

Renata Bunoiu and Radu Precup\*

# Localization and multiplicity in the homogenization of nonlinear problems

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**Abstract:** We propose a method for the localization of solutions for a class of nonlinear problems arising in the homogenization theory. The method combines concepts and results from the linear theory of PDEs, linear periodic homogenization theory, and nonlinear functional analysis. Particularly, we use the Moser-Harnack inequality, arguments of fixed point theory and Ekeland's variational principle. A significant gain in the homogenization theory of nonlinear problems is that our method makes possible the emergence of finitely or infinitely many solutions.

**Keywords:** Nonlinear elliptic problem, homogenization, localization, positive solution, multiple solutions

**MSC:** 35B27, 35J25

## 1 Introduction

The aim of homogenization theory is that, starting from an initial problem depending on a small positive parameter  $\varepsilon$  related to the heterogeneous character of the problem, to find an approximation of its solution which solves a problem independent of  $\varepsilon$ , called the homogenized problem. From a mathematical point of view this is obtained by passing to the limit with  $\varepsilon \rightarrow 0$  in the initial problem.

One of the main features in classical homogenization problems is the uniqueness of the solution of the problem under study. This is crucial in order to justify the convergence of the whole sequence of solutions of the microscopic problem to the solution of the homogenized problem. In the nonlinear case, there is a wide class of problems for which the uniqueness of the solution fails. When several solutions of a problem exist, it is important to be able to carry on the analysis on one particular solution and this needs its localization. A lot of results for nonlinear problems in homogenization theory have been obtained in the last years (see for instance [1], [3], [7], [8] [10], [12], [13], [18]). In most cases, in order to pass to the limit, it was necessary to obtain upper estimates of the solutions. However, up to our knowledge, none of these works had to deal with estimates from below of the solutions.

The aim of this paper is to develop a method of localization in bounded sets for the solutions of nonlinear homogenization problems, which allows the emergence of multiple solutions for a given problem. The localization which will be given on the initial problem, for each  $\varepsilon$  small enough, will be preserved after passage to the limit with  $\varepsilon \rightarrow 0$ . We will illustrate this method on a simple model, namely the homogenization of an elliptic nonlinear Dirichlet problem stated in a fixed domain, with periodic fast oscillating coefficients depending on the small parameter  $\varepsilon$ , which expresses the heterogeneous properties of the medium. We emphasize that our method applies to more involved problems in nonlinear homogenization theory.

The results are then extended to systems, for which we are able to localize each component of the solution, independently. Moreover, the solution appears as a Nash equilibrium with respect to the energy functionals associated to the equations of the system. In this case again, for a given nonlinear system, by the

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Renata Bunoiu, University of Lorraine, CNRS, IECL, F-57000, Metz, France. E-mail: renata.bunoiu@univ-lorraine.fr

\*Corresponding Author: Radu Precup, Faculty of Mathematics and Computer Science, Babeş-Bolyai University, Cluj, Romania. E-mail: r.precup@math.ubbcluj.ro

localization technique, we are able to prove the existence of finitely or infinitely many local Nash equilibrium solutions.

The starting point of our analysis is a linear microscopic pivot problem with a right-hand side  $g \in L^\infty(\Omega)$  which allows to estimate the solutions  $u^\varepsilon$  both from below and from above, independently of  $\varepsilon$ , via two constants  $m_g$  and  $M_g$  respectively. At this point, we underline the role of the Moser-Harnack inequality in establishing the lower bound  $m_g$  (see Lemma 3.1), which, up to our knowledge, is new in the frame of homogenization theory.

The question we answer in this paper is how the source term  $g$  can be modified by a nonlinear source factor  $f(u^\varepsilon)$  with a feedback action, such that the solution  $u^\varepsilon$  of the new problem remains  $L^\infty$ -bounded independently of  $\varepsilon$ , between two given bounds  $r, R$ . The same question is also addressed for nonlinear systems.

Our study is motivated by real-world applications in physics, engineering and biology, from where at least two physical requirements emerge. In these cases the solution corresponds to a density rate.

**First requirement:** Find a suitable nonlinear state-dependent source factor  $f(u^\varepsilon)$ , in order to guarantee that the density rate  $u^\varepsilon$  stays bounded between two *a priori* given bounds  $r, R, 0 < r < R$ . Such a requirement is natural in problems arising from medicine, where for instance the concentration rate of a drug in a tissue with a nonhomogeneous structure must remain between some prescribed limits. The same question can be addressed in problems of thermodynamics, when the heat diffusion in strongly heterogeneous media has to be limited.

**Second requirement:** The nonlinear state-dependent source factor  $f$  being given, find the bounds of the corresponding density rate  $u^\varepsilon$ .

Our approach makes use of methods and results from the linear theory of PDEs (weak and strong maximum principle, Poincaré and Moser-Harnack inequalities, compact embedding theorems), linear periodic homogenization theory, and concepts and techniques of nonlinear functional analysis (Banach’s contraction principle and its vector version owed to Perov, Ekeland variational principle, Nash equilibrium).

The paper is organized as follows: In Section 2 we give the statement of the problem and we recall some basic results of periodic linear homogenization theory. In Section 3 we state and prove the main existence and localization result giving a variational characterization to the solution and, as a consequence, we obtain a multiplicity result for nonlinearities with a repeated suitable behavior. These solutions are local minima of the associated energy functionals. Section 4 is devoted to the study of a nonlinear system, for which we are able to localize componentwise the solution.

## 2 Statement of the problem and preliminaries results

We are concerned with the following nonlinear problem with homogeneous Dirichlet boundary conditions, and strongly periodic oscillating coefficients:

$$\begin{cases} -\operatorname{div}(A^\varepsilon(x)\nabla u^\varepsilon) = g(x)f(u^\varepsilon) & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Here  $\Omega \subset \mathbb{R}^n$  is an open domain with sufficiently smooth boundary  $\partial\Omega$ . We make the following assumptions on the data:

Let  $Y = (0, 1)^n$ . For  $\alpha, \beta \in \mathbb{R}$ , with  $0 < \alpha \leq \beta$ , let  $M(\alpha, \beta, Y)$  be the set of all the matrices  $A \in (L^\infty(Y))^{n \times n}$  such that for any  $\xi \in \mathbb{R}^n$ ,

$$\alpha|\xi|^2 \leq (A(y)\xi, \xi) \leq \beta|\xi|^2, \tag{2.2}$$

almost everywhere in  $Y$ . For a  $Y$ -periodic symmetric matrix  $A \in M(\alpha, \beta, Y)$ , we set

$$A^\varepsilon(x) = A\left(\left\{\frac{x}{\varepsilon}\right\}\right) \text{ a.e. in } \Omega.$$

The function  $g$  belongs to the space  $L^\infty(\Omega)$  and the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous.

