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Research Article

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Existence and non-existence results for Kirchhoff-type problems with convolution nonlinearity

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Abstract: This paper is concerned with the following Kirchhoff-type problem with convolution nonlinearity:

$$-\Big(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\Big)\Delta u+V(x)u=(I_\alpha * F(u))f(u),\quad x\in\mathbb{R}^3,\ u\in H^1(\mathbb{R}^3),$$

where $a, b > 0$, $I_\alpha\colon \mathbb{R}^3 \to \mathbb{R}$, with $\alpha \in (0, 3)$, is the Riesz potential, $V \in \mathcal{C}(\mathbb{R}^3, [0, \infty))$, $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and $F(t) = \int_0^t f(s)\,\mathrm{d}s$. By using variational and some new analytical techniques, we prove that the above problem admits ground state solutions under mild assumptions on V and f. Moreover, we give a non-existence result. In particular, our results extend and improve the existing ones, and fill a gap in the case where $f(u) = |u|^{q-2}u$, with $q \in (1 + \alpha/3, 2]$.

Keywords: Kirchhoff-type problem, Choquard equation, convolution nonlinearity, ground state solutions

MSC 2010: 35J20, 35Q55

1 Introduction and main results

In this paper, we consider the following Kirchhoff-type problem with convolution nonlinearity:

$$-\Big(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\Big)\Delta u+V(x)u=(I_\alpha * F(u))f(u),\quad x\in\mathbb{R}^3,\ u\in H^1(\mathbb{R}^3),\tag{1.1}$$

where $a, b > 0$, $I_\alpha\colon \mathbb{R}^3 \to \mathbb{R}$, with $\alpha \in (0, 3)$, is the Riesz potential defined by

$$I_\alpha(x)=\frac{\Gamma(\frac{3-\alpha}{2})}{\Gamma(\frac{\alpha}{2})2^\alpha\pi^{3/2}|x|^{3-\alpha}},\quad x\in\mathbb{R}^3\setminus\{0\},$$

$V \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$, $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and $F(t) = \int_0^t f(s)\,\mathrm{d}s$.

Such a problem is often referred to as being nonlocal due to the appearance of the terms $(\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x)\Delta u$ or $(I_\alpha * F(u))f(u)$ which implies that (1.1) is no longer a pointwise identity. In particular, if $b = 0$, then (1.1) reduces to the following generalized Choquard equation:

$$-\Delta u+V(x)u=(I_\alpha * F(u))f(u),\quad u\in H^1(\mathbb{R}^3).\tag{1.2}$$

When $\alpha = 2$, $V(x) \equiv 1$ and $f(u) = u$, (1.2) is known as the Choquard–Pekar equation or the stationary nonlinear Hartree equation, which was introduced in 1954, in a work by Pekar [29] describing the quantum

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mechanics of a polaron at rest; for more details and applications, we refer to [19, 27]. For the case where $V(x) \equiv 1$ and $f(u) = |u|^{p-2}u$, (1.2) is known to have a solution if and only if $1 + \alpha/3 < p < 3 + \alpha$ (see [24, p. 457], [27, Theorem 1]; see also [11, Lemma 2.7]). As described Moroz and Van Schaftingen in [28], since the Hardy–Littlewood–Sobolev inequality [20] implies

$$\int_{\mathbb{R}^3} (I_\alpha * h_1)h_2(x)\,dx \le C(\alpha)\|h_1\|_{6/(3+\alpha)}\|h_2\|_{6/(3+\alpha)} \quad \text{for all } h_1, h_2 \in L^{6/(3+\alpha)}, \tag{1.3}$$

where $3 + \alpha$ and $1 + \alpha/3$ are the upper and lower critical exponents, which appear as extensions of the exponents 6 and 2 for the corresponding local problem.

Inspired by [24, 27, 28], we introduce the following basic assumption on f:

(F1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, $f(t) = o(|t|^{\alpha/3})$ as $|t| \to 0$ and $f(t) = o(|t|^{2+\alpha})$ as $|t| \to \infty$.

If we let $\alpha \to 0$ in (1.1), then it becomes formally the following Kirchhoff-type problem with local nonlinearity $g = Ff$:

$$-\Big(a + b\int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big)\Delta u + V(x)u = g(u), \quad x \in \mathbb{R}^3,\ u \in H^1(\mathbb{R}^3), \tag{1.4}$$

which is related to the stationary analogue of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2\,dx\Big)\frac{\partial^2 u}{\partial x^2} = 0. \tag{1.5}$$

Equation (1.5) is proposed by Kirchhoff [15] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. For more details on the physical aspects, we refer the readers to [2, 3, 7, 8, 26].

After Lions [21] proposed an abstract functional analysis framework to (1.4), it has received more and more attention from the mathematical community; there have been many works about the existence of nontrivial solutions to (1.4) and its fractional version by using variational methods, for example, see [4, 6, 9, 12, 13, 16–18, 22, 30, 31, 34, 35, 37, 40–42] and the references therein. A typical way to deal with (1.4) is to use the mountain-pass theorem. For this purpose, one usually assumes that $g(t)$ is superlinear at $t = 0$ and super-cubic at $t = \infty$. In this case, if g further satisfies the monotonicity condition

(G1) $g(t)/|t|^3$ is increasing for $t \in \mathbb{R} \setminus \{0\}$,

via the Nehari manifold approach, He and Zou [13] obtained the first existence result on ground state solutions of (1.4). For the case where $g(t)$ is not super-cubic at $t = \infty$, Li and Ye [17] proved that (1.4), with special forms $V = 1$ and $g(u) = |u|^{p-2}u$ for $3 < p < 6$, has a ground state positive solution by using a minimizing argument on a new manifold that is defined by a condition which is a combination of the Nehari equation and the Pohožaev equality. This idea comes from Ruiz [33], in which the nonlinear Schrödinger–Poisson system was studied. Later, by introducing another suitable manifold differing from [17], Guo [12] and Tang and Chen [37] improved the above result to (1.4), where V satisfies

(V1) $V \in \mathcal{C}(\mathbb{R}^3, [0, \infty))$ and $V_\infty := \lim_{|y|\to\infty} V(y) \ge V(x)$ for all $x \in \mathbb{R}^3$,

(V2)′ $V \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R})$ and there exists $\theta' \in (0, 1)$ such that $\nabla V(x) \cdot x \le \frac{a\theta'}{2|x|^2}$ for all $x \in \mathbb{R}^3 \setminus \{0\}$,

and g satisfies

(G2) $g \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{R})$ and $\frac{g(t)}{t}$ is increasing on $(0, \infty)$,

and

(G3) $g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and $\frac{g(t)t+6G(t)}{|t|t}$ is nondecreasing on $(-\infty, 0) \cup (0, \infty)$, where $G(t) = \int_0^t g(s)\,ds$.

respectively, and some standard growth assumptions.

Compared with (1.2) and (1.4), it is more difficult to deal with (1.1) which involves two nonlocal terms. In [23], Lü investigated the following special form of (1.1):

$$-\Big(a + b\int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big)\Delta u + (1 + \mu g(x))u = (I_\alpha * |u|^q)|u|^{q-2}u, \quad x \in \mathbb{R}^3,\ u \in H^1(\mathbb{R}^3), \tag{1.6}$$

where $q \in (2, 3+\alpha)$, $\mu > 0$ is a parameter and $g(x)$ is a nonnegative steep potential well function. By using the Nehari manifold and the concentration compactness principle, Lü proved the existence of ground state solutions for (1.6) if the parameter μ is large enough. It is worth pointing out that the same result is not available in the case where $q \in (1+\alpha/3, 2]$, even when $g(x) = 0$, since both the mountain pass theorem and the Nehari manifold argument do not work. In fact, in this case, it is more difficult to get a bounded (PS) sequence and to prove that the (PS) sequence converges weakly to a critical point of the corresponding functional in $H^1(\mathbb{R}^3)$. To the best of our knowledge, there seem to be no results dealt with this case in the literature. As for the related study of problem (1.1) involving the critical exponents, we refer to [25, 32] for more details.

Motivated by the above-mentioned papers, we shall deal with the existence of ground state solutions for (1.1) under (V1) and (F1). It is standard to check, according to (1.3), that under (V1) and (F1), the energy functional defined in $H^1(\mathbb{R}^3)$ by

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^3} [a|\nabla u|^2 + V(x)u^2]\,dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big)^2 - \frac{1}{2}\int_{\mathbb{R}^3} (I_\alpha * F(u))F(u)\,dx \tag{1.7}$$

is continuously differentiable and its critical points correspond to the weak solutions of (1.1). We say a weak solution to (1.1) is a ground state solution if it minimizes the functional Φ among all nontrivial weak solutions.

In addition to (F1), we also need the following assumptions on f:

(F2) $\lim_{|t|\to\infty} \frac{F(t)}{|t|^{1-\alpha}} = \infty$,

(F3) the function $|t|^\alpha[f(t)t + (3+\alpha)F(t)]/t$ is nondecreasing on $(-\infty, 0) \cup (0, +\infty)$.

Remark 1.1. (F3) is weaker than the following assumption, which is easier to verify:

(F4) the function $|t|^\alpha f(t)$ is nondecreasing on $(-\infty, 0) \cup (0, +\infty)$.

It is easy to see that there are many functions which satisfy (F1), (F2) and (F4). In addition, there are some functions which satisfy (F3), but not (F4), for example,

$$f(t) = \frac{\alpha+2}{(2\alpha+1)(2\alpha+3)}|t|^\alpha t + (\alpha+1)|t|^{\alpha-1}t + \alpha|t|^{\alpha-2}t\sin t + |t|^\alpha \cos t.$$

To overcome the lack of compactness of Sobolev embeddings in unbounded domains, different from [23] in which a steep potential well was considered, we assume that V satisfies (V1) and the decay condition:

(V2) $V \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R})$ and either of the following cases holds:

(i) $\nabla V(x)\cdot x \le \frac{a}{2|x|^2}$ for all $x \in \mathbb{R}^3 \setminus \{0\}$,

(ii) $\|\max\{\nabla V(x)\cdot x, 0\}\|_{3/2} \le (\frac{3}{4})^{1/3}\pi^2 a$.

Now we are in a position to state the first main result.

Theorem 1.2. *Assume that V and f satisfy* (V1), (V2) *and* (F1)–(F3). *Then problem* (1.1) *has a ground state solution $\hat{u} \in H^1(\mathbb{R}^3)$.*

Next, we further provide a minimax characterization of the ground state energy. To this end, we introduce a new monotonicity condition on V as follows:

(V3) $V \in \mathcal{C}^1(\mathbb{R}^3, \mathbb{R})$ and there exists $\theta \in [0, 1)$ such that

$$t \mapsto 4V(tx) + \nabla V(tx)\cdot(tx) + \frac{\theta a}{2t^2|x|^2}$$

is nonincreasing on $(0, +\infty)$ for every $x \in \mathbb{R}^3 \setminus \{0\}$.

