

Tej-Eddine Ghoul, Van Tien Nguyen*, and Hatem Zaag

Construction of type I blowup solutions for a higher order semilinear parabolic equation

<https://doi.org/10.1515/anona-2020-0006>

Received May 17, 2018; accepted December 13, 2018.

Abstract: We consider the higher-order semilinear parabolic equation

$$\partial_t u = -(-\Delta)^m u + u|u|^{p-1},$$

in the whole space \mathbb{R}^N , where $p > 1$ and $m \geq 1$ is an odd integer. We exhibit type I non self-similar blowup solutions for this equation and obtain a sharp description of its asymptotic behavior. The method of construction relies on the spectral analysis of a non self-adjoint linearized operator in an appropriate scaled variables setting. In view of known spectral and sectorial properties of the linearized operator obtained by Galaktionov [15], we revisit the technique developed by Merle-Zaag [23] for the classical case $m = 1$, which consists in two steps: the reduction of the problem to a finite dimensional one, then solving the finite dimensional problem by a classical topological argument based on the index theory. Our analysis provides a rigorous justification of a formal result in [15].

Keywords: Higher order parabolic equation, Blowup solution, Blowup profile, Stability

MSC: Primary: 35K50, 35B40; Secondary: 35K55, 35K57

1. Introduction

We are interested in the semilinear parabolic equation

$$\begin{cases} \partial_t u &= \mathbf{A}_m u + u|u|^{p-1}, \\ u(0) &= u_0 \end{cases} \quad \mathbf{A}_m \equiv -(-\Delta)^m, \quad (1.1)$$

where $u(t) : \mathbb{R}^N \rightarrow \mathbb{R}$ with $N \geq 1$, Δ stands for the standard Laplace operator in \mathbb{R}^N , and the exponents p and m are fixed,

$$p > 1 \quad \text{and} \quad m \in \mathbb{N}, \quad m \geq 1 \text{ odd}.$$

The higher-order semilinear parabolic equation (1.1) is a natural generation of the classical semilinear heat equation ($m = 1$). It arises in many physical applications such as theory of thin film, lubrication, convection-explosion, phase translation, or applications to structural mechanics (see the Petetier-Troy book [28] and references therein).

By standard results the local Cauchy problem for equation (1.1) can be solved in $L^1 \cap L^\infty$ thanks to the integral representation

$$u(t) = \mathcal{K}_m(t) * u_0 + \int_0^t \mathcal{K}_m(t-s) * u(s)|u(s)|^{p-1} ds, \quad (1.2)$$

Tej-Eddine Ghoul, New York University in Abu Dhabi, P.O. Box 129188, Abu Dhabi, United Arab Emirates, E-mail: teg6@nyu.edu

***Corresponding Author: Van Tien Nguyen**, New York University in Abu Dhabi, P.O. Box 129188, Abu Dhabi, United Arab Emirates, Email: Tien.Nguyen@nyu.edu

Hatem Zaag, Université Paris 13, Sorbonne Paris Cité, LAGA, CNRS (UMR 7539), F-93430, Villetaneuse, France, E-mail: Hatem.Zaag@univ-paris13.fr

where $\mathcal{K}_m(t)$ is the fundamental solution of the linear parabolic $\partial_t \mathcal{K}_m = \mathbf{A}_m \mathcal{K}_m$, defined via the inverse Fourier transform

$$\mathcal{K}_m(x, t) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-|\xi|^{2m} t - i(\xi \cdot x)} d\xi, \quad \mathcal{K}_m(x, 0) = \delta(x).$$

From Fujita [12] ($m = 1$) and Galaktionov-Pohozaev [14] ($m > 1$), we know that

$$p_F \triangleq 1 + \frac{2m}{N},$$

is the *critical Fujita exponent* for the problem in the following sense. If $p > p_F$, for any sufficient small initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ the Cauchy problem (1.1) admits a global solution satisfying $u(t) \rightarrow 0$ as $t \rightarrow +\infty$ uniformly in \mathbb{R}^N . If $p \in (1, p_F]$ and the initial data $u_0 \not\equiv 0$ with $\int_{\mathbb{R}^N} u_0 dx \geq 0$ for the nonlinearity $u|u|^{p-1}$ replaced by $|u|^p$ or $u_0 \geq 0$, then the corresponding solution to problem (1.1) blows up in some finite time $T > 0$, namely that

$$\lim_{t \rightarrow T} \|u(t)\|_{L^\infty(\mathbb{R}^N)} = +\infty.$$

Here T is called the blowup time, and a point $a \in \mathbb{R}^N$ is called a blowup point if and only if there exists a sequence $(a_n, t_n) \rightarrow (a, T)$ such that $|u(a_n, t_n)| \rightarrow +\infty$ as $n \rightarrow +\infty$. A solution of (1.1) is called Type I blowup if it satisfies

$$c(T-t)^{-\frac{1}{p-1}} \leq \|u(t)\|_{L^\infty} \leq C(T-t)^{-\frac{1}{p-1}}, \quad (1.3)$$

otherwise, it is of Type II blowup. In addition, we call a blowup solution *self-similar* if it is of the form

$$u(x, t) = (T-t)^{-\frac{1}{p-1}} \Phi(y), \quad y = \frac{x}{(T-t)^{\frac{1}{2m}}}, \quad (1.4)$$

where Φ is not identically constant. Obviously, the *self-similar* blowup solution is of Type I.

When $m = 1$, problem (1.1) reduces to the classical semilinear heat equation

$$\partial_t u = \Delta u + u|u|^{p-1}, \quad (1.5)$$

which has been extensively studied in the last four decades, and no review can be exhaustive. Given our interest in the construction of solutions with a prescribed blowup behavior, we only mention previous work in this direction. The first constructive result was given by Bricmont-Kupiainen [5] who showed the existence of type I blowup solution to equation (1.5) according to the asymptotic dynamic

$$\sup_{x \in \mathbb{R}^N} \left| (T-t)^{-\frac{1}{p-1}} u(x, t) - \kappa \left(1 + \frac{(p-1)}{4p} \frac{|x|^2}{(T-t)|\log(T-t)|} \right)^{-\frac{1}{p-1}} \right| \rightarrow 0 \quad \text{as } t \rightarrow T, \quad (1.6)$$

for some universal positive constant $\kappa = \kappa(p)$. Note that the authors of [5] also exhibited finite time blowup solutions that verify other asymptotic behaviors which are expected to be unstable. Note also that Bressan [3, 4] made a similar construction in the case of an exponential nonlinearity. Later, Merle-Zaag [23] suggested a modification of the argument of [5] and obtained the stability of the constructed solution verifying (1.6) under small perturbations of initial data. The stability of the asymptotic behavior (1.6) had been observed numerically by Berger-Kohn [2] (see also Nguyen [24] for other numerical analysis). In particular, Herrero-Velázquez [19] proved that the blowup dynamic (1.6) is *generic* in one dimensional case, and they announced the same for higher dimensional case (but never published it).

The method of [5] and [23] relies on the understanding of the spectral property of the linearized operator around an expected profile in the *similarity variables* setting. Roughly speaking, the linearized operator possesses a finite number of positive eigenvalues, a null eigenvalue and a negative spectrum; then they proceed in two steps:

- Reduction of an infinite dimensional problem to a finite dimensional one in the sense that the control of the error reduces to the control of the components corresponding to the positive eigenvalues.
- Solving the finite dimensional problem thanks to a classical topological argument based on the index theory.

This general two-step procedure has been extended to various situations such as the case of the complex Ginzburg-Landau equation by Masmoudi-Zaag [21], Nouaili-Zaag [27] (see also Zaag [31] for an earlier work); the complex semilinear heat equation with no variational structure by Duong [7], Nouaili-Zaag [26]; non-scaling invariant semilinear heat equations by Ebde-Zaag [9], Nguyen-Zaag [25], Duong-Nguyen-Zaag [8]. We also mention the work of Tayachi-Zaag [29, 30] and Ghoul-Nguyen-Zaag [16] dealing with a nonlinear heat equation with a double source depending on the solution and its gradient in some critical setting. In [17, 18], we successfully adapted the method to construct a stable blowup solution for a non variational semilinear parabolic system.

As for the present paper, we aim at extending the above mentioned method to construct for problem (1.1) finite time blowup solutions satisfying some prescribed asymptotic behavior. Although the general idea is the same as for the classical case (1.5), we would like to emphasize that the above mentioned strategy is heavy and its implementation never being straightforward, the context and difficulties are different for each specific problem. As a step forward to better understanding the blowup dynamics for (1.1), we obtain the following result.

Theorem 1.1 (Type I blowup solutions for (1.1) with a prescribed behavior). *Let $m \geq 1$ be an odd integer. There exists initial data $u_0 \in L^\infty(\mathbb{R}^N)$ such that the corresponding solution $u(t)$ to equation (1.1) satisfies*

(i) *$u(t)$ blows up in a finite time $T = T(u_0)$ only at the origin.*

(ii) *(Asymptotic behavior)*

$$\sup_{x \in \mathbb{R}^N} \left| (T-t)^{\frac{1}{p-1}} u(x, t) - \Phi \left(\frac{x}{[(T-t)|\log(T-t)|]^{\frac{1}{2m}}} \right) \right| \leq \frac{C}{|\log(T-t)|^{\frac{1}{2m}}}, \quad (1.7)$$

where

$$\Phi(\xi) = \kappa \left(1 + B_{m,p} |\xi|^{2m} \right)^{-\frac{1}{p-1}}, \quad (1.8)$$

with

$$\kappa = (p-1)^{-\frac{1}{p-1}}, \quad B_{m,p} = (-1)^{m+1} \frac{2(p-1)(2m)!}{p((4m)! - 2[(2m)!]^2)}. \quad (1.9)$$

(iii) *(Final blowup profile) There exists $u^* \in C^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{R})$ such that $u(x, t) \rightarrow u^*(x)$ as $t \rightarrow T$ uniformly on compact subsets of $\mathbb{R}^N \setminus \{0\}$, where*

$$u^*(x) \sim \kappa \left(\frac{B_{m,p} |x|^{2m}}{2m |\log |x||} \right)^{-\frac{1}{p-1}} \quad \text{as } |x| \rightarrow 0. \quad (1.10)$$

Remark 1.2. We believe that such a blowup profile (1.8) exists for all $m \in \mathbb{N}^*$. We note that the constant $B_{m,p} < 0$ when m is even (see (1.9)), so the profile Φ blows up on the finite interface $|\xi| = \xi_* = (-B_{m,p})^{-\frac{1}{2m}}$. This says that the case when m is even would lead to type II blowup solutions in the sense of (1.3). Although main ideas for a full justification of such a blowup behavior with m even remains the same, the proof would be very delicate and will be addressed in a separate work.

Remark 1.3. The blowup solution described in Theorem 1.1 is not self-similar in the sense of (1.4). Note that in contrast to blowup solutions of the classical second order semilinear heat equation (1.5), Budd-Galaktionov-Williams [6] through numerical and asymptotic calculations conjectured that there are at least $2 \lfloor \frac{m}{2} \rfloor$ non-trivial self-similar blowup solutions to (1.1), and that profiles having a single maximum correspond to stable (generic) self-similar blowup solutions.

Remark 1.4. The proof of Theorem 1.1 (in dimension $N = 1$ for simplicity) involves a detailed description of the set of initial data leading to the asymptotic dynamic (1.7). In particular, our initial data is roughly of the form (see formula (3.13) below)

$$u_0(x) = e^{-s_0} \left[\Phi \left(x e^{s_0/2m} \right) + \frac{A}{s_0^2} \chi \left(2x e^{s_0/2m}, s_0 \right) \sum_{k=1}^{2m-1} d_k \psi_k(x e^{s_0/2m}) \right],$$

where A and s_0 are large fixed constants, Φ is the profile defined by (1.8), χ is some smooth cut-off function, $(d_0, \dots, d_{2m-1}) \in \mathbb{R}^{2m}$ are free parameters, ψ_k , $0 \leq k \leq 2m-1$ are the eigenfunctions of the linearized operator (see Proposition 2.1 for a precise definition) corresponding to the positive eigenvalue $\lambda_k = 1 - \frac{k}{2m}$. Through a topological argument, we show that there exists a suitable choice of parameters (d_0, \dots, d_{2m-1}) such that the solution to equation (1.1) with the initial datum u_0 satisfies the conclusion of Theorem 1.1. In some sense, our constructed solution is $2(m-1)$ -codimension stable in the following sense. The $2m$ components of the linearized solution corresponding to $\lambda_k = 1 - \frac{k}{2m}$ have the exponential growth $e^{\lambda_k s}$. However, the first two modes corresponding to λ_0 and λ_1 can be eliminated by means of the time and space translation invariance of the problem. Hence, by fixing $2(m-1)$ directions $\psi_2, \dots, \psi_{2m-1}$ and perturbing the remaining components (in L^∞), we still obtain the same asymptotic dynamic (1.7) of the perturbed solution. The proof of $2(m-1)$ -codimension stability would require some Lipschitz regularity of the considered initial data set and it would be addressed separately in another work.

Remark 1.5. According to our construction, the asymptotic dynamic (1.7) lies on the center manifold generated by eigenfunctions corresponding to the null eigenvalue $\lambda_{2m} = 0$. Our analysis can be extended to construct for equation (1.1) a finite time blowup solution having a different asymptotic dynamic from (1.7). Such solutions particularly have asymptotic dynamics laying on the stable manifold generated by eigenfunctions corresponding to the negative eigenvalue $\lambda_k = 1 - \frac{k}{2m} < 0$ with $k \geq 2m+1$. As explained in Remark 1.4, the corresponding initial data leading to such solutions would involve k parameters with $k \geq 2m+1$ (consider $N = 1$), so that a topological argument is assigned in order to control the first k components corresponding to the eigenvectors ψ_j for $0 \leq j \leq k-1$. Although the constructive method are similar for all cases, we decide to only deal with the case of the center manifold, since the proof is the most delicate in the sense that it requires a more refined analysis in the blowup region leading to some logarithmic correction of the blowup variable as shown in (1.7).

