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Existence for (p, q) critical systems in the Heisenberg group

Abstract: This paper deals with the existence of entire nontrivial solutions for critical quasilinear systems $(\mathbb{S})$ in the Heisenberg group $\mathbb{H}^n$, driven by general (p, q) elliptic operators of Marcellini types. The study of $(\mathbb{S})$ requires relevant topics of nonlinear functional analysis because of the lack of compactness. The key step in the existence proof is the concentration–compactness principle of Lions, here proved for the first time in the vectorial Heisenberg context. Finally, the constructed solution has both components nontrivial and the results extend to the Heisenberg group previous theorems on quasilinear (p, q) systems.

1 Introduction

In recent years, great attention has been focused on the study of (p, q) systems, not only for their mathematical interest, but also for their relevant physical interpretation in applied sciences. It is also well known that the Heisenberg group $\mathbb{H}^n$, $n = 1, 2, \ldots$, appears in various areas, such as quantum theory (uncertainty principle, commutation relations) cf. [1, 2], signal theory cf. [3], theory of theta functions cf. [1, 4], and number theory. For additional physical interpretations we mention [5], while for general motivations in setting problems in the Heisenberg group context we refer to [6–11] and the papers cited there.

Here we prove the existence of nontrivial solutions for quasilinear elliptic systems in the Heisenberg group $\mathbb{H}^n$, involving (p, q) operators, which generalize the ones introduced by Marcellini in [12]. In particular, we consider the system in $\mathbb{H}^n$

$$\begin{cases} -\mathrm{div}_H\big(A(|D_H u|_H)D_H u\big) + B(|u|)u = \lambda H_u(u, v) + \dfrac{\alpha}{\wp^\star}|v|^\beta|u|^{\alpha-2}u, \\ -\mathrm{div}_H\big(A(|D_H v|_H)D_H v\big) + B(|v|)v = \lambda H_v(u, v) + \dfrac{\beta}{\wp^\star}|u|^\alpha|v|^{\beta-2}v, \end{cases} \tag{$\mathbb{S}$}$$

where λ is a positive real parameter, $Q = 2n + 2$ is the homogeneous dimension of the Heisenberg group $\mathbb{H}^n$, $\alpha > 1$ and $\beta > 1$ are two exponents such that $\alpha + \beta = \wp^\star$ and $\wp^\star$ is a critical exponent associated to $\wp$, with $1 < \wp < Q$, that is

$$\wp^\star = \frac{\wp Q}{Q - \wp},$$

which is related to the (p, q) operator A in $(\mathbb{S})$. The vector

$$D_H u = (X_1 u, \cdots, X_n u, Y_1 u, \cdots, Y_n u)$$

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denotes the horizontal gradient of u, where $\{X_j, Y_j\}_{j=1}^n$ is the basis of the horizontal left invariant vector fields on $\mathbb{H}^n$, that is

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}$$

for $j = 1, \ldots, n$.

The starting point is the paper [13], where the authors studied similar and more general systems in the Euclidean context. The main novelty of the paper is indeed to properly set (S) in the Heisenberg context. In fact, several theorems have to be proved in the new framework for the first time. Indeed, the key existence argument relies on the celebrated Lemma I.1 of [14] as well on the concentration–compactness principle in the vectorial Heisenberg context, both due to *Lions*. Following [13], we require the structure conditions.

(A) *A is a strictly positive and strictly increasing function of class $C^1(\mathbb{R}^+)$,*

(B) *$B \in C(\mathbb{R}^+)$ is a strictly positive function and $t \mapsto tB(t)$ is strictly increasing in $\mathbb{R}^+$, with $tB(t) \to 0$ as $t \to 0^+$.*

For simplicity, we introduce the functions $\mathcal{A}$ and $\mathcal{B}$, which are 0 at 0 and which are obtained by integration from

$$\mathcal{A}'(t) = tA(t), \quad \mathcal{B}'(t) = tB(t) \quad \text{for all } t \in \mathbb{R}_0^+.$$

Notice that (A) implies that $tA(t) \to 0$ as $t \to 0^+$, and so $tA(t)$ and $tB(t)$ are defined to be 0 at 0. We furthermore assume

(C_1) *there exist constants $a_0, \mathfrak{a}_0, b_0, \mathfrak{b}_0$ strictly positive, with $a_0 \le 1$, $a_1, \mathfrak{a}_1, b_1, \mathfrak{b}_1$ nonnegative, with the property that $\mathfrak{a}_1 > 0$ implies $\mathfrak{b}_1 > 0$, $a_1 > 0$ and $b_1 > 0$, and there are exponents p and q, with $1 < p < q < \wp^\star$, where $1 < \wp < Q$, $\wp = p$ if $\mathfrak{a}_1 = 0$ and $\wp = q$ if $\mathfrak{a}_1 > 0$, such that for all $t \in \mathbb{R}_0^+$*

$$a_0 t^{p-1} + \mathbb{1}_{\mathbb{R}^+}(\mathfrak{a}_1)a_1 t^{q-1} \le \mathcal{A}'(t) \le \mathfrak{a}_0 t^{p-1} + \mathfrak{a}_1 t^{q-1},$$
$$b_0 t^{p-1} + \mathbb{1}_{\mathbb{R}^+}(\mathfrak{b}_1)b_1 t^{q-1} \le \mathcal{B}'(t) \le \mathfrak{b}_0 t^{p-1} + \mathfrak{b}_1 t^{q-1},$$

where $\mathbb{1}_U$ is the characteristic function of a Lebesgue measurable subset U of $\mathbb{R}$. Assumption (C_1) was introduced by *Figueiredo* in [15]. Moreover, we assume

(C_2) *there exist constants θ and ϑ, with $\wp \le \min\{\theta, \vartheta\} < \wp^\star$, such that*

$$\theta\mathcal{A}(t) \ge t\mathcal{A}'(t), \quad \vartheta\mathcal{B}(t) \ge t\mathcal{B}'(t) \quad \text{for all } t \in \mathbb{R}_0^+$$

holds.

Several general systems verify all the assumptions (A), (B), (C_1) and (C_2), and we refer to [13] for the main prototypes of the potentials $\mathcal{A}$ and $\mathcal{B}$ covered.

The functions H_u and H_v in (S) are partial derivatives of *a function H of class $C^1(\mathbb{R}^2)$, satisfying the condition*

(H) *$H > 0$ in $\mathbb{R}^+ \times \mathbb{R}^+$, $H_u(u, 0) = 0$ for all $u \in \mathbb{R}$ and $H_v(0, v) = 0$ for all $v \in \mathbb{R}$. Furthermore, there exist $\mathfrak{m}, m, \sigma$ such that $\wp < \mathfrak{m} < m < \wp^\star$, $\max\{\theta, \vartheta\} < \sigma < \wp^\star$ and for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ for which the inequality*

$$|\nabla H(u, v)| \le \mathfrak{m}\varepsilon|(u, v)|^{\mathfrak{m}-1} + mC_\varepsilon|(u, v)|^{m-1} \quad \text{for any } (u, v) \in \mathbb{R}^2, \tag{1.1}$$

where $|(u, v)| = \sqrt{u^2 + v^2}$, $\nabla H = (H_u, H_v)$, and also the inequalities

$$0 \le \sigma H(u, v) \le \nabla H(u, v) \cdot (u, v) \quad \text{for all } (u, v) \in \mathbb{R}^2,$$

hold, where θ, ϑ are given in (C_2).

Throughout the paper, $\cdot$ denotes the Euclidean inner product and $|\cdot|$ the corresponding Euclidean norm in any space $\mathbb{R}^m$, $m = 1, 2, \ldots$.

Since $\wp = p$ if $\mathfrak{a}_1 = 0$, while $\wp = q$ if $\mathfrak{a}_1 > 0$, the natural space where finding solutions of (S) is

$$W = \left(HW^{1,p}(\mathbb{H}^n) \cap HW^{1,\wp}(\mathbb{H}^n)\right) \times \left(HW^{1,p}(\mathbb{H}^n) \cap HW^{1,\wp}(\mathbb{H}^n)\right),$$

endowed with the norm

$$\|(u, v)\| = \|u\|_{HW^{1,p}} + \|v\|_{HW^{1,p}} + \mathbb{1}_{\mathbb{R}^+}(\mathfrak{a}_1)\big(\|u\|_{HW^{1,q}} + \|v\|_{HW^{1,q}}\big)$$

for all $u \in HW^{1,\mathfrak{p}}(\mathbb{H}^n)$, where $HW^{1,\mathfrak{p}}(\mathbb{H}^n)$ is the horizontal Sobolev space defined in Section 2. We are now able to state the existence result for $(\mathbb{S})$.

Theorem 1.1. *Suppose that* (A), (B), (C_1), (C_2) *and* (H) *hold. Then, there exists* $\lambda^\star > 0$ *such that for all* $\lambda \geq \lambda^\star$ *the system* $(\mathbb{S})$ *admits at least one solution* (u_λ, v_λ) *in* W. *Moreover,* (u_λ, v_λ) *has each component nontrivial and*

$$\lim_{\lambda \to \infty} \|(u_\lambda, v_\lambda)\| = 0. \tag{1.2}$$

Since the solution (u_λ, v_λ), constructed in Theorem 1.1, has both components non trivial, it is evident that it solves an actual system, which does not reduce into an equation. Moreover, Theorem 1.1 extends in several directions previous results, not only from the Euclidean to the Heisenberg setting, but also for the mild growth conditions on the main elliptic operator of $(\mathbb{S})$, cf. e.g. [16–19].

Even if assumption (C_1) allows us to treat simultaneously when either $\mathfrak{a}_1 = 0$ or $\mathfrak{a}_1 > 0$, the most interesting case is the latter, in which $\wp = q$ and so the couple (p, q) appears in its importance. Indeed, when $\mathfrak{a}_1 > 0$ in (C_1), the main elliptic operator A has a (p, q) growth. Moreover, in this case, the solution space W has a strong dependence on (p, q), since we consider existence of entire solutions in the Heisenberg group. In fact, (p, q) problems are usually settled in bounded domains Ω, so that the natural solution space is $W = HW_0^{1,p}(\Omega) \cap HW_0^{1,q}(\Omega) = HW_0^{1,q}(\Omega)$. In this paper the situation is much more delicate, since the problem is in the entire group of Heisenberg.

The importance of studying problems involving operators with non standard growth conditions, or (p, q) operators, begins with the papers of Marcellini [12] and Zhikov [20]. Since then, the topic has been attracting increasing attention on existence and qualitative properties of solutions, but the vectorial case is much harder. Indeed, $(\mathbb{S})$ has a relevant physical interpretation in applied sciences as well as a mathematical challenge in overcoming the new difficulties intrinsic to $(\mathbb{S})$. Because of the lack of compactness, the main difficulty in treating (p, q) systems in our context relies on the proof of the key Lemma 4.6, dedicated on the crucial properties of the Palais–Smale sequences at special levels. To this aim, we prove a concentration compactness principle for systems in $S = S^{1,\wp}(\mathbb{H}^n) \times S^{1,\wp}(\mathbb{H}^n)$, where $S^{1,\wp}(\mathbb{H}^n)$, $1 < \wp < Q$, is the Folland–Stein space, that is the completion of $C_c^\infty(\mathbb{H}^n)$ with respect to the norm

$$\|D_H u\|_{L^\wp(\mathbb{H}^n)} = \left(\int_{\mathbb{H}^n} |D_H u|_H^\wp d\xi \right)^{1/\wp}.$$

Theorem 1.2. *Let* $\{(u_k, v_k)\}_k$ *be a sequence in* S *and assume that there exist* $(u, v) \in S$ *and two bounded nonnegative Radon measures* μ *and* ν *on* $\mathbb{H}^n$, *such that*

$$(u_k, v_k) \rightharpoonup (u, v) \text{ in } S,$$

$$\big(|D_H u_k|_H^\wp + |D_H v_k|_H^\wp\big) d\xi \overset{\star}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\mathbb{H}^n), \tag{1.3}$$

$$|u_k|^\alpha |v_k|^\beta d\xi \overset{\star}{\rightharpoonup} \nu \quad \text{in } \mathcal{M}(\mathbb{H}^n),$$

where $\mathcal{M}(\mathbb{H}^n)$ *is the space of all bounded regular Borel measures on* $\mathbb{H}^n$. *Then, there exist an at most countable set* J, *a family of points* $\{\xi_j\}_{j \in J} \subset \mathbb{H}^n$ *and two families of nonnegative numbers* $\{\mu_j\}_{j \in J}$ *and* $\{\nu_j\}_{j \in J}$ *such that*

$$\nu = |u|^\alpha |v|^\beta d\xi + \sum_{j \in J} \nu_j \delta_{\xi_j}, \quad \mu \geq \big(|D_H u|_H^\wp + |D_H v|_H^\wp\big) d\xi + \sum_{j \in J} \mu_j \delta_{\xi_j},$$

$$\nu_j^{\wp/\wp^\star} \leq \frac{\mu_j}{\mathcal{I}} \text{ for all } j \in J, \quad \text{where} \quad \mathcal{I} = \inf_{\substack{(u,v) \in S \\ (u \neq 0) \wedge (v \neq 0)}} \frac{\|D_H u\|_\wp^\wp + \|D_H v\|_\wp^\wp}{\left(\displaystyle\int_{\mathbb{H}^n} |u|^\alpha |v|^\beta d\xi \right)^{\wp/\wp^\star}}$$

and δ_{ξ_j} *is the Dirac function at the point* ξ_j *of* $\mathbb{H}^n$.

To the best of our knowledge, the conclusions obtained in Theorem 1.2 are new in the Heisenberg context. The proof of this result follows somehow the arguments of [21] and [22, 23], but there are some technical difficulties due to the more general context, which we overcome.

Finally, the existence of solutions for problem $\mathbb{S}$ rely on a readaptation of Proposition 2.8 of [13] in the Heisenberg group. Therefore, we have to prove an extension from the Euclidean to the Heisenberg context of the celebrated Lemma I.1 in [14] due to Lions, which is given in its general statement.

Theorem 1.3. *Let $1 \le p < \infty$ and $1 < \wp < Q$, with $p \ne \wp^\star$. Assume that $(u_k)_k$ is bounded in $L^p(\mathbb{H}^n)$, $(u_k)_k$ is bounded in $S^{1,\wp}(\mathbb{H}^n)$ and there exists $R > 0$ such that*

$$\lim_{k \to \infty} \sup_{\eta \in \mathbb{H}^n} \int_{B_R(\eta)} |u_k|^p d\xi = 0. \tag{1.4}$$

Then, $u_k \to 0$ in $L^{\mathfrak{p}}(\mathbb{H}^n)$ as $k \to \infty$ for all $\mathfrak{p}$ between p and $\wp^\star$.

The paper is organized as follows. In Section 2, we recall some fundamental definitions and properties related to the Heisenberg group $\mathbb{H}^n$. Section 3 is devoted to the proof of Theorem 1.2, while Section 4 deals with some lemmas useful to the study of system ($\mathbb{S}$). In particular, we prove Theorem 1.3 and finally Theorem 1.1, adapting the strategy of [13] and extending the results there to the Heisenberg group setting.

2 Preliminaries

In this section we present the basic properties of $\mathbb{H}^n$ as a Lie group. Analysis on the Heisenberg group is very interesting because this space is topologically Euclidean, but analytically non–Euclidean, and so some basic ideas of analysis, such as dilatations, must be developed again. One of the main differences with the Euclidean case is that the homogeneous dimension $Q = 2n + 2$ of the Heisenberg group plays a role analogous to the topological dimension in the Euclidean context. For a complete treatment, we refer to [24–27].

Let $\mathbb{H}^n$ be the Heisenberg Lie group of topological dimension $2n + 1$, that is the Lie group which has $\mathbb{R}^{2n+1}$ as a background manifold, endowed with the non–Abelian group law

$$\xi \circ \xi' = \left(z + z', t + t' + 2 \sum_{i=1}^{n} (y_i x_i' - x_i y_i') \right) \tag{2.1}$$

for all $\xi, \xi' \in \mathbb{H}^n$, with

$$\xi = (z, t) = (x_1, \ldots, x_n, y_1, \ldots, y_n, t) \text{ and } \xi' = (z', t') = (x_1', \ldots, x_n', y_1', \ldots, y_n', t').$$

The inverse is given by $\xi^{-1} = -\xi$ and so $(\xi \circ \xi')^{-1} = (\xi')^{-1} \circ \xi^{-1}$.