Similar to [12], we define the Pohožaev functional related with (1.1):

$$\mathcal{P}(u) = \frac{a}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3} [3V(x) + \nabla V(x)\cdot x]u^2\,dx + \frac{b}{2}\|\nabla u\|_2^4 - \frac{3+\alpha}{2}\int_{\mathbb{R}^3} (I_\alpha * F(u))F(u)\,dx. \tag{1.8}$$

It is well known that any solution u of (1.1) satisfies $\mathcal{P}(u) = 0$. Motivated by [17, 37], we define the Nehari–Pohožaev manifold of Φ by

$$\mathcal{M} := \Big\{u \in H^1(\mathbb{R}^3) \setminus \{0\} : J(u) := \frac{1}{2}\langle \Phi'(u), u\rangle + \mathcal{P}(u) = 0\Big\}. \tag{1.9}$$

Then every nontrivial solution of (1.1) is contained in $\mathcal{M}$. Our second main result is as follows.

Theorem 1.3. *Assume that V and f satisfy* (V1), (V3) *and* (F1)–(F3). *Then problem* (1.1) *has a ground state solution $\bar{u} \in H^1(\mathbb{R}^3)$ such that*

$$\Phi(\bar{u}) = \inf_{\mathcal{M}} \Phi = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \max_{t>0} \Phi(t^{1/2}u_t) > 0,$$

where and in the sequel $u_t(x) := u(x/t)$.

Applying Theorem 1.3 to the "limiting problem" of (1.1):

$$-\Big(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \,dx\Big)\Delta u + V_\infty u = (I_\alpha * F(u))f(u), \quad x \in \mathbb{R}^3,\ u \in H^1(\mathbb{R}^3), \tag{1.10}$$

similar to (1.7) and (1.9), we define

$$\Phi^\infty(u) := \frac{1}{2}\int_{\mathbb{R}^3} (a|\nabla u|^2 + V_\infty u^2)\,dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 \,dx\Big)^2 - \frac{1}{2}\int_{\mathbb{R}^3} (I_\alpha * F(u))F(u)\,dx \tag{1.11}$$

and

$$\mathcal{M}^\infty := \Big\{v \in H^1(\mathbb{R}^3) \setminus \{0\} : J^\infty(u) = \frac{1}{2}\langle(\Phi^\infty)'(u), u\rangle + \mathcal{P}^\infty(u) = 0\Big\},$$

where $\mathcal{P}^\infty(u) = 0$ is the Pohožave type identity related with (1.10). Then we have the following corollary.

Corollary 1.4. *Assume that* (F1)–(F3) *hold. Then problem* (1.10) *has a ground state solution $\bar{u}^\infty \in H^1(\mathbb{R}^3)$ such that*

$$\Phi^\infty(\bar{u}^\infty) = \inf_{\mathcal{M}^\infty} \Phi^\infty = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \max_{t>0} \Phi^\infty(t^{1/2}u_t) > 0.$$

In the last part of this paper, we give a non-existence result for the following special form of (1.1):

$$-\Big(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \,dx\Big)\Delta u + u = (I_\alpha * |u|^q)|u|^{q-2}u, \quad x \in \mathbb{R}^3,\ u \in H^1(\mathbb{R}^3), \tag{1.12}$$

where $q > 1$.

Theorem 1.5. *If $1 < q < 1 + \alpha/3$ or $q \geq 3 + \alpha$, then problem* (1.12) *does not admit any nontrivial solution.*

Remark 1.6. There are indeed functions which satisfy (V1)–(V3). An example is given by $V(x) = V_1 - \frac{A}{|x|^2+1}$, where $V_1 > 1$ and $0 < A < a/8$ are two positive constants. Our results extend and improve the previous results on (1.1) in the literature, which are new even when $V \equiv V_\infty > 0$. In particular, Theorems 1.2 and 1.3 fill a gap on (1.1) in the case where $f(u) = |u|^{q-2}u$, with $q \in (1 + \alpha/3, 2]$.

Remark 1.7. Letting $\alpha \to 0$, our results cover the ones in [12, 17, 37], which dealt with (1.4) that can be considered as a limiting problem of (1.1) when $\alpha \to 0$. In fact, if $\alpha = 0$, then (V2)$'$ and (G2) imply case (i) of (V2) and (F3), respectively. Moreover, since (G3) and (F3) imply that $G(t)/|t|t$ and $|t|^\alpha F(t)/t$ are nondecreasing on $(-\infty, 0) \cup (0, +\infty)$, respectively (see Lemma 2.3), one can see that (F3) reduces to (G3) when $\alpha = 0$.

To prove Theorem 1.2, we will use Jeanjean's monotonicity trick [14], that is, an approximation procedure to obtain a bounded (PS)-sequence for Φ, instead of starting directly from an arbitrary (PS)-sequence. More precisely, firstly, for $\lambda \in [1/2, 1]$, we consider a family of functionals $\Phi_\lambda : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\Phi_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} (a|\nabla u|^2 + V(x)u^2)\,dx + \frac{b}{4}\|\nabla u\|_2^4 - \frac{\lambda}{2}\int_{\mathbb{R}^3} (I_\alpha * F(u))F(u)\,dx. \tag{1.13}$$

These functionals have a mountain pass geometry, and we denote the corresponding mountain pass levels by c_λ. Moreover, Φ_λ has a bounded (PS)-sequence $\{u_n(\lambda)\} \subset H^1(\mathbb{R}^3)$ at level c_λ for almost every $\lambda \in [1/2, 1]$. Secondly, we use the global compactness lemma to show that the bounded sequence $\{u_n(\lambda)\}$ converges weakly to a nontrivial critical point of Φ_λ. To do this, we have to establish the following strict inequality:

$$c_\lambda < \inf_{\mathcal{K}_\lambda^\infty} \Phi_\lambda^\infty, \tag{1.14}$$

where Φ_λ^∞ is the associated limited functional defined by

$$\Phi_\lambda^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^3} (a|\nabla u|^2 + V_\infty u^2)\,\mathrm{d}x + \frac{b}{4}\|\nabla u\|_2^4 - \frac{\lambda}{2}\int_{\mathbb{R}^3} (I_\alpha * F(u))F(u)\,\mathrm{d}x, \tag{1.15}$$

and

$$\mathcal{K}_\lambda^\infty := \{w \in H^1(\mathbb{R}^3) \setminus \{0\} : (\Phi_\lambda^\infty)'(w) = 0\}.$$

A classical way to obtain (1.14) is to find a positive function $w_\lambda^\infty \in \mathcal{K}_\lambda^\infty$ such that $\Phi_\lambda^\infty(w_\lambda^\infty) = \inf_{\mathcal{K}_\lambda^\infty} \Phi_\lambda^\infty$ when nonconstant potential $V(x) \leq V_\infty$. However, it seems to be impossible to obtain the w_λ^∞ mentioned above only under (F1)–(F3). So the usual arguments cannot be applied here to prove (1.14). To overcome this difficulty, we follow a strategy introduced in [38], that is, we first show that there exists $\bar{u}^\infty$ such that

$$\bar{u}^\infty \in \mathcal{M}^\infty, \quad \Phi^\infty(\bar{u}^\infty) = \inf_{\mathcal{M}^\infty} \Phi^\infty, \tag{1.16}$$

and then, by means of the translation invariance for $\bar{u}^\infty$ and the crucial inequality

$$\Phi_\lambda^\infty(u) \geq \Phi_\lambda^\infty(t^{1/2}u_t) + \frac{1-t^4}{4} J_\lambda^\infty(u) + \frac{a(1-t^2)^2}{4}\|\nabla u\|_2^2 \quad \text{for all } u \in H^1(\mathbb{R}^3) \text{ and } t > 0$$

established in Lemma 3.3, we can find $\bar{\lambda} \in [1/2, 1)$ such that

$$c_\lambda < m_\lambda^\infty := \inf_{\mathcal{M}_\lambda^\infty} \Phi_\lambda^\infty \quad \text{for all } \lambda \in (\bar{\lambda}, 1] \tag{1.17}$$

(see Lemma 3.5), where

$$\mathcal{M}_\lambda^\infty = \Big\{u \in H^1(\mathbb{R}^3) \setminus \{0\} : J_\lambda^\infty(u) = \frac{1}{2}\langle(\Phi_\lambda^\infty)'(u), u\rangle + \mathcal{P}_\lambda^\infty(u) = 0\Big\}$$

and $\mathcal{P}_\lambda^\infty(u) = 0$ is the corresponding Pohožave type identity. In particular, any information on sign of $\bar{u}^\infty$ is not required in our arguments. Finally, we choose two sequences $\{\lambda_n\} \in (\bar{\lambda}, 1]$ and $\{u_{\lambda_n}\} \subset H^1(\mathbb{R}^3) \setminus \{0\}$ such that $\lambda_n \to 1$ and $\Phi'_{\lambda_n}(u_{\lambda_n}) = 0$, and by using (1.17) and the global compactness lemma, we get a nontrivial critical point $\bar{u}$ of Φ.

We would like to mention that in the proof of Theorem 1.2, a crucial step is to prove (1.16), which is a corollary of Theorem 1.3. Inspired by [5, 36, 37], we shall prove Theorem 1.3 by following this scheme:
Step 1: we verify $\mathcal{M} \neq \emptyset$ and establish the minimax characterization of $m := \inf_{\mathcal{M}} \Phi > 0$,
Step 2: we prove that m is achieved,
Step 3: we show that the minimizer of Φ on $\mathcal{M}$ is a critical point.

Although we mainly follow the procedure of [36], we have to face many new difficulties due to the mutual competing effect between $(\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x)\Delta u$ and $(I_\alpha * F(u))f(u)$. These difficulties enforce the implementation of new ideas and techniques. More precisely, in step 1, we first establish a key inequality, namely,

$$\Phi(u) \geq \Phi(t^{1/2}u_t) + \frac{1-t^4}{4}J(u) + \frac{a(1-\theta)(1-t^2)^2}{4}\|\nabla u\|_2^2 \quad \text{for all } u \in H^1(\mathbb{R}^3),\ t > 0, \tag{1.18}$$

in Lemma 2.5, where some more careful analyses on the convolution nonlinearity are introduced, see Lemmas 2.1–2.4; then we construct a saddle point structure with respect to the fibre $\{t^{1/2}u_t : t > 0\} \subset H^1(\mathbb{R}^3)$ for $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, see Lemma 2.8; finally, based on these constructions, we obtain the minimax characterization of m, see Lemma 2.9. In step 2, we first choose a minimizing sequence $\{u_n\}$ of Φ on $\mathcal{M}$, and show that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$; then, with the help of the key inequality (1.18) and a concentration-compactness argument, we prove that there exist $\hat{u} \in H^1(\mathbb{R}^3) \setminus \{0\}$ and $\hat{t} > 0$ such that $u_n \rightharpoonup \hat{u}$ in $H^1(\mathbb{R}^3)$, up to translations and extraction of a subsequence, and $\hat{t}^{1/2}\hat{u}_{\hat{t}} \in \mathcal{M}$ is a minimizer of $\inf_{\mathcal{M}} \Phi$, see Lemmas 2.14 and 2.15. This step is most difficult since there is no global compactness and not any information on $\Phi'(u_n)$. Finally, in step 3, inspired by [38, Lemma 2.13], we use the key inequality (1.18), the deformation lemma and intermediary theorem for continuous functions, which overcome the difficulty that $\mathcal{M}$ may not be a $\mathcal{C}^1$-manifold of $H^1(\mathbb{R}^3)$, due to the lack of the smoothness of $f(u)$, see Lemma 2.15.