Throughout this paper  $\Omega'$  is a fixed subdomain of  $\Omega$ ,  $\Omega' \subset\subset \Omega$ . We first look for sufficient conditions on  $f$  and  $g$  such that, for given numbers  $0 < r < R$ , problem (2.1) has a solution  $u^\varepsilon$  such that the following boundedness conditions

$$0 \leq u^\varepsilon(x) \leq R \text{ for a.a. } x \in \Omega, \tag{2.3}$$

$$r \leq u^\varepsilon(x) \text{ for a.a. } x \in \Omega', \tag{2.4}$$

are satisfied for each  $\varepsilon > 0$  small enough.

Then, we prove that, after passage to the limit in (2.1) with  $\varepsilon \rightarrow 0$ , the limit  $u$  of the sequence  $u^\varepsilon$  solves uniquely the problem

$$\begin{cases} -\operatorname{div}(A^0 \nabla u) = g(x)f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

which is therefore the homogenized problem of (2.1). The solution  $u$  of the homogenized problem, satisfies the same boundedness conditions. Here the entries of the constant matrix  $A^0$  are defined in (2.8)-(2.9) below. This localization is useful for the numerical computation of the solutions, especially when an initial approximation is required.

In addition, this localization result, applied to distinct pairs of numbers  $(r_k, R_k)$  yields finitely or infinitely many solutions for problem (2.1), provided that either

$$0 < r_k < R_k < r_{k+1} < R_{k+1}$$

or

$$0 < r_{k+1} < R_{k+1} < r_k < R_k.$$

Analogously, in Section 4, we discuss the localization of solutions for nonlinear systems. In this case the localization can be obtained componentwise, which is a significant gain for the treatment of systems in comparison with the classical approach.

## 2.1 The linear homogenization problem

We start by recalling a classical result from the homogenization of linear elliptic problems. With the domain  $\Omega$  and the coefficients  $A^\varepsilon$  defined as before, we consider the linear problem (see [2, Chapter 1]):

$$\begin{cases} -\operatorname{div}(A^\varepsilon(x) \nabla u^\varepsilon) = h(x) & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

If  $h \in L^2(\Omega)$  is given, then one can prove (see for instance [5, Chapter 6] and references therein) that problem (2.6) admits a unique solution  $u^\varepsilon \in H_0^1(\Omega)$  which converges to some  $u$  weakly in  $H_0^1(\Omega)$ , and strongly in  $L^2(\Omega)$ . The limit  $u$  is the unique solution of the homogenized problem

$$\begin{cases} -\operatorname{div}(A^0 \nabla u) = h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.7}$$

with the constant homogenized coefficients given by

$$A_{ij}^0 = \int_Y \left( a_{ij} - \sum_{k=1}^n a_{ik} \frac{\partial \chi^j}{\partial y_k} \right) dy, \tag{2.8}$$

where, for  $i, j = 1, \dots, n$ ,  $a_{ij}$  are the entries of the matrix  $A$ . The  $j$ -th components of the vectorial function  $\chi$ ,  $\chi^j \in H_{\text{per}}^1(Y)$  ( $j = 1, \dots, n$ ) are the unique weak solutions of the following cell problems:

$$\begin{cases} -\operatorname{div}_Y(A(y)(\nabla_Y \chi^j - e_j)) = 0 & \text{in } Y, \\ \mathcal{M}_Y(\chi^j) = 0. \end{cases} \tag{2.9}$$

Here  $H_{per}^1(Y)$  denotes the space of  $H^1$  functions which are  $Y$ -periodic and  $\mathcal{M}_Y(v)$  represents the mean value on the set  $Y$  of a function  $v$ .

When we make additional assumptions on the right-hand side  $h$ , one has more information on the corresponding solution of problem (2.6), as illustrated by the two remarks below.

**Remark 2.1.** If  $h(x) \geq 0$  for a.e.  $x \in \Omega$ , then the weak maximum principle (see [9, Theorem 8.1]) guarantees that for every  $\varepsilon > 0$ , one has  $u^\varepsilon(x) \geq 0$  a.e. in  $\Omega$ .

**Remark 2.2.** If  $h \in L^\infty(\Omega)$  then, according to [9, Theorem 8.15], the solution  $u^\varepsilon$  belongs to  $L^\infty(\Omega)$  and there is a constant  $\Gamma_h$ , independent of  $\varepsilon$ , such that

$$\|u^\varepsilon\|_{L^\infty(\Omega)} \leq \Gamma_h.$$

In view of Remark 2.2, we define for each  $\varepsilon \in (0, 1)$ , the solution operator

$$S^\varepsilon : L^\infty(\Omega) \rightarrow V := H_0^1(\Omega) \cap L^\infty(\Omega),$$

$$h \mapsto S^\varepsilon(h) := u_h^\varepsilon,$$

where  $u_h^\varepsilon$  is the corresponding solution of problem (2.6).

The operator  $S^\varepsilon$  is linear and due to Remark 2.1, it preserves the positivity and the order, more exactly

$$\text{if } 0 \leq h_1 \leq h_2, \text{ then } 0 \leq S^\varepsilon(h_1) \leq S^\varepsilon(h_2).$$

Searching now a weak solution  $u^\varepsilon \in V$  of problem (2.1) is thus equivalent to solving the nonlinear equation

$$u^\varepsilon = S^\varepsilon(gf(u^\varepsilon)). \quad (2.10)$$

For each  $\varepsilon$ , denoting

$$N^\varepsilon(v) := S^\varepsilon(gf(v)), \quad v \in V, \quad (2.11)$$

any solution of problem (2.10) is a fixed point of the operator  $N^\varepsilon$ . The existence and the uniqueness of such a fixed point will be obtained by using Banach's contraction principle. Alternatively, the existence and the uniqueness of the solution of (2.1) can be obtained using the variational approach in connection with the energy functional associated to (2.1), namely

$$\begin{aligned} J^\varepsilon & : V \rightarrow \mathbb{R}, \\ J^\varepsilon(v) & = \frac{1}{2} \int_{\Omega} A^\varepsilon(x) \nabla v \cdot \nabla v - \int_{\Omega} g(x) \int_0^{v(x)} f(\tau) d\tau. \end{aligned}$$

Similarly, we associate to problem (2.5) the energy functional

$$\begin{aligned} J & : V \rightarrow \mathbb{R}, \\ J(v) & = \frac{1}{2} \int_{\Omega} A^0 \nabla v \cdot \nabla v - \int_{\Omega} g(x) \int_0^{v(x)} f(\tau) d\tau. \end{aligned}$$

## 3 Main results for equations

### 3.1 A lower bound lemma

Our aim in this paper being the localization of solutions of problem (2.1), as shown in (2.3)–(2.4), we need bounds from above and from below. For the upper bounds, we use classical results from the literature, as

mentioned in Remark 2.2. Getting lower bounds is a more difficult task and, up to our knowledge, it is new in the frame of the homogenization theory. To obtain such lower bound estimates, we make use of an auxiliary result, Lemma 3.1, whose proof is based on Moser-Harnack’s inequality. The key auxiliary result is the following.

