
$$+\,p_t(\xi_2(x,t),t).$$

Note that $p(x,t)=s_+(t)$ for $x\ge 2$ and $p(x,t)=s_-(t)$ for $x\le -2$. Since $|\xi_1-\xi_2|\le 2$, $\lim\limits_{x\to+\infty}U(x,t)=\mathbf{1}$ and $\lim\limits_{x\to-\infty}U(x,t)=\mathbf{0}$ uniformly in t, by (4.3), we have

$$K_1(\xi,t)\to-\mathrm{D}f(\mathbf{1},t)s_+(t)+s_+'(t)=-\mu_+s_+(t)\text{ as }\xi\to+\infty\text{ and }\delta\to 0\text{ uniformly in }t;$$
$$K_1(\xi,t)\to-\mathrm{D}f(\mathbf{0},t)s_-(t)+s_-'(t)=-\mu_-s_-(t)\text{ as }\xi\to-\infty\text{ and }\delta\to 0\text{ uniformly in }t.$$

Then, there exist $\bar\delta_0>0$ and $M\ge 2$ such that for all $0<\delta<\bar\delta_0$,

$$K_1(\xi,t)>\begin{cases}-\mu_+s_+(t)/2, & \forall\xi\ge M,\ t\ge\tau,\\ -\mu_-s_-(t)/2, & \forall\xi\le -M,\ t\ge\tau.\end{cases}\tag{6.2}$$

By Theorem 1.3, we have $U_x(\cdot,\cdot)\gg\mathbf{0}$. Thus, we can take

$$\bar\alpha=(\bar\alpha_1,\cdots,\bar\alpha_n):=\big(\min_{|x|\le M+2,t\ge\tau}(U_1)_x(x,t),\cdots,\min_{|x|\le M+2,t\ge\tau}(U_n)_x(x,t)\big)\gg\mathbf{0}$$

and choose $\sigma_1>0$ large enough such that

$$\frac12\sigma_1\beta_1\bar\alpha>\big(L_f+\beta_1+2|c|+2\max\{d_1,\cdots,d_{i_0}\}+\max_{t\ge0}\{\|s_-'(t)\|+\|s_+'(t)\|\}\big)\mathbf{1},\tag{6.3}$$

where $L_f:=\sup_{u\in[-\delta_0\mathbf{1},\mathbf{1}+\delta_0\mathbf{1}],t\ge\tau}\|\mathrm{D}f(u,t)\|$. Let's set

$$\delta_1:=\min\Big\{\delta_0,\bar\delta_0,\frac{1}{\sigma_1},\frac{\min\{\bar\alpha_1,\cdots,\bar\alpha_n\}}{4}\Big\},$$

Given any $\delta\in(0,\delta_1)$, we consider the following two cases.

Case (i): $|\xi_2(x,t)|\ge M$. We only consider the case $\xi_2(x,t)\ge M$, the proof for the case $\xi_2(x,t)\le M$ is similar. Since $M\ge 2$, $p(\xi_2(x,t),t)=s_+(t)$, we have $p_x(\xi_2(x,t),t)=p_{xx}(\xi_2(x,t),t)=0$. Then, for all $x\in\mathbb{R},\ t>\tau$, it follows from (6.1) and (6.2) that

$$\delta^{-1}e^{\beta_1(t-\tau)}\mathcal{H}[u^+](x,t)\ge K_1(\xi_2(x,t),t)-\beta_1p(\xi_2(x,t),t)\ge[-\frac{\mu_+}{2}-\beta_1]s_+(t)\ge 0.\tag{6.4}$$

Case (ii): $|\xi_2(x,t)|\le M$. For $x\in\mathbb{R}$ and $t\ge 0$, it's obvious that

$$|\xi_1(x,t)|\le M+|\xi_1-\xi_2|\le M+2,\qquad p(x,t)=\zeta_1(x)s_+(t)+(1-\zeta_1(x))s_-(t)\le\mathbf{1},$$
$$p_x(x,t)=\zeta_1'(x)s_+(t)-\zeta_1'(x)s_-(t)\le\mathbf{2},\quad p_{xx}(x,t)=\zeta_1''(x)s_+(t)-\zeta_1''(x)s_-(t)\le\mathbf{2},$$
$$p_t(x,t)=\zeta_1(x)s_+'(t)+(1-\zeta_1(x))s_-'(t)\le\max_{t\ge0}\{\|s_-'(t)\|+\|s_+'(t)\|\}\mathbf{1}.$$

Then, from (6.1), we deduce that

$$\delta^{-1}e^{\beta_1(t-\tau)}\mathcal{H}[u^+](x,t)$$

$$\begin{aligned}
&\geq \sigma_1\beta_1\bar{\alpha} - \big(L_f + \max_{t\geq 0}\{\|s_-'(t)\| + \|s_+'(t)\|\}\big)\mathbf{1} - \beta_1\mathbf{1} - 2\big(|c| + \sigma_1\delta\beta_1 + \max\{d_1,\cdots,d_{i_0}\}\big)\mathbf{1}\\
&\geq \frac{1}{2}\sigma_1\beta_1\bar{\alpha} - \big(L_f + \beta_1 + 2|c|\big)\mathbf{1} - \big(2\max\{d_1,\cdots,d_{i_0}\} + \max_{t\geq 0}\{\|s_-'(t)\| + \|s_+'(t)\|\}\big)\mathbf{1}\\
&\geq 0,\ \forall x\in\mathbb{R},\ t>\tau.
\end{aligned} \tag{6.5}$$

Therefore, we conclude that $u^+(x,t)$ is a super-solutions of (1.1) on $[\tau,+\infty)$. Using a similar argument, we can show that $u^-(x,t)$ is a sub-solutions of (1.1) on $[\tau,+\infty)$. □

Lemma 6.2. *Let $\alpha_\pm \in [-2\delta_0\mathbf{1}, \mathbf{1}+2\delta_0\mathbf{1}]$ be any given vectors in $\mathbb{R}^n$ with $\alpha_- \ll \alpha_+$ and $\zeta(\cdot)$ be the function defined in* (4.4). *Define*

$$v^\pm(x,t) := v(t;\alpha_\pm)\zeta(x+h+Ct) + v(t;\alpha_\mp)\big(1-\zeta(x+h+Ct)\big), \tag{6.6}$$

$$w^\pm(x,t) := v(t;\alpha_\mp)\zeta(x+h-Ct) + v(t;\alpha_\pm)\big(1-\zeta(x+h-Ct)\big), \tag{6.7}$$

where $v(t;\alpha)$ is the solution of (1.5) *with initial condition $v(0;\alpha)=\alpha$. Then, for any $C\gg 1$ and $h\in\mathbb{R}$, $v^\pm(x,t)$ and $w^\pm(x,t)$ are two pairs of sub- and supersolutions of* (1.1) *on* $[0,+\infty)$.