Throughout the paper we make use of the following notations:

- $H^1(\mathbb{R}^3)$ denotes the usual Sobolev space equipped with the inner product and norm

$$(u,v)=\int_{\mathbb{R}^3}(\nabla u\cdot\nabla v+uv)\,\mathrm{d}x,\quad \|u\|=(u,u)^{1/2}\quad \text{for all } u,v\in H^1(\mathbb{R}^3).$$

- $L^s(\mathbb{R}^3)$ $(1\le s<\infty)$ denotes the Lebesgue space with the norm $\|u\|_s=(\int_{\mathbb{R}^3}|u|^s\,\mathrm{d}x)^{1/s}$.
- For any $u\in H^1(\mathbb{R}^3)\setminus\{0\}$, $u_t(x):=u(x/t)$ for $t>0$.
- For any $x\in\mathbb{R}^3$ and $r>0$, $B_r(x):=\{y\in\mathbb{R}^3:|y-x|<r\}$.
- $C_1,C_2,\dots$ denote positive constants possibly different in different places.

The rest of the paper is organized as follows. In Section 2, we study the existence of ground state solutions for (1.1) by using the Nehari–Pohožaev manifold $\mathcal{M}$, and give the proof of Theorem 1.3. In Section 3, based on Jeanjean's monotonicity trick, we consider the existence of ground state solutions for (1.1), and complete the proof of Theorem 1.2. In Section 4, we study the non-existence of solutions for problem (1.12) and present the proof of Theorem 1.5.

2 Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3. To this end, we give some useful lemmas. Since $V(x)\equiv V_\infty$ satisfies (V1)–(V3), all conclusions on Φ are also true for Φ^∞ in this paper. For (1.4), we always assume that $V_\infty>0$. First, by a simple calculation, we can verify the following lemma.

Lemma 2.1. *Assume that* (V1) *and* (V3) *hold. Then one has*

$$4t^4[V(x)-V(tx)]-(1-t^4)\nabla V(x)\cdot x\ge-\frac{\theta a(1-t^2)^2}{2|x|^2}\quad \textit{for all } t\ge0,\ x\in\mathbb{R}^3\setminus\{0\}. \tag{2.1}$$

Lemma 2.2. *Assume that* (F1) *and* (F3) *hold. Then for all* $s\ge0$ *and* $t\in\mathbb{R}$,

$$g(s,t):=4s^{(3+\alpha)/2}F(s^{1/2}t)-4F(t)+(1-s^2)[f(t)t+(3+\alpha)F(t)]\ge0. \tag{2.2}$$

Proof. It is evident that (2.2) holds for all $s>0$ and $t=0$. For $t\ne0$, it follows from (F3) that

$$\frac{\mathrm{d}}{\mathrm{d}s}g(s,t)=2s|t|^{1-\alpha}\{|s^{1/2}t|^{\alpha-1}[f(s^{1/2}t)s^{1/2}t+(3+\alpha)F(s^{1/2}t)]|t|^{\alpha-1}[f(t)t+(3+\alpha)F(t)]\}\begin{cases}\ge0, & s\ge1,\\ \le0, & 0\le s<1,\end{cases}$$

which implies that $g(s,t)\ge g(1,t)=0$ for all $s\ge0$ and $t\in(-\infty,0)\cup(0,+\infty)$. □

Lemma 2.3. *Assume that* (F1) *and* (F3) *hold. Then*

$$\frac{|t|^\alpha F(t)}{t}\ \textit{is nondecreasing on } (-\infty,0)\cup(0,+\infty). \tag{2.3}$$

Proof. By (F1) and (2.2), one has

$$g(0,t)=f(t)t+(\alpha-1)F(t)\ge0\quad\text{for all } t\in\mathbb{R}. \tag{2.4}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{|t|^\alpha F(t)}{t}\Big)=|t|^{\alpha-2}[f(t)t+(\alpha-1)F(t)],$$

(2.3) follows from (2.4). □

Lemma 2.4. *Assume that* (F1) *and* (F3) *hold. Then*

$$\begin{aligned}h(t,u):=\int_{\mathbb{R}^3}\Big\{&t^{3+\alpha}(I_\alpha*F(t^{1/2}u))F(t^{1/2}u)+\frac{1-t^4}{4}(I_\alpha*F(u))f(u)u\\&-\frac{(3+\alpha)t^4+1-\alpha}{4}(I_\alpha*F(u))F(u)\Big\}\,\mathrm{d}x\ge0\quad\textit{for all } t>0,\ u\in H^1(\mathbb{R}^3).\end{aligned} \tag{2.5}$$

Proof. Note that (F1) and (2.3) imply

$$F(t) \geq 0 \quad \text{for all } t \in \mathbb{R} \tag{2.6}$$

and

$$I_\alpha * \left(\frac{t^{\alpha/2}F(t^{1/2}u)}{t^{1/2}}\right) - I_\alpha * F(u) \begin{cases} \geq 0, & t \geq 1, \\ \leq 0, & 0 < t < 1. \end{cases} \tag{2.7}$$

By (F3), (2.6) and (2.7), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} h(t,u) &= \int_{\mathbb{R}^3} \Big\{t^{2+\alpha}(I_\alpha * F(t^{1/2}u))[f(t^{1/2}u)t^{1/2}u + (3+\alpha)F(t^{1/2}u)] - t^3(I_\alpha * F(u))[f(u)u + (3+\alpha)F(u)]\Big\}\,\mathrm{d}x \\ &= t^3 \int_{\mathbb{R}^3} |u|^{1-\alpha}\bigg\{\left(\frac{t^{\alpha/2}F(t^{1/2}u)}{t^{1/2}}\right)|t^{1/2}u|^{\alpha-1}[f(t^{1/2}u)t^{1/2}u + (3+\alpha)F(t^{1/2}u)] \\ &\qquad - (I_\alpha * F(u))|u|^{\alpha-1}[f(u)u + (3+\alpha)F(u)]\bigg\}\,\mathrm{d}x \begin{cases} \geq 0, & t \geq 1, \\ \leq 0, & 0 < t < 1, \end{cases} \end{aligned}$$

which implies that $h(t,u) \geq h(1,u) = 0$ for all $u \in H^1(\mathbb{R}^3)$. This shows that (2.5) holds. □

By (1.7) and (1.8), one has

$$J(u) = a\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3} [4V(x) + \nabla V(x)\cdot x]u^2\,\mathrm{d}x + b\|\nabla u\|_2^4 - \frac{1}{2}\int_{\mathbb{R}^3} (I_\alpha * F(u))[f(u)u + (3+\alpha)F(u)]\,\mathrm{d}x. \tag{2.8}$$

Lemma 2.5. *Assume that* (V1), (V3), (F1) *and* (F3) *hold. Then*

$$\Phi(u) \geq \Phi(t^{1/2}u_t) + \frac{1-t^4}{4}J(u) + \frac{a(1-\theta)(1-t^2)^2}{4}\|\nabla u\|_2^2 \quad \textit{for all } u \in H^1(\mathbb{R}^3),\ t > 0. \tag{2.9}$$

Proof. According to the Hardy inequality, we have

$$\|\nabla u\|_2^2 \geq \frac{1}{4}\int_{\mathbb{R}^3} \frac{u^2}{|x|^2}\,\mathrm{d}x \quad \text{for all } u \in H^1(\mathbb{R}^3). \tag{2.10}$$

Note that

$$\begin{aligned} \Phi(t^{1/2}u_t) &= \frac{at^2}{2}\|\nabla u\|_2^2 + \frac{t^4}{2}\int_{\mathbb{R}^3} V(tx)u^2\,\mathrm{d}x + \frac{bt^4}{4}\|\nabla u\|_2^4 \\ &\quad - \frac{t^{3+\alpha}}{2}\int_{\mathbb{R}^3} (I_\alpha * F(t^{1/2}u))F(t^{1/2}u)\,\mathrm{d}x \quad \text{for all } u \in H^1(\mathbb{R}^3),\ t > 0. \end{aligned} \tag{2.11}$$

Thus, by (1.7), (2.5), (2.8) and (2.11), one has

$$\begin{aligned} \Phi(u) - \Phi(t^{1/2}u_t) &= \frac{a(1-t^2)}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3} [V(x) - t^4V(tx)]u^2\,\mathrm{d}x + \frac{b(1-t^4)}{4}\|\nabla u\|_2^4 \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^3} [t^{3-\alpha}(I_\alpha * F(t^{1/2}u))F(t^{1/2}u) - (I_\alpha * F(u))F(u)]\,\mathrm{d}x \\ &= \frac{1-t^4}{4}J(u) + \frac{a(1-t^2)^2}{4}\|\nabla u\|_2^2 + \frac{1}{8}\int_{\mathbb{R}^3} \{4t^4[V(x) - V(tx)] - (1-t^4)\nabla V(x)\cdot x\}u^2\,\mathrm{d}x \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^3} \bigg\{t^{3+\alpha}(I_\alpha * F(t^{1/2}u))F(t^{1/2}u) + \frac{1-t^4}{4}(I_\alpha * F(u))f(u)u \\ &\quad - \frac{(3+\alpha)t^4 + 1 - \alpha}{4}(I_\alpha * F(u))F(u)\bigg\}\,\mathrm{d}x \\ &\geq \frac{1-t^4}{4}J(u) + \frac{a(1-\theta)(1-t^2)^2}{4}\|\nabla u\|_2^2. \end{aligned}$$

This shows that (2.9) holds. □

Note that

$$J^\infty(u) = a\|\nabla u\|_2^2 + 2V_\infty\|u\|_2^2 + b\|\nabla u\|_2^4 - \frac{1}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u))[f(u)u + (3+\alpha)F(u)]\,dx. \tag{2.12}$$

From Lemma 2.5, we have the following two corollaries.

