DE GRUYTER Adv. Nonlinear Anal. 2021; 10: 1071–1085

Research Article

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Optimality of Serrin type extension criteria to the Navier-Stokes equations

https://doi.org/10.1515/anona-2020-0130
Received May 10, 2020; accepted January 15, 2021.

Abstract: We prove that a strong solution u to the Navier-Stokes equations on $(0, T)$ can be extended if either $u \in L^\theta(0, T; \dot{U}^{-\alpha}_{\infty,1/\theta,\infty})$ for $2/\theta + \alpha = 1,\ 0 < \alpha < 1$ or $u \in L^2(0, T; \dot{V}^0_{\infty,\infty,2})$, where $\dot{U}^s_{p,\beta,\sigma}$ and $\dot{V}^s_{p,q,\theta}$ are Banach spaces that may be larger than the homogeneous Besov space $\dot{B}^s_{p,q}$. Our method is based on a bilinear estimate and a logarithmic interpolation inequality.

Keywords: Serrin type extension criterion; Navier-Stokes equations; bilinear estimate; logarithmic interpolation inequality

MSC: 35Q30 (primary), 35B65, 46E35, 76D05

1 Introduction

The motion of a viscous incompressible fluid in $\mathbb{R}^n,\ n \geq 2$, is governed by the Navier-Stokes equations:

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla \pi = 0, & x \in \mathbb{R}^n,\ t > 0, \\ \operatorname{div} u = 0, & x \in \mathbb{R}^n,\ t > 0, \\ u|_{t=0} = u_0, \end{cases} \tag{N-S}$$

where $u = (u_1(x, t), \cdots, u_n(x, t))$ and $\pi = \pi(x, t)$ denote the velocity vector field and the pressure of the fluid at the point $x \in \mathbb{R}^n$ and time $t > 0$, respectively, while $u_0 = u_0(x)$ is the given initial vector field for u.

It is known that for every $u_0 \in H^s \equiv W^{s,2}(\mathbb{R}^n)$ $(s \geq n/2-1)$, there exists a unique solution $u \in C([0, T); H^s)$ to (N-S) for some $T > 0$. Such a solution is in fact smooth in $\mathbb{R}^n \times (0, T)$. See, for instance Fujita-Kato [9]. It is an important open question whether T may be taken as $T = \infty$ or $T < \infty$. In this direction, Giga [10] gave a Serrin type criterion, *i.e.*, if the solution u satisfies the condition

$$\int_0^T \|u(t)\|^\theta_{L^p}\, \mathrm{d}t < \infty, \qquad \frac{2}{\theta} + \frac{n}{p} = 1,\ n < p \leq \infty, \tag{1.1}$$

then u can be extended to the solution in the class $C([0, T']; H^s)$ for some $T' > T$. Later on, the condition (1.1) was relaxed from the L^p-criterion to

$$\int_0^T \|u(t)\|^\theta_{\dot{B}^{-\alpha}_{\infty,\infty}}\, \mathrm{d}t < \infty, \qquad \frac{2}{\theta} + \alpha = 1,\ 0 \leq \alpha < 1 \tag{1.2}$$

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by Kozono-Ogawa-Taniuchi [16] and Kozono-Shimada [17]. In a recent work, Nakao-Taniuchi [22] gave a new criterion, instead of (1.1) and (1.2) with $p = \infty$ and $\alpha = 0$ ($\theta = 2$), in a such way that

$$\int_0^T \|u(\tau)\|_{V_{1/2}}^2 \, d\tau < \infty. \tag{1.3}$$

Here, V_β, $\beta > 0$, is introduced by

$$V_\beta := \{f \in \mathcal{S}'; \|f\|_{V_\beta} < \infty\},$$
$$\|f\|_{V_\beta} := \sup_{N=1,2,\dots} \frac{\|\psi_N * f\|_\infty}{N^\beta},$$