Proof. We only prove that $v^+(x,t)$ is a supersolution. The other cases can be proved similarly. From Lemma 2.5, we see that $P(\alpha) = v(T;\alpha) : [\mathbf{0},\mathbf{1}] \to [\mathbf{0},\mathbf{1}]$ is strongly monotone. By the Dancer-Hess connecting orbit lemma (c.f. [31]), one can see that

$$\lim_{t\to\infty} v(t;\alpha) = \mathbf{0} \text{ for any } \alpha\in[\mathbf{0},\bar{w}) \text{ and } \lim_{t\to\infty} v(t;\alpha) = \mathbf{1} \text{ for any } \alpha\in(\bar{w},\mathbf{1}],$$

where $\bar{w} = \bar{w}(0)$. Note that $-2\delta_0\mathbf{1}\in\Omega_-$ and $\mathbf{1}+2\delta_0\mathbf{1}\in\Omega_+$, where Ω_- and Ω_+ are the domains of attraction of $\mathbf{0}$ and $\mathbf{1}$, respectively. Therefore, for any $\alpha_\pm\in[-2\delta_0\mathbf{1},\mathbf{1}+2\delta_0\mathbf{1}]$ with $\alpha_-\ll\alpha_+$, $v(t;\alpha_\pm)$ are defined for all $[0,+\infty)$ and satisfy $v(t;\alpha_-)\ll v(t;\alpha_+)$ for $t\in[0,+\infty)$.

Let $\eta(x,t) = x+h+Ct$. Since $0<\zeta(\cdot)<1$, $\zeta'(s) = \zeta(s)(1-\zeta(s))$ and $\zeta''(s) = \zeta(s)(1-\zeta(s))(1-2\zeta(s))$, by using the Taylor's expansion, we obtain

$$\begin{aligned}
&\zeta f_i(v(t;\alpha_+),t) + (1-\zeta)f_i(v(t;\alpha_-),t) - f_i(\zeta v(t;\alpha_+) + (1-\zeta)v(t;\alpha_-),t)\\
=&\zeta[f_i(v(t;\alpha_+),t) - f_i(\zeta v(t;\alpha_+) + (1-\zeta)v(t;\alpha_-),t)] +\\
&(1-\zeta)[f_i(v(t;\alpha_-),t) - f_i(\zeta v(t;\alpha_+) + (1-\zeta)v(t;\alpha_-),t)]\\
=&\frac{\zeta(1-\zeta)}{2}\sum_{j=1}^n\sum_{l=1}^n\Big[(1-\zeta)\frac{\partial^2 f_i(\bar{y}(x,t),t)}{\partial x_j x_l} + \zeta\frac{\partial^2 f_i(\tilde{y}(x,t),t)}{\partial x_j x_l}\Big]\\
&\qquad\times[v_j(t;\alpha_+) - v_j(t;\alpha_-)][v_l(t;\alpha_+) - v_l(t;\alpha_-)]\\
=&\frac{\zeta(1-\zeta)}{2}\sum_{j=1}^n\sum_{l=1}^n\frac{\partial^2 f_i(y(x,t),t)}{\partial x_j x_l}[v_j(t;\alpha_+) - v_j(t;\alpha_-)][v_l(t;\alpha_+) - v_l(t;\alpha_-)],
\end{aligned} \tag{6.8}$$

where $\tilde{y}(x,t),\ \bar{y}(x,t)\in(v(t;\alpha_-),v(t;\alpha_+))$ and $y(x,t)$ is a function satisfying

$$\frac{\partial^2 f_i(y(x,t),t)}{\partial x_j x_l} = (1-\zeta)\frac{\partial^2 f_i(\bar{y}(x,t),t)}{\partial x_j x_l} + \zeta\frac{\partial^2 f_i(\tilde{y}(x,t),t)}{\partial x_j x_l}.$$

Let

$$\begin{aligned}
C := \max\{d_1,\cdots,d_{i_0}\} + \frac{1}{2}\sup\Big\{&\sum_{j=1}^n\sum_{l=1}^n\Big|\frac{\partial^2 f_i(z,t)}{\partial x_j x_l}\Big|\frac{[v_j(t;\alpha_+) - v_j(t;\alpha_-)][v_l(t;\alpha_+) - v_l(t;\alpha_-)]}{v_i(t;\alpha_+) - v_i(t;\alpha_-)},\\
&i=1,\cdots,n, z\in(v(t;\alpha_-),v(t;\alpha_+)), t\geq 0\Big\}.
\end{aligned}$$

From (6.8), direct computation shows that

$$\mathcal{H}_i[v^+](x,t) := (v_i^+)_t - d_i(v_i^+)_{xx} - f_i(v^+,t)$$

$$
\begin{aligned}
&=\zeta(\eta(x,t))f_i(v(t;\alpha_+),t)+(1-\zeta(\eta(x,t)))f_i\big(v(t;\alpha_-),t\big)-\\
&\quad f_i\big(\zeta(\eta(x,t))v(t;\alpha_+)+(1-\zeta(\eta(x,t)))v(t;\alpha_-),t\big)+\\
&\quad \zeta(\eta(x,t))(1-\zeta(\eta(x,t)))[C-d_i(1-2\zeta(\eta(x,t)))][v_i(t;\alpha_+)-v_i(t;\alpha_-)]\\
&\geq[v_i(t;\alpha_+)-v_i(t;\alpha_-)]\zeta(\eta(x,t))(1-\zeta(\eta(x,t)))\big\{C-d_i+\\
&\quad \frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{n}\frac{\partial^2 f_i(y(x,t),t)}{\partial x_j x_l}\frac{[v_j(t;\alpha_+)-v_j(t;\alpha_-)][v_l(t;\alpha_+)-v_l(t;\alpha_-)]}{v_i(t;\alpha_+)-v_i(t;\alpha_-)}\big\}\geq 0,
\end{aligned}
\tag{6.9}
$$

for $x\in\mathbb{R}$, $t>0$. Therefore, $v^+(x,t)$ is a super-solution of (1.1) on $[0,+\infty)$. □

Lemma 6.3. *Let $\phi\in L^\infty(\mathbb{R},[-\delta_0\mathbf{1},\mathbf{1}+\delta_0\mathbf{1}])$ be such that*

$$\liminf_{x\to+\infty}\phi(x)\gg\bar{w}\ \text{and}\ \limsup_{x\to-\infty}\phi(x)\ll\bar{w}. \tag{6.10}$$

Then for any $\delta\in(0,\delta_0)$, there are $H>0$ and $\tilde{T}\geq 0$ such that

$$U(x-H+c\tilde{T},\tilde{T})-\delta\mathbf{1}\leq u(x,\tilde{T};\phi)\leq U(x+H+c\tilde{T},\tilde{T})+\delta\mathbf{1}, \tag{6.11}$$

where $u(x,t;\phi)$ is the solution of (1.1) *with $u(\cdot,0;\phi)=\phi(\cdot)$.*

Proof. For any given sufficiently small $\kappa>0$, by (6.10), there exists $M_1>0$ such that

$$\phi(x)\gg\bar{w}+\kappa\mathbf{1}\ \text{for}\ x>M_1\ \text{and}\ \phi(x)\ll\bar{w}-\kappa\mathbf{1}\ \text{for}\ x<-M_1. \tag{6.12}$$

By Lemma 6.2, for any $h\in\mathbb{R}$ and sufficiently large C,

$$
\begin{aligned}
v^+(x,t)&=v(t;\mathbf{1}+2\delta_0\mathbf{1})\zeta(x+h+Ct)+v(t;\bar{w}-\kappa\mathbf{1})\big(1-\zeta(x+h+Ct)\big),\\
v^-(x,t)&=v(t;\bar{w}+\kappa\mathbf{1})\zeta(x-h-Ct)+v(t;-2\delta_0\mathbf{1})\big(1-\zeta(x-h-Ct)\big),
\end{aligned}
$$

are super- and subsolutions of (1.1) on $[0,+\infty)$, respectively. From (6.12), we can choose $h>0$ large enough such that $v^-(x,0)\leq\phi(x)\leq v^+(x,0)$, $\forall x\in\mathbb{R}$. Hence, by Lemma 2.4,

$$v^-(x,t)\leq u(x,t;\phi)\leq v^+(x,t),\ \forall x\in\mathbb{R},\ t>0. \tag{6.13}$$