Corollary 2.6. *Assume that* (F1) *and* (F3) *hold. Then*

$$\Phi^\infty(u) \geq \Phi^\infty(t^{1/2}u_t) + \frac{1-t^4}{4}J^\infty(u) + \frac{a(1-t^2)^2}{4}\|\nabla u\|_2^2 \quad \textit{for all } u \in H^1(\mathbb{R}^3),\ t > 0. \tag{2.13}$$

Corollary 2.7. *Assume that* (V1), (V3), (F1) *and* (F3) *hold. Then*

$$\Phi(u) = \max_{t>0}\Phi(t^{1/2}u_t) \quad \textit{for all } u \in \mathcal{M}.$$

Letting $t \to 0$, (2.9) and (2.13) imply

$$\Phi(u) \geq \frac{1}{4}J(u) + \frac{a(1-\theta)}{4}\|\nabla u\|_2^2 \quad \text{for all } u \in H^1(\mathbb{R}^3) \tag{2.14}$$

and

$$\Phi^\infty(u) \geq \frac{1}{4}J^\infty(u) + \frac{a}{4}\|\nabla u\|_2^2 \quad \text{for all } u \in H^1(\mathbb{R}^3).$$

Lemma 2.8. *Assume that* (V1), (V3) *and* (F1)–(F3) *hold. Then for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u^{1/2}u_{t_u} \in \mathcal{M}$.*

Proof. Let $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ be fixed and define a function $\zeta(t) := \Phi(t^{1/2}u_t)$ on $(0,\infty)$. Clearly, by (2.8) and (2.11), we have

$$\begin{aligned}\zeta'(t) = 0 \ \Leftrightarrow\ & at^2\|\nabla u\|_2^2 + \frac{t^4}{2}\int_{\mathbb{R}^3}[4V(tx) + \nabla V(tx)\cdot tx]u^2\,dx + bt^4\|\nabla u\|_2^4 \\ & - \frac{t^{3+\alpha}}{2}\int_{\mathbb{R}^3}(I_\alpha * F(t^{1/2}u))[f(t^{1/2}u)t^{1/2}u + (3+\alpha)F(t^{1/2}u)]\,dx = 0 \\ \Leftrightarrow\ & J(t^{1/2}u_t) = 0 \ \Leftrightarrow\ t^{1/2}u_t \in \mathcal{M}.\end{aligned} \tag{2.15}$$

Note that (F1) implies that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(t)t| + |F(t)| \leq \varepsilon|t|^{1+3/\alpha} + C_\varepsilon|t|^{3+\alpha} \quad \text{for all } t \in \mathbb{R}. \tag{2.16}$$

By (2.15) and (2.16), we have $\lim_{t\to0^+}\zeta'(t) = 0$ and $\zeta'(t) > 0$ for $t > 0$ small. By (F3), one has

$$\frac{t^{\alpha/2}[4f(t^{1/2}u)t^{1/2}u + (3+\alpha)F(t^{1/2}u)]}{t^{1/2}} \geq 4f(u)u + (3+\alpha)F(u) \quad \text{for all } t \geq 1. \tag{2.17}$$

From (F2), (2.6), (2.15) and (2.17), we can deduce that $\zeta'(t) < 0$ for t large. Therefore, there exists some $\hat{t} = t_u > 0$ such that $\zeta'(\hat{t}) = 0$ and $\hat{t}^{1/2}u_{\hat{t}} \in \mathcal{M}$.

Next we claim that t_u is unique for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$. In fact, for any given $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, let $\hat{t}_1, \hat{t}_2 > 0$ be such that $\zeta'(\hat{t}_1) = \zeta'(\hat{t}_2) = 0$. Then $J(\hat{t}_1^{1/2}u_{\hat{t}_1}) = J(\hat{t}_2^{1/2}u_{\hat{t}_2}) = 0$. Jointly with (2.9), we have

$$\Phi(\hat{t}_1^{1/2}u_{\hat{t}_1}) \geq \Phi(\hat{t}_2^{1/2}u_{\hat{t}_2}) + \frac{(1-\theta)a(\hat{t}_1^2 - \hat{t}_2^2)^2}{4\hat{t}_1^2}\|\nabla u\|_2^2 \tag{2.18}$$

and

$$\Phi(\hat{t}_2^{1/2}u_{\hat{t}_2}) \geq \Phi(\hat{t}_1^{1/2}u_{\hat{t}_1}) + \frac{(1-\theta)a(\hat{t}_2^2 - \hat{t}_1^2)^2}{4\hat{t}_2^2}\|\nabla u\|_2^2. \tag{2.19}$$

Both (2.18) and (2.19) imply that $\hat{t}_1 = \hat{t}_2$. Therefore, $t_u > 0$ is unique for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$. □

Combining Corollary 2.7 with Lemma 2.8, we have the following lemma.

Lemma 2.9. *Assume that* (V1), (V3) *and* (F1)–(F3) *hold. Then*

$$\inf_{u\in\mathcal{M}} \Phi(u) = m = \inf_{u\in H^1(\mathbb{R}^3)\setminus\{0\}} \max_{t>0} \Phi(t^{1/2}u_t).$$

Lemma 2.10. *Assume that* (V1) *and* (V3) *hold. Then there exist* $\gamma_1, \gamma_2 > 0$ *such that*

$$\gamma_1\|u\|^2 \le a\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3} [4V(x) + \nabla V(x)\cdot x]u^2\,dx \le \gamma_2\|u\|^2 \quad \text{for all } u \in H^1(\mathbb{R}^3). \tag{2.20}$$

Proof. Letting $t = 0$ and $t \to \infty$ in (2.1), we have, respectively,

$$\nabla V(x)\cdot x \le \frac{\theta a}{2|x|^2} \quad \text{for all } x \in \mathbb{R}^3 \setminus \{0\} \tag{2.21}$$

and

$$4V(x) + \nabla V(x)\cdot x \ge 4V_\infty - \frac{\theta a}{2|x|^2} \quad \text{for all } x \in \mathbb{R}^3 \setminus \{0\}. \tag{2.22}$$

The last inequality in (2.20) follows from (V1), (2.10) and (2.21). By (2.10) and (2.22), we have

$$\begin{aligned} a\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3} [4V(x) + \nabla V(x)\cdot x]u^2\,dx &\ge (1-\theta)a\|\nabla u\|_2^2 + 2V_\infty\|u\|_2^2 \\ &\ge \min\{(1-\theta)a, 2V_\infty\}\|u\|^2 := \gamma_1\|u\|^2 \quad \text{for all } u \in H^1(\mathbb{R}^3), \end{aligned}$$

as desired. □

Lemma 2.11. *Assume that* (V1), (V3) *and* (F1)–(F3) *hold. Then*
(i) *there exists* $\rho > 0$ *such that* $\|u\| \ge \rho$ *for all* $u \in \mathcal{M}$,
(ii) $m = \inf_{u\in\mathcal{M}} \Phi(u) > 0$.

Proof. (i) Since $J(u) = 0$ for $u \in \mathcal{M}$, by (1.3), (2.8), (2.16), (2.20) and the Sobolev embedding theorem, one has

$$\begin{aligned} \gamma_1\|u\|^2 &\le a\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3} [4V(x) + \nabla V(x)\cdot x]u^2\,dx + b\Big(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big)^2 \\ &= \frac{1}{2}\int_{\mathbb{R}^3} (I_\alpha * F(u))[f(u)u + (3+\alpha)F(u)]\,dx \\ &\le C_1(\|u\|^{2+2\alpha/3} + \|u\|^{6+2\alpha}), \end{aligned}$$

which implies

$$\|u\| \ge \rho := \min\Big\{1, \Big(\frac{\gamma_1}{2C_1}\Big)^{3/2\alpha}\Big\} \quad \text{for all } u \in \mathcal{M}. \tag{2.23}$$

(ii) Let $\{u_n\} \subset \mathcal{M}$ be such that $\Phi(u_n) \to m$. There are two possible cases.

Case 1: $\inf_{n\in\mathbb{N}}\|\nabla u_n\|_2 := \rho_1 > 0$. In this case, by (2.14), one has

$$m + o(1) = \Phi(u_n) = \Phi(u_n) - \frac{1}{4}J(u_n) \ge \frac{a(1-\theta)}{4}\|\nabla u_n\|_2^2 \ge \frac{a(1-\theta)}{4}\rho_1^2.$$

Case 2: $\inf_{n\in\mathbb{N}}\|\nabla u_n\|_2 = 0$. By (2.23), passing to a subsequence, we have

$$\|\nabla u_n\|_2 \to 0, \quad \|u_n\|_2 \ge \frac{1}{2}\rho + o(1). \tag{2.24}$$

By (V1), there exists $R > 0$ such that $V(x) \ge V_\infty/2$ for $|x| \ge R$. This implies

$$\int_{\{|tx|\ge R\}} V(tx)u^2\,dx \ge \frac{V_\infty}{2}\int_{\{|tx|\ge R\}} u^2\,dx \quad \text{for all } t > 0,\ u \in H^1(\mathbb{R}^3). \tag{2.25}$$

Making use of the Hölder inequality and the Sobolev inequality, we get

$$\begin{aligned}\int_{\{|tx|<R\}} u^2\,\mathrm{d}x &\le \Big(\frac{4\pi R^3}{3t^3}\Big)^{2/3}\Big(\int_{\{|tx|<R\}} u^6\,\mathrm{d}x\Big)^{1/3}\\ &\le \Big(\frac{4\pi R^3}{3t^3}\Big)^{2/3} S^{-1}\|\nabla u\|_2^2 \quad \text{for all } t>0,\ u\in H^1(\mathbb{R}^3).\end{aligned} \tag{2.26}$$

Let

$$\delta_0 = \min\Big\{V_\infty, aS\Big(\frac{3}{4\pi R^3}\Big)^{2/3}\Big\}. \tag{2.27}$$

By (1.3), (2.16) and the Sobolev embedding inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^3} (I_\alpha * F(u))F(u)\,\mathrm{d}x &\le C(\alpha)\|F(u)\|_{6/(3+\alpha)}^2 \le \frac{1}{4}\delta_0\|u\|_2^{2+2\alpha/3} + C_2\|u\|_6^{6+2\alpha}\\ &\le \frac{1}{4}\delta_0\|u\|_2^{2+2\alpha/3} + C_2 S^{-(3+\alpha)}\|\nabla u\|_2^{6+2\alpha} \quad \text{for all } u\in H^1(\mathbb{R}^3).\end{aligned} \tag{2.28}$$

Let $t_n = \|u_n\|_2^{-2}$. Then (2.24) implies that $\{t_n\}$ is bounded. Since $J(u_n)=0$, it follows from (2.9), (2.11), (2.24) and (2.28) that

$$\begin{aligned} m + o(1) = \Phi(u_n) &\ge \Phi(t_n^{1/2}(u_n)_{t_n})\\ &= \frac{at_n^2}{2}\|\nabla u_n\|_2^2 + \frac{t_n^4}{2}\int_{\mathbb{R}^3} V(t_n x)u_n^2\,\mathrm{d}x + \frac{bt_n^4}{4}\|\nabla u_n\|_2^4 - \frac{t_n^{3+\alpha}}{2}\int_{\mathbb{R}^3}(I_\alpha * F(t_n^{1/2}u_n))F(t_n^{1/2}u_n)\,\mathrm{d}x\\ &\ge \frac{aS}{2}\Big(\frac{3}{4\pi R^3}\Big)^{2/3} t_n^4 \int_{\{|t_n x|<R\}} u_n^2\,\mathrm{d}x + \frac{V_\infty t_n^4}{4}\int_{\{|t_n x|\ge R\}} u_n^2\,\mathrm{d}x\\ &\quad - \frac{1}{8}\delta_0 t_n^{4+4/3\alpha}\|u_n\|_2^{2+2\alpha/3} - \frac{C_2}{2S^{3+\alpha}} t_n^{6+2\alpha}\|\nabla u_n\|_2^{6+2\alpha}\\ &\ge \frac{1}{8}\delta_0 t_n^4\|u_n\|_2^2[2-(t_n^4\|u_n\|_2^2)^{\alpha/3}] + o(1)\\ &= \frac{1}{8}\delta_0 + o(1).\end{aligned}$$

Cases 1 and 2 show that $m = \inf_{u\in\mathcal{M}}\Phi(u) > 0$. □

Combining [1, Lemma 5.1], [10, Lemma 2.2], [36, Lemmas 2.7 and 2.8], and [39], we can obtain the following Brezis–Lieb type lemma.