The vector fields for $j = 1, \ldots, n$

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t},$$

constitute a basis for the real Lie algebra of left–invariant vector fields on $\mathbb{H}^n$. This basis satisfies the Heisenberg canonical commutation relations

$$[X_j, Y_k] = -4\delta_{jk} T, \quad [Y_j, Y_k] = [X_j, X_k] = [Y_j, T] = [X_j, T] = 0.$$

A left invariant vector field X, which is in the span of $\{X_j, Y_j\}_{j=1}^n$, is called *horizontal*.

We define the *horizontal gradient* of a C^1 function $u : \mathbb{H}^n \to \mathbb{R}$ by

$$D_H u = \sum_{j=1}^{n} \left[(X_j u) X_j + (Y_j u) Y_j \right].$$

Clearly, $D_H u$ is an element of the span of $\{X_j, Y_j\}_{j=1}^n$. Furthermore, if $f \in C^1(\mathbb{R})$, then

$$D_H f(u) = f'(u) D_H u.$$

In the span of $\{X_j, Y_j\}_{j=1}^n \simeq \mathbb{R}^{2n}$ we consider the natural inner product given by

$$(X, Y)_H = \sum_{j=1}^n \left(x^j y^j + \widetilde{x}^j \widetilde{y}^j \right)$$

for $X = \{x^j X_j + \widetilde{x}^j Y_j\}_{j=1}^n$ and $Y = \{y^j X_j + \widetilde{y}^j Y_j\}_{j=1}^n$. The inner product $(\cdot, \cdot)_H$ produces the Hilbertian norm

$$|X|_H = \sqrt{(X, X)_H}$$

for the horizontal vector field X. Moreover, the Cauchy–Schwarz inequality

$$\left| (X, Y)_H \right| \leq |X|_H |Y|_H$$

holds for any horizontal vector fields X and Y.

For any horizontal vector field function $X = X(\xi)$, $X = \{x^j X_j + \widetilde{x}^j Y_j\}_{j=1}^n$, of class $C^1(\mathbb{H}^n, \mathbb{R}^{2n})$, we define the *horizontal divergence* of X by

$$\text{div}_H X = \sum_{j=1}^n [X_j(x^j) + Y_j(\widetilde{x}^j)].$$

If furthermore $u \in C^1(\mathbb{H}^n)$, then the *Leibnitz formula* continues to be valid, that is

$$\text{div}_H(uX) = u \, \text{div}_H X + \left(D_H u, X \right)_H.$$

Similarly, if $u \in C^2(\mathbb{H}^n)$, then the *Kohn–Spencer Laplacian*, or equivalently the *horizontal Laplacian in $\mathbb{H}^n$*, of u is defined as follows

$$\Delta_H u = \sum_{j=1}^n (X_j^2 + Y_j^2) u$$

$$= \sum_{j=1}^n \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j \frac{\partial^2}{\partial x_j \partial t} - 4x_j \frac{\partial^2}{\partial y_j \partial t} \right) u + 4|z|^2 \frac{\partial^2 u}{\partial t^2}.$$

According to the celebrated Theorem 1.1 due to Hörmander in [28], the operator Δ_H is *hypoelliptic*. In particular, $\Delta_H u = \text{div}_H D_H u$ for each $u \in C^2(\mathbb{H}^n)$.

A well known generalization of the Kohn–Spencer Laplacian is the *horizontal $\mathfrak{p}$–Laplacian* on the Heisenberg group, $\mathfrak{p} \in (1, \infty)$, defined by

$$\Delta_{H, \mathfrak{p}} \varphi = \text{div}_H(|D_H \varphi|_H^{\mathfrak{p}-2} D_H \varphi)$$

for all $\varphi \in C_c^\infty(\mathbb{H}^n)$.

The *Korányi norm* is given by

$$r(\xi) = r(z, t) = (|z|^4 + t^2)^{1/4} \quad \text{for all } \xi = (z, t) \in \mathbb{H}^n.$$

The corresponding distance, the so called *Korányi distance*, is

$$d_K(\xi, \xi') = r(\xi^{-1} \circ \xi') \quad \text{for all } (\xi, \xi') \in \mathbb{H}^n \times \mathbb{H}^n.$$

This distance acts like the Euclidean distance in horizontal directions and behaves like the square root of the Euclidean distance in the missing direction. Consequently, the Korányi norm is homogeneous of degree 1, with respect to the dilations $\delta_R : (z, t) \mapsto (Rz, R^2 t)$, $R > 0$, since

$$r(\delta_R(\xi)) = r(Rz, R^2 t) = (|Rz|^4 + R^4 t^2)^{1/4} = R \, r(\xi)$$

for all $\xi = (z, t) \in \mathbb{H}^n$.

Let $B_R(\xi_0) = \{\xi \in \mathbb{H}^n : d_K(\xi, \xi_0) < R\}$ be the *Korányi open ball* of radius R centered at ξ_0. For simplicity B_R denotes the ball of radius R centered at $\xi_0 = O$, where $O = (0, 0)$ is the natural origin of $\mathbb{H}^n$.

It is easy to verify that the Jacobian determinant of dilatations $\delta_R : \mathbb{H}^n \to \mathbb{H}^n$ is constant and equal to R^{2n+2}. This is why the natural number $Q = 2n + 2$ is called *homogeneous dimension* of $\mathbb{H}^n$.

We recall also the definition of *Carnot–Carathéodory distance* on $\mathbb{H}^n$ and for further details we refer to [7, 25]. A piecewise smooth curve $y : [0, 1] \to \mathbb{H}^n$ is called a *horizontal curve* if $\dot{y}(t)$ belongs to the span of $\{X_j, Y_j\}_{j=1}^n$ a.e. in $[0, 1]$. The *horizontal length* of y is defined as

$$\mathrm{L}_H(y) = \int\limits_0^1 \sqrt{\left(\dot{y}(t), \dot{y}(t)\right)_H} dt = \int\limits_0^1 |\dot{y}(t)|_H dt.$$

Now, given two arbitrary points $\xi, \eta \in \mathbb{H}^n$, by the Chow–Rashevsky theorem there is a horizontal curve between them in $\mathbb{H}^n$, see [29, 30]. Therefore, the Carnot–Carathéodory distance of two points ξ and η of $\mathbb{H}^n$ is well–defined as

$$d_{CC}(\xi, \eta) = \inf\{\mathrm{L}_H(y) : y \text{ is a horizontal curve joining } \xi \text{ and } \eta \text{ in } \mathbb{H}^n\}.$$

Clearly, d_{CC} is a left invariant metric on $\mathbb{H}^n$, and

$$d_{CC}(\xi, \eta) = d_{CC}(\eta^{-1} \circ \xi, 0)$$

for all $\xi, \eta \in \mathbb{H}^n$, see [7]. Moreover, the Carnot–Carathéodory distance is homogeneous of degree 1 with respect to dilatations δ_R, that is

$$d_{CC}(\delta_R(\xi), \delta_R(\eta)) = R \, d_{CC}(\xi, \eta)$$

for all $\xi, \eta \in \mathbb{H}^n$.

In the case of the Heisenberg group, it is easy to check that the Lebesgue measure on $\mathbb{R}^{2n+1}$ is invariant under left translations. Thus, from here on, we denote by $d\xi$ the Haar measure on $\mathbb{H}^n$ that coincides with the $(2n+1)$–Lebesgue measure, since the Haar measures on Lie groups are unique up to constant multipliers. We also denote by $|U|$ the $(2n+1)$–dimensional Lebesgue measure of any measurable set $U \subset \mathbb{H}^n$. Furthermore, the Haar measure on $\mathbb{H}^n$ is Q–homogeneous with respect to dilations δ_R. Consequently,

$$|\delta_R(U)| = R^Q|U|, \qquad d(\delta_R \xi) = R^Q d\xi.$$

In particular $|B_R| = |B_1|R^Q$.

As usual, for any measurable set $U \subset \mathbb{H}^n$ and for any general exponent $\mathfrak{p}$, with $1 \le \mathfrak{p} \le \infty$, we denote by $L^{\mathfrak{p}}(U)$ the canonical Banach space, endowed with the norm

$$\|u\|_{L^{\mathfrak{p}}(U)} = \left(\int\limits_U |u|^{\mathfrak{p}} d\xi\right)^{1/\mathfrak{p}}, \quad \text{if } 1 \le \mathfrak{p} < \infty,$$

while

$$\|u\|_{L^{\infty}(U)} = \operatorname*{ess\,sup}_U u = \inf\{M : |u(\xi)| \le M \text{ for a.e. } \xi \in U\}.$$

When $U = \mathbb{H}^n$ or when there is not ambiguity about the set considered, for simplicity we denote the norm $\|\cdot\|_{\mathfrak{p}}$. All the usual properties about the Lebesgue Banach spaces continue to be valid. In particular, $L^{\mathfrak{p}}(U)$ is a separable Banach space and $C_c(U)$ is dense in it if $1 \le \mathfrak{p} < \infty$. Moreover, $L^{\mathfrak{p}}(U)$ is a reflexive Banach space if $1 < \mathfrak{p} < \infty$.

Let us now review some classical facts about the first–order Sobolev spaces on the Heisenberg group $\mathbb{H}^n$. We restrict ourselves to the special case in which $1 \le \mathfrak{p} < \infty$ and Ω is an open set in $\mathbb{H}^n$. Denote by

$HW^{1,\mathfrak{p}}(\Omega)$ the *horizontal Sobolev space* consisting of the functions $u \in L^{\mathfrak{p}}(\Omega)$ such that $D_H u$ exists in the sense of distributions and $|D_H u|_H \in L^{\mathfrak{p}}(\Omega)$, endowed with the natural norm

$$\|u\|_{HW^{1,\mathfrak{p}}(\Omega)} = \left(\|u\|^{\mathfrak{p}}_{L^{\mathfrak{p}}(\Omega)} + \|D_H u\|^{\mathfrak{p}}_{L^{\mathfrak{p}}(\Omega)} \right)^{1/\mathfrak{p}}, \quad \|D_H u\|_{L^{\mathfrak{p}}(\Omega)} = \left(\int_{\Omega} |D_H u|^{\mathfrak{p}}_H d\xi \right)^{1/\mathfrak{p}}.$$

It is easy to check that the distributional horizontal gradient of a function $u \in HW^{1,\mathfrak{p}}(\Omega)$ is uniquely defined a.e. in Ω. Furthermore, if u is a smooth function, then its classical horizontal gradient is also the distributional horizontal gradient. For this reason, if u is a non smooth function, $D_H u$ is meant in the distributional sense.

For later purposes, let us introduce the convolution, which is useful also for density results, see [31, 32]. If $u \in L^1(\mathbb{H}^n)$ and $v \in L^{\mathfrak{p}}(\mathbb{H}^n)$, with $1 \le \mathfrak{p} \le \infty$, then for a.e. $\xi \in \mathbb{H}^n$ the function

$$\eta \mapsto u(\xi \circ \eta^{-1})v(\eta)$$

is in $L^1(\mathbb{H}^n)$. Moreover, $u \star v$, defined a.e. on $\mathbb{H}^n$ by

$$(u \star v)(\xi) = \int_{\mathbb{H}^n} u(\xi \circ \eta^{-1})v(\eta)d\eta, \tag{2.2}$$

is called *convolution* of u and v. By the analog of the Young theorem $u \star v$ belongs to $L^{\mathfrak{p}}(\mathbb{H}^n)$ and

$$\|u \star v\|_{\mathfrak{p}} \le \|u\|_1 \|v\|_{\mathfrak{p}}. \tag{2.3}$$

If $\mathfrak{p} = \infty$, then $u \star v$ is well defined and uniformly continuous in $\mathbb{H}^n$.

Using the convolution (2.2), the technique of regularization, originally introduced by *Leray* and *Friedrichs* in the Euclidean context, can be extended to the Heisenberg group $\mathbb{H}^n$. In particular, it is possible to generate a sequence of mollifiers $(\rho_k)_k$ on $\mathbb{H}^n$ with the usual properties, see the Appendix of [33]. Moreover, Proposition A.1.2 of [33] yield that if $\varphi \in C_c^{\infty}(\mathbb{H}^n)$ and $u \in L^1_{\text{loc}}(\mathbb{H}^n)$ then $u \star \varphi \in C^{\infty}(\mathbb{H}^n)$ and

$$X_j(u \star \varphi) = u \star X_j \varphi, \qquad Y_j(u \star \varphi) = u \star Y_j \varphi, \quad j = 1, \dots, n. \tag{2.4}$$

Lemma 2.1. *Let* $u \in L^1(\mathbb{H}^n)$, $v \in L^{\mathfrak{q}}(\mathbb{H}^n)$, *such that* $D_H v$ *exists in the sense of distributions and* $D_H v \in L^{\mathfrak{p}}(\mathbb{H}^n, \mathbb{R}^{2n})$, *with* $1 \le \mathfrak{p}, \mathfrak{q} < \infty$. *Then* $D_H(u \star v)$ *exists in the sense of distributions,* $D_H(u \star v) \in L^{\mathfrak{p}}(\mathbb{H}^n, \mathbb{R}^{2n})$, *and*

$$X_j(u \star v) = u \star X_j v, \qquad Y_j(u \star v) = u \star Y_j v, \quad j = 1, \dots, n,$$

in the sense of distributions. In particular, if $\mathfrak{q} = \mathfrak{p}$, *then* $u \star v \in HW^{1,\mathfrak{p}}(\mathbb{H}^n)$.

Proof. Let u and v be as in the statement and divide the proof into two cases.

Case 1. $u \in C_c^{\infty}(\mathbb{H}^n)$. Fix $\varphi \in C_c^{\infty}(\mathbb{H}^n)$ and $j = 1, \dots, n$. Since $u \in L^1(\mathbb{H}^n)$, $v \in L^{\mathfrak{q}}(\mathbb{H}^n)$ and $X_j \varphi \in L^{\mathfrak{q}'}(\mathbb{H}^n)$, Lemma A.1.3 of [33] yields

$$\int_{\mathbb{H}^n} (u \star v)X_j \varphi \, d\xi = \int_{\mathbb{H}^n} (\breve{u} \star X_j \varphi)v \, d\xi, \quad \breve{u}(\xi) = u(\xi^{-1}), \ \xi \in \mathbb{H}^n. \tag{2.5}$$

Likewise, since $\breve{u} \in L^1(\mathbb{H}^n)$, $X_j v \in L^{\mathfrak{p}}(\mathbb{H}^n)$ and $\varphi \in L^{\mathfrak{p}'}(\mathbb{H}^n)$, again Lemma A.1.3 of [33] gives

$$\int_{\mathbb{H}^n} X_j v(\breve{u} \star \varphi) \, d\xi = \int_{\mathbb{H}^n} (u \star X_j v)\varphi \, d\xi. \tag{2.6}$$

Then, by the definition of distributional derivative, by the fact that $\breve{u}$ and φ are $C_c^{\infty}(\mathbb{H}^n)$ and by (2.4),(2.5) and (2.6) we get

$$\int_{\mathbb{H}^n} X_j(u \star v)\varphi \, d\xi = -\int_{\mathbb{H}^n} (u \star v)X_j \varphi \, d\xi = -\int_{\mathbb{H}^n} (\breve{u} \star X_j \varphi)v \, d\xi = -\int_{\mathbb{H}^n} X_j(\breve{u} \star \varphi)v \, d\xi$$

$$= \int_{\mathbb{H}^n} X_j v(\check{u} \star \varphi) d\xi = \int_{\mathbb{H}^n} (u \star X_j v)\varphi d\xi.$$

In conclusion, $X_j(u \star v) = u \star X_j v \in L^{\mathfrak{p}}(\mathbb{H}^n)$ by (2.3) and similarly $Y_j(u \star v) = u \star Y_j v$ for $j = 1, \ldots, n$.