where $\psi \in \mathcal{S}$ is a radially symmetric function with $\hat{\psi}(\xi) = 1$ in $B(0,1)$ and $\hat{\psi}(\xi) = 0$ in $B(0,2)^c$ and $\psi_N(x) := 2^{nN}\psi(2^N x)$. This function space V_β is called the Vishik space and admits a continuous embedding $L^\infty \subset V_\beta$ for each $\beta > 0$. The above three criteria are important from a view point of scaling invariance. Indeed, it is easy to show that if (u, π) satisfies (N-S), then so does (u_λ, π_λ) for all $\lambda > 0$, where $u_\lambda(x,t) := \lambda u(\lambda x, \lambda^2 t)$ and $\pi_\lambda(x,t) := \lambda^2 \pi(\lambda x, \lambda^2 t)$. We call a Banach space X scaling invariant for the velocity u with respect to (N-S) if $\|u_\lambda\|_X = \|u\|_X$ holds for all $\lambda > 0$. In fact, the spaces $L^\theta(0,\infty; L^p)$ with $2/\theta + n/p = 1$, $L^\theta(0,\infty; \dot{B}^{-\alpha}_{\infty,\infty})$ with $2/\theta + \alpha = 1$ and $L^2(0,\infty; V_{1/2})$ are scaling invariant for u with respect to (N-S).

On the other hand, Beale-Kato-Majda [1] and Beirão da Veiga [2] gave a criterion by means of the vorticity, *i.e.*, if the solution u satisfies the condition

$$\int_0^T \|\mathrm{rot}\, u(t)\|_{L^p}^\theta \, dt < \infty, \qquad \frac{2}{\theta} + \frac{n}{p} = 2, \ \frac{n}{2} < p \le \infty, \tag{1.4}$$

then u can be extended to a solution in the class $C([0,T'); H^s(\mathbb{R}^n))$ for some $T' > T$. Later on, the condition (1.4) was relaxed from the L^p-criterion to

$$\int_0^T \|\mathrm{rot}\, u(t)\|_{\dot{B}^0_{p,\infty}}^\theta \, dt < \infty, \qquad \frac{2}{\theta} + \frac{n}{p} = 2, \ n \le p \le \infty \tag{1.5}$$

by Kozono-Ogawa-Taniuchi [16]. Moreover, Nakao-Taniuchi [21] gave a similar type of the criterion as (1.3), instead of (1.4) and (1.5) with $p = \infty$ ($\theta = 1$), in such a way that

$$\int_0^T \|\mathrm{rot}\, u(t)\|_{V_1} \, dt < \infty.$$

Note that V_β admits the following continuous embeddings in the case $\beta = 1$:

$$L^\infty \subset bmo \subset B^0_{\infty,\infty} \subset V_1.$$

Futhermore, the author [12] improved the $\dot{B}^0_{p,\infty}$-criterion (1.5) to

$$\int_0^T \|\mathrm{rot}\, u(t)\|_{\dot{V}^0_{p,\infty,\theta}}^\theta \, dt < \infty, \qquad \frac{2}{\theta} + \frac{n}{p} = 2, \ r \le p \le \infty \tag{1.6}$$

for L^r ($n < r < \infty$) strong solutions to (N-S). Here, $\dot{V}^s_{p,q,\theta}$ is a Banach space introduced by Definition 2.1 and has a continuous embedding $\dot{B}^0_{p,\infty} \subset \dot{V}^0_{p,\infty,\theta}$. The above criteria by means of the vorticity are also important from a view point of scaling invariance. Indeed, since $\mathrm{rot}\, u_\lambda = \lambda^2 \mathrm{rot}\, u(\lambda x, \lambda^2 t)$, the spaces $L^\theta(0,\infty; L^p)$, $L^\theta(0,\infty; \dot{B}^0_{p,\infty})$, $L^\theta(0,\infty; \dot{V}^0_{p,\infty,\theta})$ with $2/\theta + n/p = 2$ and $L^1(0,\infty; V_1)$ are scaling invariant for the vorticity with respect to (N-S).

The aim of this paper is to improve the extension criterion (1.2) to the Navier-Stokes equations by means of Banach spaces which are larger than $\dot{B}^{-\alpha}_{\infty,\infty}$ in the same way that the condition (1.5) was relaxed to (1.6). In fact, we prove that if the solution u to (N-S) on $(0, T)$ satisfies the condition either

$$\int_0^T \|u(t)\|^{\theta}_{\dot{U}^{-\alpha}_{\infty,1/\theta,\infty}} \,\mathrm{d}t < \infty, \qquad \frac{2}{\theta} + \alpha = 1,\ 0 < \alpha < 1 \tag{1.7}$$