Moreover, from the proof of Lemma 6.2, we see that

$$\lim_{t\to\infty}v(t;\mathbf{1}+2\delta_0\mathbf{1})=\lim_{t\to\infty}v(t;\bar{w}+\kappa\mathbf{1})=\mathbf{1}\ \text{and}\ \lim_{t\to\infty}v(t;\bar{w}-\kappa\mathbf{1})=\lim_{t\to\infty}v(t;-2\delta_0\mathbf{1})=\mathbf{0}.$$

Hence, for any $\delta>0$, there exists $\tilde{T}\geq 0$ such that

$$\mathbf{1}-\frac{\delta}{2}\mathbf{1}\ll v(\tilde{T};\alpha_+)\ll\mathbf{1}+\frac{\delta}{2}\mathbf{1}\ \text{and}\ -\frac{\delta}{2}\mathbf{1}\ll v(\tilde{T};\alpha_-)\ll\frac{\delta}{2}\mathbf{1}, \tag{6.14}$$

where $\alpha_+=\mathbf{1}+2\delta_0\mathbf{1}$ (or $\alpha_+=\bar{w}+\kappa\mathbf{1}$) and $\alpha_-=\bar{w}-\kappa\mathbf{1}$ (or $\alpha_-=-2\delta_0\mathbf{1}$). Since $\lim_{x\to+\infty}U(x,t)=\mathbf{1}$ and $\lim_{x\to-\infty}U(x,t)=\mathbf{0}$, it follows from (6.13)-(6.14) that there exists large $H>0$ such that

$$U(x-H+c\tilde{T},\tilde{T})-\delta\mathbf{1}\leq v^-(x,\tilde{T})\leq u(x,\tilde{T};\phi)\leq v^+(x,\tilde{T})\leq U(x+H+c\tilde{T},\tilde{T})+\delta\mathbf{1}$$

for all $x\in\mathbb{R}$. This completes the proof. □

In the sequel, let β_1, $\bar{p}$ and $p(x,t)$ be the terms defined in (5.3) and (5.5), respectively; σ_1 and δ_1 be the constants defined in Lemma 6.1; and δ_0 be the term defined in Section 1.

Lemma 6.4. *There exists $\epsilon_\star\in(0,\min\{\frac{\delta_1}{2},\frac{1}{2\sigma_1}\})$ such that if $u(x,t)$ is a solution of* (1.1) *with the property: for some $\tau\in[0,+\infty)$, $\xi\in\mathbb{R}$, $h>0$ and $\delta\in(0,\bar{p}\min\{\frac{1}{\sigma_1},\frac{\delta_1}{2}\})$,*

$$U(x+c\tau+\xi,\tau)-\delta\mathbf{1}\leq u(x,\tau)\leq U(x+c\tau+\xi+h,\tau)+\delta\mathbf{1},\ \forall x\in\mathbb{R}, \tag{6.15}$$

then, for every $t \geq \tau + 1$,

$$U(x + ct + \xi(t), t) - \delta(t)\mathbf{1} \leq u(x, t) \leq U(x + ct + \xi(t) + h(t), t) + \delta(t)\mathbf{1}, \ \forall x \in \mathbb{R}, \tag{6.16}$$

where $\xi(t) \in [\xi - \sigma_1\delta/\bar{p}, \xi + h + \sigma_1\delta/\bar{p}]$,

$$\delta(t) \leq e^{-\beta_1(t-\tau-1)}[\delta/\bar{p} + \epsilon_\star \min\{1, h\}] \ \text{ and } \ h(t) \leq h - \sigma_1\epsilon_\star \min\{1, h\} + 2\sigma_1\delta/\bar{p}.$$

Proof. Take $\bar{\delta} = \delta/\bar{p}$. Then $\bar{\delta} \in (0, \min\{\frac{1}{\sigma_1}, \frac{\delta_1}{2}\})$. From the definitions of $p(x, t)$ and $\bar{p}$, we deduce that $p(x, t) \geq \bar{p}\mathbf{1}$ for any $x \in \mathbb{R}$, $t \geq 0$. Thus, by (6.15), we have

$$U(x + c\tau + \xi, \tau) - \bar{\delta}p\big(x + c\tau + \xi, \tau\big) \leq u(x, \tau) \leq U(x + c\tau + \xi + h, \tau) + \bar{\delta}p\big(x + c\tau + \xi + h, \tau\big)$$

for $x \in \mathbb{R}$. It then follows from Lemmas 2.4 and 6.1 that

$$\begin{aligned} &U\big(\xi_-(x, t), t\big) - \bar{\delta}p\big(\xi_-(x, t), t\big)e^{-\beta_1(t-\tau)} \\ \leq &u(x, t) \leq U\big(\xi_+(x, t) + h, t\big) + \bar{\delta}p\big(\xi_+(x, t) + h, t\big)e^{-\beta_1(t-\tau)}, \end{aligned} \tag{6.17}$$

for $x \in \mathbb{R}$ and $t > \tau$, where $\xi_\pm(x, t) = x + ct + \xi \pm \sigma_1\bar{\delta}(1 - e^{-\beta_1(t-\tau)})$. Let's set

$$\bar{h} = \min\{h, 1\} \ \text{ and } \ \bar{\epsilon} = \frac{1}{2}\min\{(U_1)_x(x, t) : |x| \leq 3, t \in [\tau, \tau + T]\},$$

where U_1 is the first component of U. Fix any number $x_0 \in [-c\tau - \xi, -c\tau - \xi + 1]$, we have

$$\int_0^1 [U_1(y + x_0 + c\tau + \xi + \bar{h}, \tau) - U_1(y + x_0 + c\tau + \xi, \tau)]dy \geq 2\bar{\epsilon}\bar{h}. \tag{6.18}$$

Hence, one of the following inequalities holds

$$\int_0^1 [u_1(y + x_0, \tau) - U_1(y + x_0 + c\tau + \xi, \tau)]dy \geq \bar{\epsilon}\bar{h}, \tag{6.19}$$

or

$$\int_0^1 [U_1(y + x_0 + c\tau + \xi + \bar{h}, \tau) - u_1(y + x_0, \tau)]dy \geq \bar{\epsilon}\bar{h}. \tag{6.20}$$

We prove the lemma for the case that (6.19) holds. The other case can be treated similarly.