Case 2. $u \in L^1(\mathbb{H}^n)$. There exists a sequence $(u_k)_k$ in $C_c^\infty(\mathbb{H}^n)$ such that $u_k \to u$ in $L^1(\mathbb{H}^n)$. By the previous step $D_H(u_k \star v)$ exists in the sense of distributions, $D_H(u_k \star v) \in L^{\mathfrak{p}}(\mathbb{H}^n, \mathbb{R}^{2n})$, and for $j = 1, \ldots, n$

$$\int_{\mathbb{H}^n} (u_k \star v) X_j \varphi = - \int_{\mathbb{H}^n} (u_k \star X_j v)\varphi \quad \text{for any } \varphi \in C_c^\infty(\mathbb{H}^n). \tag{2.7}$$

By (2.3)

$$\|u_k \star v - u \star v\|_{\mathfrak{p}} \le \|u_k - u\|_1 \|v\|_{\mathfrak{p}} = o(1) \quad \text{as } k \to \infty.$$

Similarly, for $j = 1, \ldots, n$,

$$\|u_k \star X_j v - u \star X_j v\|_{\mathfrak{q}} \le \|u_k - u\|_1 \|X_j v\|_{\mathfrak{q}} = o(1) \quad \text{as } k \to \infty.$$

Thus, letting $k \to \infty$ in (2.7), we conclude that

$$\int_{\mathbb{H}^n} (u \star v) X_j \varphi = - \int_{\mathbb{H}^n} (u \star X_j v)\varphi \qquad \text{for any } \varphi \in C_c^\infty(\mathbb{H}^n),$$

which is exactly the assertion for X_j. We derive the result for Y_j like so. This completes the proof. □

As in the Euclidean case, the density theorem for the horizontal Sobolev space continues to hold in the Heisenberg group. We present the proof for the sake of completeness and for later purposes, since this result is crucial to prove the main Lemma 4.6.

Theorem 2.2. $C_c^\infty(\mathbb{H}^n)$ *is dense in* $HW^{1,\mathfrak{p}}(\mathbb{H}^n)$ *for every* $\mathfrak{p}$, *with* $1 \le \mathfrak{p} < \infty$.

Proof. Let $u \in HW^{1,\mathfrak{p}}(\mathbb{H}^n)$. Consider the sequence of mollifiers $(\rho_k)_k$ on $\mathbb{H}^n$. Thus, $\rho_k \star u \in C^\infty(\mathbb{H}^n)$ by (2.4), and $\rho_k \star u \to u$ in $L^{\mathfrak{p}}(\mathbb{H}^n)$ as $k \to \infty$, with $\|\rho_k \star u\|_{\mathfrak{p}} \le \|u\|_{\mathfrak{p}}$ for all k. Moreover, Lemma 2.1 yields that $D_{\mathbb{H}^n}(\rho_k \star u) \to D_{\mathbb{H}^n} u$ in $L^{\mathfrak{p}}(\mathbb{H}^n, \mathbb{R}^{2n})$ as $k \to \infty$. Now, fix a function $\zeta \in C_c^\infty(\mathbb{H}^n)$ such that $0 \le \zeta \le 1$ and

$$\zeta(\xi) = \begin{cases} 1, & \text{if } r(\xi) < 1, \\ 0, & \text{if } r(\xi) \ge 2, \end{cases}$$

for any $\xi \in \mathbb{H}^n$. Then, define the sequence of cut–off functions

$$\zeta_k(\xi) = \zeta(\delta_{1/k}(\xi)), \quad \xi \in \mathbb{H}^n. \tag{2.8}$$

The dominated convergence theorem implies that $\zeta_k u \to u$ in $L^{\mathfrak{p}}(\mathbb{H}^n)$ and $\zeta_k X_j u \to X_j u$ in $L^{\mathfrak{p}}(\mathbb{H}^n)$ for all $j = 1, \ldots, n$. Finally, let $u_k = \zeta_k(\rho_k \star u)$, so that $(u_k)_k \subset C_c^\infty(\mathbb{H}^n)$ for all k. The constructed sequence $(u_k)_k$ converges to u in $HW^{1,\mathfrak{p}}(\mathbb{H}^n)$. Indeed,

$$u_k - u = \zeta_k\big((\rho_k \star u) - u\big) + \zeta_k u - u$$

and thus

$$\|u_k - u\|_{\mathfrak{p}} \le \|\rho_k \star u - u\|_{\mathfrak{p}} + \|\zeta_k u - u\|_{\mathfrak{p}} = o(1) \quad \text{as } k \to \infty,$$

that is $u_k \to u$ in $L^{\mathfrak{p}}(\mathbb{H}^n)$. Next, Lemma 2.1 gives

$$X_j u_k = (X_j \zeta_k)(\rho_k \star u) + \zeta_k(\rho_k \star X_j u)$$

for all $j = 1, \ldots, n$ and k. Therefore, direct calculations show that $X_j \zeta_k = X_j \zeta(\delta_{1/k}(\xi))/k$, so that

$$\|X_j u_k - X_j u\|_{\mathfrak{p}} \le \|(X_j \zeta_k)(\rho_k \star u)\|_{\mathfrak{p}} + \|\zeta_k(\rho_k \star X_j u) - X_j u\|_{\mathfrak{p}}$$

$$\leq \frac{C}{k}\|u\|_{\mathfrak{p}} + \|\rho_k \star X_j u - X_j u\|_{\mathfrak{p}} + \|\zeta_k X_j u - X_j u\|_{\mathfrak{p}} = o(1)$$

as $k \to \infty$, where C is a positive constant depending only on ζ. Repeating the argument when Y_j replaces X_j, we get the same property, so that we conclude that $D_H u_k \to D_H u$ in $L^{\mathfrak{p}}(\mathbb{H}^n, \mathbb{R}^{2n})$. Consequently, $u_k \to u$ in $HW^{1,\mathfrak{p}}(\mathbb{H}^n)$ as $k \to \infty$, and this completes the proof. □

The basic embedding theorems for the Sobolev space $HW^{1,\mathfrak{p}}(\mathbb{H}^n)$, first established in [34] by *Folland* and *Stein* in this type of generality, have a form similar to those in the Euclidean case, but the exponent governing the transition to the supercritical case is the homogeneous dimension $Q = 2n + 2$. In particular, if $\mathfrak{p}$ is an exponent, with $1 < \mathfrak{p} < Q$, then the embedding

$$HW^{1,\mathfrak{p}}(\mathbb{H}^n) \hookrightarrow L^{\mathfrak{q}}(\mathbb{H}^n)$$

is continuous for any $\mathfrak{q} \in [\mathfrak{p}, \mathfrak{p}^\star]$.

For a complete treatment on the compactness of the embeddings $HW^{1,\mathfrak{p}}(\Omega) \hookrightarrow L^{\mathfrak{q}}(\Omega)$, when Ω is a *well behaved* domain, we refer to [25, 35, 36] and also to [37], as well as the references therein. The next definition is taken from [36].

An open set Ω of $\mathbb{H}^n$ is said to be a *Poincaré–Sobolev domain*, briefly *PS* domain, if there exist a bounded open subset $U \subset \mathbb{H}^n$, with $\Omega \subset \overline{\Omega} \subset U$, a covering $\{B\}_{B \in \mathcal{F}}$ of Ω by Carnot–Carathéodory balls B and numbers $N > 0$, $\alpha \geq 1$ and $\nu \geq 1$ such that

(*i*) $\sum_{B \in \mathcal{F}} \mathbb{1}_{(\alpha+1)B} \leq N \mathbb{1}_{\Omega}$ in U,
(*ii*) there exists a (central) ball $B_0 \in \mathcal{F}$ such that for all $B \in \mathcal{F}$ there is a finite chain $B_0, B_1, \dots, B_{s(B)}$, with $B_i \cap B_{i+1} \neq \emptyset$ and $|B_i \cap B_{i+1}| \geq \max\{|B_i|, |B_{i+1}|\}/N$, $i = 0, 1, \dots, s(B) - 1$ and moreover $B \subset \nu B_i$ for $i = 0, 1, \dots, s(B)$.

This definition is purely metric. There is a large number of *PS* domains in $\mathbb{H}^n$, as explained in details in [36].

For our purposes it is important also to recall a version of the Rellich–Kondrachov theorem in the Heisenberg context. The next result is a special case of Theorem 1.3.1 in [7].

Theorem 2.3.
(*i*) *Let Ω be a bounded PS domain in $\mathbb{H}^n$ and let $1 \leq \mathfrak{p} < Q$. Then the embedding*

$$HW^{1,\mathfrak{p}}(\Omega) \hookrightarrow\hookrightarrow L^{\mathfrak{q}}(\Omega)$$

is compact for all $\mathfrak{q}$, with $1 \leq \mathfrak{q} < \mathfrak{p}^\star$, where Q is the homogeneous dimension of the Heisenberg group and $\mathfrak{p}^\star$ is the Sobolev exponent related to $\mathfrak{p}$.

(*ii*) *The Carnot–Carathéodory balls are PS domains.*

Combining Theorem 2.3, with the fact that the Carnot–Carathéodory distance and the Korányi distance are equivalent on $\mathbb{H}^n$, we get (*i*) when Ω is any Korányi ball $B_R(\xi_0)$ centered at $\xi_0 \in \mathbb{H}^n$, with radius $R > 0$.

3 The concentration compactness principle for critical systems on $\mathbb{H}^n$

For the study of nonlinear elliptic problems, involving critical nonlinearities in the sense of the Sobolev inequality, the concentration compactness principle due to Lions has been being a fundamental tool for proving existence of solutions since its appearance. We just mention [13,38–43] and the references therein.

In this section, taking inspiration from [21] and following the basic ideas of the papers [22, 23] of Lions, we extend the vectorial concentration compactness principle to the Heisenberg group setting. This key result is one of the main tools to prove the existence Theorem 1.1. However, it is of independent interest and so we present it in a general setting, giving a detailed proof not included in the original work.

Throughout the section, we assume that $\wp$ is an exponent, with $1 < \wp < Q$, and that $\alpha > 1$ and $\beta > 1$ are such that $\alpha + \beta = \wp^\star$, where $\wp^\star = \wp Q/(Q - \wp)$. First, by [31] we know that there exists a positive constant $C_{\wp^\star}$, depending only on Q and $\wp$, such that for all $u \in S^{1,\wp}(\mathbb{H}^n)$

$$\|u\|_{\wp^\star} \le C_{\wp^\star}\|D_H u\|_{\wp} \tag{3.1}$$

holds. Then, the following best constant is well defined

$$\mathfrak{I} = \inf_{\substack{(u,v) \in S \\ (u \ne 0) \wedge (v \ne 0)}} \frac{\|D_H u\|_{\wp}^{\wp} + \|D_H v\|_{\wp}^{\wp}}{\left(\displaystyle\int_{\mathbb{H}^n} |u|^\alpha |v|^\beta d\xi\right)^{\wp/\wp^\star}}, \tag{3.2}$$

where $S = S^{1,\wp}(\mathbb{H}^n) \times S^{1,\wp}(\mathbb{H}^n)$. Indeed, the Hölder inequality and the Folland–Stein inequality (3.1) give

$$\int_{\mathbb{H}^n} |u|^\alpha |v|^\beta d\xi \le \left(\int_{\mathbb{H}^n} |u|^{\wp^\star} d\xi\right)^{\alpha/\wp^\star} \left(\int_{\mathbb{H}^n} |v|^{\wp^\star} d\xi\right)^{\beta/\wp^\star} \le C_{\wp^\star}^{\wp^\star} \|D_H u\|_{\wp}^\alpha \|D_H v\|_{\wp}^\beta \tag{3.3}$$

for all $(u, v) \in S$, since $\alpha, \beta > 1$ and $\alpha + \beta = \wp^\star$. Therefore, (3.3) and the Young inequality yield

$$\left(\int_{\mathbb{H}^n} |u|^\alpha |v|^\beta d\xi\right)^{\wp/\wp^\star} \le C_{\wp^\star}^{\wp} \|D_H u\|_{\wp}^{\alpha\wp/\wp^\star} \|D_H v\|_{\wp}^{\beta\wp/\wp^\star} \le C_{\wp^\star}^{\wp} \left(\frac{\alpha}{\wp^\star}\|D_H u\|_{\wp}^{\wp} + \frac{\beta}{\wp^\star}\|D_H v\|_{\wp}^{\wp}\right)$$

$$\le C_{\wp^\star}^{\wp} \left(\|D_H u\|_{\wp}^{\wp} + \|D_H v\|_{\wp}^{\wp}\right)$$

for all $(u, v) \in S$. Hence $\mathfrak{I} \ge 1/C_{\wp^\star}^{\wp} > 0$.

Before turning to Theorem 1.2 and its proof, let us show the next result, which is an extension and generalization of Lemma 2.1 in [44] to the Heisenberg setting.

Lemma 3.1. *Let $\{(u_k, v_k)\}_k$ be a sequence in S. Assume that $(u_k, v_k) \rightharpoonup (u, v)$ in S and $(u_k, v_k) \to (u, v)$ a.e. in $\mathbb{H}^n$. Then,*

$$\lim_{k \to \infty} \int_{\mathbb{H}^n} \left\{|u_k|^\alpha |v_k|^\beta - |u_k - u|^\alpha |v_k - v|^\beta\right\} d\xi = \int_{\mathbb{H}^n} |u|^\alpha |v|^\beta d\xi.$$

Proof. Fix a sequence $\{(u_k, v_k)\}_k$ in S, as in the statement. Put $I = [0, 1]$ and consider the functions

$$f_k(\xi, t) = |u_k - tu|^{\alpha-2}(u_k - tu)|v_k|^\beta, \quad g_k(\xi, t) = |u_k - u|^\alpha |v_k - tv|^{\beta-2}(v_k - tv),$$

defined for all $(\xi, t) \in \mathbb{H}^n \times I$. Clearly, $f_k u \in L^1(\mathbb{H}^n \times I)$ and $g_k v \in L^1(\mathbb{H}^n \times I)$ by Fubini's theorem. Then, Tonelli's theorem gives

$$
\begin{aligned}
\alpha &\iint\limits_{\mathbb{H}^n \times I} f_k u \, d\xi dt + \beta \iint\limits_{\mathbb{H}^n \times I} g_k v \, d\xi dt \\
&= \alpha \iint\limits_{\mathbb{H}^n \times I} |u_k - tu|^{\alpha-2}(u_k - tu)|v_k|^\beta u \, d\xi dt \\
&\quad + \beta \iint\limits_{\mathbb{H}^n \times I} |u_k - u|^\alpha |v_k - tv|^{\beta-2}(v_k - tv)v \, d\xi dt \\
&= \int\limits_{\mathbb{H}^n} |v_k|^\beta d\xi \int\limits_I \alpha |u_k - tu|^{\alpha-2}(u_k - tu)u \, dt \\
&\quad + \int\limits_{\mathbb{H}^n} |u_k - u|^\alpha d\xi \int\limits_I \beta |v_k - tv|^{\beta-2}(v_k - tv)v \, dt \\
&= \int\limits_{\mathbb{H}^n} |v_k|^\beta d\xi \int\limits_0^1 \left(-\frac{d}{dt}|u_k - tu|^\alpha \right) dt \\
&\quad + \int\limits_{\mathbb{H}^n} |u_k - u|^\alpha d\xi \int\limits_0^1 \left(-\frac{d}{dt}|v_k - tv|^\beta \right) dt \\
&= \int\limits_{\mathbb{H}^n} |u_k|^\alpha |v_k|^\beta d\xi - \int\limits_{\mathbb{H}^n} |u_k - u|^\alpha |v_k - v|^\beta d\xi.
\end{aligned}
\tag{3.4}
$$

Moreover, since $u_k \to u$ and $v_k \to v$ a.e. in $\mathbb{H}^n$, we get as $k \to \infty$

$$
f_k \to (1-t)^{\alpha-1}|u|^{\alpha-2}u|v|^\beta, \quad g_k \to 0 \quad \text{a.e. in } \mathbb{H}^n \times I.
$$

The Hölder inequality yields

$$
\iint\limits_{\mathbb{H}^n \times I} |f_k|^{\frac{\alpha+\beta}{\alpha+\beta-1}} d\xi dt \le \left(\iint\limits_{\mathbb{H}^n \times I} |u_k - tu|^{\alpha+\beta} d\xi dt \right)^{\frac{\alpha-1}{\alpha+\beta-1}} \left(\iint\limits_{\mathbb{H}^n \times I} |v_k|^{\alpha+\beta} d\xi dt \right)^{\frac{\beta}{\alpha+\beta-1}} \le C,
$$

since $\alpha + \beta = \wp^\star$. Similarly,

$$
\iint\limits_{\mathbb{H}^n \times I} |g_k|^{\frac{\alpha+\beta}{\alpha+\beta-1}} d\xi dt \le \left(\iint\limits_{\mathbb{H}^n \times I} |u_k - u|^{\alpha+\beta} d\xi dt \right)^{\frac{\alpha}{\alpha+\beta-1}} \left(\iint\limits_{\mathbb{H}^n \times I} |v_k - tv|^{\alpha+\beta} d\xi dt \right)^{\frac{\beta-1}{\alpha+\beta-1}} \le C.
$$

Therefore, we get at once that

$$
f_k \rightharpoonup (1-t)^{\alpha-1}|u|^{\alpha-2}u|v|^\beta, \quad g_k \rightharpoonup 0 \quad \text{weakly in } L^{\frac{\alpha+\beta}{\alpha+\beta-1}}(\mathbb{H}^n \times I).
$$

Hence,

$$
\alpha \iint\limits_{\mathbb{H}^n \times I} f_k u \, d\xi dt \to \alpha \iint\limits_{\mathbb{H}^n \times I} (1-t)^{\alpha-1}|u|^\alpha |v|^\beta d\xi dt = \int\limits_{\mathbb{H}^n} |u|^\alpha |v|^\beta d\xi
\tag{3.5}
$$

as $k \to \infty$, and

$$
\beta \iint\limits_{\mathbb{H}^n \times I} g_k v \, d\xi dt \to 0
\tag{3.6}
$$

as $k \to \infty$. In conclusion, (3.4), (3.5) and (3.6) yield as $k \to \infty$ the assertion. $\square$

Finally, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $\{(u_k, v_k)\}_k$ and (u, v) be as in the statement and divide the proof into two cases.