Lemma 2.12. *Assume that* (V1) *and* (F1) *hold and* $\nabla V(x)\cdot x \in L^\infty(\mathbb{R}^3)$. *If* $u_n \rightharpoonup \bar{u}$ *in* $H^1(\mathbb{R}^3)$, *then, along a subsequence,*

$$\begin{aligned}\Phi(u_n) &= \Phi(\bar{u}) + \Phi(u_n-\bar{u}) + \frac{b}{2}\|\nabla\bar{u}\|_2^2\|\nabla(u_n-\bar{u})\|_2^2 + o(1),\\ \langle\Phi'(u_n),u_n\rangle &= \langle\Phi'(\bar{u}),\bar{u}\rangle + \langle\Phi'(u_n-\bar{u}),u_n-\bar{u}\rangle + 2b\|\nabla\bar{u}\|_2^2\|\nabla(u_n-\bar{u})\|_2^2 + o(1),\\ J(u_n) &= J(\bar{u}) + J(u_n-\bar{u}) + 2b\|\nabla\bar{u}\|_2^2\|\nabla(u_n-\bar{u})\|_2^2 + o(1).\end{aligned}$$

Lemma 2.13. *Assume that* (V1), (V3) *and* (F1)–(F3) *hold. Then* $m \le m^\infty := \inf_{u\in\mathcal{M}^\infty}\Phi^\infty(u)$.

Proof. In view of Lemmas 2.8 and 2.11, we have $\mathcal{M}^\infty \ne \emptyset$ and $m^\infty > 0$. Arguing indirectly, we assume that $m > m^\infty$. Let $\varepsilon := m - m^\infty$. Then there exists u_ε^∞ such that

$$u_\varepsilon^\infty \in \mathcal{M}^\infty \quad \text{and} \quad m^\infty + \frac{\varepsilon}{2} > \Phi^\infty(u_\varepsilon^\infty). \tag{2.29}$$

In view of Lemma 2.8, there exists $t_\varepsilon > 0$ such that $t_\varepsilon^{1/2}(u_\varepsilon^\infty)_{t_\varepsilon} \in \mathcal{M}$. Thus, it follows from (V1), (1.7), (1.11), (2.13) and (2.29) that

$$m - \frac{\varepsilon}{2} = m^\infty + \frac{\varepsilon}{2} > \Phi^\infty(u_\varepsilon^\infty) \ge \Phi^\infty(t_\varepsilon^{1/2}(u_\varepsilon^\infty)_{t_\varepsilon}) \ge \Phi(t_\varepsilon^{1/2}(u_\varepsilon^\infty)_{t_\varepsilon}) \ge m.$$

This contradiction shows that $m^\infty \ge m$. □

Lemma 2.14. *Assume that* (V1), (V3) *and* (F1)–(F3) *hold. Then m is achieved.*

Proof. In view of Lemmas 2.8 and 2.11, we have $\mathcal{M} \neq \emptyset$ and $m > 0$. Let $\{u_n\} \subset \mathcal{M}$ be such that $\Phi(u_n) \to m$. Since $J(u_n) = 0$, it follows from (2.14) that

$$m + o(1) = \Phi(u_n) = \Phi(u_n) - \frac{1}{4}J(u_n) \geq \frac{a(1-\theta)}{4}\|\nabla u_n\|_2^2.$$

This shows that $\{\|\nabla u_n\|_2\}$ is bounded. Next, we prove that $\{\|u_n\|\}$ is also bounded. Arguing by contradiction, suppose that $\|u_n\|_2 \to \infty$. By (1.3), (2.16) and the Sobolev embedding inequality, one has, for all $u \in H^1(\mathbb{R}^3)$,

$$\begin{aligned}\int_{\mathbb{R}^3} (I_\alpha * F(u))F(u)\,dx \leq C(\alpha)\|F(u)\|_{6/(3+\alpha)}^2 &\leq \frac{\delta_0}{4}\Big(\frac{\delta_0}{16m}\Big)^{\alpha/3}\|u\|_2^{2+2\alpha/3} + C_3\|u\|_6^{6+2\alpha}\\ &\leq \frac{\delta_0}{4}\Big(\frac{\delta_0}{16m}\Big)^{\alpha/3}\|u\|_2^{2+2\alpha/3} + C_3 S^{-(3+\alpha)}\|\nabla u\|_2^{6+2\alpha},\end{aligned} \tag{2.30}$$

where $\delta_0 > 0$ is defined by (2.27). Let $\tilde{t}_n = (16m/\delta_0\|u_n\|_2^2)^{1/4}$, then $\tilde{t}_n \to 0$. Since $J(u_n) = 0$, it follows from (1.11), (2.9), (2.11), (2.25), (2.26) and (2.30) that

$$\begin{aligned}m + o(1) = \Phi(u_n) &\geq \Phi(\tilde{t}_n^{1/2}(u_n)_{\tilde{t}_n})\\ &= \frac{a\tilde{t}_n^2}{2}\|\nabla u_n\|_2^2 + \frac{\tilde{t}_n^4}{2}\int_{\mathbb{R}^3} V(\tilde{t}_n x)u_n^2\,dx + \frac{b\tilde{t}_n^4}{4}\|\nabla u_n\|_2^4 - \frac{\tilde{t}_n^{3+\alpha}}{2}\int_{\mathbb{R}^3}(I_\alpha * F(\tilde{t}_n^{1/2}u_n))F(\tilde{t}_n^{1/2}u_n)\,dx\\ &\geq \frac{aS}{2}\Big(\frac{3}{4\pi R^3}\Big)^{2/3}\tilde{t}_n^4\int_{|\tilde{t}_n x|<R} u_n^2\,dx + \frac{V_\infty \tilde{t}_n^4}{4}\int_{\{|\tilde{t}_n x|\geq R\}} u_n^2\,dx\\ &\quad - \frac{\delta_0}{8}\Big(\frac{\delta_0}{16m}\Big)^{\alpha/3}\tilde{t}_n^{4+4/3\alpha}\|u_n\|_2^{2+2\alpha/3} - \frac{C_3}{2S^{3+\alpha}}\tilde{t}_n^{6+2\alpha}\|\nabla u_n\|_2^{6+2\alpha}\\ &\geq \frac{1}{8}\delta_0\tilde{t}_n^4\|u_n\|_2^2\Big[2 - \Big(\frac{\delta_0\tilde{t}_n^4\|u_n\|_2^2}{16m}\Big)^{\alpha/3}\Big] + o(1)\\ &= 2m + o(1).\end{aligned}$$

This contradiction shows that $\{\|u_n\|_2\}$ is bounded. Hence, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Passing to a subsequence, we have $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$. Then $u_n \to \bar{u}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^3)$ for $2 \leq s < 6$ and $u_n \to \bar{u}$ a.e. in $\mathbb{R}^3$. There are two possible cases: (i) $\bar{u} = 0$ and (ii) $\bar{u} \neq 0$.

Case (i) $\bar{u} = 0$, i.e., $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. Then $u_n \to 0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^3)$ for $2 \leq s < 2^*$ and $u_n \to 0$ a.e. in $\mathbb{R}^3$. Using (V1) and (2.21), it is easy to show that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}[V_\infty - V(x)]u_n^2\,dx = \lim_{n\to\infty}\int_{\mathbb{R}^3}\nabla V(x)\cdot x u_n^2\,dx = 0. \tag{2.31}$$

From (1.7), (1.11), (2.8), (2.12) and (2.31), one can get

$$\Phi^\infty(u_n) \to m, \quad J^\infty(u_n) \to 0. \tag{2.32}$$

By (F1), for some $p \in (1 + \alpha/3, 3 + \alpha)$ and any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|f(t)t| + |F(t)| \leq \epsilon(|t|^{1+3/\alpha} + |t|^{3+\alpha}) + C_\epsilon|t|^p \quad \text{for all } t \in \mathbb{R}. \tag{2.33}$$

From (1.3), (2.12), (2.16), (2.32), (2.33) and Lemma 2.11 (i), one has

$$\begin{aligned}\min\{a, 2V_\infty\}\rho_0^2 &\leq \min\{a, 2V_\infty\}\|u_n\|^2\\ &\leq a\|\nabla u_n\|_2^2 + 2V_\infty\|u_n\|_2^2 + b\|\nabla u_n\|_2^4\\ &= \frac{1}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u_n))[f(u_n)u_n + (3+\alpha)F(u_n)]\,dx + o(1)\\ &\leq C_4\big[\epsilon(\|u_n\|_2^{1+\alpha/3} + \|u\|_6^{6+2\alpha}) + C_\varepsilon\|u_n\|_{6p/(3+\alpha)}^p\big]^2 + o(1).\end{aligned} \tag{2.34}$$

Using (2.34) and Lions' concentration compactness principle [39, Lemma 1.21], we can prove that there exist $\delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that $\int_{B_1(y_n)} |u_n|^2 \, dx > \delta$. Let $\hat{u}_n(x) = u_n(x + y_n)$. Then we have $\|\hat{u}_n\| = \|u_n\|$ and

$$J^\infty(\hat{u}_n) = o(1), \quad \Phi^\infty(\hat{u}_n) \to m, \quad \int_{B_1(0)} |\hat{u}_n|^2 \, dx > \delta. \tag{2.35}$$

Therefore, there exists $\hat{u} \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that, passing to a subsequence,

$$\begin{cases} \hat{u}_n \rightharpoonup \hat{u} & \text{in } H^1(\mathbb{R}^3), \\ \hat{u}_n \rightharpoonup \hat{u} & \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^3) \text{ for all } s \in [1, 6), \\ \hat{u}_n \to \hat{u}, & \text{a.e. on } \mathbb{R}^3. \end{cases} \tag{2.36}$$

Let $w_n = \hat{u}_n - \hat{u}$. Then (2.36) and Lemma 2.12 yield

$$\Phi^\infty(\hat{u}_n) = \Phi^\infty(\hat{u}) + \Phi^\infty(w_n) + \frac{b}{2}\|\nabla\hat{u}\|_2^2\|\nabla w_n\|_2^2 + o(1) \tag{2.37}$$

and

$$J^\infty(\hat{u}_n) = J^\infty(\hat{u}) + J^\infty(w_n) + 2b\|\nabla\hat{u}\|_2^2\|\nabla w_n\|_2^2 + o(1). \tag{2.38}$$

For $u \in H^1(\mathbb{R}^3)$, we let

$$\Psi^\infty(u) := \Phi^\infty(u) - \frac{1}{4}J^\infty(u) = \frac{a}{4}\|\nabla u\|_2^2 + \frac{1}{8}\int_{\mathbb{R}^3} (I_\alpha * F(u))[f(u)u + (\alpha - 1)F(u)] \, dx. \tag{2.39}$$

From (1.11), (2.12),(2.35), (2.37), (2.38) and (2.39), one has

$$\Psi^\infty(w_n) = m - \Psi^\infty(\hat{u}) + o(1), \quad J^\infty(w_n) \le -J^\infty(\hat{u}) + o(1). \tag{2.40}$$