**Lemma 3.1.** *For each  $h \in L^\infty(\Omega)$  with  $h \geq 0$  in  $\Omega$  and  $h > 0$  on a subset of nonzero measure of  $\Omega$ , there exist  $\varepsilon_h > 0$  and  $y_h > 0$  such that  $u^\varepsilon(x) \geq y_h$  for a.a.  $x$  in the fixed subdomain  $\Omega' \subset\subset \Omega$  and every  $\varepsilon < \varepsilon_h$ .*

*Proof.* The proof will be done by contradiction. Assume the contrary. Then for every integer  $k \geq 1$ , there exists  $\varepsilon_k < 1/k$  with

$$\inf_{\Omega'} u^{\varepsilon_k} < \frac{1}{k}. \tag{3.1}$$

Let  $p \in \mathbb{R}$  be fixed such that  $1 \leq p < n/(n-2)$  if  $n \geq 3$  and  $p = 2$  if  $n < 3$ . Using Moser-Harnack’s inequality (see [11, Theorem 12.1.2]) and covering arguments as in [15, Theorem 1.3], we have

$$\inf_{\Omega'} u^{\varepsilon_k} \geq C \|u^{\varepsilon_k}\|_{L^p(\Omega')}, \tag{3.2}$$

where the constant  $C > 0$  only depends on  $p$ ,  $\Omega$ ,  $\Omega'$ , and the ellipticity constants  $\alpha$  and  $\beta$  in (2.2). From (3.1) and (3.2) we deduce that

$$\|u^{\varepsilon_k}\|_{L^p(\Omega')} \rightarrow 0 \text{ as } k \rightarrow \infty. \tag{3.3}$$

Since  $p < n/(n-2) < 2n/(n-2) = 2^*$ , the injection of  $H_0^1(\Omega)$  in  $L^p(\Omega)$  is compact, and consequently up to a subsequence we have  $u^{\varepsilon_k} \rightarrow u$  in  $L^p(\Omega)$ , where  $u$  is the solution of the homogenized problem (2.7). Then from (3.3),  $\|u\|_{L^p(\Omega')} = 0$ , that is  $u = 0$  in  $\Omega'$ , which is impossible in view of the strong maximum principle applied to the homogenized problem (2.7) in which  $h \geq 0$  and  $h$  is not the null function.  $\square$

Coming back to our initial question, namely the localization of the solution for problem (2.1), let us define, for the given numbers  $0 < r < R$ , the set

$$D_{rR} = \{v \in V : 0 \leq v(x) \leq R \text{ a.e. in } \Omega, r \leq v(x) \text{ a.e. in } \Omega'\}.$$

Note that the set  $D_{rR}$  is a closed subset of  $H_0^1(\Omega)$ . Indeed, if  $v_k \in D_{rR}$  and  $v_k \rightarrow v$  in  $H_0^1(\Omega)$ , then  $v_k \rightarrow v$  in  $L^2(\Omega)$  and there is a subsequence  $(v_{k_j})$  of the sequence  $(v_k)$  such that  $v_{k_j}(x) \rightarrow v(x)$  for a.a.  $x \in \Omega$  (see [4, Theorem 4.9]). Then, from the definition of  $D_{rR}$ , since

$$0 \leq v_{k_j}(x) \leq R \text{ a.e. in } \Omega, \text{ and } r \leq v_{k_j}(x) \text{ a.e. in } \Omega',$$

passing to the limit we obtain

$$0 \leq v(x) \leq R \text{ a.e. in } \Omega, \text{ and } r \leq v(x) \text{ a.e. in } \Omega',$$

that is  $v \in D_{rR}$ .

### 3.2 Invariance of $D_{rR}$ through the operator $N^\varepsilon$

The first step for applying Banach’s contraction principle to the nonlinear operator (2.11) is to guarantee the invariance condition. To this aim we state our first hypothesis.

**(h1)**  $g(x) \geq 0$  a.e in  $\Omega$ ,  $g(x) = 0$  for  $x \in \Omega \setminus \Omega'$ , and  $g > 0$  on a subset of nonzero measure;  $f(\tau) \geq 0$  for every  $\tau \geq 0$ .

Let  $v \in D_{rR}$ . Since  $r \leq v(x) \leq R$  a.e. in  $\Omega'$  and  $f$  is continuous, we have

$$m_f := \min_{\tau \in [r, R]} f(\tau) \leq f(v(x)) \leq M_f := \max_{\tau \in [r, R]} f(\tau), \quad \text{a.e. in } \Omega'.$$

Clearly  $0 \leq m_f \leq M_f$ . This, together with the property of  $g$  of being nonnegative in  $\Omega$  and to be zero outside  $\Omega'$ , gives

$$g(x) m_f \leq g(x) f(v(x)) \leq g(x) M_f, \quad \text{a.e. in } \Omega.$$

Then using the positivity and monotonicity properties of  $S^\varepsilon$ , Remark 2.2, by applying  $N^\varepsilon$  to the previous inequalities we obtain

$$\begin{aligned} 0 &\leq N^\varepsilon(v)(x) = S^\varepsilon(gf(v))(x) \leq S^\varepsilon(gM_f)(x) \\ &= M_f S^\varepsilon(g)(x) \leq M_f \Gamma_g, \quad \text{for a.a. } x \in \Omega. \end{aligned}$$

For the lower estimation, we make use of Lemma 3.1, applied for  $g$ , and we obtain:

$$N^\varepsilon(v)(x) \geq S^\varepsilon(gm_f)(x) = m_f S^\varepsilon(g)(x) \geq m_f \gamma_g, \quad \text{for a.a. } x \in \Omega'. \quad (3.4)$$

We are now in position to derive a sufficient condition for  $N^\varepsilon(v) \in D_{rR}$  to hold, and thus for the invariance condition  $N^\varepsilon(D_{rR}) \subset D_{rR}$  to be verified, namely

**(h2)**  $M_f \Gamma_g \leq R$  and  $r \leq m_f \gamma_g$ .

**Remark 3.1.** If in addition the function  $f$  is nondecreasing on the interval  $[r, R]$ , then  $m_f = f(r)$ ,  $M_f = f(R)$  and condition (h2) becomes

**(h2)'**  $f(R) \Gamma_g \leq R$  and  $r \leq f(r) \gamma_g$ .

This gives an answer to the second requirement stated in the Introduction:  $f$  being given, such bounds  $r, R$  always exist if the following asymptotic conditions are satisfied:

$$\lim_{\tau \rightarrow +\infty} \frac{f(\tau)}{\tau} = 0, \quad \lim_{\tau \rightarrow 0} \frac{f(\tau)}{\tau} = +\infty.$$

### 3.3 The contraction condition

Assume that  $f$  is Lipschitz continuous on the interval  $[r, R]$ , more precisely that the following condition holds:

**(h3)** There exists a constant  $0 \leq l < \alpha \lambda_1 / \|g\|_{L^\infty(\Omega)}$  such that

$$|f(\tau) - f(\bar{\tau})| \leq l |\tau - \bar{\tau}| \quad \text{for all } \tau, \bar{\tau} \in [r, R]. \quad (3.5)$$

Here  $\lambda_1$  is the first eigenvalue of the Dirichlet problem for  $-\Delta$ .