Case 1. $u = v = 0$. Fix $\varphi \in C_c^\infty(\mathbb{H}^n)$. Then $(\varphi u_k, \varphi v_k) \in S$ for all k Clearly,

$$\lim_{k \to \infty} \left(\int_{\mathbb{H}^n} |\varphi|^{\wp^\star} |u_k|^\alpha |v_k|^\beta d\xi \right)^{1/\wp^\star} = \left(\int_{\mathbb{H}^n} |\varphi|^{\wp^\star} d\nu \right)^{1/\wp^\star}. \tag{3.7}$$

Moreover, the elementary $\wp$–norm inequality in $\mathbb{R}^{2n}$ yields

$$\left| \|(D_H(\varphi u_k), D_H(\varphi v_k))\|_\wp - \|(\varphi D_H u_k, \varphi D_H v_k)\|_\wp \right| \le \|(D_H \varphi u_k, D_H \varphi v_k)\|_\wp. \tag{3.8}$$

Both $(u_k)_k$ and $(v_k)_k$ converge to 0 in $L_{\text{loc}}^\wp(\mathbb{H}^n)$ by the Rellich Theorem 2.3, so that the right hand side of (3.8) goes to 0 as $k \to \infty$. Now, (3.2) and (3.8) give

$$\mathcal{J}^{1/\wp} \left(\int_{\mathbb{H}^n} |\varphi|^{\wp^\star} |u_k|^\alpha |v_k|^\beta d\xi \right)^{1/\wp^\star} \le \left(\int_{\mathbb{H}^n} |\varphi|^\wp \left(|D_H u_k|_H^\wp + |D_H v_k|_H^\wp \right) d\xi \right)^{1/\wp}$$
$$+ \|(D_H \varphi u_k, D_H \varphi v_k)\|_\wp \tag{3.9}$$

and then, letting $k \to \infty$ in the above inequality, we get from (3.7) and (3.8)

$$\left(\int_{\mathbb{H}^n} |\varphi|^{\wp^\star} d\nu \right)^{1/\wp^\star} \le \mathcal{J}^{-1/\wp} \left(\int_{\mathbb{H}^n} |\varphi|^\wp d\mu \right)^{1/\wp} \quad \text{for all } \varphi \in C_c^\infty(\mathbb{H}^n).$$

Finally, applying Lemma 1.4.6 of [7], we conclude the proof in Case 1.

Case 2. Either $u \ne 0$ *or* $v \ne 0$. Set $\bar{u}_k = u_k - u$ and $\bar{v}_k = v_k - v$. Clearly, $(\bar{u}_k, \bar{v}_k) \rightharpoonup (0, 0)$ in S. Moreover, from Proposition 1.202 of [45], there exist bounded nonnegative Radon measures $\widetilde{\mu}$ and $\widetilde{\nu}$ on $\mathbb{H}^n$, such that, up to a subsequence, still labelled $\{(\bar{u}_k, \bar{v}_k)\}_k$, we have

$$\left(|D_H \bar{u}_k|_H^\wp + |D_H \bar{v}_k|_H^\wp \right) d\xi \overset{\star}{\rightharpoonup} \widetilde{\mu} \text{ in } \mathcal{M}(\mathbb{H}^n),$$
$$|\bar{u}_k|^\alpha |\bar{u}_k|^\beta d\xi \overset{\star}{\rightharpoonup} \widetilde{\nu} \quad \text{in } \mathcal{M}(\mathbb{H}^n). \tag{3.10}$$

Then, from Case 1. there exist an at most countable set J, a family of points $\{\xi_j\}_{j \in J} \subset \mathbb{H}^n$ and a family of nonnegative numbers $\{\nu_j\}_{j \in J}$ such that

$$\widetilde{\nu} = \sum_{j \in J} \nu_j \delta_{\xi_j}. \tag{3.11}$$

Furthermore, Lemma 3.1 implies that for all $\varphi \in C_c^\infty(\mathbb{H}^n)$

$$\lim_{k \to \infty} \left\{ \int_{\mathbb{H}^n} |\varphi|^{\wp^\star} |u_k|^\alpha |v_k|^\beta d\xi - \int_{\mathbb{H}^n} |\varphi|^{\wp^\star} |\bar{u}_k|^\alpha |\bar{v}_k|^\beta d\xi \right\} = \int_{\mathbb{H}^n} |\varphi|^{\wp^\star} |u|^\alpha |v|^\beta d\xi, \tag{3.12}$$

since $\alpha + \beta = \wp^\star$. Thus, (1.3) and (3.12) imply $\widetilde{\nu} = \nu - |u|^\alpha |v|^\beta d\xi$ by Corollary 1.3.6 of [7]. Consequently, using (3.11), we obtain the representation of ν, that is

$$\nu = |u|^\alpha |v|^\beta d\xi + \sum_{j \in J} \nu_j \delta_{\xi_j}.$$

Now, (3.2) and (3.8) give again (3.9) for all (u_k, v_k) and all $\varphi \in C_c^\infty(\mathbb{H}^n)$. As in Case 1. the Rellich Theorem 2.3 gives that both $(u_k)_k$, $(v_k)_k$ converge to u and v in $L_{\text{loc}}^\wp(\mathbb{H}^n)$, respectively. Therefore, letting $k \to \infty$ in (3.9), we get by (3.7)

$$\mathcal{J}^{1/\wp} \left(\int_{\mathbb{H}^n} |\varphi|^{\wp^\star} d\nu \right)^{1/\wp^\star} \le \left(\int_{\mathbb{H}^n} |\varphi|^\wp d\mu \right)^{1/\wp} + \left(\int_{\mathbb{H}^n} |D_H \varphi|_H^\wp \left(|u|^\wp + |v|^\wp \right) d\xi \right)^{1/\wp}. \tag{3.13}$$

Fix a test function $\varphi \in C_c^\infty(\mathbb{H}^n)$, such that $0 \le \varphi \le 1$, $\varphi(0) = 1$ and supp $\varphi = \overline{B_1}$. Take $\varepsilon > 0$ and put $\varphi_{\varepsilon,j}(\xi) = \varphi(\delta_{1/\varepsilon}(\xi \circ \xi_j^{-1}))$, $\xi \in \mathbb{H}^n$, for any fixed $j \in J$, where $\{\xi_j\}_j$ is introduced in (3.11). Fix $j \in J$. Then, (3.13), applied to $\varphi_{\varepsilon,j} \in C_c^\infty(\mathbb{H}^n)$, the Hölder inequality and a change of variable yield

$$
\mathfrak{J}^{1/\wp}\left(\int_{B_\varepsilon(\xi_j)} |\varphi_{\varepsilon,j}|^{\wp^\star} dv\right)^{1/\wp^\star} \le \left(\int_{B_\varepsilon(\xi_j)} |\varphi_{\varepsilon,j}|^\wp d\mu\right)^{1/\wp} + \left(\int_{B_\varepsilon(\xi_j)} |D_H\varphi_{\varepsilon,j}|_H^\wp \left(|u|^\wp + |v|^\wp\right) d\xi\right)^{1/\wp}
$$

$$
= \left(\int_{B_\varepsilon(\xi_j)} |\varphi_{\varepsilon,j}|^\wp d\mu\right)^{1/\wp} + \left(\int_{B_\varepsilon(\xi_j)} |D_H\varphi_{\varepsilon,j}|_H^Q d\xi\right)^{1/Q}
$$

$$
\cdot \left(\int_{B_\varepsilon(\xi_j)} \left(|u|^\wp + |v|^\wp\right)^{\wp^\star/\wp} d\xi\right)^{1/\wp^\star}
$$

$$
\le \left(\int_{B_\varepsilon(\xi_j)} |\varphi_{\varepsilon,j}|^\wp d\mu\right)^{1/\wp} + c_\varphi \left\{ \left(\int_{B_\varepsilon(\xi_j)} |u|^{\wp^\star} d\xi\right)^{\wp/\wp^\star} \right.
$$

$$
\left. + \left(\int_{B_\varepsilon(\xi_j)} |v|^{\wp^\star} d\xi\right)^{\wp/\wp^\star} \right\}^{1/\wp},
$$

where $c_\varphi = \left(\int_{B_1} |D_H\varphi(\eta)|_H^Q d\eta\right)^{1/Q}$, since $\wp/\wp^\star + \wp/Q = 1$ and

$$
\int_{B_\varepsilon(\xi_j)} |D_H\varphi_{\varepsilon,j}(\xi)|_H^Q d\xi = \int_{B_\varepsilon(\xi_j)} \frac{1}{\varepsilon^Q} |D_H\varphi(\delta_{1/\varepsilon}(\xi \circ \xi_j^{-1}))|_H^Q d\xi = \int_{B_1} |D_H\varphi(\eta)|_H^Q d\eta.
$$

Here $\eta = \delta_{1/\varepsilon}(\xi \circ \xi_j^{-1})$ is the change of variable, with $d\eta = \varepsilon^{-Q} d\xi$, as already noted in Section 2. Then, letting $\varepsilon \to 0^+$, we get

$$
\mathfrak{J}^{1/\wp} v_j^{1/\wp^\star} \le \mu_j^{1/\wp}, \quad j \in J,
$$

where $\mu_j = \lim_{\varepsilon \to 0^+} \mu(B_\varepsilon(\xi_j))$.

It remains to show that $\mu \ge \left(|D_H u|_H^\wp + |D_H v|_H^\wp\right) d\xi + \sum_{j\in J} \mu_j \delta_{\xi_j}$. Clearly,

$$
\mu \ge \sum_{j\in J} \mu_j \delta_{\xi_j}.
$$

On the other hand, $(u_k, v_k) \rightharpoonup (u, v)$ in S and so $(D_H u_k, D_H v_k) \rightharpoonup (D_H u, D_H v)$ in $L^\wp(U, \mathbb{R}^{2n})$ for every measurable subset $U \subset \mathbb{H}^n$. Therefore, the lower semi–continuity of the norm of $L^\wp(U, \mathbb{R}^{2n})$ gives at once for all compact subset $U \subset \mathbb{H}^n$

$$
\int_U \left(|D_H u|_H^\wp + |D_H v|_H^\wp\right) d\xi \le \liminf_{k\to\infty} \int_U \left(|D_H u_k|_H^\wp + |D_H v_k|_H^\wp\right) d\xi
$$

$$
\le \limsup_{k\to\infty} \int_U \left(|D_H u_k|_H^\wp + |D_H v_k|_H^\wp\right) d\xi
$$

$$
\le \int_U d\mu
$$

by Proposition 1.203 Part (ii) on page 130 of [45]. Thus,

$$
\mu \ge \left(|D_H u|_H^\wp + |D_H v|_H^\wp\right) d\xi.
$$

Finally, since $\left(|D_H u|_H^\wp + |D_H v|_H^\wp\right) d\xi$ is orthogonal to $\sum_{j\in J} \mu_j \delta_{\xi_j}$, we get the desired conclusion. This completes the proof. $\qquad \square$

4 The system $(\mathbb{S})$

The aim of this section is to prove the existence of nontrivial solutions for $(\mathbb{S})$. From now on we assume that the structural assumptions required in Theorem 1.1 hold.

The couple (u, v) is called a (weak) *solution* of system $(\mathbb{S})$ if

$$\int_{\mathbb{H}^n} \big\{ A(|D_H u|_H)(D_H u, D_H \varphi)_H + A(|D_H v|_H)(D_H v, D_H \psi)_H \big\} d\xi$$

$$+ \int_{\mathbb{H}^n} \big\{ B(|u|)u\varphi + B(|v|)v\psi \big\} d\xi$$

$$= \lambda \int_{\mathbb{H}^n} \big\{ H_u(u, v)\varphi + H_v(u, v)\psi \big\} d\xi + \frac{\alpha}{\wp^\star} \int_{\mathbb{H}^n} |u|^{\alpha-2} u |v|^\beta \varphi \, d\xi + \frac{\beta}{\wp^\star} \int_{\mathbb{H}^n} |u|^\alpha |v|^{\beta-2} v\psi \, d\xi$$

for any $(\varphi, \psi) \in W$.

The solutions of $(\mathbb{S})$ are exactly the critical points of the Euler–Lagrange functional $I = I_\lambda : W \to \mathbb{R}$, given by

$$I(u, v) = \int_{\mathbb{H}^n} \big[\mathcal{A}(|D_H u|_H) + \mathcal{A}(|D_H v|_H) \big] d\xi + \int_{\mathbb{H}^n} \big[\mathcal{B}(|u|) + \mathcal{B}(|v|) \big] d\xi$$

$$- \lambda \int_{\mathbb{H}^n} H(u, v) d\xi - \frac{1}{\wp^\star} \int_{\mathbb{H}^n} |u|^\alpha |v|^\beta d\xi,$$

for all $(u, v) \in W$, where the functions $\mathcal{A}$ and $\mathcal{B}$ are the potentials, defined in the Introduction.

From the main properties summarised in Section 2 we easily get the next result.

Lemma 4.1. *The embedding $W \hookrightarrow L^{\mathfrak{p}}(\mathbb{H}^n) \times L^{\mathfrak{p}}(\mathbb{H}^n)$ is continuous for all $\mathfrak{p} \in [p, \wp^\star]$, and*

$$\|(u, v)\|_{\mathfrak{p}} \le 2^{\mathfrak{p}-1} \big(\|u\|_{\mathfrak{p}} + \|v\|_{\mathfrak{p}} \big) \le C_{\mathfrak{p}} \|(u, v)\| \quad \text{for all } (u, v) \in W, \tag{4.1}$$

where $C_{\mathfrak{p}}$ depends on $\mathfrak{p}$, Q, p and $\wp$. If $\mathfrak{p} \in [1, \wp^\star)$, then for all $R > 0$ the embedding

$$W \hookrightarrow\hookrightarrow L^{\mathfrak{p}}(B_R) \times L^{\mathfrak{p}}(B_R)$$

is compact.

The next lemma shows that every nontrivial solution of $(\mathbb{S})$ has both components non trivial, that is it solves the actual system $(\mathbb{S})$, which does not reduce into an equation.

Lemma 4.2. *Every nontrivial solution $(u, v) \in W$ of $(\mathbb{S})$ has both components nontrivial, that is $u \ne 0$ and $v \ne 0$ in $\mathbb{H}^n$.*

Since the proof of Lemma 4.2 is not so much different from that of Lemma 2.3 in [13], we omit it here.

For simplicity in notation, let us introduce

$$\widetilde{a} = \begin{cases} a_0, & \text{if } \mathfrak{a}_1 = 0, \\ \min\{a_0, a_1\}, & \text{if } \mathfrak{a}_1 > 0, \end{cases} \tag{4.2}$$

which gives a key bound from below on A.

Lemma 4.3. *Under assumptions (A) and (C_1) we have*

$$\big(A(|X|_H)X - A(|Y|_H)Y, X - Y \big)_H \ge \frac{\widetilde{a}}{4^{\wp-1}} |X - Y|_H^\wp \tag{4.3}$$

for all X and Y in the span of $\{X_j, Y_j\}_{j=1}^n$.

The proof of Lemma 4.3, with obvious changes in notation, proceeds exactly as in Lemma 2.1 of [13]. We leave it out.

The structural assumptions of Theorem 1.1 lead that the functional I possesses the geometric features of the mountain pass theorem of *Ambrosetti* and *Rabinowitz* at special levels.