or

$$\int_0^T \|u(t)\|^{2}_{\dot{V}^{0}_{\infty,\infty,2}} \,\mathrm{d}t < \infty, \tag{1.8}$$

then u can be extended to a solution in the class $C([0, T'); H^s(\mathbb{R}^n))$ for some $T' > T$. Here, $\dot{U}^s_{p,\beta,\sigma}$ is a Banach space introduced by Definition 2.2 and has the following continuous embeddings:

$$\dot{B}^{-\alpha}_{\infty,\infty} \subset \dot{V}^{-\alpha}_{\infty,\infty,\theta} \subset \dot{U}^{-\alpha}_{\infty,1/\theta,\infty} \qquad \frac{2}{\theta} + \alpha = 1,\ 0 \le \alpha < 1.$$

Hence, we see that (1.7) and (1.8) may be regarded as a weaker condition than (1.2). Moreover, note that the spaces $L^{\theta}(0, \infty; \dot{U}^{-\alpha}_{\infty,1/\theta,\infty})$ with $2/\theta + \alpha = 1$ and $L^2(0, \infty; \dot{V}^0_{\infty,\infty,2})$ are also scaling invariant for solutions u to (N-S). In order to obtain our extension principle, we need a logarithmic interpolation inequality by means of $\dot{U}^s_{p,\beta,\sigma}$:

$$\|f\|_{\dot{B}^s_{p,\sigma}} \le C\left(1 + \|f\|_{\dot{U}^s_{p,\beta,\sigma}} \log^{\beta}(e + \|f\|_{\dot{B}^{s_1}_{p,\infty}\cap \dot{B}^{s_2}_{p,\infty}})\right).$$

This is related to the Brezis-Gallouet-Wainger inequality given in Brezis-Gallouet [5] and Brezis-Wainger [6]. Several inequalities of Brezis-Gallouet-Wainger type were established in [1], [7], [8], [11], [12], [15], [16], [19], [20], [21], [22], [23], [24], [25]. Moreover, we prove that $\dot{U}^s_{p,\beta,\sigma}$ is the weakest normed space that satisfies such a logarithmic interpolation inequality. Thus, roughly speaking, new conditions (1.7) and (1.8) may be regarded as optimal Serrin type criteria that guarantee *a priori* estimates of H^s strong solutions to (N-S) with double exponential growth form.

The present paper is organized as follows. In the next section, we shall state our main results. In section 3 and 4, proofs of our main results are established.

2 Results

2.1 Function spaces

We first introduce some notation. Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ be the set of all Schwartz functions on $\mathbb{R}^n$, and $\mathcal{S}'$ the set of tempered distributions. The L^p-norm on $\mathbb{R}^n$ is denoted by $\|\cdot\|_p$. We recall the Littlewood-Paley decomposition and use the functions $\psi, \phi_j \in \mathcal{S}$, $j \in \mathbb{Z}$, such that

$$\hat{\psi}(\xi) = \begin{cases} 1, & |\xi| \le 1, \\ 0, & |\xi| \ge 2, \end{cases}$$

$$\hat{\phi}(\xi) := \hat{\psi}(\xi) - \hat{\psi}(2\xi), \quad \hat{\phi}_j(\xi) := \hat{\phi}(\xi/2^j).$$

Let $\mathcal{Z} := \{f \in \mathcal{S}; D^{\alpha}\hat{f}(0) = 0 \text{ for all } \alpha \in \mathbb{N}^n\}$ and $\mathcal{Z}'$ denote the dual space of $\mathcal{Z}$. We note that $\mathcal{Z}'$ can be identified with the quotient space $\mathcal{S}'/\mathcal{P}$ of $\mathcal{S}'$ with respect to the space of polynomials, $\mathcal{P}$. Furthermore, the homogeneous Besov space $\dot{B}^s_{p,q} := \{f \in \mathcal{Z}'; \|f\|_{\dot{B}^s_{p,q}} < \infty\}$ is defined by the norm

$$\|f\|_{\dot{B}^s_{p,q}} := \begin{cases} \left(\displaystyle\sum_{j\in\mathbb{Z}} 2^{jsq}\|\phi_j \star f\|_p^q\right)^{\frac{1}{q}}, & q \ne \infty, \\ \displaystyle\sup_{j\in\mathbb{Z}} 2^{js}\|\phi_j \star f\|_p, & q = \infty. \end{cases}$$