By interpolation and the boundedness of $U_{xx}(x, t)$, we deduce that $\lim\limits_{|x|\to\infty} U_x(x, t) = \mathbf{0}$ uniformly in t. Then, there exists $\bar{M} > 0$ such that

$$2\sigma_1\|U_x(x, t)\| < \bar{p} \text{ for any } x \in \mathbb{R} \text{ with } |x| \geq \bar{M} - |c| - 3 \text{ and } t \geq \tau. \tag{6.21}$$

Let's define

$$\epsilon_\star := \min\Big\{\frac{\delta_1}{2}, \frac{1}{2\sigma_1}, \min_{|x|\leq\bar{M}+|c|+3, t\in[0,T]} \frac{\bar{\theta}\bar{\epsilon}}{2\sigma_1\|U_x(x, t)\|}\Big\},$$

where $\bar{\theta} = \Theta(\bar{M})$ is given in lemma 2.5. Then, for any $x \in \mathbb{R}$ with $|x| \leq \bar{M}$,

$$|\xi_-(x + x_0, \tau + 1) + 2\epsilon_\star\sigma_1\bar{h}| = |x + x_0 + c\tau + c + \xi - \sigma_1\bar{\delta}(1 - e^{-\beta_1}) + 2\epsilon_\star\sigma_1\bar{h}| \leq \bar{M} + |c| + 3,$$

which implies that

$$U\big(\xi_-(x + x_0, \tau + 1), \tau + 1\big) - U\big(\xi_-(x + x_0, \tau + 1) + 2\epsilon_\star\sigma_1\bar{h}, \tau + 1\big) \geq -\bar{\theta}\bar{\epsilon}\bar{h}\mathbf{1}. \tag{6.22}$$

Comparing the functions

$$v(x,t) := u(x+x_0,t) \text{ and } u_-(x,t) := U\big(\xi_-(x+x_0,t),t\big) - \bar{\delta}p\big(\xi_-(x+x_0,t),t\big)e^{-\beta_1(t-\tau)}$$

and using Lemma 2.5, we have

$$\begin{aligned}
& u(x+x_0,\tau+1) - [U\big(\xi_-(x+x_0,\tau+1),\tau+1\big) - \bar{\delta}p\big(\xi_-(x+x_0,\tau+1),\tau+1\big)e^{-\beta_1}] \\
\geq & \Theta(\bar{M})\int_0^1 \{u_1(y+x_0,\tau) - [U_1\big(\xi_-(y+x_0,\tau),\tau\big) - \bar{\delta}p\big(\xi_-(y,\tau),\tau\big)]\}dy\mathbf{1} \\
\geq & \bar{\theta}\int_0^1 [u_1(y+x_0,\tau) - U_1\big(y+c\tau+x_0+\xi,\tau\big)]dy\mathbf{1} \geq \bar{\theta}\bar{\epsilon}\bar{h}\mathbf{1},
\end{aligned}$$

for any $x \in \mathbb{R}$ with $|x| \leq \bar{M}$. The above inequality and (6.22) yield that

$$u(x+x_0,\tau+1) \geq U\big(\xi_-(x+x_0,\tau+1) + 2\epsilon_\star\sigma_1\bar{h},\tau+1\big) - \bar{\delta}e^{-\beta_1}p\big(\xi_-(x+x_0,\tau+1),\tau+1\big), \tag{6.23}$$

for any $x \in \mathbb{R}$ with $|x| \leq \bar{M}$. For $x \in \mathbb{R}$ with $|x| \geq \bar{M}$, we have

$$|\xi_-(x+x_0,\tau+1) + 2\epsilon_\star\sigma_1\bar{h}| \geq |x| - |x_0 + c\tau + c + \xi - \sigma_1\bar{\delta}(1-e^{-\beta_1}) + 2\epsilon_\star\sigma_1\bar{h}| \geq \bar{M} - |c| - 3.$$

It then follows from (6.21) that for any $x \in \mathbb{R}$ with $|x| \geq \bar{M}$,

$$\begin{aligned}
& U\big(\xi_-(x+x_0,\tau+1),\tau+1\big) - U\big(\xi_-(x+x_0,\tau+1) + 2\epsilon_\star\sigma_1\bar{h},\tau+1\big) \\
= & -2\sigma_1 U_x\big(\xi_-(x+x_0,\tau+1) + 2\vartheta\epsilon_\star\sigma_1\bar{h},\tau+1\big)\epsilon_\star\bar{h} \geq -\bar{p}\epsilon_\star\bar{h}\mathbf{1} \geq -\epsilon_\star\bar{h}p\big(\xi_-(x+x_0,\tau+1),\tau+1\big),
\end{aligned}$$

where $\vartheta \in (0,1)$. On the other hand, from (6.17), we have

$$u(x+x_0,\tau+1) \geq U\big(\xi_-(x+x_0,\tau+1),\tau+1\big) - \bar{\delta}e^{-\beta_1}p\big(\xi_-(x+x_0,\tau+1),\tau+1\big). \tag{6.24}$$

Thus, for $x \in \mathbb{R}$ with $|x| \geq \bar{M}$,

$$\begin{aligned}
u(x+x_0,\tau+1) \geq & U\big(\xi_-(x+x_0,\tau+1) + 2\epsilon_\star\sigma_1\bar{h},\tau+1\big) - \\
& (\epsilon_\star\bar{h} + \bar{\delta}e^{-\beta_1})p\big(\xi_-(x+x_0,\tau+1),\tau+1\big).
\end{aligned} \tag{6.25}$$

Clearly, $\tilde{\delta} := \epsilon_\star\bar{h} + \bar{\delta}e^{-\beta_1} \in (0,\delta_1)$. Combining (6.23) and (6.25), we conclude that

$$\begin{aligned}
u(x,\tau+1) \geq & U\big(\xi_-(x,\tau+1) + 2\epsilon_\star\sigma_1\bar{h},\tau+1\big) - \tilde{\delta}p\big(\xi_-(x,\tau+1),\tau+1\big) \\
= & U\big(x + c(\tau+1) + \xi_1,\tau+1\big) - \tilde{\delta}p\big(x + c(\tau+1) + \xi_2,\tau+1\big), \ \forall x \in \mathbb{R},
\end{aligned} \tag{6.26}$$

where

$$\xi_1 = \xi - \sigma_1\bar{\delta}(1-e^{-\beta_1}) + 2\epsilon_\star\sigma_1\bar{h} \text{ and } \xi_2 = \xi - \sigma_1\bar{\delta}(1-e^{-\beta_1}) \in [\xi_1 - 2, \xi_1].$$

Note that $p(\cdot,\cdot) \leq \mathbf{1}$ by the definition of $p(\cdot,\cdot)$. It then follows from (6.26), Lemma 6.1 and Lemma 2.4 that

$$\begin{aligned}
u(x,t) \geq & U\big(x + ct + \xi_1 - \sigma_1\tilde{\delta}(1-e^{-\beta_1(t-\tau-1)}),t\big) - \\
& \tilde{\delta}p\big(x + ct + \xi_2 - \sigma_1\tilde{\delta}(1-e^{-\beta_1(t-\tau-1)}),t\big)e^{-\beta_1(t-\tau-1)} \\
\geq & U\big(x + ct + \xi_1 - \sigma_1\tilde{\delta},t\big) - \tilde{\delta}e^{-\beta_1(t-\tau-1)}\mathbf{1} = U\big(x + ct + \xi(t),t\big) - \delta(t)\mathbf{1}
\end{aligned} \tag{6.27}$$

for all $x \in \mathbb{R}$ and $t \geq \tau + 1$, where

$$\xi(t) = \xi - \sigma_1\bar{\delta} + \sigma_1\epsilon_\star\bar{h}, \text{ and } \delta(t) = \tilde{\delta}e^{-\beta_1(t-\tau-1)} = (\epsilon_\star\bar{h} + \bar{\delta}e^{-\beta_1})e^{-\beta_1(t-\tau-1)}.$$

Let $h(t) = h + 2\sigma_1\bar{\delta} - \sigma_1\epsilon_\star\bar{h}$, by (6.17), we obtain the first inequality in (6.16). This completes the proof. □

Proof of Theorem 1.4.