Lemma 4.4.
(i) *There exists a couple* $(e_1, e_2) \in C_c^\infty(\mathbb{H}^n) \times C_c^\infty(\mathbb{H}^n)$ *such that* $e_1 \geq 0$ *and* $e_2 \geq 0$ *in* $\mathbb{H}^n$, $I(e_1, e_2) < 0$, $\|(e_1, e_2)\| \geq 2$ *and* $\int_{\mathbb{H}^n} |e_1|^\alpha |e_2|^\beta d\xi > 0$ *for all* $\lambda > 0$.
(ii) *For all* $\lambda > 0$ *there exist numbers* $\jmath = \jmath(\lambda) > 0$ *and* $\rho = \rho(\lambda) \in (0, 1]$ *such that* $I(u, v) \geq \jmath$ *for all* $(u, v) \in W$, *with* $\|(u, v)\| = \rho$.

The proof of Lemma 4.4 is standard and again very similar to the demonstration which first appears in Lemma 2.4 of [13] and so there is no reason to produce here.

Lemma 4.4 arises the special level

$$c_\lambda = \inf_{y \in \Gamma} \max_{t \in [0,1]} I(y(t)) \tag{4.4}$$

of I, where $\Gamma = \{y \in C([0, 1], W) : y(0) = (0, 0), \ I((e_1, e_2)) < 0\}$ and this occurs for all $\lambda > 0$.

Obviously, $c_\lambda > 0$ for all $\lambda > 0$. Moreover, for all $\lambda > 0$ clearly $\|(e_1, e_2)\| \geq 2 > \rho$, since $\rho = \rho(\lambda) \in (0, 1]$ and $(e_1, e_2) \in W$ does not depend on λ. Then, Lemma 4.4 and the mountain pass theorem yield that there exists a Palais–Smale sequence $\{(u_k, v_k)\}_k \subset W$ of I at the special level c_λ for all $\lambda > 0$.

Now we introduce an asymptotic property of the levels c_λ as $\lambda \to \infty$, which is crucial in the proof of the key Lemma 4.6. This result was observed in the Euclidean vectorial case in [13], cf. Lemma 2.5, and also in the Euclidean scalar case in [15], cf. Lemma 2.2 and Remark 2.3.

Lemma 4.5. *The set of critical levels* $\{c_\lambda\}_\lambda$ *satisfies the following asymptotics*

$$\lim_{\lambda \to \infty} c_\lambda = 0.$$

The proof of Lemma 4.5 follows directly from that of Lemma 2.5 of [13] and it is not reported here. Taking inspiration from [13], we prove a crucial result, observing first some properties obtained straight from the structural assumptions. Indeed, (A), (B) and (C_2) imply that

$$\mathcal{A}(t) \leq t\mathcal{A}'(t) \leq \theta \mathcal{A}(t), \qquad \mathcal{B}(t) \leq t\mathcal{B}'(t) \leq \vartheta \mathcal{B}(t)$$

for all $t \in \mathbb{R}_0^+$. Moreover, for any $\varepsilon > 0$, condition (H) gives the existence of a number $C_\varepsilon > 0$ such that

$$|H(u, v)| \leq \varepsilon |(u, v)|^{\mathfrak{m}} + C_\varepsilon |(u, v)|^m \quad \text{for all } (u, v) \in \mathbb{R}^2 \tag{4.5}$$

holds. Clearly, (3.3) yields at once that for all $(u, v) \in W$

$$\int_{\mathbb{H}^n} |u|^\alpha |v|^\beta d\xi \leq C_{\wp^\star}^{\wp^\star} \|(u, v)\|^{\wp^\star}, \tag{4.6}$$

since $\alpha, \beta > 1$ are such that $\alpha + \beta = \wp^\star$.

For simplicity, in what follows we put

$$a = \begin{cases} \min\{a_0, b_0\}, & \text{if } \mathfrak{a}_1 = 0, \\ \min\{a_0, b_0, a_1, b_1\}, & \text{if } \mathfrak{a}_1 > 0, \end{cases} \tag{4.7}$$

while $\mathfrak{a} = \max\{\mathfrak{a}_0, \mathfrak{b}_0, \mathfrak{a}_1, \mathfrak{b}_1\}$ for all cases $\mathfrak{a}_1 \geq 0$. Clearly, $0 < a \leq \tilde{a}$ by (C_2).

We are going now to prove essential properties of the Palais–Smale sequences of I at the special level c_λ. In particular, the next lemma, which is relevant in the proof of the main theorem, is a special case of Lemma 2.6 in [13], when H is independent of ξ and the Hardy terms are not considered. In any case it is worth to produce the proof since it relies on delicate arguments in the Heisenberg context, as Theorems 1.2 and 2.2.

Lemma 4.6. *Let $\{(u_k, v_k)\}_k \subset W$ be a Palais–Smale sequence of I at the level c_λ for all $\lambda > 0$. Then,*

(i) up to a subsequence, $(u_k, v_k) \rightharpoonup (u_\lambda, v_\lambda)$ in W as $k \to \infty$,

(ii) there exists $\lambda^* > 0$ such that the weak limit (u_λ, v_λ) is a solution of $(\mathbb{S})$ for all $\lambda \geq \lambda^*$,

(iii) the set $\{(u_\lambda, v_\lambda)\}_{\lambda \geq \lambda^*}$ satisfies the asymptotic property (1.2).

Proof. Fix $\lambda > 0$ and a Palais–Smale sequence $\{(u_k, v_k)\}_k \subset W$ of I at level c_λ, that is

$$I(u_k, v_k) \to c_\lambda \quad \text{and} \quad I'(u_k, v_k) \to 0 \quad \text{in } W' \text{ as } k \to \infty. \tag{4.8}$$

The proof of the fact that $\{(u_k, v_k)\}_k$ is bounded in W is similar to that contained in Lemma 2.6 of [13] with obvious changes and we leave it out. Thus, since $\{(u_k, v_k)\}_k$ is bounded in the reflexive Banach space W, there exist $(u_\lambda, v_\lambda) \in W$, and nonnegative numbers ι_λ and δ_λ such that, up to a subsequence, we have

$$(u_k, v_k) \rightharpoonup (u_\lambda, v_\lambda) \text{ in } W$$
$$\|u_k\|_{HW^{1,p}}^p + \|v_k\|_{HW^{1,p}}^p + \mathbb{1}_{\mathbb{R}^+}(\mathfrak{a}_1)\left(\|u_k\|_{HW^{1,q}}^q + \|v_k\|_{HW^{1,q}}^q\right) \to \iota_\lambda, \tag{4.9}$$

and also, by (4.6) and Lemma 4.1

$$\int_{\mathbb{H}^n} |u_k - u_\lambda|^\alpha |v_k - v_\lambda|^\beta d\xi \to \delta_\lambda. \tag{4.10}$$

For simplicity, in what follows we still denote by $\{(u_k, v_k)\}_k$ every subsequence extracted from the original sequence $\{(u_k, v_k)\}_k$. Moreover, by Lemma 4.4 for all $\mathfrak{p} \in [1, \wp^\star)$, up to a subsequence, we have that

$$(u_k, v_k) \to (u_\lambda, v_\lambda) \text{ in } L^{\mathfrak{p}}(B_R) \times L^{\mathfrak{p}}(B_R),$$
$$(u_k, v_k) \to (u_\lambda, v_\lambda) \text{ a.e. in } \mathbb{H}^n, \tag{4.11}$$
$$|u_k| \leq g_R, \ |v_k| \leq g_R \text{ a.e. in } \mathbb{H}^n$$

for some $g_R \in L^\wp(B_R)$ and all $R > 0$. Furthermore, by the Folland–Stein inequality (3.1) and the Hölder inequality, since $\alpha > 1$, $\beta > 1$ and $\alpha + \beta = \wp^\star$, we obtain

$$\int_{\mathbb{H}^n} \left||u_k|^{\alpha-1}|v_k|^\beta\right|^{\frac{\wp^\star}{\wp^\star-1}} d\xi \leq \|u_k\|_{\wp^\star}^{\frac{\wp^\star(\alpha-1)}{\wp^\star-1}} \|v_k\|_{\wp^\star}^{\frac{\wp^\star(\wp^\star-\alpha)}{\wp^\star-1}} \leq C_{\wp^\star}^{\wp^\star} \|D_H u_k\|_\wp^{\frac{\wp^\star(\alpha-1)}{\wp^\star-1}} \|D_H v_k\|_\wp^{\frac{\wp^\star(\wp^\star-\alpha)}{\wp^\star-1}}$$

$$\leq C_{\wp^\star}^{\wp^\star} \|(u_k, v_k)\|^{\wp^\star} \leq C,$$

where $C > 0$ is a suitable constant. Similarly,

$$\int_{\mathbb{H}^n} \left||u_k|^\alpha |v_k|^{\beta-1}\right|^{\frac{\wp^\star}{\wp^\star-1}} d\xi \leq C.$$

Consequently, again up to a subsequence, we have

$$|u_k|^{\alpha-2} u_k |v_k|^\beta \rightharpoonup |u_\lambda|^{\alpha-2} u_\lambda |v_\lambda|^\beta \text{ in } L^{\wp^\star/(\wp^\star-1)}(\mathbb{H}^n),$$
$$|u_k|^\alpha |v_k|^{\beta-2} v_k \rightharpoonup |u_\lambda|^\alpha |v_\lambda|^{\beta-2} v_\lambda \text{ in } L^{\wp^\star/(\wp^\star-1)}(\mathbb{H}^n) \tag{4.12}$$

thanks to (4.11). In virtue of Proposition 1.202 of [45], there exist two bounded nonnegative Radon measures μ and ν on $\mathbb{H}^n$, such that, up to a subsequence, we have

$$\widetilde{a}\left(|D_H u_k|_H^\wp + |D_H v_k|_H^\wp\right) d\xi \overset{\star}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\mathbb{H}^n), \quad |u_k|^\alpha |v_k|^\beta d\xi \overset{\star}{\rightharpoonup} \nu \text{ in } \mathcal{M}(\mathbb{H}^n). \tag{4.13}$$

Therefore, Theorem 1.2 guarantees the existence of an at most countable set J, of a family of points $\{\xi_j\}_{j \in J}$ and of two families of nonnegative numbers $\{\mu_j\}_{j \in J}$ and $\{\nu_j\}_{j \in J}$ such that

$$\nu = |u_\lambda|^\alpha |v_\lambda|^\beta d\xi + \sum_{j \in J} \nu_j \delta_{\xi_j}, \quad \mu \geq \widetilde{a}\left(|D_H u_\lambda|_H^\wp + |D_H v_\lambda|_H^\wp\right) dx + \sum_{j \in J} \mu_j \delta_{\xi_j},$$

$$\nu_j^{\wp/\wp^\star} \leq \frac{\mu_j}{\widetilde{a}\,\mathfrak{I}} \text{ for all } j \in J, \tag{4.14}$$

where δ_{ξ_j} is the Dirac function at the point ξ_j of $\mathbb{H}^n$, and $\mathcal{I}$ is defined in (3.2).

Now (4.8), (4.9) and (H) give as $k \to \infty$

$$c_\lambda + o(1) \geq \ell\left\{\|u_k\|_{HW^{1,p}}^p + \|v_k\|_{HW^{1,p}}^p + \mathbb{1}_{\mathbb{R}^+}(\mathfrak{a}_1)\left(\|u_k\|_{HW^{1,q}}^q + \|v_k\|_{HW^{1,q}}^q\right)\right\}$$
$$+ \left(\frac{1}{\sigma} - \frac{1}{\wp^\star}\right)\int\limits_{\mathbb{H}^n} |u_k|^\alpha |v_k|^\beta d\xi, \tag{4.15}$$

where by (4.7)

$$\ell = \left(\frac{1}{\max\{\theta, \vartheta\}} - \frac{1}{\sigma}\right) a > 0,$$

since $\max\{\theta, \vartheta\} < \sigma$ from (H).

First we assert that

$$\lim_{\lambda \to \infty} \iota_\lambda = 0. \tag{4.16}$$

Otherwise, $\limsup_{\lambda \to \infty} \iota_\lambda = \iota > 0$. Hence, there is a sequence $k \mapsto \lambda_k \uparrow \infty$ such that $\iota_{\lambda_k} \to \iota$ as $k \to \infty$. Then, letting $k \to \infty$, we get from (4.15) and Lemma 4.5 that

$$0 \geq \ell \iota > 0,$$

which is impossible. This contradiction proves the assertion (4.16). Moreover, from the fact that $(u_k, v_k) \rightharpoonup (u_\lambda, v_\lambda)$ in W, we have

$$\|u_\lambda\|_{HW^{1,p}}^p + \|v_\lambda\|_{HW^{1,p}}^p + \mathbb{1}_{\mathbb{R}^+}(\mathfrak{a}_1)\left(\|u_\lambda\|_{HW^{1,q}}^q + \|v_\lambda\|_{HW^{1,q}}^q\right) \leq \iota_\lambda.$$

Therefore,

$$\lim_{\lambda \to \infty} \int\limits_{\mathbb{H}^n} |u_\lambda|^\alpha |v_\lambda|^\beta d\xi = \lim_{\lambda \to \infty} \|(u_\lambda, v_\lambda)\| = 0 \tag{4.17}$$

by (4.6) and (4.16).

Fix now a test function $\varphi \in C_c^\infty(\mathbb{H}^n)$, such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in B_1, while $\varphi \equiv 0$ in B_2^c, and $\|D_H\varphi\|_\infty \leq 2$. Take $\varepsilon > 0$ and put $\varphi_{\varepsilon,j}(\xi) = \varphi(\delta_{1/\varepsilon}(\xi \circ \xi_j^{-1}))$, $\xi \in \mathbb{H}^n$, for any fixed $j \in J$, where $\{\xi_j\}_j$ is introduced in (4.14). Fix $j \in J$. Then $\varphi_{\varepsilon,j}(u_k, v_k) \in W$ and so $\langle I'(u_k, v_k), \varphi_{\varepsilon,j}(u_k, v_k)\rangle = o(1)$ as $k \to \infty$ by (4.8) and (4.9). Therefore, as $k \to \infty$

$$o(1) = \int\limits_{\mathbb{H}^n}\left\{A(|D_Hu_k|_H)\left(u_kD_Hu_k, D_H\varphi_{\varepsilon,j}\right)_H\right.$$
$$\left. + A(|D_Hv_k|_H)\left(v_kD_Hv_k, D_H\varphi_{\varepsilon,j}\right)_H\right\}d\xi$$
$$+ \int\limits_{\mathbb{H}^n}\varphi_{\varepsilon,j}\left\{A(|D_Hu_k|_H)|D_Hu_k|_H^2 + A(|D_Hv_k|_H)|D_Hv_k|_H^2\right.$$
$$\left. + B(|u_k|)|u_k|^2 + B(|v_k|)|v_k|^2\right\}d\xi$$
$$- \lambda\int\limits_{\mathbb{H}^n}\varphi_{\varepsilon,j}[H_u(u_k, v_k)u_k + H_v(u_k, v_k)v_k]d\xi - \int\limits_{\mathbb{H}^n}\varphi_{\varepsilon,j}|u_k|^\alpha|v_k|^\beta d\xi, \tag{4.18}$$

since $\alpha + \beta = \wp^\star$. Moreover, by (C_2), the Hölder inequality and a change of variable

$$\limsup_{k \to \infty}\left|\int\limits_{\mathbb{H}^n}A(|D_Hu_k|_H)\left(u_kD_Hu_k, D_H\varphi_{\varepsilon,j}\right)_Hd\xi\right|$$
$$\leq \limsup_{k \to \infty}\left\{\int\limits_{B(\xi_j, 2\varepsilon)}\left\{\mathfrak{a}_0|D_Hu_k|_H^{p-1}|u_k| \cdot |D_H\varphi_{\varepsilon,j}|_H + \mathfrak{a}_1|D_Hu_k|_H^{q-1}|u_k| \cdot |D_H\varphi_{\varepsilon,j}|_H\right\}d\xi\right\}$$

$$\leq \limsup_{k\to\infty} \left\{ \mathfrak{a}_0 \|D_H u_k\|_p^{p-1} \left(\int_{B(\xi_j,2\varepsilon)} |u_k D_H \varphi_{\varepsilon,j}(\xi)|_H^p d\xi \right)^{1/p} \right\}$$

$$+ \limsup_{k\to\infty} \left\{ \mathfrak{a}_1 \|D_H u_k\|_q^{q-1} \left(\int_{B(\xi_j,2\varepsilon)} |u_k D_H \varphi_{\varepsilon,j}(\xi)|_H^q d\xi \right)^{1/q} \right\}$$

$$\leq c_0 \mathfrak{a}_0 \left(\int_{B(\xi_j,2\varepsilon)} |u_\lambda D_H \varphi_{\varepsilon,j}(\xi)|_H^p d\xi \right)^{1/p} + c_1 \mathfrak{a}_1 \left(\int_{B(\xi_j,2\varepsilon)} |u_\lambda D_H \varphi_{\varepsilon,j}(\xi)|_H^q d\xi \right)^{1/q}$$

$$\leq c_\varphi \left\{ c_0 \mathfrak{a}_0 \left(\int_{B(\xi_j,2\varepsilon)} |u_\lambda|^{p^\star} d\xi \right)^{1/p^\star} + c_1 \mathfrak{a}_1 \left(\int_{B(\xi_j,2\varepsilon)} |u_\lambda|^{q^\star} d\xi \right)^{1/q^\star} \right\},$$

where $c_0 = \sup_{k\in\mathbb{N}} \|D_H u_k\|_p^{p-1}$, $c_1 = \sup_{k\in\mathbb{N}} \|D_H u_k\|_q^{q-1}$, $c_\varphi = \left(\int_{B_2} |D_H \varphi(\eta)|_H^Q d\eta \right)^{1/Q}$, while the change of variable is given by $\eta = \delta_{1/\varepsilon}(\xi \circ \xi_j^{-1})$. Consequently,

$$\lim_{\varepsilon\to 0^+} \limsup_{k\to\infty} \left| \int_{\mathbb{H}^n} A(|D_H u_k|_H)(u_k D_H u_k, D_H \varphi_{\varepsilon,j})_H d\xi \right| = 0.$$