If there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $w_{n_i} = 0$, then we have

$$\Phi^\infty(\hat{u}) = m, \quad J^\infty(\hat{u}) = 0. \tag{2.41}$$

Next, we assume that $w_n \neq 0$. In view of Lemma 2.8, there exists $t_n > 0$ such that $t_n^{1/2}(w_n)_{t_n} \in \mathcal{M}^\infty$. We claim that $J^\infty(\hat{u}) \le 0$. Otherwise, if $J^\infty(\hat{u}) > 0$, then (2.40) implies $J^\infty(w_n) < 0$ for large n. From (1.11), (2.12), (2.13), (2.40) and Lemma 2.13, we obtain

$$\begin{aligned} m - \Psi^\infty(\hat{u}) + o(1) = \Psi^\infty(w_n) &= \Phi^\infty(w_n) - \frac{1}{4}J^\infty(w_n) \\ &\ge \Phi^\infty(t_n^{1/2}(w_n)_{t_n}) - \frac{t_n^3}{4}J^\infty(w_n) + \frac{a(1 - t_n^2)^2}{4}\|\nabla w_n\|_2^2 \\ &\ge m^\infty \ge m \quad \text{for large } n \in \mathbb{N}, \end{aligned}$$

which is a contradiction due to $\Psi^\infty(\hat{u}) > 0$. Hence, $J^\infty(\hat{u}) \le 0$. In view of Lemma 2.8, there exists $t_\infty > 0$ such that $t_\infty^{1/2}\hat{u}_{t_\infty} \in \mathcal{M}^\infty$. By (1.11), (2.4), (2.12), (2.13), (2.32), (2.35), (2.39), the weak semicontinuity of norm, Fatou's lemma and Lemma 2.13, we have

$$\begin{aligned} m = \lim_{n\to\infty} \Psi^\infty(\hat{u}_n) \ge \Psi^\infty(\hat{u}) &= \Phi^\infty(\hat{u}) - \frac{1}{4}J^\infty(\hat{u}) \\ &\ge \Phi^\infty(t_\infty^{1/2}\hat{u}_{t_\infty}) - \frac{t_\infty^4}{4}J^\infty(\hat{u}) + \frac{a(1 - t_\infty^2)^2}{4}\|\nabla\hat{u}\|_2^2 \\ &\ge m^\infty - \frac{t_\infty^4}{4}J^\infty(\hat{u}) + \frac{a(1 - t_\infty^2)^2}{4}\|\nabla\hat{u}\|_2^2 \ge m, \end{aligned}$$

which implies that (2.41) holds too. In view of Lemma 2.8, there exists $\hat{t} > 0$ such that $\hat{t}^{1/2}\hat{u}_{\hat{t}} \in \mathcal{M}$. Moreover, it follows from (V1), (1.7), (1.11), (2.13) and (2.41) that

$$m \le \Phi(\hat{t}^{1/2}\hat{u}_{\hat{t}}) \le \Phi^\infty(\hat{t}^{1/2}\hat{u}_{\hat{t}}) \le \Phi^\infty(\hat{u}) = m.$$

This shows that m is achieved at $\hat{t}^{1/2}\hat{u}_{\hat{t}} \in \mathcal{M}$.

Case (ii) $\bar{u} \neq 0$. In this case, analogous to the proof of (2.41), by using Φ and J instead of Φ^∞ and J^∞, we can deduce that

$$\Phi(\bar{u}) = m, \quad J(\bar{u}) = 0.$$

Hence, the proof is complete. □

Lemma 2.15. *Assume that* (V1), (V3) *and* (F1)–(F3) *hold. If* $\bar{u} \in \mathcal{M}$ *and* $\Phi(\bar{u}) = m$, *then* $\bar{u}$ *is a critical point of* Φ.

Proof. Following the idea of [38, Lemma 2.13], we use the deformation lemma and intermediary theorem for continuous functions to prove this lemma. Assume that $\Phi'(\bar{u}) \neq 0$. Then there exist $\delta > 0$ and $\varrho > 0$ such that

$$\|u - \bar{u}\| \leq 3\delta \implies \|\Phi'(u)\| \geq \varrho.$$

By [37, equation (2.47)], one has $\lim_{t\to 1}\|t^{1/2}\bar{u}_t - \bar{u}\| = 0$. Thus, there exists $\delta_1 > 0$ such that

$$|t^{1/2} - 1| < \delta_1 \implies \|t^{1/2}\bar{u}_t - \bar{u}\| < \delta.$$

In view of (2.15), (2.16) and (2.17), there exist $T_1 \in (0, 1)$ and $T_2 \in (1, \infty)$ such that

$$J(T_1^{1/2}\bar{u}_{T_1}) > 0, \quad J(T_2^{1/2}\bar{u}_{T_2}) < 0.$$

The rest of the proof is similar to the proof of [38, Lemma 2.13]. Indeed, we can obtain the desired conclusion by using

$$\Phi(t^{1/2}\bar{u}_t) \leq \Phi(\bar{u}) - \frac{a(1-\theta)(1-t^2)^2}{4}\|\nabla\bar{u}\|_2^2 = m - \frac{a(1-\theta)(1-t^2)^2}{4}\|\nabla\bar{u}\|_2^2 \quad \text{for all } t > 0$$

and

$$\varepsilon := \min\Big\{\frac{a(1-\theta)(1-T_1^2)^2}{12}\|\nabla\bar{u}\|_2^2, \frac{a(1-\theta)(1-T_2^2)^2}{12}\|\nabla\bar{u}\|_2^2, 1, \frac{\varrho\delta}{8}\Big\}$$

instead of [38, (2.40) and ε], respectively. □

Proof of Theorem 1.3. In view of Lemmas 2.9, 2.14 and 2.15, there exists $\bar{u} \in \mathcal{M}$ such that

$$\Phi(\bar{u}) = m = \inf_{u\in H^1(\mathbb{R}^3)\setminus\{0\}} \max_{t>0} \Phi(t^{1/2}u_t), \quad \Phi'(\bar{u}) = 0.$$

This shows that $\bar{u}$ is a ground state solution of (1.1). □

3 Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. Without loss of generality, we consider that $V(x) \not\equiv V_\infty$.

Proposition 3.1 ([14]). *Let X be a Banach space and let $K \subset \mathbb{R}^+$ be an interval. We consider a family $\{\mathcal{I}_\lambda\}_{\lambda\in K}$ of $\mathcal{C}^1$-functionals on X of the form*

$$\mathcal{I}_\lambda(u) = A(u) - \lambda B(u) \quad \textit{for all } \lambda \in K,$$

where $B(u) \geq 0$ for all $u \in X$, and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to \infty$. We assume that there are two points v_1, v_2 in X such that

$$c_\lambda := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{I}_\lambda(\gamma(t)) > \max\{\mathcal{I}_\lambda(v_1), \mathcal{I}_\lambda(v_2)\}, \quad \textit{where } \Gamma = \{\gamma \in \mathcal{C}([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then, for almost every $\lambda \in K$, there is a bounded (PS)$_{c_\lambda}$ sequence for $\mathcal{I}_\lambda$, that is, there exists a sequence such that
(i) $\{u_n(\lambda)\}$ *is bounded in X,*
(ii) $\mathcal{I}_\lambda(u_n(\lambda)) \to c_\lambda$,
(iii) $\mathcal{I}'_\lambda(u_n(\lambda)) \to 0$ *in X^*, where X^* is the dual of X.*
Moreover, c_λ is nonincreasing and left continuous on $\lambda \in [1/2, 1]$.

Lemma 3.2 ([12]). *Assume that* (V1), (V2) *and* (F1) *hold. Let u be a critical point of Φ_λ in $H^1(\mathbb{R}^3)$. Then we have the following Pohožave type identity:*

$$\mathcal{P}_\lambda(u) := \frac{a}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3}[3V(x) + \nabla V(x)\cdot x]u^2\,dx + \frac{b}{2}\|\nabla u\|_2^4 - \frac{3+\alpha}{2}\lambda\int_{\mathbb{R}^3}(I_\alpha * F(u))F(u)\,dx = 0. \tag{3.1}$$

We set $J_\lambda(u) := \frac{1}{2}\langle\Phi'_\lambda(u), u\rangle + \mathcal{P}_\lambda(u)$. Then

$$J_\lambda(u) = a\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3}[4V(x) + \nabla V(x)\cdot x]u^2\,dx + b\|\nabla u\|_2^4 - \frac{\lambda}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u))[f(u)u + (3+\alpha)F(u)]\,dx \tag{3.2}$$

for $\lambda \in [1/2, 1]$. Correspondingly, we also let

$$J_\lambda^\infty(u) = a\|\nabla u\|_2^2 + 2V_\infty\|u\|_2^2 + b\|\nabla u\|_2^4 - \frac{\lambda}{2}\int_{\mathbb{R}^3}(I_\alpha * F(u))[f(u)u + (3+\alpha)F(u)]\,dx \tag{3.3}$$

for $\lambda \in [1/2, 1]$. Set

$$\mathcal{M}_\lambda^\infty = \{u \in H^1(\mathbb{R}^3)\setminus\{0\} : J_\lambda^\infty(u) = 0\}, \quad m_\lambda^\infty = \inf_{u\in\mathcal{M}_\lambda^\infty}\Phi_\lambda^\infty(u).$$

By Corollary 2.6, we have the following lemma.

Lemma 3.3. *Assume that* (F1) *and* (F3) *hold. Then*

$$\Phi_\lambda^\infty(u) \geq \Phi_\lambda^\infty(t^{1/2}u_t) + \frac{1-t^4}{4}J_\lambda^\infty(u) + \frac{a(1-t^2)^2}{4}\|\nabla u\|_2^2 \quad \textit{for all } u \in H^1(\mathbb{R}^3),\ t > 0. \tag{3.4}$$

In view of Theorem 1.2, $\Phi_1^\infty = \Phi^\infty$ has a minimizer $u^\infty \neq 0$ on $\mathcal{M}_1^\infty = \mathcal{M}^\infty$, i.e.,

$$u^\infty \in \mathcal{M}_1^\infty, \quad (\Phi_1^\infty)'(u^\infty) = 0 \quad \text{and} \quad m_1^\infty = \Phi_1^\infty(u^\infty). \tag{3.5}$$

Since (1.10) is autonomous, $V \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$ and $V(x) \leq V_\infty$ but $V(x) \not\equiv V_\infty$, there exist $\bar{x} \in \mathbb{R}^3$ and $\bar{r} > 0$ such that

$$V_\infty - V(x) > 0, \quad |u^\infty(x)| > 0 \quad \text{for a.e. } x, \text{ with } |x - \bar{x}| \leq \bar{r}. \tag{3.6}$$

Lemma 3.4. *Assume that* (V1), (V2) *and* (F1)–(F3) *hold. Then*
(i) *there exists $T > 0$ independent of λ such that $\Phi_\lambda(T^{1/2}(u^\infty)_T) < 0$ for all $\lambda \in [1/2, 1]$;*
(ii) *there exists a positive constant κ_0 independent of λ such that for all $\lambda \in [1/2, 1]$,*

$$c_\lambda := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\Phi_\lambda(\gamma(t)) \geq \kappa_0 > \max\{\Phi_\lambda(0), \Phi_\lambda(T^{1/2}(u^\infty)_T)\},$$

where

$$\Gamma = \{\gamma \in \mathcal{C}([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, \gamma(1) = T^{1/2}(u^\infty)_T\};$$

(iii) *c_λ and m_λ^∞ are nonincreasing on $\lambda \in [1/2, 1]$.*

The proof of Lemma 3.4 is standard, so we omit it.