Strategy of the proof.

Let us sketch the main ideas of the proof of Theorem 1.1.

- Similarity variables and linearized problem. According to the scaling invariance of the problem (1.1), we introduce the *similarity variables*

$$u(x, t) = (T-t)^{-\frac{1}{p-1}} w(y, s), \quad y = \frac{x}{(T-t)^{\frac{1}{2m}}}, \quad s = -\ln(T-t), \quad (1.11)$$

which leads to the new equation

$$\begin{aligned} \partial_s w &= \mathbf{A}_m w - \frac{1}{2m} y \cdot \nabla w - \frac{1}{p-1} w + w|w|^{p-1} \\ &\equiv \mathcal{L}_m w - \frac{p}{p-1} w + w|w|^{p-1}, \end{aligned} \quad (1.12)$$

where the linear operator \mathcal{L}_m is given by

$$\mathcal{L}_m = \mathbf{A}_m - \frac{1}{2m} y \cdot \nabla + \text{Id}. \quad (1.13)$$

Note that the change of variables (1.11) allows us to reduce the finite time blowup problem to a long time behavior one at the cost of an extra scaling term in the new equation (1.12). In the setting (1.11), proving the asymptotic dynamic (1.7) is equivalent to proving that

$$\sup_{y \in \mathbb{R}^N} \left| w(y, s) - \Phi \left(y s^{-\frac{1}{2m}} \right) \right| \rightarrow 0 \quad \text{as } s \rightarrow +\infty. \quad (1.14)$$

It is then natural to linearize equation (1.12) around the expected profile Φ by introducing

$$q = w - \Phi.$$

which leads to the equation of the form

$$\partial_s q = (\mathcal{L}_m + V(y, s))q + B(q) + R(y, s), \quad (1.15)$$

where $B(q)$ is built to be quadratic, R measures the error generated by Φ and is uniformly bounded by $\mathcal{O}(s^{-1})$, and V is the potential defined as

$$V(y, s) = p \left(\Phi^{p-1} - \kappa^{p-1} \right).$$

Our goal is to construct for equation (1.15) a solution q defined for all $(y, s) \in \mathbb{R}^N \times [s_0, +\infty)$ such that

$$\sup_{y \in \mathbb{R}^N} |q(y, s)| \rightarrow 0 \quad \text{as } s \rightarrow +\infty.$$

- *Properties of the linearized operator.* In view of equation (1.15), we see that the nonlinear quadratic and the error term are small and can be negligible in comparison with the linear term. Roughly speaking, the linear part will play an important role in the dynamic of the solution. It is essential to determine the spectrum and corresponding eigenfunctions of both \mathcal{L}_m and its adjoint \mathcal{L}_m^* . According to [15], the spectrum of the linear operator \mathcal{L}_m comprises real simple eigenvalues only,

$$\text{spec}(\mathcal{L}_m) = \left\{ \lambda_\beta = 1 - \frac{|\beta|}{2m}, \beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n, |\beta| = \beta_1 + \dots + \beta_n \right\},$$

and the corresponding eigenfunction ψ_β with $|\beta| = n$ is polynomial of order n (see Proposition 2.1 below). Moreover, the family of the eigenfunctions $\{\psi_\beta\}_{\beta \in \mathbb{N}^n}$ forms a complete subset in $L^2_\rho(\mathbb{R}^N)$ where ρ is some exponentially decaying weight function.

Depending on the asymptotic behavior of the potential V , we observe that

- Inside the blowup region, $|y| \leq Ks^{\frac{1}{2m}}$ for some K large, the effect of V is regarded as a perturbation of \mathcal{L}_m .
- Outside the blowup region, $|y| \geq Ks^{\frac{1}{2m}}$, the full linear part $\mathcal{L}_m + V$ behaves like $\mathcal{L}_m - \frac{p}{p-1}$, which has a purely negative spectrum. Hence, the control of the solution in this region is simple.

- *Decomposition of the solution and reduction to a finite dimensional problem.* According to the spectrum of \mathcal{L}_m , we decompose

$$q(y, s) = \sum_{|\beta|=0}^{2m} q_\beta(s) \psi_\beta(y) + q_-(y, s),$$

where q_- is the projection of q on the subspace of \mathcal{L}_m corresponding to strictly negative eigenvalues. Since the spectrum of the linear part of the equation satisfied by q_- is negative, it is controllable to zero. We would like to notice that we do not use the Feymann-Kac representation¹ as for the case $m = 1$ treated in [23], because of its complicated implementation for higher order cases $m \geq 2$. To avoid such a formula, we further decompose

$$q_-(y, s) = \sum_{|\beta|=2m+1}^M q_\beta(s) \psi_\beta + q_{M,\perp}(y, s),$$

for some M large enough (typically $M \geq 4\|V\|_{L^\infty_{y,s}}$). A direct projection yields

$$q'_\beta = \left(1 - \frac{|\beta|}{2m} \right) q_\beta + \mathcal{O} \left(s^{-\frac{|\beta|+2}{2m}} \right), \quad |\beta| = 2m+1, \dots, M,$$

¹ In [5] and [23], the kernel \mathcal{K} of the heat semigroup associated to the linear operator $\mathcal{L}_1 + V$ is defined through the Feymann-Kac representation

$$\mathcal{K}(s, \sigma, y, x) = e^{(s-\sigma)\mathcal{L}_1}(y, x) \int d\mu_{y,x}^{s-\sigma}(\omega) e^{\int_0^{s-\sigma} V(\omega(\tau), \sigma+\tau) d\tau},$$

where $e^{t\mathcal{L}_1}(y, x)$ is given by Mehler's formula and $d\mu_{y,x}^t$ is the oscillator measure on the continuous path: $\omega : [0, t] \rightarrow \mathbb{R}^N$ with $\omega(0) = x$ and $\omega(t) = y$.

from which we obtain the rough bound $|\theta_\beta(s)| = \mathcal{O}(s^{-\frac{|\beta|+2}{2m}})$ for $|\beta| = 2m+1, \dots, M$. For the infinite part $q_{M,\perp}$, we explore the properties of the semigroup $e^{s\mathcal{L}_m}$ and a standard Gronwall inequality to close the estimate for this part.

The control of the null mode q_{2m} is delicate since the potential has in some sense a critical size in our analysis. In particular, we need a careful refinement of the asymptotic behavior of V to derive the sharp ODE

$$q'_\beta = -\frac{2}{s}q_\beta + \mathcal{O}\left(\frac{1}{s^3}\right) \quad \text{for } |\beta| = 2m,$$

which shows a negative spectrum after changing the variable $\tau = \ln s$, hence, the rough bound $|q_\beta(s)| = \mathcal{O}\left(\frac{\log s}{s^2}\right)$ for $|\beta| = 2m$. Here the precise value of $B_{m,p}$ given in (1.9) is crucial for many algebraic identities to derive this sharp ODE. At this stage, we reduce the infinite dimensional problem to a finite dimensional one in the sense that it remains to control a finite number of positive modes q_β for $|\beta| \leq 2m-1$. This is done through a classical topological argument based on the index theory.

The rest of the paper is organized as follows: In Section 2 we recall basic spectral properties of the linearized operator \mathcal{L} and its adjoint \mathcal{L}^* , then we perform a formal spectral analysis to derive an approximate blowup profile served for our analysis later. In Section 3 we give all arguments for the proof of Theorem 1.1 without going to technical details (the reader who is not interested in the technical details can stop at this section). Section 4 is the hearth of our analysis: it is devoted to the study of the dynamic of the linearized problem from which we reduce the problem to a finite dimensional one.

Acknowledgments: The authors would like to thank the anonymous referees for their careful reading and suggestions to improve the presentation of the paper.

2. A formal approach via spectral analysis.

In this section we first recall basic spectral properties of the linearized operator, then present a formal approach based on a spectral analysis to derive the blowup profile given in Theorem 1.1.

2.1. Spectral properties of the linearized operator.

In this subsection, we recall from [15] the basic spectral properties of the linear operator \mathcal{L}_m and its adjoint \mathcal{L}_m^* . The case $m = 1$ is well known, since we can rewrite

$$\mathcal{L}_1 = \frac{1}{\rho_1} \nabla \cdot (\rho_1 \nabla) + \text{Id} \quad \text{with} \quad \rho_1(y) = (4\pi)^{-\frac{N}{2}} e^{-\frac{|y|^2}{4}},$$

which is a self-adjoint operator in the weighted Hilbert space $L^2(e^{-|y|^2/4} dy)$ with the domain $\mathcal{D}(\mathcal{L}) = H^2(e^{-|y|^2/4} dy)$. It has a real discrete spectrum and the corresponding eigenfunctions are derived from Hermite polynomials.

For $m \geq 2$, the operator \mathcal{L}_m is not symmetric and does not admit a self-adjoint extension. Then we denote \mathcal{L}_m^* the formal adjoint of \mathcal{L}_m as

$$\mathcal{L}_m^* f = \mathbf{A}_m f + \frac{1}{2m} \nabla \cdot (y f) + f, \quad (2.1)$$

in the sense that

$$\langle \mathcal{L}_m f, g \rangle = \langle f, \mathcal{L}_m^* g \rangle \quad \text{for } (f, g) \in \mathcal{D}(\mathcal{L}_m) \times \mathcal{D}(\mathcal{L}_m^*). \quad (2.2)$$

From [10] and [11], we know that the following elliptic equation

$$\mathbf{A}_m F + \frac{1}{2m} \nabla \cdot (y F) = 0 \quad \text{in } \mathbb{R}^N \quad \text{with} \quad \int_{\mathbb{R}^N} F(y) dy = 1. \quad (2.3)$$

has a unique radial solution given by the explicit formula

$$F(y) = (2\pi)^{-\frac{N}{2}} \int_0^\infty e^{-s^{2m}} s^{\frac{N}{2}} J_{\frac{N-2}{2}}(s|y|) ds, \quad (2.4)$$

with J_ν being the Bessel function. In particular, F satisfies the estimate

$$|F(y)| < D e^{-d|y|^\nu} \quad \text{in } \mathbb{R}^N, \quad \text{where } \nu = \frac{2m}{2m-1}, \quad (2.5)$$

for some positive constants D and d depending on m and N .

We introduce the weight functions

$$\rho^* = \rho^{-1}, \quad \rho(y) = e^{-a|y|^\nu}, \quad (2.6)$$

where $a = a(m, N) \in (0, d]$ is a small constant, d and ν are introduced in (2.5).

Proposition 2.1 (Spectral properties of \mathcal{L}_m and \mathcal{L}_m^* , [15]). *Let $m \in \mathbb{N}^*$, we have*

(i) $\mathcal{L}_m : H_\rho^{2m}(\mathbb{R}^N) \rightarrow L_\rho^2(\mathbb{R}^N)$ is a bounded linear operator with the spectrum

$$\text{spec}(\mathcal{L}_m) = \left\{ \lambda_\beta = 1 - \frac{|\beta|}{2m}, \quad \beta = (\beta_1, \dots, \beta_N) \in \mathbb{N}^N, \quad |\beta| = 0, 1, 2, \dots \right\}. \quad (2.7)$$

The set of eigenfunctions $\{\psi_\beta\}_{|\beta| \geq 0}$ is complete in $L_\rho^2(\mathbb{R}^N)$, where ψ_β has the separable decomposition

$$\psi_\beta(y) = \psi_{\beta_1}(y_1) \cdots \psi_{\beta_N}(y_N) \quad \text{with} \quad \psi_k(\xi) = \frac{1}{\sqrt{k!}} \sum_{j=0}^{\lfloor \frac{k}{2m} \rfloor} \frac{(-1)^{jm}}{j!} \partial_\xi^{2jm} \xi^k. \quad (2.8)$$

(ii) $\mathcal{L}_m^* : H_{\rho^*}^{2m}(\mathbb{R}^N) \rightarrow L_{\rho^*}^2(\mathbb{R}^N)$ is a bounded linear operator with the spectrum

$$\text{spec}(\mathcal{L}_m^*) = \left\{ \lambda_\beta^* = 1 - \frac{|\beta|}{2m}, \quad \beta = (\beta_1, \dots, \beta_N) \in \mathbb{N}^N, \quad |\beta| = 0, 1, 2, \dots \right\}. \quad (2.9)$$

The set of eigenfunctions $\{\psi_\beta^*\}_{|\beta| \geq 0}$ is complete in $L_{\rho^*}^2(\mathbb{R}^N)$, where ψ_β^* 's are given by

$$\psi_\beta^*(y) = \frac{(-1)^{|\beta|}}{\sqrt{\beta!}} \partial_y^\beta F(y), \quad (2.10)$$

where $\beta! = \beta_1! \cdots \beta_N!$, $\partial_y^\beta = \frac{\partial^{\beta_1}}{\partial y_1} \cdots \frac{\partial^{\beta_N}}{\partial y_N}$ and the function F is defined by (2.4).

(iii) (Orthogonality)

$$\langle \psi_\beta, \psi_\gamma^* \rangle = \delta_{\beta, \gamma}, \quad (2.11)$$

where $\delta_{\beta, \gamma}$ is the Kronecker delta.

Remark 2.2. We note from the orthogonality (2.11) and the definition of ψ_k that for all polynomials $P_n(y)$ of degree $n < |y|$, we have $\langle P_n, \psi_y^* \rangle = 0$.