Similarly, in the v component, and so as $k \to \infty$ and $\varepsilon \to 0^+$

$$\left| \int_{\mathbb{H}^n} \left\{ A(|D_H u_k|_H)(u_k D_H u_k, D_H \varphi_{\varepsilon,j})_H \right. \right.$$

$$\left. \left. + A(|D_H v_k|_H)(v_k D_H v_k, D_H \varphi_{\varepsilon,j})_H \right\} d\xi \right| = o_{k,\varepsilon}(1). \tag{4.19}$$

Clearly, by (C_1), the properties of φ and (4.11), as $k \to \infty$

$$\int_{\mathbb{H}^n} \varphi_{\varepsilon,j} \left\{ B(|u_k|)|u_k|^2 + B(|v_k|)|v_k|^2 \right\} d\xi \leq \int_{B(x_j,2\varepsilon)} \left\{ \mathfrak{b}_0 (|u_k|^p + |v_k|^p) + \mathfrak{b}_1 (|u_k|^q + |v_k|^q) \right\} d\xi$$

$$\to \int_{B(x_j,2\varepsilon)} \left\{ \mathfrak{b}_0 (|u_\lambda|^p + |v_\lambda)|^p) + \mathfrak{b}_1 (|u_\lambda|^q + |v_\lambda)|^q) \right\} d\xi,$$

since $1 < p < q < \wp^\star$. Hence,

$$\lim_{\varepsilon\to 0^+} \lim_{k\to\infty} \int_{\mathbb{H}^n} \varphi_{\varepsilon,j} \left\{ B(|u_k|)|u_k|^2 + B(|v_k|)|v_k|^2 \right\} d\xi = 0. \tag{4.20}$$

Similarly, by (H) and (4.11), as $k \to \infty$

$$\int_{\mathbb{H}^n} \varphi_{\varepsilon,j} [H_u(u_k, v_k)u_k + H_v(u_k, v_k)v_k] d\xi \leq \int_{B(x_j,2\varepsilon)} \left(\mathfrak{m}|(u_k, v_k)|^{\mathfrak{m}} + m\, C_1 |(u_k, v_k)|^m \right) d\xi$$

$$\to \int_{B(x_j,2\varepsilon)} \left(\mathfrak{m}|(u_\lambda, v_\lambda))|^{\mathfrak{m}} + m\, C_1 |(u_\lambda, v_\lambda))|^m \right) d\xi,$$

since $1 < \wp < \mathfrak{m} < m < \wp^\star$, and then

$$\lim_{\varepsilon\to 0^+} \lim_{k\to\infty} \int_{\mathbb{H}^n} \varphi_{\varepsilon,j} \left\{ H_u(u_k, v_k)u_k + H_v(u_k, v_k)v_k \right\} d\xi = 0. \tag{4.21}$$

In conclusion, (C_1), (4.7), (4.18)–(4.21) give for all $j \in J$

$$\int_{\mathbb{H}^n} \varphi_{\varepsilon,j} d\mu + o(1) \le \int_{\mathbb{H}^n} \varphi_{\varepsilon,j} d\nu \tag{4.22}$$

as $\varepsilon \to 0^+$.

Now, by Lemma 4.5 there exists $\lambda^* = \lambda^*(Q, \wp) > 0$ such that

$$c_\lambda < \left(\frac{1}{\sigma} - \frac{1}{\wp^*} \right) (\widetilde{a} \, \mathfrak{I})^{Q/\wp} \quad \text{for all } \lambda \ge \lambda^*. \tag{4.23}$$

Notice that, (4.14) and (4.22) yield $\widetilde{a} \, \mathfrak{I} \, \nu_j^{\wp/\wp^*} \le \mu_j \le \nu_j$ for all $j \in J$. Assume by contradiction that $\nu_j > 0$ for some $j \in J$. Then, $\nu_j \ge (\widetilde{a} \, \mathfrak{I})^{Q/\wp}$ and so (4.15) implies

$$c_\lambda + o(1) \ge \left(\frac{1}{\sigma} - \frac{1}{\wp^*} \right) \int_{\mathbb{H}^n} |u_k|^\alpha |v_k|^\beta d\xi \ge \left(\frac{1}{\sigma} - \frac{1}{\wp^*} \right) \int_{\mathbb{H}^n} \varphi_{\varepsilon,j} d\nu$$

as $k \to \infty$. On the other hand, as $k \to \infty$ and $\varepsilon \to 0^+$ we have

$$c_\lambda \ge \left(\frac{1}{\sigma} - \frac{1}{\wp^*} \right) \nu_j \ge \left(\frac{1}{\sigma} - \frac{1}{\wp^*} \right) (\widetilde{a} \, \mathfrak{I})^{Q/\wp} > 0,$$

and this contradicts (4.23). Hence, $\nu_j = 0$ for all $j \in J$ and for all $\lambda \ge \lambda^*$.

Consequently, there exists $\lambda^* > 0$ such that for all $\lambda \ge \lambda^*$

$$|u_k|^\alpha |v_k|^\beta d\xi \overset{*}{\rightharpoonup} \nu = |u_\lambda|^\alpha |v_\lambda|^\beta d\xi \quad \text{in } \mathcal{M}(\mathbb{H}^n)$$

as $k \to \infty$, by (4.13) and (4.14). In particular, for all $\phi \in C_c^\infty(\mathbb{H}^n)$

$$\lim_{k \to \infty} \int_{\mathbb{H}^n} \phi |u_k|^\alpha |v_k|^\beta d\xi = \int_{\mathbb{H}^n} \phi |u_\lambda|^\alpha |v_\lambda|^\beta d\xi. \tag{4.24}$$

From now on in the proof we fix $\lambda \ge \lambda^*$.

Take $R > 0$ and $\varphi \in C_c^\infty(\mathbb{H}^n)$ such that $0 \le \varphi \le 1$ in $\mathbb{H}^n$, $\varphi \equiv 1$ in B_R, $\varphi \equiv 0$ in B_{2R}^c and $\|D_H \varphi\|_\infty \le 2$. By Lemma 4.3 we have

$$\frac{\widetilde{a}}{4^{\wp-1}} \int_{B_R} |D_H u_k - D_H u_\lambda|_H^\wp d\xi$$

$$\le \int_{B_R} \left(A(|D_H u_k|_H) D_H u_k - A(|D_H u_\lambda|) D_H u_\lambda, D_H u_k - D_H u_\lambda \right)_H d\xi$$

$$\le \int_{\mathbb{H}^n} \left(A(|D_H u_k|_H) D_H u_k - A(|D_H u_\lambda|_H) D_H u_\lambda, D_H u_k - D_H u_\lambda \right)_H \varphi \, d\xi \tag{4.25}$$

$$= \int_{\mathbb{H}^n} \varphi A(|D_H u_k|_H) |D_H u_k|_H^2 d\xi - \int_{\mathbb{H}^n} \varphi A(|D_H u_k|_H) \left(D_H u_k, D_H u_\lambda \right)_H d\xi + o(1)$$

as $k \to \infty$ by (4.9). Similarly, we obtain (4.25) also in the v variable. Now, we can estimate the right hand side of (4.25) as

$$
\int_{\mathbb{H}^n} \varphi A(|D_H u_k|_H) \big\{ |D_H u_k|_H^2 - (D_H u_k, D_H u_\lambda)_H \big\} d\xi
$$

$$
+ \int_{\mathbb{H}^n} \varphi A(|D_H v_k|_H) \big\{ |D_H v_k|_H^2 - (D_H v_k, D_H v_\lambda)_H \big\} d\xi
$$

$$
= \langle I'(u_k, v_k), \varphi(u_k, v_k) \rangle - \langle I'(u_k, v_k), \varphi(u_\lambda, v_\lambda) \rangle
$$

$$
- \int_{\mathbb{H}^n} \big\{ A(|D_H u_k|_H)(u_k - u_\lambda)(D_H u_k, D_H \varphi)_H
$$

$$
+ A(|D_H v_k|_H)(v_k - v_\lambda)(D_H v_k, D_H \varphi)_H \big\} d\xi \tag{4.26}
$$

$$
- \int_{\mathbb{H}^n} \varphi \{ B(|u_k|)u_k(u_k - u_\lambda) + B(|v_k|)v_k(v_k - v_\lambda) \} d\xi
$$

$$
+ \lambda \int_{\mathbb{H}^n} \varphi [H_u(u_k, v_k)(u_k - u_\lambda) + H_v(u_k, v_k)(v_k - v_\lambda)] d\xi
$$

$$
+ \int_{\mathbb{H}^n} \varphi \left\{ |u_k|^\alpha |v_k|^\beta - \frac{\alpha}{\wp^\star}|u_k|^{\alpha-2}u_k|v_k|^\beta u_\lambda - \frac{\beta}{\wp^\star}|u_k|^\alpha|v_k|^{\beta-2}v_k v_\lambda \right\} d\xi.
$$

Clearly,

$$
\langle I'(u_k, v_k), \varphi(u_k, v_k) \rangle - \langle I'(u_k, v_k), \varphi(u_\lambda, v_\lambda) \rangle = o(1) \quad \text{as } k \to \infty.
$$

Moreover, by (C_1) and the Hölder inequality

$$
\left| \int_{\mathbb{H}^n} A(|D_H u_k|_H)(u_k - u_\lambda)(D_H u_k, D_H \varphi)_H d\xi \right|
$$

$$
\leq 2 \left\{ \mathfrak{a}_0 \|D_H u_k\|_p^{p-1} \left(\int_{B_{2R}} |u_k - u_\lambda|^p d\xi \right)^{1/p} + \mathfrak{a}_1 \|D_H u_k\|_q^{q-1} \left(\int_{B_{2R}} |u_k - u_\lambda|^q d\xi \right)^{1/q} \right\},
$$

and similarly in v component. Therefore, by (4.9)

$$
\lim_{k \to \infty} \int_{\mathbb{H}^n} \big\{ A(|D_H u_k|_H)(u_k - u_\lambda)(D_H u_k, D_H \varphi)_H
$$

$$
+ A(|D_H v_k|_H)(v_k - v_\lambda)(D_H v_k, D_H \varphi)_H \big\} d\xi = 0. \tag{4.27}
$$

Again by (C_1) and the Hölder inequality

$$
\left| \int_{\mathbb{H}^n} \varphi B(|u_k|)u_k(u_k - u_\lambda) d\xi \right| \leq \left\{ \mathfrak{b}_0 \|u_k\|_p^{p-1} \left(\int_{B_{2R}} |u_k - u_\lambda|^p d\xi \right)^{1/p} \right.
$$

$$
\left. + \mathfrak{b}_1 \|u_k\|_q^{q-1} \left(\int_{B_{2R}} |u_k - u_\lambda|^q d\xi \right)^{1/q} \right\},
$$

which yields by (4.9), also in v component,

$$
\lim_{k \to \infty} \int_{\mathbb{H}^n} \varphi \{ B(|u_k|)u_k(u_k - u_\lambda) + B(|v_k|)v_k(v_k - v_\lambda) \} d\xi = 0. \tag{4.28}
$$

Likewise, by (H), the Hölder inequality, (4.5) with $\varepsilon = 1$ and (4.11)

$$
0 \leq \int_{\mathbb{H}^n} \varphi [H_u(u_k, v_k)(u_k - u_\lambda) + H_v(u_k, v_k)(v_k - v_\lambda)] d\xi
$$

$$\leq \int\limits_{B_{2R}} \varphi |(u_k, v_k) - (u_\lambda, v_\lambda)| \big(\mathfrak{m}|(u_k, v_k)|^{m-1} + m C_1 |(u_k, v_k)|^{m-1}\big) d\xi.$$

Thus, as $k \to \infty$

$$0 \leq \int\limits_{\mathbb{H}^n} \varphi [H_u(u_k, v_k)(u_k - u_\lambda) + H_v(u_k, v_k)(v_k - v_\lambda)] d\xi$$

$$\leq C\big(\|(u_k, v_k) - (u_\lambda, v_\lambda)\|_{L^{\mathfrak{m}}(B_{2R})} + \|(u_k, v_k) - (u_\lambda, v_\lambda)\|_{L^m(B_{2R})}\big) \to 0, \tag{4.29}$$

where

$$C = \mathfrak{m} \sup_{k \in \mathbb{N}} \|(u_k, v_k)\|^{\mathfrak{m}-1} + m\, C_1 \sup_{k \in \mathbb{N}} \|(u_k, v_k)\|^{m-1} < \infty,$$

since $1 < \wp < \mathfrak{m} < m < \wp^\star$. Finally, $\alpha + \beta = \wp^\star$ gives as $k \to \infty$

$$\int\limits_{\mathbb{H}^n} \varphi |u_k|^\alpha |v_k|^\beta d\xi - \frac{\alpha}{\wp^\star} \int\limits_{\mathbb{H}^n} \varphi |u_k|^{\alpha-2} u_k |v_k|^\beta u_\lambda d\xi - \frac{\beta}{\wp^\star} \int\limits_{\mathbb{H}^n} \varphi |u_k|^\alpha |v_k|^{\beta-2} v_k v_\lambda d\xi \to 0, \tag{4.30}$$

by (4.12) and (4.24). Therefore, combining (4.25)–(4.30), we have

$$\frac{\widetilde{a}}{4^{\wp-1}} \int\limits_{B_R} \big(|D_H u_k - D_H u_\lambda|_H^\wp + |D_H v_k - D_H v_\lambda|_H^\wp\big) d\xi \leq o(1) \quad \text{as } k \to \infty.$$

Thus, $D_H u_k \to D_H u_\lambda$ and $D_H v_k \to D_H v_\lambda$ in $L^\wp(B_R, \mathbb{R}^{2n})$ for all $R > 0$. Consequently, up to subsequences, still labelled $\{(u_k, v_k)\}_k$, we get

$$D_H u_k \to D_H u_\lambda \quad \text{and} \quad D_H v_k \to D_H v_\lambda \quad \text{a.e. in } \mathbb{H}^n, \tag{4.31}$$

and for all $R > 0$ there exists a function $h_R \in L^\wp(B_R)$ such that $|D_H u_k|_H \leq h_R$ and $|D_H v_k|_H \leq h_R$ a.e. in B_R and for all $k \in \mathbb{N}$.