We recall that imposing on  $f$  the Lipschitz continuity condition is natural for getting uniqueness in non-linear problems. Nevertheless, in our case, this condition is assumed only locally, as is often the case in real-world applications, and mathematically it makes the possibility to study by localization problems having multiple solutions.

For any fixed  $\varepsilon > 0$ , we prove that, under hypothesis (h3), the operator  $N^\varepsilon$  satisfies the contraction property on  $D_{rR}$ . To this aim, let  $v_1, v_2 \in D_{rR}$ , and denote  $w_i = N^\varepsilon(v_i)$ ,  $i = 1, 2$ . Then

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla w_i) = gf(v_i) & \text{in } \Omega \\ w_i = 0 & \text{on } \partial\Omega \end{cases}$$

in the weak sense, for  $i = 1, 2$ . These give

$$\begin{cases} -\operatorname{div} (A^\varepsilon \nabla (w_1 - w_2)) = g [f(v_1) - f(v_2)] & \text{in } \Omega \\ w_1 - w_2 = 0 & \text{on } \partial\Omega \end{cases}$$

in the weak sense. Thus, multiplying by  $w_1 - w_2$  and integrating over  $\Omega$  yield

$$\int_{\Omega} A^\varepsilon \nabla (w_1 - w_2) \cdot \nabla (w_1 - w_2) = \int_{\Omega} g [f(v_1) - f(v_2)] (w_1 - w_2). \tag{3.6}$$

For the left-hand side integral in (3.6), by the ellipticity condition (2.2) on  $A^\varepsilon$ , one has

$$\alpha \|w_1 - w_2\|_{H_0^1(\Omega)}^2 \leq \int_{\Omega} A^\varepsilon \nabla (w_1 - w_2) \cdot \nabla (w_1 - w_2). \tag{3.7}$$

For the right-hand side integral in (3.6), since  $g$  is zero outside  $\Omega'$ ,  $v_1(x), v_2(x) \in [r, R]$  for a.a.  $x \in \Omega'$ , and by using (h3), we have

$$\begin{aligned} \int_{\Omega} g [f(v_1) - f(v_2)] (w_1 - w_2) &= \int_{\Omega'} g [f(v_1) - f(v_2)] (w_1 - w_2) \\ &\leq l \int_{\Omega'} g |v_1 - v_2| |w_1 - w_2| \\ &\leq l \|g\|_{L^\infty(\Omega)} \|v_1 - v_2\|_{L^2(\Omega)} \|w_1 - w_2\|_{L^2(\Omega)}. \end{aligned}$$

Two times application of Poincaré’s inequality with the sharp constant  $1/\sqrt{\lambda_1}$  yields

$$\int_{\Omega} g [f(v_1) - f(v_2)] (w_1 - w_2) \leq \frac{l \|g\|_{L^\infty(\Omega)}}{\lambda_1} \|v_1 - v_2\|_{H_0^1(\Omega)} \|w_1 - w_2\|_{H_0^1(\Omega)}. \tag{3.8}$$

Now (3.6)-(3.8) imply

$$\|w_1 - w_2\|_{H_0^1(\Omega)} \leq \frac{l \|g\|_{L^\infty(\Omega)}}{\alpha \lambda_1} \|v_1 - v_2\|_{H_0^1(\Omega)}, \tag{3.9}$$

which in view of (h3) shows that  $N^\varepsilon$  is a contraction on  $D_{rR}$ , with the contraction constant

$$L := \frac{l \|g\|_{L^\infty(\Omega)}}{\alpha \lambda_1} < 1.$$

### 3.4 Existence and localization result

We are now in position to state the existence and localization result for the nonlinear problem (2.1).

**Theorem 3.2.** *Assume that conditions (h1)-(h3) hold. Then there exists  $\varepsilon_0 > 0$  such that*

(i) *for any  $\varepsilon < \varepsilon_0$ , problem (2.1) has a unique solution  $u^\varepsilon$  in  $D_{rR}$ , which is in  $D_{rR}$  the unique minimum point of the energy functional  $J^\varepsilon$ .*

(ii)  *$u^\varepsilon \rightarrow u$  as  $\varepsilon \rightarrow 0$ , weakly in  $H_0^1(\Omega)$  and strongly in  $L^2(\Omega)$ , where  $u$  is the unique function in  $D_{rR}$  which solves the homogenized problem (2.5). Also  $u$  is in  $D_{rR}$  the unique minimum point of the energy functional  $J$ , and*

$$J^\varepsilon(u^\varepsilon) \rightarrow J(u) \text{ as } \varepsilon \rightarrow 0.$$

*Proof.* (a) The existence and uniqueness in  $D_{rR}$  of  $u^\varepsilon$  follows from Banach’s contraction principle, in view of the previous considerations. In order to prove the convergence of  $u^\varepsilon$  to  $u$ , note that, due to the boundedness in  $L^\infty(\Omega)$  of the set  $\{gf(u^\varepsilon)\}$ , the set  $\{u^\varepsilon\}$  is bounded in  $H_0^1(\Omega)$ , and so relatively compact in  $L^2(\Omega)$ . Thus there is a subsequence  $(u^{\varepsilon_k})$  with  $\varepsilon_k \rightarrow 0$  which is weakly convergent in  $H_0^1(\Omega)$  to some  $u \in H_0^1(\Omega)$ , and strongly

convergent in  $L^2(\Omega)$  to  $u$ . Obviously, the limit  $u$  belongs to  $D_{rR}$ . Now we are in position to pass to the limit in problem (2.1). The limit of the left-hand side of (2.1) is found as in the linear case (see [5, Chapter 9]), while for the right-hand side of (2.1) we use the uniform boundedness of  $(u^{\varepsilon_k})$  in  $L^\infty(\Omega)$ , its strong convergence in  $L^2(\Omega)$ , and the continuity of Nemytskii’s operator associated to  $f$  from  $L^2(\Omega)$  to itself. Thus, the limit  $u$  satisfies problem (2.5). The uniqueness in  $D_{rR}$  of the solution of the homogenized problem, can be proved with similar arguments as above, by means of Banach’s contraction principle, and implies that the entire sequence  $(u^\varepsilon)$  converges to  $u$ , weakly in  $H_0^1(\Omega)$ , and strongly in  $L^2(\Omega)$ .