Based on lemmas 6.1-6.4, the proof is similar to those of [21, Theorem 3.1] and [24, Theorem 5.1]. For completeness, we present it in the sequel.

(1) Let δ_1, σ_1 and β_1 be as in Lemma 6.1. Given any $\epsilon > 0$. By the uniformly boundedness of $U_x(x,t)$ and the periodicity of $U(x,\cdot)$, there exists $\nu_0 \in (0,1)$ such that for any $\nu \in (0,\nu_0]$,

$$\|U(x+\nu,t) - U(x,t)\| < \frac{\epsilon}{2} \text{ for all } x \in \mathbb{R},\ t \geq 0. \tag{6.28}$$

Choose $\delta \in (0, \bar{p}\min\{\frac{\epsilon}{2}, \frac{\delta_1}{2}, \frac{\nu_0}{\sigma_1}\})$. For any $\phi \in L^\infty(\mathbb{R})$ with $\|\phi(\cdot) - U(\cdot,0)\|_{L^\infty(\mathbb{R})} < \delta$, we have

$$U(x,0) - \bar{\delta}p(x,0) \leq U(x,0) - \delta\mathbf{1} \leq \phi(x) \leq U(x,0) + \delta\mathbf{1} \leq U(x,0) + \bar{\delta}p(x,0),\ \forall x \in \mathbb{R},$$

where $\bar{\delta} = \delta/\bar{p} \in (0, \min\{\frac{\epsilon}{2}, \frac{\delta_1}{2}, \frac{\nu_0}{\sigma_1}\})$. Then it follows from Lemmas 6.1 and 2.4 that

$$\begin{aligned}&U\big(x+ct-\sigma_1\bar{\delta}(1-e^{-\beta_1 t}),t\big) - \bar{\delta}p\big(x+ct-\sigma_1\bar{\delta}(1-e^{-\beta_1 t}),t\big)e^{-\beta_1 t}\\ \leq& u(x,t;\phi) \leq U\big(x+ct+\sigma_1\bar{\delta}(1-e^{-\beta_1 t}),t\big) + \bar{\delta}p\big(x+ct+\sigma_1\bar{\delta}(1-e^{-\beta_1,t}),t\big)e^{-\beta_1 t}\end{aligned}$$

for $x \in \mathbb{R}$, $t > 0$, which implies that

$$U\big(x+ct-\sigma_1\bar{\delta},t\big) - \bar{\delta}\mathbf{1} \leq u(x,t;\phi) \leq U\big(x+ct+\sigma_1\bar{\delta},t\big) + \bar{\delta}\mathbf{1}. \tag{6.29}$$

Combining (6.28) and (6.29), we deduce that $\|u(\cdot,t;\phi) - U(\cdot+ct,t)\|_{L^\infty(\mathbb{R})} < \epsilon$ for all $t > 0$.

(2) Let $\epsilon_\star$ be as in Lemma 6.4, $\delta^\star = \min\{\frac{\bar{p}\delta_1}{2}, \frac{\bar{p}}{\sigma_1}, \delta_0, \frac{\bar{p}\epsilon_\star}{4}\}$ and $\kappa^\star = \sigma_1\epsilon^\star - 2\sigma_1\delta^\star/\bar{p}$. Clearly, $1 > \kappa^\star \geq \sigma_1\epsilon^\star/2 > 0$. Fix a number $t_\star \geq 2$ such that

$$e^{-\beta_1(t_\star-1)}[\frac{1}{\bar{p}} + \frac{\epsilon^\star}{\delta^\star}] \leq 1 - \kappa^\star.$$

By Lemma 6.3, there exist $\xi_0 \in \mathbb{R}$, $T_0 > 0$ and $h_0 > 1$ such that (6.15) holds for $(\tau,\xi,h,\delta) = (T_0,\xi_0,h_0,\delta^\star)$, that is,

$$U(x+cT_0+\xi_0,T_0) - \delta^\star\mathbf{1} \leq u(x,T_0;\phi) \leq U(x+cT_0+\xi_0+h_0,T_0) + \delta^\star\mathbf{1},\ \forall x \in \mathbb{R}. \tag{6.30}$$

We first prove the following two claims.

Claim 1. *There exist* $\hat{T} > T_0$, $\hat{\xi} \in \mathbb{R}$ *such that* (6.15) *holds for* $(\tau,\xi,h,\delta) = (\hat{T},\hat{\xi},1,\delta^\star)$, *i.e.*

$$U(x+c\hat{T}+\hat{\xi},\hat{T}) - \delta^\star\mathbf{1} \leq u(x,\hat{T};\phi) \leq U(x+c\hat{T}+\hat{\xi}+1,\hat{T}) + \delta^\star\mathbf{1},\ \forall x \in \mathbb{R}. \tag{6.31}$$

Let $N \in \mathbb{Z}_+$ such that $0 < h_0 - N\kappa^\star \leq 1$. With (6.30), by Lemma 6.4, we conclude that

$$U(x+c\tilde{T}_1+\xi(\tilde{T}_1),\tilde{T}_1) - \delta(\tilde{T}_1)\mathbf{1} \leq u(x,\tilde{T}_1;\phi) \leq U(x+c\tilde{T}_1+\xi(\tilde{T}_1)+h(\tilde{T}_1),\tilde{T}_1) + \delta(\tilde{T}_1)\mathbf{1},$$

$\forall x \in \mathbb{R}$, where $\tilde{T}_1 := T_0 + t_\star$, and

$$\begin{aligned}\tilde{\xi}_1 &:= \xi(\tilde{T}_1) \in [\xi_0 - \sigma_1\delta^\star/\bar{p}, \xi_0 + h_0 + \sigma_1\delta^\star/\bar{p}],\\ \tilde{\delta}_1 &:= \delta(\tilde{T}_1) \leq e^{-\beta_1(t_\star-1)}[\delta^\star/\bar{p} + \epsilon_\star\min\{1,h_0\}] = \delta^\star e^{-\beta_1(t_\star-1)}[1/\bar{p} + \epsilon_\star/\delta^\star] < \delta^\star,\\ \tilde{h}_1 &:= h(\tilde{T}_1) \leq h_0 - \sigma_1\epsilon_\star\min\{1,h_0\} + 2\sigma_1\delta^\star/\bar{p} = h_0 - \kappa^\star.\end{aligned}$$

Hence, we have

$$U(x+c\tilde{T}_1+\tilde{\xi}_1,\tilde{T}_1) - \tilde{\delta}_1\mathbf{1} \leq u(x,\tilde{T}_1;\phi) \leq U(x+c\tilde{T}_1+\tilde{\xi}_1+\tilde{h}_1,\tilde{T}_1) + \tilde{\delta}_1\mathbf{1},\ \forall x \in \mathbb{R}.$$

Repeating the same process, we can show that

$$U(x+c\tilde{T}_N+\tilde{\xi}_N,\tilde{T}_N) - \tilde{\delta}_N\mathbf{1} \leq u(x,\tilde{T}_N;\phi) \leq U(x+c\tilde{T}_N+\tilde{\xi}_N+\tilde{h}_N,\tilde{T}_N) + \tilde{\delta}_N\mathbf{1}, \tag{6.32}$$

where $x \in \mathbb{R}$, $\tilde{T}_N := T_0 + Nt_\star$, and

$$\tilde{\xi}_N \in [\xi_{N-1} - \sigma_1 \tilde{\delta}_{N-1}/\bar{p}, \xi_{N-1} + h_{N-1} + \sigma_1 \tilde{\delta}_{N-1}/\bar{p}],\ \tilde{\delta}_N < \delta^\star,\ \tilde{h}_N \le h_0 - N\kappa^\star \le 1.$$

Let $\hat{T} = \tilde{T}_N = T_0 + Nt_\star$ and $\hat{\xi} = \tilde{\xi}_N$, then the Claim 1 follows from (6.32) and the monotonicity of $U(x,t)$ with respect to $x \in \mathbb{R}$.