For $N = 1$, we have

$$\begin{aligned} \psi_k(y) &= a_k y^k \quad \text{for } 0 \leq k < 2m, \quad a_k = \frac{1}{\sqrt{k!}}, \\ \psi_{2mj}(y) &= \sum_{i=0}^j c_{2mi} y^{2mi} \quad \text{and} \quad y^{2mj} = \sum_{i=0}^j c'_{2mi} \psi_{2mi}(y) \quad \text{for } j \in \mathbb{N}. \end{aligned}$$

We end this subsection by recalling basic properties of the semigroup $e^{s\mathcal{L}_m}$ for $s > 0$.

Lemma 2.3 (Properties of the semigroup $e^{s\mathcal{L}_m}$). *The kernel of the semigroup $e^{s\mathcal{L}_m}$ is given by*

$$e^{s\mathcal{L}_m}(y, x) = \frac{e^s}{[(1 - e^{-s})]^{\frac{N}{2m}}} F\left(\frac{ye^{-\frac{s}{2m}} - x}{[2m(1 - e^{-s})]^{\frac{1}{2m}}}\right), \quad \forall s > 0, \quad (2.12)$$

where F is defined as in (2.4). The action of $e^{s\mathcal{L}_m}$ is defined by

$$e^{s\mathcal{L}_m}g(y) = \int_{\mathbb{R}^N} e^{s\mathcal{L}_m}(y, x)g(x)dx. \quad (2.13)$$

We also have the following estimates:

- (i) $\|e^{s\mathcal{L}_m}g\|_{L^\infty} \leq Ce^s\|g\|_{L^\infty}$ for all $g \in L^\infty(\mathbb{R}^N)$.
- (ii) $\|e^{s\mathcal{L}_m}\operatorname{div}(g)\|_{L^\infty} \leq Ce^s(1 - e^{-s})^{-\frac{1}{2m}}\|g\|_{L^\infty}$ for all $g \in L^\infty(\mathbb{R}^N)$.
- (iii) If $|f(x)| \leq \eta(1 + |x|^{M+1})$ for all $x \in \mathbb{R}^N$, then

$$\left|e^{s\mathcal{L}_m}\Pi_{M,\perp}f(y)\right| \leq C\eta e^{-\frac{Ms}{2m}}(1 + |y|^{M+1}), \quad \forall y \in \mathbb{R}^N, \quad (2.14)$$

where $\Pi_{M,\perp}$ is defined as in (3.10).

Proof. The formula (2.12) can be verified by a direct computation thanks to equation (2.3). The estimates (i)-(iii) are straightforward from the definitions (2.12) and (2.13). \square

2.2. Approximate blowup profile.

In this subsection we recall the formal approach of [13] (see also [6]) to figure out an appropriate blowup profile for our analysis later. This approach had been used in several problems involving the second order Laplacian, see for example [5], [30], [16–18]. The argument relies on the basis of the known spectral properties of the rescaled operator \mathcal{L}_m and its adjoint \mathcal{L}_m^* given in the previous subsection. For simplicity, we consider the one dimensional case and symmetric positive solutions.

Let us introduce

$$\bar{w} = w - \kappa,$$

where $\kappa = (p-1)^{-\frac{1}{p-1}}$ is the constant equilibrium to equation (1.12). This yields the following perturbed equation

$$\partial_s \bar{w} = \mathcal{L}_m \bar{w} + \frac{p}{2\kappa} \bar{w}^2 + R(\bar{w}), \quad (2.15)$$

where $|R(\bar{w})| \leq C|\bar{w}|^3$ for $|\bar{w}| \ll 1$.

From Proposition 2.1, we know that ψ_n with $n \geq 2m+1$ correspond to negative eigenvalues of \mathcal{L}_m . Therefore, we may consider

$$\bar{w}(y, s) = \sum_{i=0}^m \bar{w}_{2i}(s)\psi_{2i}(y), \quad (2.16)$$

where $\sum_{i=0}^m |\bar{w}_{2i}(s)| \rightarrow 0$ as $s \rightarrow +\infty$. Plugging this ansatz to equation (2.15), taking the scalar product with ψ_{2j}^* and using Proposition 2.1, we find that for $0 \leq j \leq m$,

$$\bar{w}'_{2j}(s) = \left(1 - \frac{j}{m}\right) w_{2j}(s) + \frac{p}{2\kappa} \left\langle \left(\sum_{i=0}^m \bar{w}_{2i}(s)\psi_{2i}(y)\right)^2, \psi_{2j}^* \right\rangle + \mathcal{O}\left(\sum_{i=0}^m |\bar{w}_{2i}(s)|^3\right). \quad (2.17)$$

Assuming that \bar{w}_{2m} is dominant, i.e.

$$\text{for } 0 \leq j \leq m-1, \quad |\bar{w}_{2j}(s)| \ll |\bar{w}_{2m}(s)| \quad \text{as } s \rightarrow +\infty, \quad (2.18)$$

then system (2.17) reduces to

$$\begin{aligned}\bar{w}'_{2j}(s) &= \left(1 - \frac{j}{m}\right) w_{2j}(s) + \mathcal{O}(|\bar{w}_{2m}(s)|^2) \quad \text{for } 0 \leq j \leq m-1, \\ \bar{w}'_{2m}(s) &= \frac{p}{2\kappa} \bar{\mu}_{2m} \bar{w}_{2m}^2 + o(|\bar{w}_{2m}(s)|^2), \quad \text{where } \bar{\mu}_{2m} = \langle \psi_{2m}^2, \psi_{2m}^* \rangle.\end{aligned}$$

Solving this system yields

$$\bar{w}_{2m}(s) = -\frac{2\kappa}{p\bar{\mu}_{2m}} \frac{1}{s} + \mathcal{O}\left(\frac{\ln^2 s}{s}\right), \quad \sum_{i=0}^{m-1} |\bar{w}_{2i}(s)| = \mathcal{O}\left(\frac{1}{s^2}\right) \quad \text{as } s \rightarrow +\infty,$$

which is in agreement with the assumption (2.18).

Hence, from (2.16) and the definition of ψ_{2m} , we derive the following asymptotic behavior

$$w(y, s) = \kappa - \frac{2\kappa}{p\bar{\mu}_{2m}s} \left(\frac{y^{2m} + (-1)^m (2m)!}{\sqrt{(2m)!}} \right) + \mathcal{O}\left(\frac{\ln s}{s^2}\right), \quad (2.19)$$

where the convergence takes place in $L^2_\rho(\mathbb{R})$ as well as uniformly in compact sets by standard parabolic regularity.

The expansion (2.19) provides a relevant variable for blowup, namely $z = ys^{-\frac{1}{2m}}$, that governs the behavior in the intermediate region. In particular, we try to search formally a solution w of equation (1.12) of the form

$$w(y, s) = \Phi(z) + \frac{(-1)^{m+1} 2\kappa \sqrt{(2m)!}}{p\bar{\mu}_{2m}s} + \mathcal{O}\left(\frac{1}{s^{1+\epsilon}}\right), \quad (2.20)$$

for some $\epsilon > 0$, with the boundary condition $\Phi(0) = \kappa$.

Plugging this ansatz to equation (1.12) and comparing the leading order terms, we arrive at

$$-\frac{z}{2m} \Phi' - \frac{1}{p-1} \Phi + \Phi^p = 0, \quad \Phi(0) = \kappa. \quad (2.21)$$

Solving this ODE yields

$$\Phi(z) = \kappa \left(1 + B_{m,p} z^{2m}\right)^{-\frac{1}{p-1}} \quad \text{for some } B_{m,p} \in \mathbb{R}.$$

By matching expansions (2.19) and (2.20), we find that

$$B_{m,p} = \frac{2(p-1)}{p\bar{\mu}_{2m}\sqrt{(2m)!}}, \quad \bar{\mu}_{2m} = \langle \psi_{2m}^2, \psi_{2m}^* \rangle.$$

Let us compute $\bar{\mu}_{2m}$ in the following. Using (2.8), we write

$$\begin{aligned}\psi_{2m}(y) &= \frac{1}{\sqrt{(2m)!}} (y^{2m} + (-1)^m (2m)!), \\ \psi_{4m}(y) &= \frac{1}{\sqrt{(4m)!}} \left(y^{4m} + (-1)^m \frac{(4m)!}{(2m)!} y^{2m} + (4m)! \right), \\ \psi_{2m}^2(y) &= \frac{1}{(2m)!} \left[\sqrt{(4m)!} \psi_{4m} + (-1)^{m+1} \sqrt{(2m)!} \left(\frac{(4m)!}{(2m)!} - 2(2m)! \right) \psi_{2m} + c_0(m) \psi_0 \right].\end{aligned}$$

Using the orthogonality relation (2.11), the definition (2.10) of ψ_{2m}^* and the fact that $\int_{\mathbb{R}} F(y) dy = 1$, we compute by an integration by parts,

$$\begin{aligned}\bar{\mu}_{2m} &= \frac{(-1)^{m+1}}{\sqrt{(2m)!}} \left(\frac{(4m)!}{(2m)!} - 2(2m)! \right) \int_{\mathbb{R}} \psi_{2m}(y) \psi_{2m}^*(y) dy \\ &= \frac{(-1)^{m+1}}{\sqrt{(2m)!}} \left(\frac{(4m)!}{[(2m)!]^2} - 2 \right) \int_{\mathbb{R}} (y^{2m} + c_{2m,0}) \partial_y^{2m} F(y) dy \\ &= (-1)^{m+1} \frac{(2m)}{\sqrt{(2m)!}} \left(\frac{(4m)!}{[(2m)!]^2} - 2 \right) \int_{\mathbb{R}} F(y) dy \\ &= (-1)^{m+1} \sqrt{(2m)!} \left(\frac{(4m)!}{[(2m)!]^2} - 2 \right).\end{aligned} \quad (2.22)$$

In conclusion, we have derived the following candidate for the blowup profile in the similarity variables:

$$w(y, s) \sim \varphi(y, s) := \kappa \left(1 + B_{m,p} \frac{y^{2m}}{s} \right)^{-\frac{1}{p-1}} + \frac{A_{m,p}}{s}, \quad (2.23)$$

where

$$B_{m,p} = \frac{2(p-1)}{p\bar{\mu}_{2m}\sqrt{(2m)!}}, \quad A_{m,p} = \frac{(-1)^{m+1}2\kappa\sqrt{(2m)!}}{p\bar{\mu}_{2m}}. \quad (2.24)$$

Since we want the profile φ bounded, this requests $B_{m,p} > 0$ or $\bar{\mu}_{2m} > 0$, which only happens for m odd.

3. Proof of Theorem 1.1 without technical details.

In this section we give all arguments of the proof of Theorem 1.1. We only deal with the one dimensional case $N = 1$ for simplicity since the analysis for the higher dimensional cases $N \geq 2$ is exactly the same up to some complicated calculation of the projection of (1.15) on the eigenspaces of \mathcal{L}_m . The proof of Theorem 1.1 is completed in three parts:

- In the first part we formulate the problem by linearizing the rescaled equation (1.12) around the approximate profile φ given by (2.23). We also introduce a shrinking set in which the constructed solution of the linearized equation is trapped.
- In the second part we exhibit an explicit formula of the initial data and show that the corresponding solution belongs to the shrinking set. In particular, the reader can find how to reduce the problem to a finite dimensional one (all technical details will be left to the next section) and the use of a topological argument based on index theory to conclude.
- The last part is devoted to the proof of items (i) and (iii) of Theorem 1.1. Although the argument of the proof is almost the same as for the classical case $m = 1$, we would like to sketch the main ideas for the reader convenience.

3.1. Formulation of the problem.

According to the formal analysis given in Section 2.2, we introduce

$$q(y, s) = w(y, s) - \varphi(y, s), \quad (3.1)$$

and write from (1.12) the equation driving by q ,

$$\partial_s q = (\mathcal{L}_m + V(y, s))q + B(q) + R(y, s), \quad (3.2)$$

where \mathcal{L}_m is the linearized operator defined by (1.13) and

$$V(y, s) = p \left(\varphi^{p-1} - \kappa^{p-1} \right), \quad (3.3)$$

$$B(q) = (q + \varphi)|q + \varphi|^{p-1} - \varphi^p - p\varphi^{p-1}q, \quad (3.4)$$

$$R(y, s) = -\partial_s \varphi + \mathbf{A}_m \varphi - \frac{1}{2m} y \cdot \nabla \varphi - \frac{\varphi}{p-1} + \varphi^p. \quad (3.5)$$

Let $\chi_0 \in \mathcal{C}_0^\infty(\mathbb{R}_+)$ be a cut-off function with $\text{supp}(\chi_0) \subset [0, 2]$ and $\chi_0 \equiv 1$ on $[0, 1]$. We introduce

$$\chi(y, s) = \chi_0 \left(\frac{|y|}{Ks^{\frac{1}{2m}}} \right) \quad \text{with } K \gg 1 \text{ fixed}, \quad (3.6)$$

and define

$$q_e(y, s) = q(y, s)(1 - \chi(y, s)). \quad (3.7)$$

We decompose

$$q(y, s) = \sum_{k=0}^M q_k(s) \psi_k(y) + q_{M,\perp}(y, s), \quad (3.8)$$

where

- $q_k = \Pi_k(q)$ is the projection of q on the eigenmode corresponding to eigenvalue $\lambda_k = 1 - \frac{k}{2m}$, defined as

$$q_k(s) = \langle q, \psi_k^* \rangle \quad \text{with } \psi_k^* \text{ being introduced in (2.10)}. \quad (3.9)$$

- $q_{M,\perp} = \Pi_{M,\perp}(q)$ is called the infinite dimensional part of q , where $\Pi_{M,\perp}(q)$ is the projection of q on the eigensubspace where the spectrum of \mathcal{L}_m is lower than $\frac{1-M}{2m}$. Note that we have the orthogonality

$$\langle q_{M,\perp}, \psi_k^* \rangle = 0 \quad \text{for } k \leq M. \quad (3.10)$$

- M is typically a large constant

$$M = 4mN \quad \text{with } N \in \mathbb{N}^*, N \geq \|V\|_{L_{y,s}^\infty}. \quad (3.11)$$

which allows us to successfully apply a standard Gronwall's inequality to the control of the infinite dimensional part $q_{M,\perp}$.