(b) To show that for each fixed  $\varepsilon$ ,  $u^\varepsilon$  minimizes  $J^\varepsilon$  in  $D_{rR}$ , let us first note that  $J^\varepsilon$  is a  $C^1$  functional on  $V$  and under the standard identification of  $H^{-1}(\Omega)$  to  $H_0^1(\Omega)$ , one has

$$(J^\varepsilon)'(v) = v - N^\varepsilon(v). \tag{3.10}$$

Hence the fixed points of  $N^\varepsilon$  are the critical points of  $J^\varepsilon$ . It is easy to see that  $J^\varepsilon$  is bounded from below on  $D_{rR}$ . Then, using Ekeland’s variational principle (see, e.g., [6], [17]), there is a minimizing sequence  $(v_k)$  of elements from  $D_{rR}$  such that

$$J^\varepsilon(v_k) \leq \inf_{D_{rR}} J^\varepsilon + \frac{1}{k} \tag{3.11}$$

and

$$J^\varepsilon(v_k) \leq J^\varepsilon(v) + \frac{1}{k} \|v - v_k\|_{H_0^1(\Omega)} \tag{3.12}$$

for all  $v \in D_{rR}$ . For any fixed index  $k$ , choose

$$v_t = v_k - t (J^\varepsilon)'(v_k), \quad 0 < t < 1.$$

Using (3.10), one has

$$v_t = (1 - t)v_k + tN^\varepsilon(v_k).$$

Here, one has  $v_k \in D_{rR}$  and by the invariance property in Section 3.2,  $N^\varepsilon(v_k)$  also belongs to  $D_{rR}$ . Since  $D_{rR}$  is convex, it follows that  $v_t \in D_{rR}$  for every  $t \in (0, 1)$ . Replacing  $v$  by  $v_t$  into (3.12) and then dividing by  $t$ , yields

$$t^{-1} \left( J^\varepsilon(v_k) - J^\varepsilon(v_k - t (J^\varepsilon)'(v_k)) \right) \leq \frac{1}{k} \left\| (J^\varepsilon)'(v_k) \right\|_{H_0^1(\Omega)},$$

whence letting  $t$  go to zero, one finds

$$\left\langle (J^\varepsilon)'(v_k), (J^\varepsilon)'(v_k) \right\rangle \leq \frac{1}{k} \left\| (J^\varepsilon)'(v_k) \right\|_{H_0^1(\Omega)},$$

that is

$$\left\| (J^\varepsilon)'(v_k) \right\|_{H_0^1(\Omega)} \leq \frac{1}{k}.$$

Hence

$$w_k := (J^\varepsilon)'(v_k) \rightarrow 0 \text{ in } H_0^1(\Omega). \tag{3.13}$$

Using (3.10), we have  $w_k = v_k - N^\varepsilon(v_k)$  and by the contraction property of  $N^\varepsilon$ , we find

$$\begin{aligned} \|v_{k+p} - v_k\|_{H_0^1(\Omega)} &\leq \|N^\varepsilon(v_{k+p}) - N^\varepsilon(v_k)\|_{H_0^1(\Omega)} + \|w_{k+p} - w_k\|_{H_0^1(\Omega)} \\ &\leq L \|v_{k+p} - v_k\|_{H_0^1(\Omega)} + \|w_{k+p} - w_k\|_{H_0^1(\Omega)}, \end{aligned}$$

whence

$$\|v_{k+p} - v_k\|_{H_0^1(\Omega)} \leq \frac{1}{1 - L} \|w_{k+p} - w_k\|_{H_0^1(\Omega)}.$$

This ensures that  $(v_k)$  is a Cauchy sequence, so it converges in  $H_0^1(\Omega)$  to some  $v^\varepsilon$ . Clearly  $v^\varepsilon \in D_{rR}$  and from (3.11) and (3.13), passing to the limit with  $k \rightarrow \infty$  we obtain that  $v^\varepsilon$  minimizes  $J^\varepsilon$  in  $D_{rR}$  and  $(J^\varepsilon)'(v^\varepsilon) = 0$ . Hence  $v^\varepsilon$  is a fixed point in  $D_{rR}$  of  $N^\varepsilon$ . The uniqueness of the fixed point, guaranteed by Banach’s contraction principle, implies that  $v^\varepsilon = u^\varepsilon$  as desired. The proof of the fact that the limit  $u$ , solution of the homogenized problem, minimizes  $J$  in  $D_{rR}$  is analogous.

Finally the convergence  $J^\varepsilon(u^\varepsilon) \rightarrow J(u)$  follows from a standard result in homogenization theory (see [5, Section 8.2]) and the properties of Nemytskii’s operator.  $\square$

**Remark 3.2.** (*Multiplicity*) If the conditions (h1)-(h3) are satisfied for two pairs of numbers  $(r, R)$ , let them be  $(r_1, R_1)$  and  $(r_2, R_2)$ , with  $0 < r_1 < R_1 < r_2 < R_2$ , then the sets  $D_{r_1 R_1}$  and  $D_{r_2 R_2}$  are disjoint, and according to Theorem 3.2, problem (2.1), as well as the corresponding homogenized problem (2.5), have two distinct solutions, one in  $D_{r_1 R_1}$  and the other in  $D_{r_2 R_2}$ . This argument, extended to a finite or infinite sequence of pairs  $(r, R)$ , can be used in order to obtain multiple (finitely or infinitely many) solutions of problem (2.1) and of the corresponding homogenized problem (2.5).

**Remark 3.3.** Under the assumption that  $g$  vanishes in  $\Omega \setminus \Omega'$ , we were able to localize the solutions assuming the Lipschitz continuity condition for  $f$  only locally, in the interval  $[r, R]$ . It is precisely this local Lipschitz condition which allows us to obtain, via Theorem 3.2, multiple solutions. If the assumption that  $g$  vanishes in  $\Omega \setminus \Omega'$  does not hold, one can however localize the solutions in the set  $D_{rR}$ , but under the stronger condition that  $f$  is Lipschitz continuous in  $[0, R]$  instead of  $[r, R]$ . Correspondingly, in this case  $M_f$  is

$$M_f = \max_{\tau \in [0, R]} f(\tau)$$

and the estimation from below (3.4) should be driven as follows:

$$\begin{aligned} N^\varepsilon(v)(x) &\geq S^\varepsilon(1_{\Omega'} g f(v))(x) \geq S^\varepsilon(1_{\Omega'} g m_f)(x) \\ &= m_f S^\varepsilon(1_{\Omega'} g)(x) \geq m_f y_{1_{\Omega'} g}. \end{aligned}$$

Here  $1_{\Omega'}$  is the characteristic function of  $\Omega'$ , i.e.,  $1_{\Omega'}(x) = 1$  if  $x \in \Omega'$ ,  $1_{\Omega'}(x) = 0$  for  $x \in \Omega \setminus \Omega'$ . This implies that  $y_g$  in (3.4) should be replaced by  $y_{1_{\Omega'} g}$ .

Nevertheless, under this stronger Lipschitz condition on  $f$ , Theorem 3.2 cannot produce multiple solutions. Indeed, if Theorem 3.2 would apply to two pairs of numbers, let them be  $(r_1, R_1)$  and  $(r_2, R_2)$ , with  $0 < r_1 < R_1 < r_2 < R_2$ , then the sets  $D_{r_1 R_1}$  and  $D_{r_2 R_2}$  being disjoint,  $N^\varepsilon$  would have two distinct fixed points, one in  $D_{r_1 R_1}$  and other in  $D_{r_2 R_2}$ . On the other hand,  $f$  should be Lipschitz continuous on  $[0, R_2]$  with the Lipschitz constant  $l < \alpha \lambda_1 / \|g\|_{L^\infty(\Omega)}$ . Then it is easy to see that the contraction condition on  $N^\varepsilon$ , namely inequality (3.9), holds true for every  $v_1, v_2 \in B_{R_2} := \{v \in V : 0 \leq v(x) \leq R_2 \text{ for a.a. } x \in \Omega\}$ , and this guarantees that  $N^\varepsilon$  has at most one fixed point in  $B_{R_2}$ . Since both disjoint sets  $D_{r_1 R_1}$  and  $D_{r_2 R_2}$  are included in  $B_{R_2}$ , we get a contradiction. This justifies our claim about multiplicity.