Claim 2. *Let $T_k^\star = \hat{T} + kt_\star$, $\delta_k^\star = (1-\kappa^\star)^k \delta^\star$, $h_k^\star = (1-\kappa^\star)^k$, $k \in \mathbb{N}$. There exits a sequence $\{\xi_k^\star\}_{k=1}^\infty \subseteq \mathbb{R}$ with $\xi_0^\star = \hat{\xi}$ be such that $|\xi_{k+1}^\star - \xi_k^\star| \le h_k^\star(1 + 2\sigma_1\delta^\star/\bar{p})$*, and

$$U(x + cT_k^\star + \xi_k^\star, T_k^\star) - \delta_k^\star \mathbf{1} \le u(x, T_k^\star; \phi) \le U(x + cT_k^\star + \xi_k^\star + h_k^\star, T_k^\star) + \delta_k^\star \mathbf{1},\ \forall x \in \mathbb{R},\ k \ge 0. \tag{6.33}$$

In fact, by Claim 1, we see that (6.33) holds for $k = 0$. By using the mathematical induction and Lemma 6.4, one can easily show that the Claim 2 holds.

Then, for each $k \ge 0$, (6.33) implies

$$\begin{aligned} &U(x + cT_k^\star + \xi_k^\star, T_k^\star) - \bar{\delta}_k^\star p(x + cT_k^\star + \xi_k^\star, T_k^\star) \le u(x, T_k^\star; \phi) \\ \le &U(x + cT_k^\star + \xi_k^\star + h_k^\star, T_k^\star) + \bar{\delta}_k^\star p(x + cT_k^\star + \xi_k^\star, T_k^\star),\ \forall x \in \mathbb{R}, \end{aligned} \tag{6.34}$$

where $\bar{\delta}_k^\star = \delta_k^\star/\bar{p} = (1-\kappa^\star)^k \delta^\star/\bar{p} < \delta_1$. Note that $h_k^\star \in [0,1]$, $\forall k \in \mathbb{N}$. By (6.34) and Lemmas 6.1 and 2.4, we have

$$\begin{aligned} &U(x + ct + \xi_k^\star - \sigma_1\bar{\delta}_k^\star(1 - e^{-\beta_1(t-T_k^\star)}), t) - \bar{\delta}_k^\star p(x + ct + \xi_k^\star - \sigma_1\bar{\delta}_k^\star(1 - e^{-\beta_1(t-T_k^\star)}), t)e^{-\beta_1(t-T_k^\star)} \\ \le &u(x, t; \phi) \\ \le &U(x + ct + \xi_k^\star + h_k^\star + \sigma_1\bar{\delta}_k^\star(1 - e^{-\beta_1(t-T_k^\star)}), t) \\ &+ \bar{\delta}_k^\star p(x + ct + \xi_k^\star + \sigma_1\bar{\delta}_k^\star(1 - e^{-\beta_1(t-T_k^\star)}), t)e^{-\beta_1(t-T_k^\star)},\ \forall x \in \mathbb{R},\ t \ge T_k^\star. \end{aligned}$$

Using the fact $p(\cdot,\cdot) \le \mathbf{1}$ and the monotonicity of $U(x,t)$ with respect to x, we obtain

$$U(x + ct + \xi_k^\star - \sigma_1\bar{\delta}_k^\star, t) - \bar{\delta}_k^\star \le u(x, t; \phi) \le U(x + ct + \xi_k^\star + h_k^\star + \sigma_1\bar{\delta}_k^\star, t) + \bar{\delta}_k^\star \tag{6.35}$$

for all $x \in \mathbb{R}$, $t \ge T_k^\star$. Moreover, for any $t \ge \hat{T}$, let $k = [\frac{t-\hat{T}}{t_\star}]$ be the largest integer not greater than $(t - \hat{T})/t_\star$,

$$\delta(t) := \bar{\delta}_k^\star,\ \xi(t) = \xi_k^\star - \sigma_1\bar{\delta}_k^\star \text{ and } h(t) := h_k^\star + 2\sigma_1\bar{\delta}_k^\star.$$

It then follows from (6.35) that

$$U(x + ct + \xi(t), t) - \delta(t) \le u(x, t; \phi) \le U(x + ct + \xi(t) + h(t), t) + \delta(t). \tag{6.36}$$

Take $\mu_0 := \frac{1}{t_\star}\ln(1-\kappa^\star)$, one can easily show that

$$\delta(t) \le \frac{\delta^\star}{\bar{p}} e^{-\mu_0(t-\hat{T}-t_\star)},\ h(t) \le (1 + 2\sigma_1\delta^\star/\bar{p})e^{-\mu_0(t-\hat{T}-t_\star)},\ \xi(\infty) := \lim_{t\to\infty} \xi(t) \text{ exists}$$

and

$$|\xi(t) - \xi(\infty)| \le \left[\frac{1}{\kappa^\star}(1 + 2\sigma_1\delta^\star/\bar{p}) + 2\sigma_1\delta^\star/\bar{p}\right] e^{-\mu_0(t-\hat{T}-t_\star)}.$$

Hence the assertion of Theorem 1.4 follows.

As a direct consequence of the asymptotic stability, we can prove the uniqueness of the smooth periodic traveling waves.

Proof of Theorem 1.5.

Without loss of generality, we assume that $U^{(2)}(x,t)$ is smooth. By Theorem 1.3, we know that $U_x^{(2)}(\cdot,\cdot) \gg \mathbf{0}$. Since $U^{(2)}(+\infty, t) = \mathbf{1}$ and $U^{(2)}(-\infty, t) = \mathbf{0}$ uniformly in t, we see that $\mathbf{0} \ll U^{(2)}(\cdot,\cdot) \ll \mathbf{1}$. By Theorem 1.4 (ii), there exists $\xi^\star \in \mathbb{R}$ such that

$$\sup_{x\in\mathbb{R}} \|U^{(1)}(x + c_1 t, t) - U^{(2)}(x + c_2 t + \xi^\star, t)\| \to 0 \text{ as } t \to \infty.$$

Hence,

$$\sup_{x\in\mathbb{R}} \|U^{(1)}(x+(c_1-c_2)t-\xi^\star, t) - U^{(2)}(x,t)\| \to 0 \text{ as } t\to\infty.$$

In particular,

$$\sup_{x\in\mathbb{R}} \|U^{(1)}(x+(c_1-c_2)nT-\xi^\star, nT) - U^{(2)}(x,nT)\| \to 0 \text{ as } n\to\infty.$$

By the periodicity of the traveling waves, we obtain

$$\sup_{x\in\mathbb{R}} \|U^{(1)}(x+(c_1-c_2)nT-\xi^\star, 0) - U^{(2)}(x,0)\| \to 0 \text{ as } n\to\infty. \tag{6.37}$$

Then, fix any $x_1\in\mathbb{R}$, it follows from (6.37) that

$$\|U^{(1)}(x_1+(c_1-c_2)nT-\xi^\star, 0) - U^{(2)}(x_1,0)\| \to 0 \text{ as } n\to\infty.$$

Since $U^{(1)}(+\infty,0)=\mathbf{1}$, $U^{(1)}(-\infty,0)=\mathbf{0}$ and $\mathbf{0}\ll U^{(2)}(x_1,0)\ll\mathbf{1}$, we deduce that $c:=c_1=c_2$. Thus, it follows from (6.37) that $U^{(1)}(\cdot,0) = U^{(2)}(\cdot+\xi^\star,0)$. Therefore, $U^{(1)}(x+ct,t) = U^{(2)}(x+ct+\xi^\star,t)$, $\forall x,\ t\in\mathbb{R}$. This completes the proof of Theorem 1.5.