We aim at constructing for equation (1.15) a global in time solution q such that

$$\|q(s)\|_{L^\infty(\mathbb{R}^N)} \rightarrow 0 \quad \text{as } s \rightarrow +\infty. \quad (3.12)$$

According to the decomposition (3.8), it is enough to show that there exists a solution q belonging to the following set.

Definition 3.1 (Shrinking set to trap solutions). For each $A > 0$, for each $s > 0$, we denote $\mathcal{V}_A(s)$ the set of all functions $q(y, s)$ in $L^\infty(\mathbb{R})$ such that

$$\begin{aligned} |q_k(s)| &\leq \frac{A}{s^2} \quad \text{for } 0 \leq k \leq 2m-1, & |q_{2m}(s)| &\leq \frac{A^2 \log s}{s^2}, \\ |q_k(s)| &\leq A^k s^{-\frac{k+1}{2m}} \quad \text{for } 2m+1 \leq k \leq M, & \|q_e(s)\|_{L^\infty(\mathbb{R})} &\leq A^{M+2} s^{-\frac{1}{2m}}, \\ \forall y \in \mathbb{R}, & |q_{M,\perp}(y, s)| &\leq A^{M+1} s^{-\frac{M+2}{2m}} (|y|^{M+1} + 1), \end{aligned}$$

where q_k , $q_{M,\perp}$ and q_e are defined as in the decomposition (3.8) and (3.7).

Remark 3.2. By definition, we see that if $q(s) \in \mathcal{V}_A(s)$, then the following bound holds

$$\begin{aligned} |q(y, s)| &\leq CA^{M+1} \left(\sum_{k=0}^{M+1} (|y|^k + 1) s^{-\frac{k+1}{2m}} \right) \mathbf{1}_{\{|y| \leq 2Ks^{\frac{1}{2m}}\}} + \|q_e\|_{L^\infty(\mathbb{R}^N)} \\ &\leq CA^{M+2} s^{-\frac{1}{2m}} \quad \text{for all } y \in \mathbb{R}. \end{aligned}$$

Hence, our goal (3.12) reduces to constructing for equation (1.15) a solution $q(s)$ belonging to the shrinking set $\mathcal{V}_A(s)$ for all $s \in [s_0, +\infty)$.

3.2. Existence of solutions trapped in \mathcal{V}_A .

In this step we aim at proving that there actually exists initial data $\phi(y) = q(y, s_0)$ such that the corresponding solution $q(y, s)$ to (1.15) belongs to the shrinking set $\mathcal{V}_A(s)$. Given $A \geq 1$ and $s_0 \geq e$, we consider initial data of the form

$$\phi_{A,s_0,d}(y) = \frac{A}{s_0^2} \chi(2y, s_0) \sum_{k=0}^{2m-1} d_k \psi_k(y), \quad (3.13)$$

where $\mathbf{d} = (d_0, d_1, \dots, d_{2m-1}) \in \mathbb{R}^{2m}$ are real parameters to be fixed later, χ is introduced in (3.6). In particular, the initial data (3.13) belongs to $\mathcal{V}_A(s_0)$ as shown in the following proposition.

Proposition 3.3 (Properties of initial data (3.13)). *For each $A \gg 1$, there exist $s_0 = s_0(A) \gg 1$ and a cuboid $\mathcal{D}_{s_0} \subset [-A, A]^{2m}$ such that for all $(d_0, \dots, d_{2m-1}) \in \mathcal{D}_{s_0}$, the following properties hold:*

- (i) *The initial data $\phi_{A,s_0,\mathbf{d}}$ defined in (3.13) belongs to $\mathcal{V}_A(s_0)$ with strict inequalities except for the positive modes ϕ_k with $0 \leq k \leq 2m-1$, where $\phi_k = \Pi_k(\phi_{A,s_0,\mathbf{d}})$.*
- (ii) *The map $\Gamma : \mathcal{D}_{s_0} \rightarrow \mathbb{R}^{2m}$, defined as $\Gamma(d_0, \dots, d_{2m-1}) = (\phi_0, \dots, \phi_{2m-1})$, is linear, one to one from \mathcal{D}_{s_0} to $[-As_0^{-2}, As_0^{-2}]^{2m}$, and maps $\partial\mathcal{D}_{s_0}$ into $\partial([-As_0^{-2}, As_0^{-2}]^{2m})$. Moreover, the degree of Γ on the boundary is different from zero.*

Proof. The proof directly follows from the definition of the projection $\Pi_k(\phi_{A,s_0,\mathbf{d}})$ and it is straightforward, so we omit it here. \square

Starting with initial data $\phi_{A,s_0,\mathbf{d}}$ belonging to $\mathcal{V}_A(s_0)$, we claim that we can fine-tune the parameters $(d_0, \dots, d_{2m-1}) \in \mathcal{D}_{s_0}$ such that the corresponding solution $q(s)$ to equation (1.15) stays in $\mathcal{V}_A(s)$ for all $s \geq s_0$. More precisely, we claim the following.

Proposition 3.4 (Existence of solutions trapped in $\mathcal{V}_A(s)$). *There exist $A \gg 1$, $s_0 = s_0(A) \gg 1$ and $(d_0, \dots, d_{2m-1}) \in \mathcal{D}_{s_0}$ such that if $q(s)$ is the solution to equation (1.15) with initial data (3.13), then $q(s) \in \mathcal{V}_A(s)$ for all $s \geq s_0(A)$.*

Proof. From the local Cauchy problem of (1.1) in $L^\infty(\mathbb{R})$, we see that for each initial data $\phi_{A,s_0,\mathbf{d}} \in \mathcal{V}_A(s_0)$, equation (1.15) has a unique solution $q(s) \in \mathcal{V}_A(s)$ for all $s \in [s_0, s_*)$ with $s_* = s_*(\mathbf{d})$. If $s_* = +\infty$ for some $\mathbf{d} = (d_0, \dots, d_{2m-1}) \in \mathcal{D}_{s_0}$, we are done. Otherwise, we proceed by contradiction and assume that $s_*(\mathbf{d}) < +\infty$ for all $\mathbf{d} \in \mathcal{D}_{s_0}$. By continuity and the definition of s_* , we remark that $q(s_*) \in \partial\mathcal{V}_A(s_*)$. We claim the following.

Proposition 3.5 (Finite dimensional reduction). *There exist $A \gg 1$, $s_0 = s_0(A) \gg 1$ and $(d_0, \dots, d_{2m-1}) \in \mathcal{D}_{s_0}$ such that the following properties hold: If $q(s)$ is the solution to equation (1.15) with initial data (3.13) and $q(s) \in \mathcal{V}_A(s)$ for all $s \in [s_0, s_1]$ and $q(s_1) \in \partial\mathcal{V}_A(s_1)$, then*

- (i) (Reduction to a finite dimensional problem) $(q_0, \dots, q_{2m-1})(s_1) \in \partial([-As_1^{-2}, As_1^{-2}]^{2m})$.
- (ii) (Transversality) *There exists $\mu_0 > 0$ such that $q(s_1 + \mu) \notin \mathcal{V}_A(s_1 + \mu)$ for all $\mu \in (0, \mu_0)$.*

Let us postpone the proof of Proposition 3.5 to the next section and continue our argument. From (i) of Proposition 3.5, we obtain $(q_0, \dots, q_{2m-1})(s_*) \in \partial([-As_*^{-2}, As_*^{-2}]^{2m})$, and the following mapping is well defined

$$\begin{aligned} \Theta : \mathcal{D}_{s_0} &\rightarrow \partial([-1, 1]^{2m}) \\ (d_0, \dots, d_{2m-1}) &\mapsto \frac{s_*^2}{A} (q_0, \dots, q_{2m-1})(s_*). \end{aligned}$$

From the transversality given in item (ii) of Proposition 3.5, (q_0, \dots, q_{2m-1}) actually crosses its boundary at $s = s_*$, resulting in the continuity of s_* and Θ . Applying again the transversality, we see that if $(d_0, \dots, d_{2m-1}) \in \partial\mathcal{D}_{s_0}$, then $q(s)$ leaves $\mathcal{V}_A(s)$ at $s = s_0$, thus, $s_* = s_0$ and $\Theta|_{\partial\mathcal{D}_{s_0}} = \Gamma$, the map defined in item (ii) of Proposition 3.3. Using that item, we see that the degree of Θ is not zero. Since Θ is continuous, this is a contradiction. This concludes the proof of Proposition 3.4, assuming that Proposition 3.5 holds. \square

3.3. Conclusion of Theorem 1.1.

From Proposition 3.4, there exists a solution q to equation (1.15) such that $q(s) \in \mathcal{V}_A(s)$ for all $s \geq s_0$. From Remark 3.2, we deduce that $\|q(s)\|_{L^\infty(\mathbb{R})} \leq C(A)s^{-\frac{1}{2m}}$ for all $s \geq s_0$. The conclusion of item (ii) then follows from

(1.11) and (3.1). Item (i) of Theorem 1.1 is just a direct consequence of items (ii) and (iii). Because the proof of item (iii) is similar to the classical case $m = 1$, we only sketch the main ideas for the reader's convenience. The existence of the final blowup profile $u^* \in C^\infty(\mathbb{R} \setminus \{0\})$ follows from the technique of Merle [22]. Here we focus on a precise description of the final blowup profile u^* in a neighborhood of the singularity. To do so, we follow the technique of Herrero-Velázquez [20] (see also Bebernes-Bricher [1], Zaag [31] for a similar approach) by introducing the auxiliary function

$$h(\xi, \tau; x_0) = (T - t_0(x_0))^{\frac{1}{p-1}} u(x, t), \quad (3.14)$$

where

$$\xi = \frac{x - x_0}{[T - t_0(x_0)]^{\frac{1}{2m}}}, \quad \tau = \frac{t - t_0(x_0)}{T - t_0(x_0)},$$

and $t_0(x_0)$ is uniquely determined by

$$|x_0| = K[(T - t_0(x_0))|\log(T - t_0(x_0))|]^{\frac{1}{2m}}, \quad K \gg 1 \text{ fixed}. \quad (3.15)$$

We note that $h(\xi, \tau; x_0)$ is also a solution to (1.1) because of the invariance of (1.1) under dilations. From (1.7), we have

$$\sup_{|\xi| \leq |\log(T - t_0(x_0))|^{\frac{1}{4m}}} |h(\xi, 0, x_0) - \Phi(K)| \leq \frac{C}{(T - t_0(x_0))^{\frac{1}{2m}}} \rightarrow 0 \quad \text{as } |x_0| \rightarrow 0.$$

Let $\hat{h}_K(\tau)$ be the solution to (1.1) with the constant initial datum $\Phi(K)$, defined as

$$\hat{h}_K(\tau) = \kappa \left(1 - \tau + B_{m,p} K^{2m} \right)^{-\frac{1}{p-1}}, \quad \tau \in [0, 1).$$

By the continuity with respect to initial data for equation (1.1), one can show that

$$\sup_{|\xi| \leq |\log(T - t_0(x_0))|^{\frac{1}{4m}}, 0 \leq \tau < 1} |h(\xi, \tau; x_0) - \hat{h}_K(\tau)| \leq \epsilon(x_0) \rightarrow 0, \quad \text{as } |x_0| \rightarrow 0.$$

Passing to the limit $\tau \rightarrow 1$ yields

$$u^*(x_0) = (T - t_0(x_0))^{-\frac{1}{p-1}} \lim_{\tau \rightarrow 1} h(0, \tau; x_0) \sim (T - t_0(x_0))^{-\frac{1}{p-1}} \hat{h}_K(1).$$

From the definition (3.15), we compute

$$|\log(T - t_0(x_0))| \sim 2m |\log |x_0||, \quad T - t_0(x_0) \sim \frac{|x_0|^{2m}}{2m K^{2m} |\log |x_0||} \quad \text{as } |x_0| \rightarrow 0,$$

from which we obtain the asymptotic behavior

$$u^*(x_0) \sim \kappa \left(\frac{B_{m,p} |x_0|^{2m}}{2m |\log |x_0||} \right)^{-\frac{1}{p-1}} \quad \text{as } |x_0| \rightarrow 0.$$

This concludes the proof of Theorem 1.1, assuming that Proposition 3.5 holds.

4. Reduction to a finite dimensional problem.

This section is the central part in our analysis where we give all details of the proof of Proposition 3.5, completing hence the proof of Theorem 1.1. The essential idea is to project equation (1.15) onto different components according to the decomposition (3.8). In particular, we claim that Proposition 3.5 is a direct consequence of the following.