## 4 Componentwise localization for systems

The technique used in Section 3 for one equation can be extended to systems, for the localization of each component of the solution, individually. For simplicity we discuss only a system of two equations in  $\Omega$ , namely

$$\begin{cases} -\operatorname{div}(A_1^\varepsilon(x) \nabla u_1^\varepsilon) = g_1(x) f_1(u_1^\varepsilon, u_2^\varepsilon) \\ -\operatorname{div}(A_2^\varepsilon(x) \nabla u_2^\varepsilon) = g_2(x) f_2(u_1^\varepsilon, u_2^\varepsilon) \end{cases} \quad (4.1)$$

under the Dirichlet boundary condition  $u_i^\varepsilon = 0$  on  $\partial\Omega$ ,  $i = 1, 2$ .

We assume on the matrices  $A_i^\varepsilon$ , and on the functions  $g_i, f_i$ , similar hypotheses as in Section 2, allowing the ellipticity constants for the two equations to be different, namely  $\alpha_i$  and  $\beta_i$ ,  $i = 1, 2$ . Then, solving the Dirichlet problem for system (4.1) in the weak sense is equivalent to finding in  $V \times V$  the fixed point of the problem

$$\begin{aligned} u_1^\varepsilon &= N_1^\varepsilon(u_1^\varepsilon, u_2^\varepsilon) \\ u_2^\varepsilon &= N_2^\varepsilon(u_1^\varepsilon, u_2^\varepsilon) \end{aligned}$$

where  $N_i^\varepsilon = S_i^\varepsilon(g_i f_i)$ ,  $i = 1, 2$ . Also, in terms of the energy functionals associated to the equations of the system, solving problem (4.1) is equivalent to solving the critical point problem

$$\begin{aligned} J_{11}^\varepsilon(u_1^\varepsilon, u_2^\varepsilon) &= 0 \\ J_{22}^\varepsilon(u_1^\varepsilon, u_2^\varepsilon) &= 0, \end{aligned}$$

where by  $J_{ii}^\varepsilon(v_1, v_2)$  we have denoted the Fréchet derivative of  $J_i^\varepsilon(v_1, v_2)$  with respect to the variable  $v_i$ , for  $i = 1, 2$ , and the expressions of the functionals  $J_i^\varepsilon(v_1, v_2)$  are

$$\begin{aligned} J_1^\varepsilon(v_1, v_2) &= \frac{1}{2} \int_{\Omega} A_1^\varepsilon(x) \nabla v_1 \cdot \nabla v_1 - \int_{\Omega} g_1(x) \int_0^{v_1(x)} f_1(\tau, v_2(x)) d\tau, \\ J_2^\varepsilon(v_1, v_2) &= \frac{1}{2} \int_{\Omega} A_2^\varepsilon(x) \nabla v_2 \cdot \nabla v_2 - \int_{\Omega} g_2(x) \int_0^{v_2(x)} f_2(v_1(x), \tau) d\tau. \end{aligned}$$

We are interested to guarantee the existence and uniqueness of a solution  $(u_1^\varepsilon, u_2^\varepsilon)$  in a bounded subset of  $V \times V$ , more exactly  $(u_1^\varepsilon, u_2^\varepsilon) \in D := D_{r_1 R_1} \times D_{r_2 R_2}$ , where  $0 < r_1 < R_1$  and  $0 < r_2 < R_2$  are given numbers.

Here are the hypotheses that duplicate the assumptions made in Section 3 for a single equation:

**(H1)**  $g_i(x) \geq 0$  a.e in  $\Omega$ ,  $g_i(x) = 0$  for  $x \in \Omega \setminus \Omega'$ , and  $g_i > 0$  on a subset of nonzero measure;  $f_i(\tau_1, \tau_2) \geq 0$  for every  $\tau_1, \tau_2 \geq 0$ .

**(H2)**  $M_i \Gamma_{g_i} \leq R_i$  and  $r_i \leq m_i \gamma_{g_i}$ , where

$$M_i = \max_{\substack{\tau_1 \in [r_1, R_1] \\ \tau_2 \in [r_2, R_2]}} f_i(\tau_1, \tau_2), \quad m_i = \min_{\substack{\tau_1 \in [r_1, R_1] \\ \tau_2 \in [r_2, R_2]}} f_i(\tau_1, \tau_2).$$

**(H3)** There exist constants  $0 \leq l_{ij}$  ( $i, j = 1, 2$ ) such that

$$\begin{aligned} |f_i(\tau_1, \tau_2) - f_i(\bar{\tau}_1, \bar{\tau}_2)| &\leq l_{i1} |\tau_1 - \bar{\tau}_1| + l_{i2} |\tau_2 - \bar{\tau}_2| \\ \text{for all } \tau_1, \bar{\tau}_1 &\in [r_1, R_1] \text{ and } \tau_2, \bar{\tau}_2 \in [r_2, R_2], \end{aligned}$$

and the spectral radius of the matrix  $L = [L_{ij}]_{i,j=1,2}$ , where

$$L_{ij} := \frac{l_{ij} \|g_i\|_{L^\infty(\Omega)}}{\alpha_i \lambda_1} \quad (i, j = 1, 2)$$

is less than one.

**Theorem 4.1.** Assume that conditions (H1)-(H3) hold. Then there exists  $\varepsilon_0 > 0$  such that for any  $\varepsilon < \varepsilon_0$  :

(i) system (4.1) has a unique solution  $(u_1^\varepsilon, u_2^\varepsilon) \in D$ , which is in  $D$  the unique Nash equilibrium point with respect to the energy functionals  $J_1^\varepsilon, J_2^\varepsilon$ , i.e.,

$$J_1^\varepsilon(u_1^\varepsilon, u_2^\varepsilon) = \min_{D_{r_1 R_1}} J_1^\varepsilon(\cdot, u_2^\varepsilon) \quad \text{and} \quad J_2^\varepsilon(u_1^\varepsilon, u_2^\varepsilon) = \min_{D_{r_2 R_2}} J_2^\varepsilon(u_1^\varepsilon, \cdot).$$

(ii)  $u_i^\varepsilon \rightarrow u_i$  as  $\varepsilon \rightarrow 0$ , weakly in  $H_0^1(\Omega)$  and strongly in  $L^2(\Omega)$ , where  $(u_1, u_2)$  is the unique couple of functions in  $D$  which solves the homogenized system

$$\begin{cases} -\operatorname{div}(A_1^0 \nabla u_1) = g_1(x) f_1(u_1, u_2) \\ -\operatorname{div}(A_2^0 \nabla u_2) = g_2(x) f_2(u_1, u_2). \end{cases} \quad (4.2)$$