Lemma 3.5. *Assume that* (V1), (V2) *and* (F1)–(F3) *hold. Then there exists $\bar{\lambda} \in [1/2, 1)$ such that $c_\lambda < m_\lambda^\infty$ for $\lambda \in (\bar{\lambda}, 1]$.*

Proof. It is easy to see that $\Phi_\lambda(t^{1/2}(u^\infty)_t)$ is continuous on $t \in (0, \infty)$. Hence, for any $\lambda \in [1/2, 1)$, we can choose $t_\lambda \in (0, T)$ such that $\Phi_\lambda(t_\lambda^{1/2}(u^\infty)_{t_\lambda}) = \max_{t\in(0,T]}\Phi_\lambda(t^{1/2}(u^\infty)_t)$. By (2.3) and (2.7), one has

$$\left(I_\alpha * \frac{F(t_\lambda^{1/2}u^\infty)}{t_\lambda^{(1-\alpha)/2}}\right)\frac{F(t_\lambda^{1/2}u^\infty)}{t_\lambda^{(1-\alpha)/2}} \leq \left(I_\alpha * \frac{F(T^{1/2}u^\infty)}{T^{(1-\alpha)/2}}\right)\frac{F(T^{1/2}u^\infty)}{T^{(1-\alpha)/2}}. \tag{3.7}$$

Set

$$\gamma_0(t) = \begin{cases}(tT)^{1/2}(u^\infty)_{(tT)} & \text{for } t > 0,\\ 0 & \text{for } t = 0.\end{cases}$$

Then $\gamma_0 \in \Gamma$, defined by Lemma 3.4 (ii). Moreover,

$$\Phi_\lambda(t_\lambda^{1/2}(u^\infty)_{t_\lambda}) = \max_{t\in[0,1]} \Phi_\lambda(\gamma_0(t)) \geq c_\lambda. \tag{3.8}$$

Let

$$\zeta_0 := \min\{3\bar{r}/8(1+|\bar{x}|), 1/4\}. \tag{3.9}$$

Then it follows from (3.6) and (3.9) that

$$|x-\bar{x}| \leq \frac{\bar{r}}{2} \quad \text{and} \quad s \in [1-\zeta_0, 1+\zeta_0] \ \Rightarrow\ |sx-\bar{x}| \leq \bar{r}. \tag{3.10}$$

Let

$$\begin{aligned}\bar{\lambda} := \max\Bigg\{\frac{1}{2}, 1 - \frac{(1-\zeta_0)^4 \min_{s\in[1-\zeta_0,1+\zeta_0]} \int_{\mathbb{R}^3}[V_\infty - V(sx)]|u^\infty|^2\,dx}{T^{3+\alpha}\int_{\mathbb{R}^3}(I_\alpha * F(T^{1/2}u^\infty))F(T^{1/2}u^\infty)\,dx},\\ 1 - \frac{a(1-\theta)\zeta_0^2\|\nabla u^\infty\|_2^2}{2T^{3+\alpha}\int_{\mathbb{R}^3}(I_\alpha * F(T^{1/2}u^\infty))F(T^{1/2}u^\infty)\,dx}\Bigg\}.\end{aligned} \tag{3.11}$$

Then it follows from (3.6) and (3.10) that $1/2 \leq \bar{\lambda} < 1$. We have two cases to distinguish:

Case (i) $t_\lambda \in [1-\zeta_0, 1+\zeta_0]$. By (1.13), (1.15), (3.4), (3.7)–(3.11) and Lemma 3.4 (iii), we have

$$\begin{aligned}m_\lambda^\infty &\geq m_1^\infty = \Phi_1^\infty(u^\infty) \geq \Phi_1^\infty(t_\lambda^{1/2}(u^\infty)_{t_\lambda})\\ &= \Phi_\lambda(t_\lambda^{1/2}(u^\infty)_{t_\lambda}) - \frac{1-\lambda}{2}t_\lambda^{3+\alpha}\int_{\mathbb{R}^3}(I_\alpha * F(t_\lambda^{1/2}u^\infty))F(t_\lambda^{1/2}u^\infty)\,dx + \frac{t_\lambda^4}{2}\int_{\mathbb{R}^3}[V_\infty - V(t_\lambda x)]|u^\infty|^2\,dx\\ &\geq c_\lambda - \frac{1-\lambda}{2}T^{3+\alpha}\int_{\mathbb{R}^3}(I_\alpha * F(T^{1/2}u^\infty))F(T^{1/2}u^\infty)\,dx + \frac{(1-\zeta_0)^4}{2}\min_{s\in[1-\zeta_0,1+\zeta_0]}\int_{\mathbb{R}^3}[V_\infty - V(sx)]\,|u^\infty|^2\,dx\\ &> c_\lambda \quad \text{for all } \lambda \in (\bar{\lambda}, 1].\end{aligned}$$

Case (ii) $t_\lambda \in (0, 1-\zeta_0) \cup (1+\zeta_0, T]$. Since $V_\infty \geq V(x)$ for all $x \in \mathbb{R}^3$, it follows from (1.13), (1.15), (3.4), (3.5), (3.7), (3.8), (3.11) and Lemma 3.4 (iii) that

$$\begin{aligned}m_\lambda^\infty &\geq m_1^\infty = \Phi_1^\infty(u^\infty) \geq \Phi_1^\infty(t_\lambda^{1/2}(u^\infty)_{t_\lambda}) + \frac{a(1-\theta)(1-t_\lambda^2)^2}{4}\|\nabla u^\infty\|_2^2\\ &= \Phi_\lambda(t_\lambda^{1/2}(u^\infty)_{t_\lambda}) - \frac{1-\lambda}{2}t_\lambda^{3+\alpha}\int_{\mathbb{R}^3}(I_\alpha * F(t_\lambda^{1/2}u^\infty))F(t_\lambda^{1/2}u^\infty)\,dx\\ &\quad + \frac{t_\lambda^4}{2}\int_{\mathbb{R}^3}[V_\infty - V(t_\lambda x)]|u^\infty|^2\,dx + \frac{a(1-\theta)(1-t_\lambda^2)^2}{4}\|\nabla u^\infty\|_2^2\\ &\geq c_\lambda - \frac{1-\lambda}{2}T^{3+\alpha}\int_{\mathbb{R}^3}(I_\alpha * F(T^{1/2}u^\infty))F(T^{1/2}u^\infty)\,dx + \frac{a(1-\theta)\zeta_0^2}{4}\|\nabla u^\infty\|_2^2\\ &> c_\lambda \quad \text{for all } \lambda \in (\bar{\lambda}, 1].\end{aligned}$$

In both cases, we obtain $c_\lambda < m_\lambda^\infty$ for $\lambda \in (\bar{\lambda}, 1]$. □

Lemma 3.6. *Assume that* (V1), (V2) *and* (F1)–(F3) *hold. Let* $\{u_n\}$ *be a bounded* $(PS)_{c_\lambda}$ *sequence for* Φ_λ *with* $\lambda \in [1/2, 1]$. *Then there exist a subsequence of* $\{u_n\}$, *still denoted it by* $\{u_n\}$, *and* $u_0 \in H^1(\mathbb{R}^3)$ *such that*

(i) $A_\lambda^2 := \lim_{n\to\infty}\|\nabla u_n\|_2^2$ *exists*, $u_n \rightharpoonup u_\lambda$ *in* $H^1(\mathbb{R}^3)$ *and* $\mathcal{E}_\lambda'(u_\lambda) = 0$;
(ii) $w^k \neq 0$ *and* $(\mathcal{E}_\lambda^\infty)'(w^k) = 0$ *for* $1 \leq k \leq l$;
(iii) *we have*

$$c + \frac{bA_\lambda^4}{4} = \mathcal{E}_\lambda(u_\lambda) + \sum_{k=1}^{l}\mathcal{E}_\lambda^\infty(w^k)$$

and

$$A_\lambda^2 = \|\nabla u_\lambda\|_2^2 + \sum_{k=1}^{l}\|\nabla w^k\|_2^2, \tag{3.12}$$

where

$$\mathcal{E}_\lambda(u) = \frac{a + bA_\lambda^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} V(x)u^2 \, dx - \frac{\lambda}{2} \int_{\mathbb{R}^3} (I_\alpha * F(u))F(u) \, dx \tag{3.13}$$

and

$$\mathcal{E}_\lambda^\infty(u) = \frac{a + bA_\lambda^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{V_\infty}{2} \int_{\mathbb{R}^3} u^2 \, dx - \frac{\lambda}{2} \int_{\mathbb{R}^3} (I_\alpha * F(u))F(u) \, dx. \tag{3.14}$$

We agree that in the case $l = 0$, the above holds without w^k.

Lemma 3.7. *Assume that* (V1), (V2) *and* (F1)–(F3) *hold. Then, for almost every $\lambda \in (\bar{\lambda}, 1]$, there exists $u_\lambda \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that*

$$\Phi_\lambda'(u_\lambda) = 0, \quad \Phi_\lambda(u_\lambda) = c_\lambda.$$

Proof. Lemma 3.4 implies that $\Phi_\lambda(u)$ satisfies the assumptions of Proposition 3.1, with $X = H^1(\mathbb{R}^3)$ and $\mathcal{I}_\lambda = \Phi_\lambda$. So, for almost every $\lambda \in [1/2, 1]$, there exists a bounded sequence $\{u_n(\lambda)\} \subset H^1(\mathbb{R}^3)$ (for simplicity, we denote it by $\{u_n\}$ instead of $\{u_n(\lambda)\}$) such that

$$\Phi_\lambda(u_n) \to c_\lambda > 0, \quad \|\Phi_\lambda'(u_n)\| \to 0.$$

By Lemma 3.6, there exist a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and $u_\lambda \in H^1(\mathbb{R}^3)$ such that $A_\lambda^2 := \lim_{n\to\infty} \|\nabla u_n\|_2^2$ exists, $u_n \rightharpoonup u_\lambda$ in $H^1(\mathbb{R}^3)$ and $\mathcal{E}_\lambda'(u_\lambda) = 0$, and there exist $l \in \mathbb{N} \cup \{0\}$ and $w^1, \dots, w^l \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that $(\mathcal{E}_\lambda^\infty)'(w^k) = 0$ for $1 \le k \le l$,

$$c_\lambda + \frac{bA_\lambda^4}{4} = \mathcal{E}_\lambda(u_\lambda) + \sum_{k=1}^{l} \mathcal{E}_\lambda^\infty(w^k)$$

and

$$A_\lambda^2 = \|\nabla u_\lambda\|_2^2 + \sum_{k=1}^{l} \|\nabla w^k\|_2^2.$$

Since $\mathcal{E}_\lambda'(u_\lambda) = 0$, we have the following Pohožaev identity:

$$\tilde{\mathcal{P}}_\lambda(u_\lambda) := \frac{a + bA_\lambda^2}{2} \|\nabla u_\lambda\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} [3V(x) + \nabla V(x) \cdot x] u_\lambda^2 \, dx - \frac{3 + \alpha}{2} \lambda \int_{\mathbb{R}^3} (I_\alpha * F(u))F(u) \, dx = 0. \tag{3.15}$$