Now, fix ϕ and ψ in $C_c^\infty(\mathbb{H}^n)$ and let $R > 0$ so large that $\operatorname{supp} \phi \subset B_R$ and $\operatorname{supp} \psi \subset B_R$. By the above construction and (C_1) we have a.e. in B_R

$$\big|A(|D_H u_k|_H)\big(D_H u_k, D_H \phi\big)_H + A(|D_H v_k|_H)\big(D_H v_k, D_H \psi\big)_H\big|$$

$$\leq \big(\mathfrak{a}_0 |D_H u_k|_H^{p-1} + \mathfrak{a}_1 |D_H u_k|_H^{q-1}\big) |D_H \phi|_H$$

$$+ \big(\mathfrak{a}_0 |D_H v_k|_H^{p-1} + \mathfrak{a}_1 |D_H v_k|_H^{q-1}\big) |D_H \psi|_H$$

$$\leq \big(\mathfrak{a}_0 h_R^{p-1} + \mathfrak{a}_1 h_R^{q-1}\big) \big(|D_H \phi|_H + |D_H \psi|_H\big) = \mathfrak{h},$$

where $\mathfrak{h} \in L^1(B_R)$. Then, the dominated convergence theorem yields as $k \to \infty$

$$\int\limits_{\mathbb{H}^n} \big\{A(|D_H u_k|_H)\big(D_H u_k, D_H \phi\big)_H + A(|D_H v_k|_H)\big(D_H v_k, D_H \psi\big)_H\big\} d\xi$$

$$= \int\limits_{B_R} \big\{A(|D_H u_k|_H)\big(D_H u_k, D_H \phi\big)_H + A(|D_H v_k|_H)\big(D_H v_k, D_H \psi\big)_H\big\} d\xi$$

$$\to \int\limits_{\mathbb{H}^n} \big\{A(|D_H u_\lambda|_H)\big(D_H u_\lambda, D_H \phi\big)_H + A(|D_H v_\lambda|_H)\big(D_H v_\lambda, D_H \psi\big)_H\big\} d\xi.$$

Similarly, again (C_1) and (4.11) give a.e. in B_R

$$\big|B(|u_k|)u_k \phi + B(|v_k|)v_k \psi\big| \leq \big(\mathfrak{b}_0 g_R^{p-1} + \mathfrak{b}_1 g_R^{q-1}\big)\big(|\phi| + |\psi|\big) = \mathfrak{g},$$

where $\mathfrak{g} \in L^1(B_R)$. Then, the dominated convergence theorem gives as $k \to \infty$

$$\int\limits_{\mathbb{H}^n} \big(B(|u_k|)u_k \phi + B(|v_k|)v_k \psi\big) d\xi \to \int\limits_{\mathbb{H}^n} \big(B(|u_\lambda|)u_\lambda \phi + B(|v_\lambda|)v_\lambda \psi\big) d\xi.$$

Moreover, by (H)

$$|H_u(u_k, v_k)\phi + H_v(u_k, v_k)\psi| \leq \mathfrak{m}|(u_k, v_k)|^{m-1}\phi + m\, C_1|(u_k, v_k)|^{m-1}\psi \leq \mathfrak{G},$$

where $\mathfrak{G} \in L^1(B_R)$, and so, again by the dominated convergence theorem, as $k \to \infty$ we obtain

$$\int_{\mathbb{H}^n}\big[H_u(u_k, v_k)\phi + H_v(u_k, v_k)\psi\big]\, d\xi \to \int_{\mathbb{H}^n}\big[H_u(u_\lambda, v_\lambda)\phi + H_v(u_\lambda, v_\lambda)\psi\big]\, d\xi.$$

Finally, since $\langle I'(u_k, v_k), (\phi, \psi)\rangle = o(1)$ as $k \to \infty$, we have

$$\int_{\mathbb{H}^n}\Big\{A(|D_H u_k|_H)\big(D_H u_k, D_H\phi\big)_H + A(|D_H v_k|_H)\big(D_H v_k, D_H\psi\big)_H\Big\}\, d\xi$$

$$+ \int_{\mathbb{H}^n}\Big\{B(|u_k|)u_k\phi + B(|v_k|)v_k\psi\Big\}\, d\xi$$

$$= \lambda\int_{\mathbb{H}^n}[H_u(x, u_k, v_k)\phi + H_v(x, u_k, v_k)\psi]\, d\xi$$

$$+ \frac{\alpha}{\wp^\star}\int_{\mathbb{H}^n}|u_k|^{\alpha-2}u_k|v_k|^\beta\phi\, d\xi + \frac{\beta}{\wp^\star}\int_{\mathbb{H}^n}|u_k|^\alpha|v_k|^{\beta-2}v_k\psi\, d\xi + o(1).$$

Thus, from what we proved above, we get as $k \to \infty$

$$\int_{\mathbb{H}^n}\Big\{A(|D_H u_\lambda|_H)\big(D_H u_\lambda, D_H\phi\big)_H + A(|D_H v_\lambda|_H)\big(D_H v_\lambda, D_H\psi\big)_H\Big\}\, d\xi$$

$$+ \int_{\mathbb{H}^n}\big(B(|u_\lambda|)u_\lambda\phi + B(|v_\lambda|)v_\lambda\psi\big)\, d\xi$$

$$= \lambda\int_{\mathbb{H}^n}\big[H_u(u_\lambda, v_\lambda)\phi + H_v(u_\lambda, v_\lambda)\psi\big]\, d\xi \qquad (4.32)$$

$$+ \frac{\alpha}{\wp^\star}\int_{\mathbb{H}^n}|u_\lambda|^{\alpha-2}u_\lambda|v_\lambda|^\beta\phi\, d\xi + \frac{\beta}{\wp^\star}\int_{\mathbb{H}^n}|u_\lambda|^\alpha|v_\lambda|^{\beta-2}v_\lambda\psi\, d\xi$$

for all ϕ and ψ in $C_c^\infty(\mathbb{H}^n)$.

Fix now $(\Phi, \Psi) \in W$ and put $\phi_k = \zeta_k(\rho_k \star \Phi)$ and $\psi_k = \zeta_k(\rho_k \star \Psi)$, where $(\rho_k)_k$ is the sequence of mollifiers introduced in Section 2 and $(\zeta_k)_k$ is a sequence of cut–off functions defined as in (2.8). Then, from the proof of Theorem 2.2, it is evident that the sequences $(\phi_k)_k$ and $(\psi_k)_k$ are in $C_c^\infty(\mathbb{H}^n)$ and have the properties that $\phi_k \to \Phi$, $\psi_k \to \Psi$ in $HW^{1,p}(\mathbb{H}^n) \cap HW^{1,\wp}(\mathbb{H}^n)$ and $\phi_k \to \Phi$, $\psi_k \to \Psi$, $D_H\phi_k \to D_H\Phi$, $D_H\psi_k \to D_H\Psi$ a.e. in $\mathbb{H}^n$ as $k \to \infty$. Moreover, (4.32) holds along $(\phi_k)_k$ and $(\psi_k)_k$ for all k. Then, passing to the limit as $k \to \infty$ under the sign of integrals by the dominated convergence theorem, we obtain the validity of (4.32) for all $(\Phi, \Psi) \in W$. In conclusion,

$$\langle I'(u_\lambda, v_\lambda), (\Phi, \Psi)\rangle = 0 \quad \text{for all } (\Phi, \Psi) \in W, \qquad (4.33)$$

that is (u_λ, v_λ) is a solution of ($\mathbb{S}$) for all $\lambda \geq \lambda^\star$. This completes the proof of part (ii).

As already noted (iii) is a direct consequence of (4.17). □

The next result is an adaptation of Lemma I.1 in [14] for the scalar case, where the space $\mathbb{R}^n$ is replaced by the Heisenberg group $\mathbb{H}^n$. We present it in the generality given in its statement, that is the exponents p and $\wp$ are not related as in (C_1).

Proof of Theorem 1.3. Let $(u_k)_k$ be as in the statement. Thus, the Folland–Stein inequality (3.1) gives that the sequence $(u_k)_k$ is bounded also in $L^{\wp^\star}(\mathbb{H}^n)$. We divide the proof into two cases.

Case 1. $(u_k)_k$ *is bounded also in* $L^\infty(\mathbb{H}^n)$. Take $\mathfrak{q}$ such that $\mathfrak{q} > \min\{p, \wp^*\}$. Then, (1.4) implies

$$\sup_{\eta \in \mathbb{H}^n} \int\limits_{B_R(\eta)} |u_k|^{\mathfrak{q}} d\xi = o(1) \tag{4.34}$$

as $k \to \infty$. Indeed, if $p < \wp^*$, then $\mathfrak{q} > p$ and so

$$\sup_{\eta \in \mathbb{H}^n} \int\limits_{B_R(\eta)} |u_k|^{\mathfrak{q}} d\xi = \sup_{\eta \in \mathbb{H}^n} \int\limits_{B_R(\eta)} |u_k|^{\mathfrak{q}-p} |u_k|^p d\xi \le \|u_k\|_\infty^{\mathfrak{q}-p} \sup_{\eta \in \mathbb{H}^n} \int\limits_{B_R(\eta)} |u_k|^p d\xi = o(1)$$

as $k \to \infty$, since $(u_k)_k$ is bounded in $L^\infty(\mathbb{H}^n)$. Similarly, if $p > \wp^*$ then $\mathfrak{q} > \wp^*$ and so

$$\sup_{\eta \in \mathbb{H}^n} \int\limits_{B_R(\eta)} |u_k|^{\mathfrak{q}} d\xi = \sup_{\eta \in \mathbb{H}^n} \int\limits_{B_R(\eta)} |u_k|^{\mathfrak{q}-\wp^*} |u_k|^{\wp^*} d\xi \le \|u_k\|_\infty^{\mathfrak{q}-\wp^*} \sup_{\eta \in \mathbb{H}^n} \int\limits_{B_R(\eta)} |u_k|^{\wp^*} d\xi$$

$$\le \|u_k\|_\infty^{\mathfrak{q}-\wp^*} c_R \sup_{\eta \in \mathbb{H}^n} \int\limits_{B_R(\eta)} |u_k|^p d\xi = o(1)$$

as $k \to \infty$, where $c_R = |B_R(\eta)|^{(p-\wp^*)/p} = R^{Q(p-\wp^*)/p}$, since $(u_k)_k$ is bounded in $L^\infty(\mathbb{H}^n)$. Fix now $\bar p > 1$ such that $p < \bar p$ and $p < (\bar p - 1)\wp' < \infty$, where $\wp'$ is the Hölder conjugate of $\wp$. It follows from the definition of $\bar p$, that (4.34) holds for $\mathfrak{q} = \bar p$ and $\mathfrak{q} = (\bar p - 1)\wp'$, that is

$$\sup_{\eta \in \mathbb{H}^n} \int\limits_{B_R(\eta)} |u_k|^{\bar p} d\xi = o(1), \quad \sup_{\eta \in \mathbb{H}^n} \int\limits_{B_R(\eta)} |u_k|^{(\bar p - 1)\wp'} d\xi = o(1) \tag{4.35}$$

as $k \to \infty$. Therefore, the Hölder inequality gives

$$\sup_{\eta \in \mathbb{H}^n} \int\limits_{B_R(\eta)} |u_k|^{\bar p - 1} |D_H u_k|_H d\xi$$

$$\le \sup_{\eta \in \mathbb{H}^n} \left\{ \left(\int\limits_{B_R(\eta)} |u_k|^{(\bar p - 1)\wp'} d\xi \right)^{1/\wp'} \cdot \left(\int\limits_{B_R(\eta)} |D_H u_k|_H^\wp d\xi \right)^{1/\wp} \right\} \tag{4.36}$$

$$\le \|D_H u_k\|_\wp \cdot \sup_{\eta \in \mathbb{H}^n} \left(\int\limits_{B_R(\eta)} |u_k|^{(\bar p - 1)\wp'} d\xi \right)^{1/\wp'} = o(1)$$

as $k \to \infty$, since $(D_H u_k)_k$ is bounded in $L^\wp(\mathbb{H}^n, \mathbb{R}^{2n})$ and (4.35) holds. Consequently, from (4.35) and (4.36), we get the existence of a sequence $(\varepsilon_k)_k$, independent of η, such that $\varepsilon_k \to 0$ as $k \to \infty$ and

$$\int\limits_{B_R(\eta)} |u_k|^{\bar p} d\xi + \int\limits_{B_R(\eta)} \bar p |u_k|^{\bar p - 1} |D_H u_k|_H d\xi \le \varepsilon_k \tag{4.37}$$

for all $k \in \mathbb{N}$. Clearly, for all $\eta \in \mathbb{H}^n$, we have $|u_k|^{\bar p} \in L^1(B_R(\eta))$. Furthermore, the Hölder inequality yields

$$|D_H(|u_k|^{\bar p})|_H = \bar p |u_k|^{\bar p - 1} |D_H u_k|_H \in L^1(B_R(\eta))$$

for all $\eta \in \mathbb{H}^n$. Consequently, $|u_k|^{\bar p} \in HW^{1,1}(B_R(\eta))$. Fix $\mathfrak{r} \in (1, Q/(Q-1))$. Then the embedding Theorem 2.3, yields the existence of a constant C_R, independent of η, such that

$$\int\limits_{B_R(\eta)} |u_k|^{\bar p \mathfrak{r}} d\xi \le C_R^{\mathfrak{r}} \left(\int\limits_{B_R(\eta)} |u_k|^{\bar p} d\xi + \int\limits_{B_R(\eta)} \bar p |u_k|^{\bar p - 1} |D_H u_k|_H d\xi \right)^{\mathfrak{r}}$$

$$\le C_R^{\mathfrak{r}} \varepsilon_k^{\mathfrak{r}-1} \left(\int\limits_{B_R(\eta)} |u_k|^{\bar p} d\xi + \int\limits_{B_R(\eta)} \bar p |u_k|^{\bar p - 1} |D_H u_k|_H d\xi \right), \tag{4.38}$$

where ε_k is introduced in (4.37). Moreover, $(u_k)_k$ is bounded in $L^{\bar{p}}(\mathbb{H}^n)$ and in $L^{(\bar{p}-1)\wp'}(\mathbb{H}^n)$ by the interpolation theorem, since $(u_k)_k$ is bounded in $L^p(\mathbb{H}^n)$ and in $L^\infty(\mathbb{H}^n)$. Therefore, since $(D_H u_k)_k$ is bounded in $L^\wp(\mathbb{H}^n, \mathbb{R}^{2n})$, the Hölder inequality gives

$$
\int_{\mathbb{H}^n} \left\{ |u_k|^{\bar{p}} + \bar{p}|u_k|^{\bar{p}-1}|D_H u_k|_H \right\} d\xi
$$

$$
\leq \int_{\mathbb{H}^n} |u_k|^{\bar{p}} d\xi + \bar{p} \left(\int_{\mathbb{H}^n} |u_k|^{(\bar{p}-1)\wp'} d\xi \right)^{1/\wp'} \cdot \left(\int_{\mathbb{H}^n} |D_H u_k|_H^\wp d\xi \right)^{1/\wp} \leq c, \tag{4.39}
$$

where c is a number independent of k. Now, from Lemma 2.3 in [46], there exists a sequence $(\eta_j)_j \subset \mathbb{H}^n$ such that $\mathbb{H}^n = \bigcup_{j=1}^\infty B_R(\eta_j)$ and each $\xi \in \mathbb{H}^n$ is covered by at most 24^Q balls $B_R(\eta_j)$. Hence, from (4.38) and (4.39), we have

$$
\int_{\mathbb{H}^n} |u_k|^{\bar{p}\mathfrak{r}} d\xi \leq \sum_{j=1}^\infty \int_{B_R(\eta_j)} |u_k|^{\bar{p}\mathfrak{r}} d\xi \leq (24)^Q C_R^{\mathfrak{r}} \varepsilon_k^{\mathfrak{r}-1} \left(\int_{\mathbb{H}^n} \left\{ |u_k|^{\bar{p}} + \bar{p}|u_k|^{\bar{p}-1}|D_H(u_k)|_H \right\} d\xi \right)
$$

$$
\leq C \varepsilon_k^{\mathfrak{r}-1} = o(1)
$$

as $k \to 0$, where $C = (24)^Q C_R^{\mathfrak{r}} c$. Consequently,

$$
u_k \to 0 \text{ in } L^{\bar{p}\mathfrak{r}}(\mathbb{H}^n) \tag{4.40}
$$

for any $\mathfrak{r} \in (1, Q/(Q-1))$ and any $\bar{p}$, with $p < \bar{p}$ and $p < (\bar{p}-1)\wp' < \infty$.