Also  $(u_1, u_2)$  is in  $D$  the unique Nash equilibrium with respect to the energy functionals  $J_1, J_2$  associated to the equations of the homogenized system (4.2), and

$$J_i^\varepsilon(u_1^\varepsilon, u_2^\varepsilon) \rightarrow J_i(u_1, u_2) \quad \text{as } \varepsilon \rightarrow 0, \quad \text{for } i = 1, 2.$$

*Proof.* (a) Existence, uniqueness and localization: Using similar arguments as in the case of a single equation, we obtain for any  $(v_1, v_2), (\bar{v}_1, \bar{v}_2) \in D$  and  $i \in \{1, 2\}$  :

$$0 \leq N_i^\varepsilon(v_1, v_2)(x) \leq M_i S_i^\varepsilon(g_i)(x) \leq M_i \Gamma_{g_i}, \quad \text{for a.a. } x \in \Omega,$$

and

$$N_i^\varepsilon(v_1, v_2)(x) \geq m_i S_i^\varepsilon(g_i)(x) \geq m_i y_{g_i}, \text{ for a.a. } x \in \Omega', \quad (4.3)$$

which in view of (H2) show that  $N^\varepsilon(D) \subset D$ , where  $N^\varepsilon = (N_1^\varepsilon, N_2^\varepsilon)$ .

Also

$$\|N_i^\varepsilon(v_1, v_2) - N_i^\varepsilon(\bar{v}_1, \bar{v}_2)\|_{H_0^1(\Omega)} \leq L_{i1} \|v_1 - \bar{v}_1\|_{H_0^1(\Omega)} + L_{i2} \|v_2 - \bar{v}_2\|_{H_0^1(\Omega)}, \quad i = 1, 2$$

which, using the matrix  $L$  defined in (H3), can be written in the matrix form

$$\begin{bmatrix} \|N_1^\varepsilon(v_1, v_2) - N_1^\varepsilon(\bar{v}_1, \bar{v}_2)\|_{H_0^1(\Omega)} \\ \|N_2^\varepsilon(v_1, v_2) - N_2^\varepsilon(\bar{v}_1, \bar{v}_2)\|_{H_0^1(\Omega)} \end{bmatrix} \leq L \begin{bmatrix} \|v_1 - \bar{v}_1\|_{H_0^1(\Omega)} \\ \|v_2 - \bar{v}_2\|_{H_0^1(\Omega)} \end{bmatrix}.$$

Then, from Perov's fixed point theorem (see [14, Theorem 1]), the operator  $N^\varepsilon$  has in  $D$  a unique fixed point  $(u_1^\varepsilon, u_2^\varepsilon)$ , solution of system (4.1).

(b) Nash equilibrium: In order to prove that the solution  $(u_1^\varepsilon, u_2^\varepsilon)$  is a Nash equilibrium, we use an iterative approximation scheme. We start with a fixed element  $u_{2,0}^\varepsilon$  in  $D_{r_2 R_2}$ . At each step  $k$  ( $k \geq 1$ ),  $u_{2,k-1}^\varepsilon$  been determined at step  $k-1$ , first we apply Ekeland's principle to  $J_1^\varepsilon(\cdot, u_{2,k-1}^\varepsilon)$  and find an  $u_{1,k}^\varepsilon \in D_{r_1 R_1}$  such that

$$J_1^\varepsilon(u_{1,k}^\varepsilon, u_{2,k-1}^\varepsilon) \leq \inf_{D_{r_1 R_1}} J_1^\varepsilon(\cdot, u_{2,k-1}^\varepsilon) + \frac{1}{k}, \quad \|J_{11}^\varepsilon(u_{1,k}^\varepsilon, u_{2,k-1}^\varepsilon)\|_{H_0^1(\Omega)} \leq \frac{1}{k}. \quad (4.4)$$

Next we apply Ekeland's principle to  $J_2^\varepsilon(u_{1,k}^\varepsilon, \cdot)$  and obtain an  $u_{2,k}^\varepsilon \in D_{r_2 R_2}$  with

$$J_2^\varepsilon(u_{1,k}^\varepsilon, u_{2,k}^\varepsilon) \leq \inf_{D_{r_2 R_2}} J_2^\varepsilon(u_{1,k}^\varepsilon, \cdot) + \frac{1}{k}, \quad \|J_{22}^\varepsilon(u_{1,k}^\varepsilon, u_{2,k}^\varepsilon)\|_{H_0^1(\Omega)} \leq \frac{1}{k}. \quad (4.5)$$

Let

$$v_k := J_{11}^\varepsilon(u_{1,k}^\varepsilon, u_{2,k-1}^\varepsilon) \quad \text{and} \quad w_k := J_{22}^\varepsilon(u_{1,k}^\varepsilon, u_{2,k}^\varepsilon).$$

From (4.4) and (4.5), one has  $v_k, w_k \rightarrow 0$  in  $H_0^1(\Omega)$ . Also, as in the case of one equation, we have

$$\begin{aligned} u_{1,k}^\varepsilon - N_1^\varepsilon(u_{1,k}^\varepsilon, u_{2,k-1}^\varepsilon) &= v_k \\ u_{2,k}^\varepsilon - N_2^\varepsilon(u_{1,k}^\varepsilon, u_{2,k}^\varepsilon) &= w_k. \end{aligned} \quad (4.6)$$

The first equality in (4.6) written for  $k$  and  $k+p$ , with  $p \in \mathbb{N} \setminus \{0\}$ , yields

$$\begin{aligned} \|u_{1,k+p}^\varepsilon - u_{1,k}^\varepsilon\|_{H_0^1(\Omega)} &\leq \|N_1^\varepsilon(u_{1,k+p}^\varepsilon, u_{2,k+p-1}^\varepsilon) - N_1^\varepsilon(u_{1,k}^\varepsilon, u_{2,k-1}^\varepsilon)\|_{H_0^1(\Omega)} + \|v_{k+p} - v_k\|_{H_0^1(\Omega)} \\ &\leq L_{11} \|u_{1,k+p}^\varepsilon - u_{1,k}^\varepsilon\|_{H_0^1(\Omega)} + L_{12} \|u_{2,k+p-1}^\varepsilon - u_{2,k-1}^\varepsilon\|_{H_0^1(\Omega)} + \|v_{k+p} - v_k\|_{H_0^1(\Omega)} \\ &\leq L_{11} \|u_{1,k+p}^\varepsilon - u_{1,k}^\varepsilon\|_{H_0^1(\Omega)} + L_{12} \|u_{2,k+p}^\varepsilon - u_{2,k}^\varepsilon\|_{H_0^1(\Omega)} \\ &\quad + L_{12} \left( \|u_{2,k+p-1}^\varepsilon - u_{2,k-1}^\varepsilon\|_{H_0^1(\Omega)} - \|u_{2,k+p}^\varepsilon - u_{2,k}^\varepsilon\|_{H_0^1(\Omega)} \right) + \|v_{k+p} - v_k\|_{H_0^1(\Omega)}. \end{aligned} \quad (4.7)$$