If case (i) of (V2) holds, then it follows from the Hardy inequality that

$$\int_{\mathbb{R}^3} \nabla V(x) \cdot x u^2 \, dx \le \frac{a}{2} \int_{\mathbb{R}^3} \frac{u^2}{|x|^2} \, dx \le 2a \|\nabla u\|_2^2 \quad \text{for all } u \in H^1(\mathbb{R}^3). \tag{3.16}$$

If case (ii) of (V2) holds, then it follows from the Sobolev embedding inequality that

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla V(x) \cdot x u^2 \, dx &\le \Big(\int_{\mathbb{R}^3} |\max\{\nabla V(x) \cdot x, 0\}|^{3/2} \, dx \Big)^{2/3} \Big(\int_{\mathbb{R}^3} u^6 \, dx \Big)^{1/3} \\ &\le \frac{\|\max\{\nabla V(x) \cdot x, 0\}\|_{3/2}}{S} \|\nabla u\|_2^2 \\ &\le 2a \|\nabla u\|_2^2 \quad \text{for all } u \in H^1(\mathbb{R}^3), \end{aligned} \tag{3.17}$$

where $S = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \|\nabla u\|_2^2 / \|u\|_6^2 = (\frac{3}{4})^{1/3} \pi^2$. It follows from (2.4), (3.13), (3.15) and either of (3.16) and (3.17) that

$$\begin{aligned} \mathcal{E}_\lambda(u_\lambda) &= \mathcal{E}_\lambda(u_\lambda) - \frac{1}{4} \Big[\frac{1}{2} \langle (\mathcal{E}_\lambda)'(u_\lambda), u_\lambda \rangle + \tilde{\mathcal{P}}_\lambda(u_\lambda) \Big] \\ &= \frac{a + bA_\lambda^2}{4} \|\nabla u_\lambda\|_2^2 - \frac{1}{8} \int_{\mathbb{R}^3} \nabla V(x) \cdot x u_\lambda^2 \, dx + \frac{\lambda}{8} \int_{\mathbb{R}^3} (I_\alpha * F(u_\lambda))[f(u_\lambda)u_\lambda + (\alpha - 1)F(u_\lambda)] \, dx \\ &\ge \frac{bA_\lambda^2}{4} \|\nabla u_\lambda\|_2^2. \end{aligned} \tag{3.18}$$

Since $(\mathcal{E}_\lambda^\infty)'(w^k)=0$, we have $\tilde{\mathcal{P}}_\lambda^\infty(w^k)=0$. Thus, from (3.3), (3.12), (3.14) and (3.15), it follows that

$$\begin{aligned}0&=\frac{1}{2}\langle(\mathcal{E}_\lambda^\infty)'(w^k),w^k\rangle+\tilde{\mathcal{P}}_\lambda^\infty(w^k)\\&=(a+bA_\lambda^2)\|\nabla w^k\|_2^2+2V_\infty\|w^k\|_2^2-\frac{\lambda}{2}\int_{\mathbb{R}^3}(I_\alpha*F(w^k))[f(w^k)w^k+(3+\alpha)F(w^k)]\,dx\\&\geq J_\lambda^\infty(w^k).\end{aligned}\tag{3.19}$$

Since $w^k\in H^1(\mathbb{R}^3)\setminus\{0\}$, in view of Lemma 2.8, there exists $t_k>0$ such that $t_k^{1/2}(w^k)_{t_k}\in\mathcal{M}_\lambda^\infty$. From (1.15), (2.13), (3.3),(3.14) and (3.19), one has

$$\begin{aligned}\mathcal{E}_\lambda^\infty(w^k)&=\mathcal{E}_\lambda^\infty(w^k)-\frac{1}{4}\Big[\frac{1}{2}\langle(\mathcal{E}_\lambda^\infty)'(w^k),w^k\rangle+\tilde{\mathcal{P}}_\lambda^\infty(w^k)\Big]\\&=\frac{a+bA_\lambda^2}{4}\|\nabla w^k\|_2^2+\frac{\lambda}{8}\int_{\mathbb{R}^3}(I_\alpha*F(u_\lambda))[f(u_\lambda)u_\lambda+(\alpha-1)F(u_\lambda)]\,dx\\&=\frac{bA_\lambda^2}{4}\|\nabla w^k\|_2^2+\Phi_\lambda^\infty(w^k)-\frac{1}{4}J_\lambda^\infty(w^k)\\&\geq\frac{bA_\lambda^2}{4}\|\nabla w^k\|_2^2+\Phi_\lambda^\infty(t_k^{1/2}(w^k)_{t_k})-\frac{t_k^4}{4}J_\lambda^\infty(w^k)\\&\geq\frac{bA_\lambda^2}{4}\|\nabla w^k\|_2^2+m_\lambda^\infty.\end{aligned}\tag{3.20}$$

It follows from (2.4), (3.12), (3.18) and (3.20) that

$$\begin{aligned}c_\lambda+\frac{bA_\lambda^4}{4}&=\mathcal{E}_\lambda(u_\lambda)+\sum_{k=1}^{l}\mathcal{E}_\lambda^\infty(w^k)\\&\geq lm_\lambda^\infty+\frac{bA_\lambda^2}{4}\Big[\|\nabla u_\lambda\|_2^2+\sum_{k=1}^{l}\|\nabla w^k\|_2^2\Big]\\&\geq lm_\lambda^\infty+\frac{bA_\lambda^4}{4}\quad\text{for all }\lambda\in(\bar{\lambda},1],\end{aligned}$$

which, together with Lemma 3.5, implies that $l=0$ and $\mathcal{E}_\lambda(u_\lambda)=c_\lambda+\frac{bA_\lambda^4}{4}$. Hence, $u_n\to u_\lambda$ in $H^1(\mathbb{R}^3)$ and $\Phi_\lambda(u_\lambda)=c_\lambda$. □

Lemma 3.8. *Assume that* (V1), (V2) *and* (F1)–(F3) *hold. Then there exists* $\bar{u}\in H^1(\mathbb{R}^3)\setminus\{0\}$ *such that*

$$\Phi'(\bar{u})=0,\quad \Phi(\bar{u})=c_1>0.\tag{3.21}$$

Proof. In view of Lemma 3.7, there exist two sequences $\{\lambda_n\}\subset(\bar{\lambda},1]$ and $\{u_{\lambda_n}\}\subset H^1(\mathbb{R}^3)$, denoted by $\{u_n\}$, such that

$$\lambda_n\to1,\quad \Phi'_{\lambda_n}(u_n)=0,\quad \Phi_{\lambda_n}(u_n)=c_{\lambda_n}.\tag{3.22}$$

By Lemma 3.4 (iii), (1.13), (3.1), (3.22) and either of (3.16) and (3.17), one has

$$\begin{aligned}c_{1/2}\geq c_{\lambda_n}&=\Phi_{\lambda_n}(u_n)-\frac{1}{3+\alpha}\mathcal{P}_{\lambda_n}(u_n)\\&=\frac{1}{2(3+\alpha)}\Big\{a(2+\alpha)\|\nabla u_n\|_2^2+\int_{\mathbb{R}^3}[\alpha V(x)-\nabla V(x)\cdot x]u_n^2\,dx\Big\}+\frac{b(1+\alpha)}{4(3+\alpha)}\|\nabla u_n\|_2^4\\&\geq\frac{\alpha}{2(3+\alpha)}\int_{\mathbb{R}^3}V(x)u_n^2\,dx+\frac{b(1+\alpha)}{4(3+\alpha)}\|\nabla u_n\|_2^4.\end{aligned}$$

This shows that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. In view of the proof of Lemma 3.7, we can show that there exists $\bar{u}\in H^1(\mathbb{R}^3)\setminus\{0\}$ such that (3.21) holds. □

Proof of Theorem 1.2. Let

$$\mathcal{K} := \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : \Phi'(u) = 0\}, \quad \hat{m} := \inf_{u \in \mathcal{K}} \Phi(u).$$

Then Lemma 3.8 shows that $\mathcal{K} \neq \emptyset$ and $\hat{m} \leq c_1$. For any $u \in \mathcal{K}$, (2.8), (3.2) and Lemma 3.2 imply $\mathcal{P}(u) = 0$. By (3.16) or (3.17), we have $\Phi(u) = \Phi(u) - \frac{1}{3}\mathcal{P}(u) > 0$ for any $u \in \mathcal{K}$, and so $\hat{m} \geq 0$. Let $\{u_n\} \subset \mathcal{K}$ be such that

$$\Phi'(u_n) = 0, \quad \Phi(u_n) \to \hat{m}.$$

In view of Lemma 3.5, $\hat{m} \leq c_1 < m_1^\infty$. Arguing as in the proof of Lemma 3.8, we can prove that there exists $\hat{u} \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that

$$\Phi'(\hat{u}) = 0, \quad \Phi(\hat{u}) = \hat{m}.$$

This shows that $\hat{u} \in H^1(\mathbb{R}^3)$ is a ground state solution of (1.1). □

4 Proof of Theorem 1.5

In this section, we give the proof of Theorem 1.5. In view of (1.7), the energy functional corresponding to (1.12) is defined in $H^1(\mathbb{R}^3)$ by

$$\hat{\Phi}(u) = \frac{1}{2}\int_{\mathbb{R}^3} [a|\nabla u|^2 + u^2]\,\mathrm{d}x + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x\Big)^2 - \frac{1}{2q^2}\int_{\mathbb{R}^3} (I_\alpha * |u|^q)|u|^q\,\mathrm{d}x.$$

From Lemma 3.2, if $\hat{\Phi}'(u) = 0$, then u satisfies the following Pohožaev type identity:

$$\hat{P}(u) := \frac{a}{2}\|\nabla u\|_2^2 + \frac{3}{2}\|u\|_2^2 + \frac{b}{2}\|\nabla u\|_2^4 - \frac{3+\alpha}{2q^2}\int_{\mathbb{R}^3} (I_\alpha * |u|^q)|u|^q\,\mathrm{d}x = 0. \tag{4.1}$$

Proof of Theorem 1.5. Let $v \in H^1(\mathbb{R}^3)$ be a solution to (1.12). Then

$$\langle \hat{\Phi}(v), v\rangle = a\|\nabla v\|_2^2 + \|v\|_2^2 + b\|\nabla v\|_2^4 - \frac{1}{q}\int_{\mathbb{R}^3} (I_\alpha * |v|^q)|v|^q\,\mathrm{d}x = 0. \tag{4.2}$$

By (4.1) and (4.2), one has

$$0 = \hat{P}(v) - \frac{3+\alpha}{2q}\langle \hat{\Phi}(v), v\rangle = \frac{q-(3+\alpha)}{2q}a\|\nabla v\|_2^2 + \frac{3q-(3+\alpha)}{2q}\|v\|_2^2 + \frac{q-(3+\alpha)}{4q}b\|\nabla v\|_2^4. \tag{4.3}$$

If $1 < q < 1 + \alpha/3$ or $q \geq 3 + \alpha$, then (4.3) implies $v = 0$. This completes the proof. □

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