Proposition 4.1 (Dynamics of equation (1.15)). *For all $A \gg 1$, there exists $s_0 = s_0(A) \gg 1$ such that if $q(s) \in \mathcal{V}_A(s)$ for all $s \in [\tau, \tau_1]$ with $\tau_1 \geq \tau \geq s_0$, then the following estimates hold for all $s \in [\tau, \tau_1]$:*

(i) (Control of the finite dimensional part)

$$\left| q'_k(s) - \left(1 - \frac{k}{2m} \right) q_k \right| \leq \frac{C}{s^2} \quad \text{for } 0 \leq k \leq 2m-1. \quad (4.1)$$

$$\left| q'_{2m}(s) + \frac{2}{s} q_{2m} \right| \leq \frac{CA^3}{s^3}, \quad (4.2)$$

$$|q_j(s)| \leq C e^{-\left(\frac{j}{2m}-1\right)(s-\tau)} |q_j(\tau)| + \frac{CA^{j-1}}{s^{\frac{j+1}{2m}}} \quad \text{for } 2m+1 \leq j \leq M. \quad (4.3)$$

(ii) (Control of the infinite dimensional and outer part)

$$\left\| \frac{q_{M,\perp}(y, s)}{1 + |y|^{M+1}} \right\|_{L^\infty(\mathbb{R})} \leq C e^{-\frac{M(s-\tau)}{2m}} \left\| \frac{q_{M,\perp}(y, \tau)}{1 + |y|^{M+1}} \right\|_{L^\infty(\mathbb{R})} + \frac{CA^M}{s^{\frac{M+2}{2}}}. \quad (4.4)$$

$$\|q_e(s)\|_{L^\infty(\mathbb{R})} \leq C e^{-\frac{(s-\tau)}{2(p-1)}} \|q_e(\tau)\|_{L^\infty(\mathbb{R})} + \frac{CA^{M+1}}{s^{\frac{1}{2m}}} (1 + s - \tau). \quad (4.5)$$

Remark 4.2. Note that the sharp ODE (4.2) comes from the precise choice of the constant $B_{m,p}$ appearing in the profile Φ defined in (1.8). Other choices would give the error of size $\mathcal{O}\left(\frac{1}{s^2}\right)$ which is too large to close the estimate for q_{2m} .

Let us postpone the proof of Proposition 4.1 and proceed with the proof of Proposition 3.5.

Proof of Proposition 3.5 assuming Proposition 4.1. Since the argument of the proof is similar to what was done in [23], we only sketch the proof for the reader's convenience. We recall from the assumption that

$$q(s) \in \mathcal{V}_A(s) \quad \text{for all } s \in [s_0, s_1] \quad \text{and} \quad q(s_1) \in \partial \mathcal{V}_A(s_1). \quad (4.6)$$

Thus, part (i) of Proposition 3.5 will be proved if we show that all the bounds given in Definition 3.1 can be improved, except for the first $2m$ components q_k with $0 \leq k \leq 2m-1$, in the sense that for all $s \in [s_0, s_1]$:

$$|q_{2m}(s)| < \frac{A^2 \log s}{s^2}, \quad |q_k(s)| < \frac{A^k}{s^{\frac{k+1}{2m}}} \quad \text{for } 2m+1 \leq k \leq M, \quad (4.7)$$

$$\left\| \frac{q_{M,\perp}(y, s)}{1 + |y|^{M+1}} \right\|_{L^\infty} < \frac{A^{M+1}}{s^{\frac{M+2}{2m}}}, \quad \|q_e(s)\|_{L^\infty} < \frac{A^{M+2}}{s^{\frac{1}{2m}}}. \quad (4.8)$$

We first deal with q_{2m} . We argue by contradiction by assuming that there is $s_* \in [s_0, s_1]$ such that for all $s \in [s_0, s_*)$,

$$|q_{2m}(s)| < \frac{A^2 \log s}{s^2}, \quad |q_{2m}(s_*)| = \frac{A^2 \log s_*}{s_*^2},$$

(note that the existence of s_* is guaranteed by item (i) of Proposition 3.3). Considering $q_{2m}(s_*) > 0$ and using minimality (the case $q_{2m}(s_*) < 0$ is similar), we have on the one hand,

$$q'_{2m}(s_*) \geq A^2 \frac{d}{ds} \left(\frac{\log s_*}{s_*^2} \right) = \frac{A^2}{s_*^3} - \frac{2A^2 \log s_*}{s_*^3},$$

which holds thanks to (4.6). On the other hand, we obtain from the ODE (4.2) that

$$q'_{2m}(s_*) \leq \frac{-2A^2 \log s_*}{s_*^3} + \frac{CA}{s_*^3},$$

and the contradiction follows if $A \geq 2C + 1$.

As for q_j with $2m+1 \leq j \leq M$ and $q_{M,\perp}$ and q_e , we argue as follows: Let $\lambda = \log A$ and assume that $s_0 \geq \lambda$ such that for all $\tau \geq s_0$ and $s \in [\tau, \tau + \lambda]$ we have

$$\tau \leq s \leq \tau + \lambda \leq \tau + s_0 \leq 2\tau, \quad \text{hence,} \quad \frac{1}{\tau} \sim \frac{1}{s}.$$

We distinguish into two cases:

- For $s - s_0 \leq \lambda$, we use estimates (4.3), (4.4) and (4.5) with $\tau = s_0$ together with Proposition 3.3 to find that

$$\begin{aligned} |q_j(s)| &\leq \frac{C(1 + A^{j-1})}{s^{\frac{j+1}{2m}}} < \frac{A^j}{s^{\frac{j+1}{2m}}} \quad \text{for } 2m+1 \leq j \leq M, \\ \left\| \frac{q_{M,\perp}(y, s)}{1 + |y|^{M+1}} \right\|_{L^\infty} &\leq \frac{C(1 + A^M)}{s^{\frac{M+2}{2m}}} < \frac{A^{M+1}}{s^{\frac{M+2}{2m}}}, \\ \|q_e(s)\|_{L^\infty} &\leq \frac{C(1 + A^{M+1} \log A)}{s^{\frac{1}{2m}}} < \frac{A^{M+2}}{s^{\frac{1}{2m}}}, \end{aligned}$$

for A large enough. Hence, estimates (4.7) and (4.8) hold for $s - s_0 \leq \lambda$.

- For $s - s_0 > \lambda$, we use estimates (4.3), (4.4) and (4.5) with $\tau = s - \lambda > s_0$ together with (4.6) to write (remember that $s \geq \lambda/2$)

$$\begin{aligned} |q_j(s)| &\leq e^{-\left(\frac{j}{2m}-1\right)\lambda} \frac{A^j}{(s/2)^{\frac{j+1}{2m}}} + \frac{CA^{j-1}}{s^{\frac{j+1}{2m}}} < \frac{A^j}{s^{\frac{j+1}{2m}}} \quad \text{for } 2m+1 \leq j \leq M, \\ \left\| \frac{q_{M,\perp}(y, s)}{1 + |y|^{M+1}} \right\|_{L^\infty} &\leq Ce^{-\frac{M\lambda}{4m}} \frac{A^{M+1}}{(s/2)^{\frac{M+2}{2m}}} + \frac{CA^M}{s^{\frac{M+2}{2}}} < \frac{A^{M+1}}{s^{\frac{M+2}{2m}}}, \\ \|q_e(s)\|_{L^\infty} &\leq Ce^{-\frac{\lambda}{2(p-1)}} \frac{A^{M+2}}{(s/2)^{\frac{1}{2m}}} + \frac{CA^{M+1}(1 + \log A)}{s^{\frac{1}{2m}}} < \frac{A^{M+2}}{s^{\frac{1}{2m}}}. \end{aligned}$$

Therefore, estimates (4.7) and (4.8) hold for all $s \in [s_0, s_1]$, hence, the conclusion of part (i) of Proposition 3.5 follows.

Part (ii) of Proposition 3.5 is a direct consequence of the dynamics of q_k given in (4.1) and (4.6). Indeed, from part (i) of Proposition 4.1, we know that $q_k(s_1) = \epsilon \frac{A}{s_1^2}$ for some $k \in \{0, \dots, 2m-1\}$ and $\epsilon = \pm 1$. Using estimate (4.1) yields

$$\epsilon q'_k(s_1) \geq \left(1 - \frac{k}{2m}\right) \epsilon q_k(s_1) - \frac{C}{s_1^2} \geq \left(1 - \frac{k}{2m}\right) A - C \geq \frac{1}{s_1^2}.$$

Thus, for $0 \leq k \leq \frac{k}{2m}$ and A large enough, we have $\epsilon q'_k(s_1) > 0$. Therefore, q_k is traversal outgoing to the boundary curve $s \mapsto \epsilon A s^{-2}$ at $s = s_1$. This concludes the proof of Proposition 3.5, assuming that Proposition 4.1 holds. \square

We now give the proof of Proposition 4.1 to complete the proof of Proposition 3.5. We divide the proof into two subsections according to the two parts of Proposition 4.1.

4.1. Control of the finite dimensional part.

We prove item (i) of Proposition 4.1 in this part. A direct projection of equation (1.15) on the eigenfunction ψ_k for $0 \leq k \leq M$ yields

$$q'_k(s) - \left(1 - \frac{k}{2m}\right) q_k(s) = \Pi_k(Vq + B(q) + R). \quad (4.9)$$

Estimate of $\Pi_k(Vq)$.

We claim the following.

Lemma 4.3 (Expansion of V). *The potential V defined by (3.3) satisfies the estimate*

$$|V(y, s)| \leq \frac{C(1 + |y|^{2m})}{s}, \quad \forall y \in \mathbb{R}, \quad s \geq 1, \quad (4.10)$$

and admits the following uniform expansion for all $n \in \mathbb{N}^*$,

$$V(y, s) = \sum_{j=1}^n \frac{1}{s^j} V_j(y) + \mathcal{O}\left(\frac{1 + |y|^{2m(n+1)}}{s^{n+1}}\right), \quad \forall |y| \leq s^{\frac{1}{2m}}, \quad s > 1, \quad (4.11)$$

where V_j 's are even polynomial of degree $2mj$. More precisely, we have

$$V_1(y) = -\frac{2}{\bar{\mu}_{2m}} \psi_{2m} \quad \text{with} \quad \bar{\mu}_{2m} = \langle \psi_{2m}^2, \psi_{2m}^* \rangle. \quad (4.12)$$

Proof. Estimate (4.10) is trivial. Estimate (4.11) follows from a Taylor expansion in the variable $z = \frac{|y|^{2m}}{s}$. Formula (4.12) comes from the definitions (2.23) and (2.8) of φ and ψ_{2m} . \square

From Lemma 4.3, we obtain the following estimate for $\Pi_k(Vq)$.

Lemma 4.4 (Estimate of $\Pi_k(Vq)$). *Under the assumption of Proposition 4.1, we have*

$$\begin{aligned} |\Pi_k(Vq)| &\leq \frac{C}{s^2} \quad \text{for} \quad 0 \leq k \leq 2m-1, \\ \left| \Pi_{2m}(Vq) + \frac{2}{s} q_{2m} \right| &\leq \frac{CA}{s^3}, \\ |\Pi_k(Vq)| &\leq \frac{CA^{k-2}}{s^{\frac{k+1}{2m}}} \quad \text{for} \quad 2m+1 \leq k \leq M. \end{aligned}$$

Proof. By Definition 3.1, we have

$$|q(y, s)| \leq \frac{CA^{M+1}}{s^{1+\frac{1}{m}}} (1 + |y|^{M+1}) \quad \text{for all } y \in \mathbb{R}. \quad (4.13)$$

Using this, the estimate (4.10) and noting from the definition (2.10) and (2.5) that ψ_k^* is exponentially decaying, we obtain for $0 \leq k \leq 2m-1$,

$$|\Pi_k(Vq)| \leq \frac{CA^{M+1}}{s^{2+\frac{1}{m}}} \int_{\mathbb{R}} (1 + |y|^{M+1+2m}) \psi_k^*(y) dy \leq \frac{C}{s^2}.$$

For $2m+1 \leq k \leq M$, we write from the decomposition (3.8),

$$\Pi_k(Vq) = \sum_{i=0}^M q_i(s) \int_{\mathbb{R}} V(y, s) \psi_i(y) \psi_k^*(y) dy + \int_{\mathbb{R}} V(y, s) q_{M,\perp}(y, s) \psi_k^*(y) dy.$$

Using (4.10) and the bound of $q_{M,\perp}$ given in Definition 3.1 yields

$$\left| \int_{\mathbb{R}} V(y, s) q_{M,\perp}(y, s) \psi_k^*(y) dy \right| \leq \frac{CA^{M+1}}{s^{1+\frac{M+1}{2m}}} \int_{\mathbb{R}} (1 + |y|^{M+1+2m}) \psi_k^*(y) dy \leq \frac{CA^{M+1}}{s^{1+\frac{M+1}{2m}}}.$$

From the orthogonality (2.11), we note that $\langle P_n, \psi_k^* \rangle = 0$ for all polynomial P_n of degree $n \leq k-1$. We then use (4.11) and (4.13) to estimate

$$\begin{aligned} \sum_{i=0}^M q_i(s) \int_{\mathbb{R}} V(y, s) \psi_i(y) \psi_k^*(y) dy &= \sum_{i=0}^M \sum_{j=1}^n \frac{q_i(s)}{s^j} \int_{|y| \leq s^{\frac{1}{2m}}} V_j(y) \psi_i(y) \psi_k^*(y) dy \\ &\quad + \mathcal{O}\left(\frac{A^{M+1}}{s^{n+1+\frac{1}{m}}} \int_{|y| \leq s^{\frac{1}{2m}}} (1 + |y|^{2m(n+1)+M+1}) |\psi_k^*(y)| dy\right) \\ &\quad + \mathcal{O}\left(\frac{A^{M+1}}{s^{2+\frac{1}{m}}} \int_{|y| \geq s^{\frac{1}{2m}}} (1 + |y|^{2m+M+1}) |\psi_k^*(y)| dy\right). \end{aligned}$$