Fix now $\mathfrak{p}$ between p and $\wp^\star$ and $\mathfrak{r} \in (1, Q/(Q-1))$. In the case $p < \wp^\star$, we can choose $\bar{p}$ sufficiently big so that $\bar{p}\mathfrak{r} > \wp^\star$. Then, by the interpolation theorem applied to p, $\mathfrak{p}$, and $\bar{p}\mathfrak{r}$, since $p < \mathfrak{p} < \wp^\star < \bar{p}\mathfrak{r}$, we get for a suitable $\tau \in (0, 1)$

$$
\|u_k\|_{\mathfrak{p}} \leq \|u_k\|_p^\tau \|u_k\|_{\bar{p}\mathfrak{r}}^{1-\tau} = o(1) \text{ as } k \to \infty,
$$

since $(u_k)_k$ is bounded in $L^p(\mathbb{H}^n)$ and (4.40) holds. Similarly, in the case $p > \wp^\star$, we choose $\bar{p}$ sufficiently big so that $\bar{p}\mathfrak{r} > p$ and we apply the interpolation theorem to $\wp^\star$, $\mathfrak{p}$, and $\bar{p}\mathfrak{r}$. Thus, we obtain for a suitable $\tau \in (0, 1)$

$$
\|u_k\|_{\mathfrak{p}} \leq \|u_k\|_{\wp^\star}^\tau \|u_k\|_{\bar{p}\mathfrak{r}}^{1-\tau} = o(1) \text{ as } k \to \infty,
$$

since (4.40) holds, $\wp^\star < \mathfrak{p} < p < \bar{p}\mathfrak{r}$ and $(u_k)_k$ is bounded in $L^{\wp^\star}(\mathbb{H}^n)$ by the Folland–Stein inequality (3.1). In conclusion, in all the cases, $u_k \to 0$ in $L^\mathfrak{p}(\mathbb{H}^n)$ as $k \to \infty$ for all $\mathfrak{p}$ between p and $\wp^\star$, and this concludes the proof of Case 1.

Case 2. General case. Fix $N \in \mathbb{N}$ and put $v_k = \min\{|u_k|, N\}$ for all $k \in \mathbb{N}$. Clearly, $(v_k)_k$ is a bounded sequence in $L^\infty(\mathbb{H}^n)$. Then, from Case 1, it results

$$
v_k \to 0 \text{ in } L^\mathfrak{p}(\mathbb{H}^n) \tag{4.41}
$$

for all $\mathfrak{p}$ between p and $\wp^\star$. Fix now $\mathfrak{p}$ and $\mathfrak{q}_1$ between p and $\wp^\star$, with $\mathfrak{q}_1 > \mathfrak{p}$. By the interpolation theorem, $(u_k)_k$ is bounded in $L^{\mathfrak{q}_1}(\mathbb{H}^n)$, since $(u_k)_k$ is bounded in $L^p(\mathbb{H}^n)$ and also in $L^{\wp^\star}(\mathbb{H}^n)$ by the Folland–Stein inequality. Then, by the definition of v_k,

$$
\int_{\mathbb{H}^n} |u_k|^\mathfrak{p} d\xi = \int_{|u_k| \leq N} |u_k|^\mathfrak{p} d\xi + \int_{|u_k| \geq N} |u_k|^\mathfrak{p} d\xi = \int_{\mathbb{H}^n} |v_k|^\mathfrak{p} d\xi + \int_{|u_k| \geq N} |u_k|^{\mathfrak{p}-\mathfrak{q}_1} |u_k|^{\mathfrak{q}_1} d\xi
$$

$$
\leq \int_{\mathbb{H}^n} |v_k|^\mathfrak{p} d\xi + \frac{1}{N^{\mathfrak{q}_1-\mathfrak{p}}} \int_{\mathbb{H}^n} |u_k|^{\mathfrak{q}_1} d\xi \leq \int_{\mathbb{H}^n} |v_k|^\mathfrak{p} d\xi + \frac{C}{N^{\mathfrak{q}_1-\mathfrak{p}}},
$$

where C is a nonnegative constant independent of k. Consequently, from (4.41) we get

$$
\limsup_{k \to \infty} \int_{\mathbb{H}^n} |u_k|^\mathfrak{p} d\xi \leq \frac{C}{N^{\mathfrak{q}_1-\mathfrak{p}}} \quad \text{for all } N \in \mathbb{N}. \tag{4.42}
$$

Finally, passing to the limit as $N \to \infty$ in (4.42), we conclude the proof. $\qquad\square$

Theorem 1.3 holds in particular if we require that p and $\wp$ are such that $1 < p < \wp^\star$, and that the sequence $(u_k)_k$ is bounded in $HW^{1,p}(\mathbb{H}^n) \cap HW^{1,\wp}(\mathbb{H}^n)$. We shall apply Theorem 1.3 in this special case in the next Proposition 4.7, which is an alternative of Lions–type. The result we give is however a readaptation of Proposition 2.8 of [13] in the Heisenberg group setting.

Proposition 4.7. *For any $\lambda > 0$ let $\{(u_k, v_k)\}_k \subset W$ be a Palais–Smale sequence of I at level c_λ in (4.4) such that $(u_k, v_k) \rightharpoonup (0, 0)$ in W as $k \to \infty$. Then, either*

(i) $(u_k, v_k) \to (0, 0)$ in W, or

(ii) there exists $R > 0$ and a sequence $(\eta_k)_k \in \mathbb{H}^n$ such that

$$\limsup_{k \to \infty} \int_{B_R(\eta_k)} \left(|u_k|^p + |v_k|^p \right) d\xi > 0.$$

Moreover, $(\eta_k)_k$ is not bounded in $\mathbb{H}^n$.

Proof. Assume that (ii) does not occur. Then, for all $R > 0$

$$\lim_{k \to \infty} \sup_{\eta \in \mathbb{H}^n} \int_{B_R(\eta)} \left(|u_k|^p + |v_k|^p \right) d\xi = 0.$$

First, note that $(u_k)_k$ and $(v_k)_k$ are bounded in $L^p(\mathbb{H}^n)$, while $(D_H u_k)_k$ and $(D_H v_k)_k$ are bounded in $L^\wp(\mathbb{H}^n, \mathbb{R}^{2n})$. Therefore, Theorem 1.3 implies that $u_k \to 0$ and $v_k \to 0$ in $L^{\mathfrak{p}}(\mathbb{H}^n)$ as $k \to \infty$ for all $\mathfrak{p} \in (p, \wp^\star)$. Consequently, by (H) and (1.1), with $\varepsilon = 1$, we have

$$0 \leq \int_{\mathbb{H}^n} \left(H_u(u_k, v_k)u_k + H_v(u_k, v_k)v_k \right) d\xi \leq \int_{\mathbb{H}^n} (\mathfrak{m}|(u_k, v_k)|^{\mathfrak{m}} + m\, C_1|(u_k, v_k)|^m) d\xi \to 0$$

as $k \to \infty$, since $1 < p \leq \wp < \mathfrak{m} < m < \wp^\star$. Moreover, since $\{(u_k, v_k)\}_k \subset W$ is a Palais–Smale sequence of I at level c_λ, arguing as in the proof of Lemma 4.6, part (i), we know that there exists a number δ_λ such that (4.10) holds, that is in this casen

$$\int_{\mathbb{H}^n} |u_k|^\alpha |v_k|^\beta d\xi \to \delta_\lambda$$

as $k \to \infty$. Therefore,

$$\int_{\mathbb{H}^n} \left(A(|D_H u_k|_H)|D_H u_k|_H^2 + A(|D_H v_k|_H)|D_H v_k|_H^2 \right) d\xi + \int_{\mathbb{H}^n} \left(B(|u_k|)|u_k|^2 + B(|v_k|)|v_k|^2 \right) d\xi$$

$$= \int_{\mathbb{H}^n} |u_k|^\alpha |v_k|^\beta d\xi + o(1) = \delta_\lambda + o(1)$$

as $k \to \infty$. Then, (C_2) yields as $k \to \infty$ and $\lambda \to \infty$

$$a\left\{ \|u_k\|_{HW^{1,p}}^p + \|v_k\|_{HW^{1,p}}^p + \mathbb{1}_{\mathbb{R}^+}(\mathfrak{a}_1)\left(\|u_k\|_{HW^{1,q}}^q + \|v_k\|_{HW^{1,q}}^q \right) \right\}$$

$$\leq \int_{\mathbb{H}^n} \left(A(|D_H u_k|_H)|D_H u_k|_H^2 + A(|D_H v_k|_H)|D_H v_k|_H^2 \right) d\xi$$

$$+ \int_{\mathbb{H}^n} \left(B(|u_k|)|u_k|^2 + B(|v_k|)|v_k|^2 \right) d\xi + o_k(1) = o_{k,\lambda}(1),$$

where a is introduced in (4.7). Thus, $\|(u_k, v_k)\| \to 0$ as $k \to \infty$, and then (i) holds.

In order to prove the last claim, assume by contradiction that $(\eta_k)_k$ is bounded in $\mathbb{H}^n$. Consequently, there exists $M > 0$ so large that $B_R(\eta_k) \subset B_M$ for all k. Now, since $(u_k, v_k) \rightharpoonup (0, 0)$ in W as $k \to \infty$, and since the

embedding $W \hookrightarrow\hookrightarrow L^{\mathfrak{p}}(B_R) \times L^{\mathfrak{p}}(B_R)$ is compact for all $\mathfrak{p} \in [1, \wp^{\star})$ and all $R > 0$ thanks to Lemma 4.1, we have $(u_k, v_k) \to (0, 0)$ in $L^{\mathfrak{p}}(B_R) \times L^{\mathfrak{p}}(B_R)$ for all $\mathfrak{p} \in [1, \wp^{\star})$ and all $R > 0$. Therefore,

$$0 = \lim_{k \to \infty} \int_{B_M} \left(|u_k|^p + |v_k|^p\right) d\xi \geq \limsup_{k \to \infty} \int_{B_R(\eta_k)} \left(|u_k|^p + |v_k|^p\right) d\xi > 0,$$

which gives the required contradiction. Hence, $(\eta_k)_k$ is not bounded in $\mathbb{H}^n$ as stated. $\qquad\square$

Finally, thanks to Proposition 4.7, we are ready to prove the existence of nontrivial solutions for system $(\mathbb{S})$.

Proof of Theorem 1.1. First, thanks to Lemmas 4.4 and 4.6, for any $\lambda > 0$ the functional I has the geometry of the mountain pass theorem, and then I admits a Palais–Smale sequence $\{(u_k, v_k)\}_k$ at level c_λ which, up to a subsequence, still denoted by $\{(u_k, v_k)\}_k$, weakly converges to some limit $(u_\lambda, v_\lambda) \in W$. Moreover, as asserted in Lemma 4.6, part (*ii*), there exists a threshold $\lambda^\star > 0$ and the weak limit (u_λ, v_λ) is a critical point of I for all $\lambda \geq \lambda^\star$, namely a weak solution of $(\mathbb{S})$. Furthermore, as stated in Lemma 4.6, part (*iii*), the solution has the asymptotic property (1.2). It remains to show that the constructed solution (u_λ, v_λ) is nontrivial.

Assume by contradiction that $(u_\lambda, v_\lambda) = (0, 0)$. Clearly $\{(u_k, v_k)\}_k$ cannot converge strongly to $(0, 0)$ in W, since otherwise $I'(u_\lambda, v_\lambda) = 0$ and $0 = I(u_\lambda, v_\lambda) = c_\lambda > 0$ by Lemma 4.4. Therefore, by Proposition 4.7 there exist $R > 0$ and a sequence $(\eta_k)_k \in \mathbb{H}^n$ such that

$$\limsup_{k \to \infty} \int_{B_R(\eta_k)} \left(|u_k|^p + |v_k|^p\right) d\xi > 0. \tag{4.43}$$

Now, define a new sequence $\{(\widetilde{u}_k, \widetilde{v}_k)\}_k$, where $\widetilde{u}_k(\xi) = u_k(\xi \circ \eta_k)$, $\widetilde{v}_k(\xi) = v_k(\xi \circ \eta_k)$, for all $\xi \in \mathbb{H}^n$, where $\circ$ is the product in $\mathbb{H}^n$ defined in (2.1). Therefore, $I(\widetilde{u}_k, \widetilde{v}_k) = I(u_k, v_k)$ by the left invariance of the horizontal gradient and of the Haar measure. Moreover, for all $(\varphi, \psi) \in W$, with $\|(\varphi, \psi)\| = 1$, putting $\varphi_k(\xi) = \varphi(\xi \circ \eta_k^{-1})$ and $\psi_k(\xi) = \psi(\xi \circ \eta_k^{-1})$, $\xi \in \mathbb{H}^n$, by the change of variable $\widetilde{\xi} = \xi \circ \eta_k$ we have

$$\left| \int_{\mathbb{H}^n} \left\{ A(|D_H \widetilde{u}_k|_H)(D_H \widetilde{u}_k, D_H \varphi)_H + A(|D_H \widetilde{v}_k|_H)(D_H \widetilde{v}_k, D_H \psi)_H \right\} d\xi \right.$$

$$+ \int_{\mathbb{H}^n} \left\{ B(|\widetilde{u}_k|)\widetilde{u}_k \varphi + B(|\widetilde{v}_k|)\widetilde{v}_k \psi \right\} d\xi - \lambda \int_{\mathbb{H}^n} \left\{ H_u(\widetilde{u}_k, \widetilde{v}_k)\varphi + H_v(\widetilde{u}_k, \widetilde{v}_k)\psi \right\} d\xi$$

$$\left. - \frac{\alpha}{\wp^\star} \int_{\mathbb{H}^n} |\widetilde{u}_k|^{\alpha-2}\widetilde{u}_k|\widetilde{v}_k|^\beta \varphi d\xi + \frac{\beta}{\wp^\star} \int_{\mathbb{H}^n} |\widetilde{u}_k|^\alpha |v_k|^{\beta-2}\widetilde{v}_k \psi d\xi \right|$$

$$= \left| \int_{\mathbb{H}^n} \left\{ A(|D_H u_k|_H)(D_H u_k, D_H \varphi_k)_H + A(|D_H v_k|_H)(D_H v_k, D_H \psi_k)_H \right\} d\widetilde{\xi} \right.$$

$$+ \int_{\mathbb{H}^n} \left\{ B(|u_k|)u_k \varphi_k + B(|v_k|)v_k \psi_k \right\} d\widetilde{\xi} - \lambda \int_{\mathbb{H}^n} \left\{ H_u(u_k, v_k)\varphi_k + H_v(u_k, v_k)\psi_k \right\} d\widetilde{\xi}$$

$$\left. - \frac{\alpha}{\wp^\star} \int_{\mathbb{H}^n} |u_k|^{\alpha-2}u_k|v_k|^\beta \varphi_k d\widetilde{\xi} + \frac{\beta}{\wp^\star} \int_{\mathbb{H}^n} |u_k|^\alpha |v_k|^{\beta-2}v_k \psi_k d\widetilde{\xi} \right|$$

$$= \left| \langle I'(u_k, v_k), (\varphi_k, \psi_k) \rangle \right| \leq \|I'(u_k, v_k)\|_{W'} \|(\varphi_k, \psi_k)\| = \|I'(u_k, v_k)\|_{W'},$$

since $1 = \|(\varphi, \psi)\| = \|(\varphi_k, \psi_k)\|$. Then, as $k \to \infty$

$$\|I'(\widetilde{u}_k, \widetilde{v}_k)\|_{W'} = \sup_{\substack{(\varphi, \psi) \in W \\ \|(\varphi, \psi)\| = 1}} \left| \langle I'(\widetilde{u}_k, \widetilde{v}_k), (\varphi, \psi) \rangle \right| \leq \|I'(u_k, v_k)\|_{W'} = o(1).$$

Therefore, the sequence $\{(\widetilde{u}_k, \widetilde{v}_k)\}_k$ is again a Palais–Smale sequence at level c_λ in (4.4). Thus $\{(\widetilde{u}_k, \widetilde{v}_k)\}_k$, up to a subsequence, weakly converges to some $(\widetilde{u}_\lambda, \widetilde{v}_\lambda)$ in W by Lemma 4.6. Furthermore, (4.43) yields

$$0 < \limsup_{k \to \infty} \int_{B_R(y_k)} \left(|u_k|^p + |v_k|^p\right) d\xi = \lim_{k \to \infty} \int_{B_R} \left(|\widetilde{u}_k|^p + |\widetilde{v}_k|^p\right) d\widetilde{\xi} = \int_{B_R} \left(|\widetilde{u}_\lambda|^p + |\widetilde{v}_\lambda|^p\right) d\widetilde{\xi}.$$

Hence, $(\widetilde{u}_\lambda, \widetilde{v}_\lambda) \neq (0, 0)$. Finally, Lemma 4.2 gives that both components of $(\widetilde{u}_\lambda, \widetilde{v}_\lambda)$ are nontrivial, and this concludes the proof. □

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