See Bergh-Löfström [3, Chapter 6.3] and Triebel [26, Chapter 5] for details. Let $C_0^\infty(\mathbb{R}^n)$ denote the set of all C^∞ functions with compact support in $\mathbb{R}^n$ and $C_{0,\sigma}^\infty := \{\phi \in (C_0^\infty(\mathbb{R}^n))^n; \operatorname{div}\phi = 0\}$. Concerning Sobolev spaces we use the notation $H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$. Then H_σ^s is the closure of $C_{0,\sigma}^\infty$ with respect to H^s-norm. In Section 4 we will also use homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^n)$ and note that $\dot{H}^s = \dot{B}^s_{2,2}$ for all $s \in \mathbb{R}$.

We now introduce Banach spaces $\dot{V}^s_{p,q,\theta}$ and $\dot{U}^s_{p,\beta,\sigma}$ which are larger than the homogeneous Besov spaces $\dot{B}^s_{p,q}$. These spaces may be regarded as modified versions of spaces defined by Nakao-Taniuchi [22] and Vishik [27].

Definition 2.1. *Let $s \in \mathbb{R}$, $1 \le p, q, \theta \le \infty$ and let $\{\phi_j\}_{j=-\infty}^\infty$ be the Littlewood-Paley decomposition. Then, $\dot{V}^s_{p,q,\theta}(\mathbb{R}^n) := \{f \in \mathcal{Z}'; \|f\|_{\dot{V}^s_{p,q,\theta}} < \infty\}$ is introduced by the norm*

$$\|f\|_{\dot{V}^s_{p,q,\theta}} := \begin{cases} \sup\limits_{N=1,2,\cdots} \dfrac{\left(\sum_{|j|\le N} 2^{js\theta}\|\phi_j \star f\|_p^\theta\right)^{\frac{1}{\theta}}}{N^{\frac{1}{\theta}-\frac{1}{q}}}, & \theta \ne \infty, \\ \sup\limits_{N=1,2,\cdots} N^{\frac{1}{q}} \max\limits_{|j|\le N} 2^{js}\|\phi_j \star f\|_p, & \theta = \infty. \end{cases}$$

Definition 2.2. *Let $s, \beta \in \mathbb{R}$, $1 \le p, \sigma \le \infty$ and let $\{\phi_j\}_{j=-\infty}^\infty$ be the Littlewood-Paley decomposition. Then, $\dot{U}^s_{p,\beta,\sigma}(\mathbb{R}^n) := \{f \in \mathcal{Z}'; \|f\|_{\dot{U}^s_{p,\beta,\sigma}} < \infty\}$ is equipped with the norm*

$$\|f\|_{\dot{U}^s_{p,\beta,\sigma}} := \begin{cases} \sup\limits_{N=1,2,\cdots} \dfrac{\left(\sum_{|j|\le N} 2^{js\sigma}\|\phi_j \star f\|_p^\sigma\right)^{\frac{1}{\sigma}}}{N^\beta}, & \sigma \ne \infty, \\ \sup\limits_{N=1,2,\cdots} \dfrac{\max_{|j|\le N} 2^{js}\|\phi_j \star f\|_p}{N^\beta}, & \sigma = \infty. \end{cases}$$

We see from the following proposition that $\dot{V}^s_{p,q,\theta}$ and $\dot{U}^s_{p,\beta,\sigma}$ are extensions of $\dot{B}^s_{p,q}$ and $\dot{V}^s_{p,q,\theta}$, respectively.

Proposition 2.3.
(i) Let $s \in \mathbb{R}$, $1 \le p, q \le \infty$ and $1 \le \theta_1 \le \theta_2 \le q < \theta_3$. Then, it holds that

$$\{0\} = \dot{V}^s_{p,q,\theta_3} \subset \dot{B}^s_{p,q} = \dot{V}^s_{p,q,q} \subset \dot{V}^s_{p,q,\theta_2} \subset \dot{V}^s_{p,q,\theta_1}.$$

(ii) Let $s \in \mathbb{R}$, $1 \le p, \sigma \le \infty$ and $\beta_1 < 0 \le \beta_2 \le \beta_3$. Then, it holds that

$$\{0\} = \dot{U}^s_{p,\beta_1,\sigma} \subset \dot{B}^s_{p,\sigma} = \dot{U}^s_{p,0,\sigma} \subset \dot{U}^s_{p,\beta_2,\sigma} \subset \dot{U}^s_{p,\beta_3,\sigma}.$$