The last term is bounded by $\mathcal{O}(e^{-cs})$ because of the exponential decay of ψ_k^* . By taking $n \in \mathbb{N}^*$ such that $n + 1 + \frac{1}{m} \geq \frac{k+2}{2m}$, the second term is bounded by $\mathcal{O}\left(A^{M+1}s^{-\frac{k+2}{2m}}\right)$. Using the fact that $V_j\psi_i$ is polynomial of degree $2mj + i$, we see that $\langle V_j\psi_i, \psi_k^* \rangle = 0$ for $2mj + i \leq k - 1$. Combining this with the bounds of q_i given in Definition 3.1 of \mathcal{V}_A , we obtain the rough estimate

$$\begin{aligned} \left| \sum_{i=0}^M \sum_{j=1}^n \frac{q_i(s)}{s^j} \int_{|y| \leq s^{\frac{1}{2m}}} V_j(y) \psi_i(y) \psi_k^*(y) dy \right| &\leq \sum_{i=0}^M \sum_{j=1, 2jm+i \geq k}^n \frac{CA^i}{s^{j+\frac{i+1}{2m}}} \\ &\leq \frac{CA^{k-2}}{s^{\frac{k+1}{2m}}} + \sum_{i=k-1}^M \frac{CA^M}{s^{1+\frac{i+1}{2m}}} \leq \frac{CA^{k-2}}{s^{\frac{k+1}{2m}}}. \end{aligned}$$

As for $k = 2m$, we need to use the precise definition (4.12) of V_1 and process similarly as for $k \geq 2m + 1$. Indeed, from decomposition (3.8) and expansion (4.11) with $n = 2$, we have

$$\Pi_{2m}(Vq) - \frac{q_{2m}}{s} \int_{|y| \leq s^{\frac{1}{2m}}} V_1(y) \psi_{2m} \psi_{2m}^* dy = \sum_{i=0, i \neq 2m}^M \frac{q_i(s)}{s} \int_{|y| \leq s^{\frac{1}{2m}}} V_1 \psi_i \psi_{2m}^* + \mathcal{O}\left(\frac{A^{M+1}}{s^{3+\frac{1}{m}}}\right).$$

Using the definition (4.12) of V_1 , the bound of q_i given in Definition 3.1 and the fact coming from the definition (2.8) of ψ_β and the orthogonality (2.11) that

$$\int_{\mathbb{R}} \psi_{2m} \psi_i \psi_{2m}^* dy = \int_{\mathbb{R}} (c_0 \psi_{2m+i} + c_1 \psi_i + c_2 \psi_{i-2m}) \psi_{2m}^* = 0 \quad \text{for } 2m+1 \leq i \leq 4m-1,$$

we arrive at

$$\left| \Pi_{2m}(Vq) + \frac{2}{s} q_{2m} \right| \leq \sum_{i=0}^{2m-1} \frac{CA}{s^3} + \sum_{4m}^M \frac{CA^M}{s^{1+\frac{i+1}{2m}}} + \frac{CA^{M+1}}{s^{3+\frac{1}{m}}} \leq \frac{CA}{s^3}.$$

This concludes the proof of Lemma 4.4. \square

We now turn to the estimate of the main contribution of the nonlinear term $B(q)$ under the projection Π_k . We begin with the following expansion.

Lemma 4.5 (Expansion of $B(q)$). *For all $|q| < 1$, $s \geq s_0 \gg 1$ and $|y| \leq s^{\frac{1}{2m}}$, the function $B(q)$ defined by (3.4) admits the uniform expansion*

$$\left| B(q) - \sum_{j=2}^{M+1} q^j \sum_{l=0}^M \frac{1}{s^l} \left(B_{j,l}(ys^{-\frac{1}{2m}}) + \tilde{B}_{j,l}(y, s) \right) \right| \leq C|q|^{M+2} + \frac{C}{s^{M+1}}, \quad (4.14)$$

where $B_{j,l}(z)$'s are even polynomials of degree l and

$$|\tilde{B}_{j,l}(y, s)| \leq \frac{C(1 + |y|^{M+1})}{s^{\frac{M+1}{2m}}}.$$

Furthermore, we have the estimate for all $y \in \mathbb{R}$ and $s \geq 1$,

$$|B(q)| \leq C|q|^{\bar{p}} \quad \text{with } \bar{p} = \min\{2, p\}. \quad (4.15)$$

Proof. We first note from the definition (2.23) of φ that there exist two positive constants c_0, C_0 such that $c_0 \leq \varphi(y, s) \leq C_0$ for $|y| \leq s^{\frac{1}{2m}}$. Thus, a Taylor expansion of $B(q, \varphi)$ in terms of q yields

$$\left| B(q) - \sum_{j=2}^{M+1} B_j(\varphi) q^j \right| \leq C|q|^{M+1},$$

where $B_j(\varphi)$ admits the following expansion in terms of $\frac{1}{s}$,

$$\left| B_j(\varphi) - \sum_{l=0}^M \frac{1}{s^l} B_{j,l}(\Phi) \right| \leq \frac{C}{s^{M+1}}.$$

Now, a Taylor expansion of $B_{j,l}(\Phi)$ in terms of the variable $z = ys^{-\frac{1}{2m}}$ yields the desired result. This concludes the proof of Lemma 4.5. \square

With Lemma 4.5 at hand, we estimate the main contribution of $B(q)$ under the projection Π_k . We claim the following.

Lemma 4.6 (Estimate for $\Pi_k(B(q))$). *Under the assumption of Proposition 4.1, we have*

$$\begin{aligned} |\Pi_k(B(q))| &\leq \frac{C}{s^2} \quad \text{for } 0 \leq k \leq 2m-1, \quad |\Pi_{2m}(B(q))| \leq \frac{C}{s^3}, \\ |\Pi_k(B(q))| &\leq \frac{CA^k}{s^{\frac{k+2}{2m}}} \quad \text{for } 2m+1 \leq k \leq M. \end{aligned}$$

Proof. From (4.14), the exponential decay of ψ_k^* and (4.13), we have

$$\Pi_k(B(q)) = \sum_{j=2}^{M+1} \sum_{l=0}^M \frac{1}{s^l} \int_{|y| \leq s^{\frac{1}{2m}}} q^j B_{j,l} \psi_k^* dy + \mathcal{O}\left(\frac{A^{M+1}}{s^{2(1+1/m)+\frac{M+1}{2m}}}\right) + \mathcal{O}(e^{-cs}).$$

From the decomposition (3.8), we write

$$q^j = \left(\sum_{i=0}^M q_i \psi_i + q_{M,\perp} \right)^j \equiv (q_{<} + q_{M,\perp})^j \quad \text{for } j \geq 2.$$

From Definition 3.1 of \mathcal{V}_A and the uniform boundedness of q , we obtain the bound for $j \geq 2$,

$$|q^j - q_{<}^j| \leq C \left(|q_{M,\perp}|^j + |q_{<}|^{j-1} |q_{M,\perp}| \right) \leq \frac{CA^{j(M+1)}}{s^{\frac{M+4+2m}{2m}}} (1 + |y|^{j(M+1)}).$$

Therefore,

$$\Pi_k(B(q)) = \sum_{j=2}^{M+1} \sum_{l=0}^M \frac{1}{s^l} \int_{|y| \leq s^{\frac{1}{2m}}} q_{<}^j B_{j,l} \psi_k^* dy + \mathcal{O}\left(\frac{A^{j(M+1)}}{s^{\frac{M+4+2m}{2m}}}\right).$$

We only handle the case $k = 2m$, which is the most delicate. The case $k \neq 2m$ can be processed similarly and we omit it. From (4.13), we have

$$\sum_{j=3}^M \sum_{l=0}^M \frac{1}{s^l} \int_{|y| \leq s^{\frac{1}{2m}}} |q_{<}^j B_{j,l} \psi_{2m}^*| dy + \sum_{l=1}^M \frac{1}{s^l} \int_{|y| \leq s^{\frac{1}{2m}}} |q_{<}^2 B_{j,l} \psi_{2m}^*| dy \leq \frac{CA^{j(M+1)}}{s^{3+\frac{1}{m}}} \leq \frac{C}{s^3}.$$

It remains to control the term $I := \int_{|y| \leq s^{\frac{1}{2m}}} q_{<}^2 \psi_{2m}^* dy$. By the definition of $q_{<}$, we expand

$$\begin{aligned} q_{<}^2 &= \left(\sum_{i=0}^{2m} q_i \psi_i \right)^2 + 2 \left(\sum_{i=0}^{2m} q_i \psi_i \right) \left(\sum_{i'=2m+1}^M q_{i'} \psi_{i'} \right) + \left(\sum_{i'=2m+1}^M q_{i'} \psi_{i'} \right)^2 \\ &= I_1 + I_2 + I_3. \end{aligned}$$

From Definition 3.1, we have $|I_1| \leq \frac{CA^4 \log^2}{s^4} (1 + |y|^{4m})$ and $|I_2| \leq \frac{CA^{M+4} \log s}{s^{3+1/m}} (1 + |y|^{M+2m})$. Hence,

$$\int_{|y| \leq s^{\frac{1}{2m}}} |(I_1 + I_2) \psi_{2m}^*| dy \leq \frac{CA^4 \log^2}{s^4} + \frac{CA^{M+4} \log s}{s^{3+1/m}} \leq \frac{1}{s^3}.$$

Note from the orthogonality relation (2.11) that if $i' + l' \neq J2m$ for $J \in \mathbb{N}^*$, we have

$$\begin{aligned} \langle \psi_{i'} \psi_{l'}, \psi_{2m}^* \rangle &= \left\langle \left(\sum_{i=0}^{\lfloor \frac{i'}{2m} \rfloor} c_{i',i} \partial_y^{2im} y^{i'} \right) \left(\sum_{l=0}^{\lfloor \frac{l'}{2m} \rfloor} c_{l',l} \partial_y^{2lm} y^{l'} \right), \psi_{2m}^* \right\rangle \\ &= \left\langle \sum_{j=0}^{\lfloor \frac{i'}{2m} \rfloor + \lfloor \frac{l'}{2m} \rfloor} \hat{c}_{i'+l',j} y^{i'+l'-2jm}, \psi_{2m}^* \right\rangle = \left\langle \sum_{j=0}^{\lfloor \frac{i'}{2m} \rfloor + \lfloor \frac{l'}{2m} \rfloor} \tilde{c}_{i'+l',j} \psi_{i'+l'-2jm}, \psi_{2m}^* \right\rangle = 0. \end{aligned}$$

Therefore, we estimate

$$\begin{aligned} \int_{|y| \leq s^{\frac{1}{2m}}} I_3 \psi_{2m}^* dy &= \sum_{i', l'=2m+1}^M c_{i',l'} q_{i'}(s) q_{l'}(s) \int_{\mathbb{R}} \psi_{i'} \psi_{l'} \psi_{2m}^* dy + \mathcal{O}(e^{-cs}) \\ &= \sum_{i', l'=2m+1, i'+l'=J2m}^M c_{i',l'} q_{i'}(s) q_{l'}(s) \int_{\mathbb{R}} \psi_{i'} \psi_{l'} \psi_{2m}^* dy + \mathcal{O}(e^{-cs}). \end{aligned}$$

for $J \in \mathbb{N}^*$. Since $i' + l' \geq 4m + 2$, it follows that $J \geq 3$. From Definition 3.1, we have the bound $|q_{i'} q_{l'}| \leq \frac{A^{i'+l'}}{s^{\frac{i'+l'+2}{2m}}} \leq \frac{A^{i'+l'}}{s^{3+\frac{1}{m}}} \leq \frac{1}{s^3}$. Therefore, $\left| \int_{|y| \leq s^{\frac{1}{2m}}} I_3 \psi_{2m}^* dy \right| \leq \frac{1}{s^3}$. This concludes the proof of Lemma 4.6. \square

We now deal with the generated error term R . We begin with the following expansion.

Lemma 4.7 (Expansion of R). *The function R defined by (3.5) satisfies*

$$\|R(s)\|_{L^\infty} \leq \frac{C}{s},$$

and admits the following expansion: for all $n \in \mathbb{N}^*$, $|y| \leq s^{\frac{1}{2m}}$ and $s \geq 1$:

$$\left| R(y, s) - \sum_{j=1}^{n-1} \frac{1}{s^{j+1}} R_j(y) \right| \leq \frac{C(1 + |y|^{2mn})}{s^{n+1}}, \quad (4.16)$$

where R_j 's are polynomials of degree $2mj$ of the form $R_j(y) = \sum_{i=0}^j d_i y^{2mi}$. Moreover, the coefficient of degree $2m$ of R_1 is identically zero, hence, $\langle R_1, \psi_{2m}^* \rangle = 0$.

Proof. Let $z = ys^{-\frac{1}{2m}}$ and note that $\Phi(z)$ satisfies equation (2.21). Therefore, we rewrite (3.5) as follows:

$$R(y, s) = \frac{z}{2ms} \cdot \nabla \Phi + \frac{A_{m,p}}{s^2} + \frac{1}{s} \mathbf{A}_m \Phi - \frac{A_{m,p}}{(p-1)s} + Q \left(\Phi + \frac{A_{m,p}}{s} \right) - Q(\Phi), \quad (4.17)$$

where $Q(h) = h^p$. Since $c_0 \leq \Phi(z) \leq C_0$ for all $|z| \leq 1$ with c_0, C_0 some positive constants, we have the following expansion of Q ,

$$\left| Q \left(\Phi + \frac{A_{m,p}}{s} \right) - Q(\Phi) - \sum_{j=1}^J \frac{1}{s^j} Q_j(\Phi) \right| \leq \frac{C}{s^{J+1}} \quad \text{for } J \in \mathbb{N}^*.$$

Then, we expand Q_j and all the remaining terms in (4.17) in power series of $Z = z^{2m}$ to obtain the desired result. Note that the coefficient of $\frac{1}{s}$ in the expansion of R (after an elementary computation) is given by

$$A_{m,p} - \frac{(2m)! \kappa B_{m,p}}{p-1} = 0,$$

where we used (2.24). Moreover, $R_1(y) = C_1 y^{2m} + C_2$, where $C_2 = C_2(p, m)$ and

$$C_1 = \frac{\kappa B_{m,p}}{p-1} \left(\frac{(4m)!}{(2m)!} \frac{p B_{m,p}}{2(p-1)} - \frac{p A_{m,p}}{\kappa} - 1 \right) = 0.$$

Again, the precise values of $B_{m,p}$ and $A_{m,p}$ given in (2.24) are crucial in deriving that C_1 is identically zero. Therefore, the orthogonality relation (2.11) yields $\langle R_1, \psi_{2m}^* \rangle = 0$. This concludes the proof of Lemma 4.7. \square

As a direct consequence of Lemma 4.7, we have the following.