7 Applications

In this section, we will apply our main results developed in Sections 2-3 to some epidemic and population models.

7.1 An epidemic model

For simplicity, we consider the following dimensionless epidemic system as (1.3), i.e.,

$$\begin{cases} u_t = du_{xx} - u(x,t) + \alpha v(x,t), \\ v_t = -\beta v(x,t) + g(u(x,t),t), \end{cases} \tag{7.1}$$

where $\alpha := a_{12}/a_{11}^2$ and $\beta := a_{22}/a_{11}$. Let us denote $f(w,t) := (-w_1+\alpha w_2, -\beta w_2 + g(w_1,t))$ and assume that

(M1) $d,\alpha,\beta>0$, $g(\cdot,\cdot)\in C^1([-1,\infty)\times[0,\infty))$, $g(0,\cdot)\equiv 0$, $g_u(u,t) := \dfrac{\partial g(u,t)}{\partial u} > 0$, $\forall (u,t)\in[-1,\infty)\times[0,\infty)$, and $g(\cdot,t+T)=g(\cdot,t)$ for some $T>0$.

(M2) The Poincaré map $P(\alpha) := w(T;\alpha)$, where $w(t;\alpha)$ is the solution to the ODE:

$$w'(t) = f(w(t),t),\ w(0)=\alpha\in\mathbb{R}^2, \tag{7.2}$$

has exactly three fixed points w^-, $\bar{w}$ and w^+ satisfying $w^- < \bar{w} < w^+$, $r(\mathrm{D}P(w^\pm))<1$ and $r(\mathrm{D}P(\bar{w}))>1$.

It is easy to verify that

$$Df(w,t) = \begin{pmatrix} -1 & \alpha \\ g_{w_1}(w_1,t) & -\beta \end{pmatrix}.$$

Thus, by the assumptions (M1) and (M2), we see that the assumptions (C1)–(C4) and (C4)′ hold for (7.1) hold. Therefore, the conclusions of Theorems 1.2–1.5 are valid for system (7.1).

7.2 A population model

Let us consider the population model (1.4) by taking $y_1(t)\equiv y_1$, $y_2(t)\equiv y_2$, and $f(u)=y(t)h(u)$ with $h(u) = u(1-u)(u-\frac{1}{2})$, i.e.,

$$\begin{cases} u_t = d_1u_{xx} + y(t)h(u) - y_1u(x,t) + y_2v(x,t), \\ v_t = y_1u(x,t) - y_2v(x,t), \end{cases} \tag{7.3}$$

where $d_1, \gamma_1, \gamma_2 > 0$, $\gamma(\cdot + T) = \gamma(\cdot) > 0$ for some $T > 0$.

For convenience, we denote $F(w, t) = (F_1(w, t), F_2(w, t))$, where $w = (w_1, w_2)$,

$$F_1(w, t) := \gamma(t)h(w_1) - \gamma_1 w_1 + \gamma_2 w_2 \text{ and } F_2(w, t) := \gamma_1 w_1 - \gamma_2 w_2.$$

Then the reaction system of (7.3) can be rewritten by

$$w'(t) = F(w(t), t). \tag{7.4}$$

It is clear that $(0, 0)$, $(1/2, \gamma_1/(2\gamma_2))$ and $(1, \gamma_1/\gamma_2)$ are T–periodic solutions of (7.4). Let $P(\alpha) := w(T; \alpha)$, where $w(t; \alpha)$ is the solution of system (7.4) with initial value $w(0; \alpha) = \alpha$. We can obtain the following result.

Theorem 7.1. *Assume $d_1 > 0$, $\gamma(\cdot + T) = \gamma(\cdot) > 0$ for some $T > 0$. Then the conclusions of* Theorems 1.2–1.5 *are valid for system* (7.3)*, that is,* (7.3) *admits a unique (up to translation) and globally stable (with phase shift) bistable time-periodic traveling front connecting* $(0, 0)$ *and* $(1, \gamma_1/\gamma_2)$.

Proof. By direct computations, we have $h'(w_1) = 3w_1 - 3w_1^2 - \frac{1}{2}$ and

$$DF(w, t) = \begin{pmatrix} \gamma(t)h'(w_1) - \gamma_1 & \gamma_2 \\ \gamma_1 & -\gamma_2 \end{pmatrix}.$$

Then, it is easy to see that (C1), (C3), (C4) and (C4)$'$ hold for system (7.3).

Next, we verify the bistable assumption (C2) for (7.3). We first prove that system (7.4) has exactly three T–periodic solutions: $(0, 0)$, $(1/2, \gamma_1/(2\gamma_2))$ and $(1, \gamma_1/\gamma_2)$. Suppose that $w_\star(t) = (u_\star(t), v_\star(t))$ is a T–periodic solution of system (7.4). It is clear that

$$\begin{cases} u_\star'(t) = \gamma(t)h(u_\star(t)) - \gamma_1 u_\star(t) + \gamma_2 v_\star(t), \\ v_\star'(t) = \gamma_1 u_\star(t) - \gamma_2 v_\star(t), \end{cases}$$

which implies that $u_\star'(t) + v_\star'(t) = \gamma(t)h(u_\star(t))$. Hence, we obtain

$$\int_0^T \gamma(t)h(u_\star(t))dt = 0. \tag{7.5}$$

Then we have the following

Claim: $0 \le u_\star(t) \le 1$ for all $t \in \mathbb{R}$.

We only prove that $u_\star(t) \le 1$ for all $t \in \mathbb{R}$, since the other assertion can be treated similarly. Suppose that there exists $t_1 \in \mathbb{R}$ such that $u_\star(t_1) > 1$. Take $\bar{w}(t) = (\bar{u}(t), \bar{v}(t))$, where

$$\bar{u}(t) \equiv M \text{ and } \bar{v}(t) \equiv M\gamma_1/\gamma_2, \ \forall t \in \mathbb{R}.$$

We can choose $M > 1$ sufficiently large such that $\bar{w}(t) \gg w_\star(t)$, $\forall t \in \mathbb{R}$. Moreover, it is clear that $\bar{w}(t)$ is an upper constant solution of (7.4). Let $w(t; \bar{w}(0)) = (u(t; \bar{w}(0)), v(t; \bar{w}(0)))$ be the solution of (7.4) with initial value $w(0; \bar{w}(0)) = \bar{w}(0)$. Since (7.4) is a monotone system, $\bar{w}(0) \ge (1, \gamma_1/\gamma_2)$ and $(1, \gamma_1/\gamma_2)$ is a solution of (7.4), it follows that

$$w(t; \bar{w}(0)) \ge w_\star(t) \text{ and } w(t; \bar{w}(0)) \ge (1, \gamma_1/\gamma_2), \ \forall t \in \mathbb{R}.$$