(iii) Let $s, \beta \in \mathbb{R}$, $1 \le p, q, \theta \le \infty$, $\tilde{\beta} = \frac{1}{\theta} - \frac{1}{q}$ and $1 \le \sigma_1 \le \sigma_2 \le \infty$. Then, it holds that

$$\dot{V}^s_{p,q,\theta} = \dot{U}^s_{p,\tilde{\beta},\theta} \quad \text{and} \quad \dot{U}^s_{p,\beta,\sigma_1} \subset \dot{U}^s_{p,\beta,\sigma_2}.$$

Proof. We easily prove $\dot{V}^s_{p,q,\theta_2} \subset \dot{V}^s_{p,q,\theta_1}$ in (i) by the standard and the reverse Hölder's inequality. The others follow from the definitions of $\dot{B}^s_{p,q}$, $\dot{V}^s_{p,q,\theta}$ and $\dot{U}^s_{p,\beta,\sigma}$. □

It follows by Proposition 2.3 (i) and (iii) that

$$\dot{B}^s_{\infty,\infty} \subset \dot{V}^s_{\infty,\infty,\theta} \subset \dot{U}^s_{\infty,1/\theta,\infty} \tag{2.1}$$

for $s \in \mathbb{R}$ and $1 \le \theta < \infty$. We observe from the following examples that the continuous embeddings (2.1) are proper if $s > -n$ and $1 \le \theta < \infty$, which is important in terms of Theorem 2.9.

Example 2.4. (1) The continuous embedding $\dot{B}^s_{\infty,\infty} \subset \dot{V}^s_{\infty,\infty,\theta}$ is proper if $s > -n$ and $1 \le \theta < \infty$. We now introduce a distribution $f \in \dot{V}^s_{\infty,\infty,\theta} \setminus \dot{B}^s_{\infty,\infty}$ for $s > -n$ and $1 \le \theta < \infty$. Let $f \in \mathcal{Z}'$ defined as

$$\hat{f}(\xi) := \begin{cases} k2^{-(n+s)[k^{\theta+1}]}, & 2^{[k^{\theta+1}]-1} \le |\xi| \le 2^{[k^{\theta+1}]+1} \quad (k = 1, 2, \cdots), \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, since $\hat{f} \in L^\infty$ holds, we obtain $f \in \mathcal{Z}'$. We easily see that

$$\|\phi_j \star f\|_\infty = \int_{\mathbb{R}^n} \hat{\phi}_j(\xi)\hat{f}(\xi)\,\mathrm{d}\xi = \int_{2^{j-1}\le|\xi|\le 2^{j+1}} \hat{\phi}_j(\xi)\hat{f}(\xi)\,\mathrm{d}\xi$$

$$\begin{cases} = 2^{-s[k^{\theta+1}]}k\|\hat{\phi}\|_1 & \text{for } j = [k^{\theta+1}] \quad (k = 1, 2, \cdots), \\ \le 2^{-s[k^{\theta+1}]}2^n k\|\hat{\phi}\|_1 & \text{for } j = [k^{\theta+1}] \pm 1 \quad (k = 1, 2, \cdots), \\ = 0 & \text{for } j \in \mathbb{Z} \setminus \bigcup_{k=1,2,\cdots}\{[k^{\theta+1}], [k^{\theta+1}] \pm 1\}. \end{cases}$$

Hence, it holds that

$$\|f\|_{\dot{B}^s_{\infty,\infty}} \ge \sup_{k=1,2,\cdots} 2^{s[k^{\theta+1}]}\|\phi_{[k^{\theta+1}]} \star f\|_\infty = \sup_{k=1,2,\cdots} k\|\hat{\phi}\|_1 = \infty. \tag{2.2}$$