Lemma 4.8 (Estimate of $\Pi_k(R)$). *Under the assumption of Proposition 4.1, we have*

$$\begin{aligned} \Pi_k(R) &= \mathcal{O}(s^{-M}) \quad \text{for } (k \bmod 2m) \neq 0, \\ |\Pi_0(R)| &\leq \frac{C}{s^2}, \quad |\Pi_{2m}(R)| \leq \frac{C}{s^3}, \quad |\Pi_{2mi}(R)| \leq \frac{C}{s^{i+1}} \quad \text{for } i \in \mathbb{N}^*. \end{aligned}$$

Proof. The proof directly follows from the expansion (4.16) and the fact that

$$\langle y^{2mj}, \psi_k^* \rangle = \sum_{i=0}^j a_i \langle \psi_{2mi}, \psi_k^* \rangle = 0 \quad \text{for } (k \bmod 2m) \neq 0.$$

For $k \in \mathbb{N}$ with $(k \bmod 2m) \neq 0$, we use the expansion (4.16) with $n = M - 1$ (we can replace M by any positive integer $L \gg 1$) and write

$$\begin{aligned} |\Pi_k(R)| &= \left| \int_{\mathbb{R}} R(y, s) \psi_k^*(y) dy \right| = \left| \int_{|y| \leq s^{\frac{1}{2m}}} R(y, s) \psi_k^*(y) dy + \int_{|y| \geq s^{\frac{1}{2m}}} R(y, s) \psi_k^*(y) dy \right| \\ &\leq \sum_{j=1}^{M-2} \int_{|y| \geq s^{\frac{1}{2m}}} \frac{|R_j(y)|}{s^{j+1}} |\psi_k^*(y)| dy + \frac{C}{s^M} \int_{\mathbb{R}} (1 + |y|^{2m(M-1)}) |\psi_k^*(y)| dy \\ &\quad + \frac{C}{s} \int_{|y| \geq s^{\frac{1}{2m}}} |\psi_k^*(y)| dy \leq \frac{C}{s^M} + C e^{-cs} \leq \frac{C}{s^M}. \end{aligned}$$

For $(k \bmod 2m) \neq 0$, i.e. $k = 2mi$ for some $i \in \mathbb{N}$, we use (4.16) with $n = i + 1$ to get the conclusion. This ends the proof of Lemma 4.8. \square

A collection of all estimates given in Lemmas 4.4, 4.6, 4.8 and equation (4.9) yields the conclusion of part (i) of Proposition 4.1.

4.2. Control of the infinite dimensional and the outer part.

We prove item (ii) of Proposition in this part. We first deal with the infinite dimensional part $q_{M,\perp}$, then the outer part q_e .

Control of $q_{M,\perp}$:

Applying $\Pi_{M,\perp}$ to equation (1.15) and using the fact that $\Pi_{M,\perp} \psi_n = 0$ for all $n \leq M$ (see (3.10)), we obtain

$$\partial_s q_{M,\perp} = \mathcal{L}_m q_{M,\perp} + \Pi_{M,\perp} (Vq + B(q) + R). \quad (4.18)$$

By definition, we have $\Pi_{M,\perp}(y^k) = 0$ for all $k \leq M$. Using the uniform bound $\|R(s)\|_{L^\infty} \leq \frac{C}{s}$ and the expansion (4.16) with $n = \frac{M}{2m}$ (see (3.11) for the definition of M) and noting that $\Pi_{M,\perp}(R_j) = 0$ for $j \leq \frac{M}{2m}$, we obtain

$$|\Pi_{M,\perp}(R)| \leq \frac{C(1 + |y|^{M+2m})}{s^{\frac{M}{2m}+2}} \mathbf{1}_{|y| \leq s^{\frac{1}{2m}}} + |R(y, s)| \mathbf{1}_{|y| \geq s^{\frac{1}{2m}}} \leq \frac{C(1 + |y|^{M+1})}{s^{\frac{M+1}{2m}+1}}. \quad (4.19)$$

As for the estimate of $\Pi_{M,\perp}(Vq)$, we claim the following.

Lemma 4.9 (Estimate of $\Pi_{M,\perp}(Vq)$). *Under the assumption of Proposition 4.1, we have*

$$\left\| \frac{\Pi_{M,\perp}(Vq)}{1 + |y|^{M+1}} \right\|_{L^\infty(\mathbb{R})} \leq \left(\|V(s)\|_{L^\infty(\mathbb{R})} + \frac{C}{s} \right) \left\| \frac{q_{M,\perp}}{1 + |y|^{M+1}} \right\|_{L^\infty(\mathbb{R})} + \frac{CA^M}{s^{\frac{M+2}{2m}}}. \quad (4.20)$$

Proof. By definition, we write

$$\Pi_{M,\perp}(Vq) = Vq_{M,\perp} - \Pi_{M,<}(Vq_{M,\perp}) + \Pi_{M,\perp}(Vq_{M,<}),$$

where $q_{M,<} = \Pi_{M,<}(q) = (\text{Id} - \Pi_{M,\perp})(q)$. Using estimate (4.10), we derive

$$\left\| \frac{Vq_{M,\perp}}{1 + |y|^{M+1}} \right\|_{L^\infty} + \left\| \frac{\Pi_{M,<}(Vq_{M,\perp})}{1 + |y|^{M+1}} \right\|_{L^\infty} \leq \left(\|V(s)\|_{L^\infty(\mathbb{R})} + \frac{1}{s} \right) \left\| \frac{q_{M,\perp}}{1 + |y|^{M+1}} \right\|_{L^\infty}.$$

As for the control of $\Pi_{M,\perp}(Vq_{M,<}) = \sum_{i \leq M} \Pi_{M,\perp}(q_i V\psi_i)$, we argue as follows:

- If $M - i = 0 \bmod 2m$, we take $n = \frac{M-i}{2m}$ in the expansion (4.11) and write

$$\Pi_{M,\perp}(Vq_i\psi_i) = \sum_{j=1}^n \frac{q_i}{s^j} \Pi_{M,\perp}(V_j\psi_i) + \Pi_{M,\perp}(\tilde{V}_n q_i\psi_i),$$

where $|\tilde{V}_n(y, s)| \leq \frac{C(1+|y|^{2m(n+1)})}{s^{n+1}}$. Using the fact that $\Pi_{M,\perp}(y^n) = 0$ for $n \leq M$, we have $\sum_{j=1}^n \Pi_{M,\perp}(V_j q_i) = 0$. Hence,

$$|\Pi_{M,\perp}(Vq_i\psi_i)| \leq \frac{C|q_i|}{s^{\frac{M-i}{2m}+1}} \leq \frac{CA^M}{s^{\frac{M+2}{2m}}}.$$

- If $M - i = l \bmod 2m$ for some $l \in (1, 2m)$, we take $n = \frac{M-i-l}{2m}$ in the expansion (4.11) to deduce that

$$|\Pi_{M,\perp}(Vq_i\psi_i)| \leq \frac{C|q_i|}{s^{\frac{M-i-l}{2m}+1}} \leq \frac{CA^M}{s^{\frac{M+2}{2m}}}.$$

This concludes the proof of Lemma 4.9. □

Concerning the control of $\Pi_{M,\perp}(B(q))$, we claim the following.

Lemma 4.10 (Estimate of $\Pi_{M,\perp}(B(q))$). *Under the assumption of Proposition 4.1, we have*

$$\left\| \frac{\Pi_{M,\perp}(B(q))}{1 + |y|^{M+1}} \right\|_{L^\infty(\mathbb{R})} \leq \frac{CA^{(M+2)^2}}{s^{\frac{M+1+\tilde{p}}{2m}}} \quad \text{with } \tilde{p} = \min\{2, p\}. \quad (4.21)$$

Proof. Let B_M be defined by

$$B_M(y, s) = \Pi_{M,<} \left[\sum_{j=2}^{M+1} q^j \sum_{l=0}^M \frac{1}{s^l} B_{j,l}(ys^{-\frac{1}{2m}}) \right], \quad (4.22)$$

where $\Pi_{M,<} = \text{Id} - \Pi_{M,\perp}$ and $B_{j,l}$'s are introduced in Lemma 4.5. Then we claim the following: for all $y \in \mathbb{R}$ and $s \geq 1$,

$$|B(q) - B_M| \leq \frac{CA^{(M+2)^2}}{s^{\frac{M+1+\tilde{p}}{2m}}} (1 + |y|^{M+1}). \quad (4.23)$$

The estimate (4.21) simply follows from (4.23) since $\Pi_{M,\perp}(B_M) = 0$. Let us prove (4.23). In the region $|y| \geq s^{\frac{1}{2m}}$, we use (4.15) and the bounds given in Definition 3.1 to estimate

$$|B(q)| \leq \frac{CA^{\tilde{p}(M+2)}}{s^{\frac{\tilde{p}}{2m}}} \leq \frac{CA^{\tilde{p}(M+2)}}{s^{\frac{M+1+\tilde{p}}{2m}}} (1 + |y|^{M+1}) \quad \text{for } |y| \geq s^{\frac{1}{2m}}.$$

In the region $|y| \leq s^{\frac{1}{2m}}$, we write from Lemma 4.5 the expansion

$$\begin{aligned} & \left| B(q) - \Pi_{M,\leq} \left(\sum_{j=2}^{M+1} q^j \sum_{l=0}^M \frac{1}{s^l} B_{j,l}(ys^{-\frac{1}{2m}}) \right) \right| \\ & \leq \left| \Pi_{M,\perp} \left(\sum_{j=2}^{M+1} q^j \sum_{l=0}^M \frac{1}{s^l} B_{j,l}(ys^{-\frac{1}{2m}}) \right) \right| + \left| \sum_{j=2}^{M+1} q^j \sum_{l=0}^M \frac{1}{s^l} \tilde{B}_{j,l}(y, s) \right| + |q|^{M+2} + \frac{C}{s^{M+1}}. \end{aligned}$$

A similar argument as in the proof of Lemma 4.6 shows that the coefficient of degree $k \geq M+1$ of the polynomial $\sum_{j=0}^{M+1} q^j \sum_{l=0}^M \frac{1}{s^l} B_{j,l}(ys^{-\frac{1}{2m}})$ is bounded by $\frac{A^k}{s^{\frac{k+2}{2m}}}$, hence,

$$\left| \Pi_{M,\perp} \left(\sum_{j=2}^{M+1} q^j \sum_{l=0}^M \frac{1}{s^l} B_{j,l}(ys^{-\frac{1}{2m}}) \right) \right| \leq \frac{CA^{2(M+1)}}{s^{\frac{M+3}{2m}}} (1 + |y|^{M+1}), \quad \forall |y| \leq s^{\frac{1}{2m}}.$$

From (4.13) and Remark 3.2, we obtain

$$\left| \sum_{j=2}^{M+1} q^j \sum_{l=0}^M \frac{1}{s^l} \tilde{B}_{j,l}(y, s) \right| \leq \frac{CA^{(M+2)(M+1)}}{s^{\frac{1}{m} + \frac{M+1}{2m}}} (1 + |y|^{M+1}) \leq \frac{CA^{(M+2)^2}}{s^{\frac{M+1+p}{2m}}} (1 + |y|^{M+1}),$$

and

$$|q|^{M+2} \leq C \left(\frac{A^{M+2}}{s^{\frac{1}{2m}}} \right)^{M+1} \frac{A^{M+2}}{s^{1+\frac{1}{m}}} (1 + |y|^{M+1}) \leq \frac{CA^{(M+2)^2}}{s^{\frac{M+1+p}{2m}}} (1 + |y|^{M+1}).$$

This completes the proof of (4.23) as well as the proof of Lemma 4.10. \square

Plugging (4.19), (4.20) and (4.21) into equation (4.18) yields

$$\partial_s q_{M,\perp} = \mathcal{L}_m q_{M,\perp} + G_{M,\perp},$$

where $G_{M,\perp}$ satisfies for s large enough,

$$\left\| \frac{G_{M,\perp}(y, s)}{1 + |y|^{M+1}} \right\|_{L^\infty(\mathbb{R})} \leq \|V(s)\|_{L^\infty(\mathbb{R})} \left\| \frac{q_{M,\perp}(y, s)}{1 + |y|^{M+1}} \right\|_{L^\infty(\mathbb{R})} + \frac{CA^M}{s^{\frac{M+2}{2m}}}.$$

Using the semigroup representation by \mathcal{L}_m , we write for all $s \in [\tau, \tau_1]$,

$$q_{M,\perp}(s) = e^{(s-\tau)\mathcal{L}_m} q_{M,\perp}(\tau) + \int_{\tau}^s e^{(s-s')\mathcal{L}_m} G_{M,\perp}(s') ds',$$

where $e^{s\mathcal{L}_m}$ is defined in Lemma 2.3. Letting $\lambda(s) = \left\| \frac{q_{M,\perp}(s)}{1 + |y|^{M+1}} \right\|_{L^\infty(\mathbb{R})}$ and using (iii) of Lemma (2.3), we have

$$\begin{aligned} \lambda(s) & \leq e^{-\frac{M}{2m}(s-\tau)} \lambda(\tau) + \int_{\tau}^s e^{-\frac{M}{2m}(s-s')} \left\| \frac{G_{M,\perp}(s')}{1 + |y|^{M+1}} \right\|_{L^\infty(\mathbb{R})} ds' \\ & \leq e^{-\frac{M}{2m}(s-\tau)} \lambda(\tau) + \int_{\tau}^s e^{-\frac{M}{2m}(s-s')} \left(\|V(s')\|_{L^\infty(\mathbb{R})} \lambda(s') + \frac{CA^M}{s'^{\frac{M+2}{2m}}} \right) ds'. \end{aligned}$$

Since we have fixed M large such that $\|V\|_{L_{y,s}^\infty} \leq \frac{M}{4m}$ (see (3.11)), a standard Gronwall's argument applied to the function $e^{\frac{Ms}{2m}} \lambda(s)$ yields

$$e^{\frac{Ms}{2m}} \lambda(s) \leq e^{\frac{M(s-\tau)}{4m}} e^{\frac{M\tau}{2m}} \lambda(\tau) + Ce^{\frac{Ms}{2m}} \frac{A^M}{s^{\frac{M+2}{2}}},$$

from which we conclude the proof of (4.4).