The second equality in (4.6) written for  $k$  and  $k+p$  gives

$$\|u_{2,k+p}^\varepsilon - u_{2,k}^\varepsilon\|_{H_0^1(\Omega)} \leq L_{21} \|u_{1,k+p}^\varepsilon - u_{1,k}^\varepsilon\|_{H_0^1(\Omega)} + L_{22} \|u_{2,k+p}^\varepsilon - u_{2,k}^\varepsilon\|_{H_0^1(\Omega)} + \|w_{k+p} - w_k\|_{H_0^1(\Omega)}. \quad (4.8)$$

Denote

$$\begin{aligned} a_{k,p} &= \|u_{1,k+p}^\varepsilon - u_{1,k}^\varepsilon\|_{H_0^1(\Omega)}, \quad b_{k,p} = \|u_{2,k+p}^\varepsilon - u_{2,k}^\varepsilon\|_{H_0^1(\Omega)}, \\ c_{k,p} &= \|v_{k+p} - v_k\|_{H_0^1(\Omega)}, \quad d_{k,p} = \|w_{k+p} - w_k\|_{H_0^1(\Omega)}. \end{aligned}$$

Clearly,

$$c_{k,p} \rightarrow 0 \text{ and } d_{k,p} \rightarrow 0 \text{ as } k \rightarrow \infty, \text{ uniformly with respect to } p. \quad (4.9)$$

With these notations, (4.7) and (4.8) become

$$\begin{aligned} a_{k,p} &\leq L_{11}a_{k,p} + L_{12}b_{k,p} + c_{k,p} + L_{12}(b_{k-1,p} - b_{k,p}), \\ b_{k,p} &\leq L_{21}a_{k,p} + L_{22}b_{k,p} + d_{k,p}, \end{aligned}$$

which can be expressed in the matrix form

$$\begin{bmatrix} a_{k,p} \\ b_{k,p} \end{bmatrix} \leq L \begin{bmatrix} a_{k,p} \\ b_{k,p} \end{bmatrix} + \begin{bmatrix} c_{k,p} + L_{12}(b_{k-1,p} - b_{k,p}) \\ d_{k,p} \end{bmatrix}. \quad (4.10)$$

Under the assumption that the spectral radius of the matrix  $L$  is less than one, the matrix  $I - L$  (where  $I$  is the unit matrix), is invertible and its inverse contains only nonnegative elements (see [14, Lemma 2]). Thus (4.10) gives

$$\begin{bmatrix} a_{k,p} \\ b_{k,p} \end{bmatrix} \leq (I - L)^{-1} \begin{bmatrix} c_{k,p} + L_{12}(b_{k-1,p} - b_{k,p}) \\ d_{k,p} \end{bmatrix}.$$

Let  $(I - L)^{-1} = [\rho_{ij}]$ . Then

$$\begin{aligned} a_{k,p} &\leq \rho_{11}(c_{k,p} + L_{12}(b_{k-1,p} - b_{k,p})) + \rho_{12}d_{k,p} \\ b_{k,p} &\leq \rho_{21}(c_{k,p} + L_{12}(b_{k-1,p} - b_{k,p})) + \rho_{22}d_{k,p}. \end{aligned} \quad (4.11)$$

From the second inequality, one has

$$b_{k,p} \leq \frac{\rho_{21}L_{12}}{1 + \rho_{21}L_{12}}b_{k-1,p} + \frac{\rho_{21}c_{k,p} + \rho_{22}d_{k,p}}{1 + \rho_{21}L_{12}}. \quad (4.12)$$

Note that the sequence  $(b_{k,p})_{k \geq 1}$  is bounded uniformly with respect to  $p$ . Indeed, from (4.6),

$$u_{2,k}^\varepsilon - w_k = N_2^\varepsilon(u_{1,k}^\varepsilon, u_{2,k}^\varepsilon),$$

whence

$$\begin{cases} -\operatorname{div}(A^\varepsilon(x) \nabla(u_{2,k}^\varepsilon - w_k)) = g_2(x)f_2(u_{1,k}^\varepsilon, u_{2,k}^\varepsilon) & \text{in } \Omega \\ u_{2,k}^\varepsilon - w_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Since the right-hand side is bounded in  $L^\infty(\Omega)$  independently of  $k$ , this implies

$$\|u_{2,k}^\varepsilon - w_k\|_{H_0^1(\Omega)} \leq c_0$$

for every  $k$ . As a result, the sequence  $(u_{2,k}^\varepsilon)_{k \geq 1}$  is bounded in the norm of  $H_0^1(\Omega)$  by some constant  $c$ . Then, clearly  $b_{k,p} \leq 2c$ , as claimed. Now we recall a lemma proved in [16, Lemma 3.2].

**Lemma 4.2.** *Let  $(x_{k,p}), (y_{k,p})$  be two sequences of real numbers depending on a parameter  $p$ , such that*

$$(x_{k,p}) \text{ is bounded uniformly with respect to } p,$$

and

$$0 \leq x_{k,p} \leq \lambda x_{k-1,p} + y_{k,p} \quad (4.13)$$

for all  $k, p$  and some  $\lambda \in [0, 1)$ . If  $y_{k,p} \rightarrow 0$  as  $k \rightarrow \infty$  uniformly with respect to  $p$ , then  $x_{k,p} \rightarrow 0$  uniformly with respect to  $p$ .

We apply the previous lemma for

$$x_{k,p} = b_{k,p}, \quad y_{k,p} = \frac{\rho_{21}c_{k,p} + \rho_{22}d_{k,p}}{1 + \rho_{21}L_{12}} \quad \text{and} \quad \lambda = \frac{\rho_{21}L_{12}}{1 + \rho_{21}L_{12}}.$$

According to (4.12), one has (4.13), while due to (4.9), one has  $y_{k,p} \rightarrow 0$  as  $k \rightarrow \infty$ , uniformly with respect to  $p$ . Also, obviously,  $0 \leq \lambda < 1$ . It follows that  $b_{k,p} \rightarrow 0$  as  $k \rightarrow \infty$ , uniformly with respect to  $p$ , that is  $(u_{2,k}^\varepsilon)$  is a Cauchy sequence. Next, the first inequality in (4.11) implies that  $(u_{1,k}^\varepsilon)$  is also a Cauchy sequence. Let  $v_1^\varepsilon, v_2^\varepsilon$  be the limits of the sequences  $(u_{1,k}^\varepsilon), (u_{2,k}^\varepsilon)$  as  $k \rightarrow \infty$ , respectively. Clearly  $(v_1^\varepsilon, v_2^\varepsilon) \in D$ , and passing to the limit in (4.4), (4.5) we obtain that  $(v_1^\varepsilon, v_2^\varepsilon)$  solves system (4.1). The uniqueness of the solution in  $D$  implies that  $(v_1^\varepsilon, v_2^\varepsilon) = (u_1^\varepsilon, u_2^\varepsilon)$ .  $\square$

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