Let us define a sequence of functions $\{w_k(t)\}_{k\in\mathbb{N}}$ as follows:

$$w_k(t) = (u_k(t), v_k(t)) := w(t + kT; \bar{w}(0)), \ \forall t \in \mathbb{R}.$$

Since $\bar{w}(t)$ is an upper constant solution of (7.4), we have $w(T; \bar{w}(0)) \le \bar{w}(T)$. Then, for $k \in \mathbb{N}$,

$$w_{k+1}(t) = w(t + (k + 1)T; \bar{w}(0)) = w(t + kT; w(T; \bar{w}(0)))$$

$$\leq w(t+kT;\bar{w}(T)) = w(t+kT;\bar{w}(0)) = w_k(t),\ \forall t \in \mathbb{R}.$$

By this monotonicity, we can define $\tilde{w}(t) = (\tilde{u}(t), \tilde{v}(t)) := \lim_{k\to\infty} w_k(t), \forall t \in \mathbb{R}$. It is easy to see that $w_k(t)$ satisfies

$$\begin{cases} u_k'(t) = y(t)h(u_k(t)) - y_1u_k(t) + y_2v_k(t), \\ v_k'(t) = y_1u_k(t) - y_2v_k(t), \end{cases}$$

which yields that

$$\begin{cases} u_k(t) = u_k(0) + \int_0^t [y(s)h(u_k(s)) - y_1u_k(s) + y_2v_k(s)]ds, \\ v_k(t) = v_k(0) + \int_0^t [y_1u_k(s) - y_2v_k(s)]ds. \end{cases}$$

Taking $k \to \infty$, we obtain

$$\begin{cases} \tilde{u}(t) = \tilde{u}(0) + \int_0^t [y(s)h(\tilde{u}(s)) - y_1\tilde{u}(s) + y_2\tilde{v}(s)]ds, \\ \tilde{v}(t) = \tilde{v}(0) + \int_0^t [y_1\tilde{u}(s) - y_2\tilde{v}(s)]ds. \end{cases}$$

This deduces that $\tilde{w}(t)$ is a solution of (7.4). Moreover,

$$\tilde{w}(T) = \lim_{k\to\infty} w_k(T) = \lim_{k\to\infty} w((k+1)T;\bar{w}(0)) = \lim_{k'\to\infty} w(k'T;\bar{w}(0)) = \tilde{w}(0).$$

Hence $\tilde{w}(t)$ is a T–periodic solutions of (7.4) which satisfies

$$\tilde{w}(t) \geq w_\star(t) \text{ and } \tilde{w}(t) \geq (1, y_1/y_2), \forall t \in \mathbb{R}.$$

Since $\int_0^T y(t)h(\tilde{u}(t))dt = 0$, $y(\cdot) > 0$ and $h(u) = u(1-u)(u-\frac{1}{2})$, we conclude that $\tilde{w}(t) \equiv (1, y_1/y_2)$, and hence $u_\star(t) \leq \tilde{u}(t) \leq 1$ for all $t \in \mathbb{R}$, which contradicts to $u_\star(t_1) > 1$. Therefore, the claim holds.

Next, we prove that $w_\star(t) = (0,0)$, or $(1/2, y_1/(2y_2))$ or $(1, y_1/y_2)$. By (7.5), we see that if $h(u_\star(t))$ does not change sign on $[0,T]$, then $h(u_\star(t)) \equiv 0$. Further, there exists $t_0 \in [0,T]$ such that $u_\star(t_0) = 0$ or $1/2$ or 1. We consider the following three cases.

Case 1: $u_\star(t_0) = 1/2$. If $v_\star(t_0) > y_1/(2y_2)$, then $w_\star(t) \geq (1/2, y_1/(2y_2))$ which yields that $1/2 \leq u_\star(t) \leq 1$ for all $t \in \mathbb{R}$. It then follows from (7.5) that $u_\star(t) \equiv 1/2$, and hence $w_\star(t) = (1/2, y_1/(2y_2))$. If $v_\star(t_0) \leq y_1/(2y_2)$, then $w_\star(t) \leq (1/2, y_1/(2y_2))$, and hence $0 \leq u_\star(t) \leq 1/2$. Using (7.5) again, we obtain $u_\star(t) \equiv 1/2$. Thus, $w_\star(t) = (1/2, y_1/(2y_2))$.

Case 2: $u_\star(t_0) = 0$. If there exists $t_1 \in \mathbb{R}$ such that $u_\star(t_1) = 1/2$, then follows from Case 1 that $w_\star(t) = (1/2, y_1/(2y_2))$. Otherwise, $0 \leq u_\star(t) < 1/2$ for all $t \in \mathbb{R}$. Hence, (7.5) implies that $w_\star(t) = (0,0)$.

Case 3: $u_\star(t_0) = 1$. Similar to Case 2, we can show that $w_\star(t) = (1/2, y_1/(2y_2))$ or $(1, y_1/y_2)$.

From above discussions, we conclude that $w_\star(t) = (0,0)$, or $(1/2, y_1/(2y_2))$ or $(1, y_1/y_2)$, that is, system (7.4) has exactly three T–periodic solutions: $(0,0)$, $(1/2, y_1/(2y_2))$ and $(1, y_1/y_2)$.

In the sequel, we set $\omega^- := (0,0)$, $\bar{\omega} := (1/2, y_1/(2y_2))$, $\omega^+ := (1, y_1/y_2)$. Note that $h'(0) = h'(1) = -1/2$. Then, $r(\mathrm{D}P(\omega^-) = r(\mathrm{D}P(\omega^+))$ is the principal eigenvalue of the strongly positive matrix $S(T)$, where $S(t)$ is the fundamental matrix solution of the following linear system with $S(0) = I$:

$$\begin{cases} u' = -(\frac{1}{2}y(t) + y_1)u + y_2v, \\ v' = y_1u - y_2v. \end{cases} \tag{7.6}$$

Let $\mu_1 := \frac{1}{T}\ln r(\mathrm{D}P(\omega^-))$. Form [33, Lemma 2.1], there exists a positive, T–periodic function $\hat{w}(t) = (\hat{w}_1(t), \hat{w}_2(t))$ such that $e^{\mu_1 t}\hat{w}(t)$ is a solution of system (7.6). Thus, we have

$$\begin{cases} \mu_1\hat{w}_1(t) + \hat{w}_1'(t) = -(\frac{1}{2}y(t) + y_1)\hat{w}_1(t) + y_2\hat{w}_2(t), \\ \mu_1\hat{w}_2(t) + \hat{w}_2'(t) = y_1\hat{w}_1(t) - y_2\hat{w}_2(t), \end{cases}$$

which yields that

$$\mu_1(\hat{w}_1(t) + \hat{w}_2(t)) + \hat{w}_1'(t) + \hat{w}_2'(t) = -\frac{1}{2}y(t)\hat{w}_1(t).$$

We deduce that

$$\mu_1\int_0^T(\hat{w}_1(t) + \hat{w}_2(t))dt = -\frac{1}{2}\int_0^T y(t)\hat{w}_1(t)dt < 0.$$

Hence, $\mu_1 = \frac{1}{T}\ln r(\mathrm{D}P(\omega^-)) < 0$ and therefore $r(\mathrm{D}P(\omega^-) = r(\mathrm{D}P(\omega^+)) < 1$. Similarly, we can show that $r(\mathrm{D}P(\bar{\omega})) > 1$. This verifies (C2) for system (7.3). The proof is completed. □

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