Control of q_e .

We write from (1.15) the equation satisfied by $q_e = (1 - \chi(y, s))q$,

$$\partial_s q_e = \left(\mathcal{L}_m - \frac{p}{p-1} \right) q_e + (1 - \chi)(Q + R) + E_1 + E_2,$$

where R is defined by (3.5),

$$Q = |q + \varphi|^{p-1}(q + \varphi) - \varphi^p, \quad E_1 = -q(\partial_s \chi + \mathbf{A}_m \chi + \frac{1}{2m} y \cdot \nabla \chi),$$

$$E_2 = \mathbf{A}_m(\chi q) + q \mathbf{A}_m \chi - \chi \mathbf{A}_m q = \sum_{j=1}^{2m-1} c_j \nabla^j (q \nabla^{2m-j} \chi).$$

Using the semigroup representation by \mathcal{L}_m and items (i) – (ii) of Lemma 2.3, we write for all $s \in [\tau, \tau_1]$,

$$\begin{aligned} \|q_e(s)\|_{L^\infty} &\leq e^{-\frac{s-\tau}{p-1}} \|q_e(\tau)\|_{L^\infty} + \int_{\tau}^s e^{-\frac{s-s'}{p-1}} \|(1 - \chi)(Q(s') + R(s')) + E_1(s')\|_{L^\infty} ds' \\ &\quad + C \sum_{j=1}^{2m-1} \int_{\tau}^s e^{-\frac{s-s'}{p-1}} (1 - e^{-(s-s')})^{-\frac{j}{2m}} \|q(s') \nabla^{2m-j} \chi(s')\|_{L^\infty} ds'. \end{aligned}$$

For K large enough, we have

$$\|(1 - \chi(y, s')Q(s'))\|_{L^\infty} \leq C \|\varphi(s')\|_{L^\infty}^{p-1} \|q_e(s')\|_{L^\infty} \leq \frac{1}{2(p-1)} \|q_e(s')\|_{L^\infty}.$$

Recall from Lemma 4.7 that $\|R(s')\|_{L^\infty} \leq \frac{C}{s'}$. As for E_1 , we use the definition of χ given in (3.6) and the bounds given in Definition 3.1 to obtain the estimate

$$\|E_1(s')\|_{L^\infty} \leq C \|q(s')\|_{L^\infty} \left(K s'^{\frac{1}{2m}} \leq |y| \leq 2K s'^{\frac{1}{2m}} \right) \leq \frac{C A^{M+1}}{s'^{\frac{1}{2m}}}.$$

Since $\|\nabla^{2m-j} \chi(s')\|_{L^\infty} \leq C s'^{-\frac{2m-j}{2m}}$, using the fact that $\nabla^{2m-j} \chi$ is compactly supported, we estimate

$$\sum_{j=1}^{2m-1} \|q(s') \nabla^{2m-j} \chi(s')\|_{L^\infty} \leq \frac{C A^{M+1}}{s'^{\frac{1}{m}}}.$$

A collection of these above estimates yields

$$\begin{aligned} \|q_e(s)\|_{L^\infty} &\leq e^{-\frac{s-\tau}{p-1}} \|q_e(\tau)\|_{L^\infty} \\ &\quad + \int_{\tau}^s e^{-\frac{s-s'}{p-1}} \left(\frac{1}{2(p-1)} \|q_e(s')\|_{L^\infty} + \frac{C A^{M+1}}{s'^{\frac{1}{2m}}} + \frac{C A^{M+1}}{\left[s' \sqrt{1 - e^{-(s-s')}} \right]^{\frac{1}{m}}} \right) ds'. \end{aligned}$$

Applying the standard Gronwall's inequality to the function $e^{\frac{s}{p-1}} \|q_e(s)\|_{L^\infty}$ yields

$$e^{\frac{s}{p-1}} \|q_e(s)\|_{L^\infty} \leq e^{\frac{s-\tau}{2(p-1)}} e^{\frac{\tau}{p-1}} \lambda(\tau) + \frac{C A^{M+1}}{s^{\frac{1}{2m}}} (s - \tau + 1),$$

from which the estimate (4.5) follows. This completes the proof of Proposition 4.1. \square

References

- [1] J. Bebernes and S. Bricher. Final time blowup profiles for semilinear parabolic equations via center manifold theory. *SIAM J. Math. Anal.*, 23(4):852–869, 1992. ISSN 0036-1410. 10.1137/0523045. URL <http://dx.doi.org/10.1137/0523045>.

- [2] M. Berger and R. V. Kohn. A rescaling algorithm for the numerical calculation of blowing-up solutions. *Comm. Pure Appl. Math.*, 41(6):841–863, 1988. ISSN 0010-3640. 10.1002/cpa.3160410606. URL <http://dx.doi.org/10.1002/cpa.3160410606>.
- [3] A. Bressan. On the asymptotic shape of blow-up. *Indiana Univ. Math. J.*, 39(4):947–960, 1990. ISSN 0022-2518. 10.1512/iumj.1990.39.39045. URL <http://dx.doi.org/10.1512/iumj.1990.39.39045>.
- [4] A. Bressan. Stable blow-up patterns. *J. Differential Equations*, 98(1):57–75, 1992. ISSN 0022-0396. 10.1016/0022-0396(92)90104-U. URL [http://dx.doi.org/10.1016/0022-0396\(92\)90104-U](http://dx.doi.org/10.1016/0022-0396(92)90104-U).
- [5] J. Bricmont and A. Kupiainen. Universality in blow-up for nonlinear heat equations. *Nonlinearity*, 7(2):539–575, 1994. ISSN 0951-7715. URL <http://stacks.iop.org/0951-7715/7/539>.
- [6] C. J. Budd, V. A. Galaktionov, and J. F. Williams. Self-similar blow-up in higher-order semilinear parabolic equations. *SIAM J. Appl. Math.*, 64(5):1775–1809, 2004. ISSN 0036-1399. 10.1137/S003613990241552X. URL <http://dx.doi.org/10.1137/S003613990241552X>.
- [7] G. K. Duong. Profile for the imaginary part of a blowup solution for a complex-valued semilinear heat equation. *arXiv:1712.07183*. URL <https://arxiv.org/abs/1712.07183>.
- [8] G. K. Duong, V. T. Nguyen, and H. Zaag. Construction of a stable blowup solution with a prescribed behavior for a non-scaling invariant semilinear heat equation. *Tunisian Journal of Mathematics (to appear)*, 2018. URL <https://arxiv.org/abs/1704.08580>.
- [9] M. A. Ebde and H. Zaag. Construction and stability of a blow up solution for a nonlinear heat equation with a gradient term. *SéMA J.*, (55):5–21, 2011. ISSN 1575-9822.
- [10] S. D. Eĭ del'man. *Parabolic systems*. Translated from the Russian by Scripta Technica, London. North-Holland Publishing Co., Amsterdam-London; Wolters-Noordhoff Publishing, Groningen, 1969.
- [11] M. V. Fedorjuk. Singularities of the kernels of Fourier integral operators, and the asymptotic behavior of the solution of a mixed problem. *Uspehi Mat. Nauk*, 32(6(198)):67–115, 287, 1977. ISSN 0042-1316.
- [12] H. Fujita. On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo Sect. I*, 13: 109–124 (1966), 1966. ISSN 0040-8980.
- [13] V. A. Galaktionov. Five types of blow-up in a semilinear fourth-order reaction-diffusion equation: an analytic-numerical approach. *Nonlinearity*, 22(7):1695–1741, 2009. ISSN 0951-7715. 10.1088/0951-7715/22/7/012. URL <http://dx.doi.org/10.1088/0951-7715/22/7/012>.
- [14] V. A. Galaktionov and S. I. Pohozaev. Existence and blow-up for higher-order semilinear parabolic equations: majorizing order-preserving operators. *Indiana Univ. Math. J.*, 51(6):1321–1338, 2002. ISSN 0022-2518. URL <https://doi.org/10.1512/iumj.2002.51.2131>.
- [15] Victor A. Galaktionov. On a spectrum of blow-up patterns for a higher-order semilinear parabolic equation. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 457(2011):1623–1643, 2001. ISSN 1364-5021. 10.1098/rspa.2000.0733. URL <http://dx.doi.org/10.1098/rspa.2000.0733>.
- [16] T. Ghoul, V. T. Nguyen, and H. Zaag. Blowup solutions for a nonlinear heat equation involving a critical power nonlinear gradient term. *J. Differential Equations*, 263(8):4517 – 4564, 2017. ISSN 0022-0396. <http://dx.doi.org/10.1016/j.jde.2017.05.023>. URL <http://www.sciencedirect.com/science/article/pii/S0022039617302838>.
- [17] T. Ghoul, V. T. Nguyen, and H. Zaag. Blowup solutions for a reaction-diffusion system with exponential nonlinearities. *J. Differential Equations*, 264(12):7523–7579, 2018. ISSN 0022-0396. 10.1016/j.jde.2018.02.022. URL <https://doi.org/10.1016/j.jde.2018.02.022>.
- [18] T. Ghoul, V. T. Nguyen, and H. Zaag. Construction and stability of blowup solutions for a non-variational semilinear parabolic system. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 35(6):1577 – 1630, 2018. ISSN 0294-1449. <https://doi.org/10.1016/j.anihpc.2018.01.003>. URL <http://www.sciencedirect.com/science/article/pii/S0294144918300076>.
- [19] M. A. Herrero and J. J. L. Velázquez. Generic behaviour of one-dimensional blow up patterns. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 19(3):381–450, 1992. ISSN 0391-173X. URL http://www.numdam.org/item?id=ASNSP_1992_4_19_3_381_0.
- [20] M. A. Herrero and J. J. L. Velázquez. Blow-up behaviour of one-dimensional semilinear parabolic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(2):131–189, 1993. ISSN 0294-1449.
- [21] N. Masmoudi and H. Zaag. Blow-up profile for the complex Ginzburg-Landau equation. *J. Funct. Anal.*, 255(7):1613–1666, 2008. ISSN 0022-1236. 10.1016/j.jfa.2008.03.008. URL <http://dx.doi.org/10.1016/j.jfa.2008.03.008>.
- [22] F. Merle. Solution of a nonlinear heat equation with arbitrarily given blow-up points. *Comm. Pure Appl. Math.*, 45(3): 263–300, 1992. ISSN 0010-3640. 10.1002/cpa.3160450303. URL <http://dx.doi.org/10.1002/cpa.3160450303>.
- [23] F. Merle and H. Zaag. Stability of the blow-up profile for equations of the type $u_t = \Delta u + |u|^{p-1}u$. *Duke Math. J.*, 86(1):143–195, 1997. ISSN 0012-7094. 10.1215/S0012-7094-97-08605-1. URL <http://dx.doi.org/10.1215/S0012-7094-97-08605-1>.
- [24] V. T. Nguyen. Numerical analysis of the rescaling method for parabolic problems with blow-up in finite time. *Phys. D*, 339: 49–65, 2017. ISSN 0167-2789. URL <https://doi.org/10.1016/j.physd.2016.09.002>.
- [25] V. T. Nguyen and H. Zaag. Construction of a stable blow-up solution for a class of strongly perturbed semilinear heat equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 16(4):1275–1314, 2016. 10.2422/2036-2145.201412_001. URL http://dx.doi.org/10.2422/2036-2145.201412_001.

- [26] N. Nouaili and H. Zaag. Profile for a simultaneously blowing up solution to a complex valued semilinear heat equation. *Comm. Partial Differential Equations*, 40(7):1197–1217, 2015. ISSN 0360-5302. 10.1080/03605302.2015.1018997. URL <http://dx.doi.org/10.1080/03605302.2015.1018997>.
- [27] N. Nouaili and H. Zaag. Construction of a blow-up solution for the Complex Ginzburg-Landau equation in some critical case. *Arch. Ration. Mech. Anal.*, to appear, 2018. URL <https://arxiv.org/abs/1703.00081>.
- [28] L. A. Peletier and W. C. Troy. *Spatial patterns*, volume 45 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 2001. ISBN 0-8176-4110-6. URL <https://doi.org/10.1007/978-1-4612-0135-9>. Higher order models in physics and mechanics.
- [29] S. Tayachi and H. Zaag. Existence and stability of a blow-up solution with a new prescribed behavior for a heat equation with a critical nonlinear gradient term. *Actes du Colloque EDP-Normandie, Le Havre*, *arXiv:1610.01289*, 2015.
- [30] S. Tayachi and H. Zaag. Existence of a stable blow-up profile for the nonlinear heat equation with a critical power nonlinear gradient term. *arXiv:1506.08306*, 2016.
- [31] H. Zaag. Blow-up results for vector-valued nonlinear heat equations with no gradient structure. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 15(5):581–622, 1998. ISSN 0294-1449. 10.1016/S0294-1449(98)80002-4. URL [http://dx.doi.org/10.1016/S0294-1449\(98\)80002-4](http://dx.doi.org/10.1016/S0294-1449(98)80002-4).