On the other hand, for any $N = 1, 2, \cdots$, there exists $k_N \in \mathbb{N}$ such that $k_N^{\theta+1} \le N < (k_N + 1)^{\theta+1}$. Therefore, we obtain

$$\begin{aligned} \sum_{|j|\le N} 2^{js\theta}\|\phi_j \star f\|^\theta_\infty &\le \sum_{k=1}^{k_N+1} \sum_{j=[k^{\theta+1}]-1}^{[k^{\theta+1}]+1} 2^{js\theta}\|\phi_j \star f\|^\theta_\infty \\ &\le \sum_{k=1}^{k_N+1} \sum_{j=[k^{\theta+1}]-1}^{[k^{\theta+1}]+1} 2^{js\theta}(2^{-s[k^{\theta+1}]}2^n k\|\hat{\phi}\|_1)^\theta \\ &= C\sum_{k=1}^{k_N+1} k^\theta \le C(k_N + 1)^{\theta+1} \le Ck_N^{\theta+1} \le CN, \end{aligned}$$

where C is dependent only on n, s and θ. Thus, it follows that

$$\|f\|_{\dot{V}^s_{\infty,\infty,\theta}} = \sup_{N=1,2,\cdots} \frac{\left(\sum_{|j|\le N} 2^{js\theta}\|\phi_j \star f\|^\theta_\infty\right)^{\frac{1}{\theta}}}{N^{\frac{1}{\theta}}} \le \sup_{N=1,2,\cdots} \frac{C^{\frac{1}{\theta}}N^{\frac{1}{\theta}}}{N^{\frac{1}{\theta}}} < \infty. \tag{2.3}$$

From (2.2) and (2.3), we get $f \in \dot{V}^s_{\infty,\infty,\theta} \setminus \dot{B}^s_{\infty,\infty}$.

(2) The continuous embedding $\dot{V}^s_{\infty,\infty,\theta} = \dot{U}^s_{\infty,1/\theta,\theta} \subset \dot{U}^s_{\infty,1/\theta,\infty}$ is also proper if $s > -n$ and $1 \le \theta < \infty$. We now introduce a distribution $g \in \dot{U}^s_{\infty,1/\theta,\infty} \setminus \dot{V}^s_{\infty,\infty,\theta}$ for $s > -n$ and $1 \le \theta < \infty$. Let $g \in \mathcal{Z}'$ defined as

$$\hat{g}(\xi) := \begin{cases} k^{\frac{\theta+1}{\theta}}2^{-(n+s)[k^{\theta+1}]}, & 2^{[k^{\theta+1}]-1} \le |\xi| \le 2^{[k^{\theta+1}]+1} \quad (k = 1, 2, \cdots), \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, since $\hat{g} \in L^\infty$ holds, we obtain $g \in \mathcal{Z}'$. We easily see that

$$\|\phi_j \star g\|_\infty = \int_{\mathbb{R}^n} \hat{\phi}_j(\xi)\hat{g}(\xi)\,\mathrm{d}\xi = \int_{2^{j-1}\le|\xi|\le 2^{j+1}} \hat{\phi}_j(\xi)\hat{g}(\xi)\,\mathrm{d}\xi$$

$$\begin{cases} = 2^{-s[k^{\theta+1}]}k^{\frac{\theta+1}{\theta}}\|\hat{\phi}\|_1 & \text{for } j = [k^{\theta+1}] \quad (k = 1, 2, \cdots), \\ \le 2^{-s[k^{\theta+1}]}2^n k^{\frac{\theta+1}{\theta}}\|\hat{\phi}\|_1 & \text{for } j = [k^{\theta+1}] \pm 1 \quad (k = 1, 2, \cdots), \\ = 0 & \text{for } j \in \mathbb{Z} \setminus \bigcup_{k=1,2,\cdots}\{[k^{\theta+1}], [k^{\theta+1}] \pm 1\}. \end{cases}$$

For any $N = 1, 2, \cdots$, we take $k_N \in \mathbb{N}$ such that $k_N^{\theta+1} \le N < (k_N + 1)^{\theta+1}$. Then, it holds that

$$\sum_{|j|\le N} 2^{js\theta}\|\phi_j \star g\|_\infty^\theta \ge \sum_{1\le k\le k_N} 2^{s[k^{\theta+1}]\theta}\|\phi_{[k^{\theta+1}]} \star g\|_\infty^\theta = C_1 \sum_{1\le k\le k_N} k^{\theta+1}$$
$$\ge C_1 k_N^{\theta+2} \ge C_1(k_N+1)^{\theta+2} \ge C_1 N^{\frac{\theta+2}{\theta+1}},$$

where C_1 is dependent only on n and θ. Hence, we have

$$\|f\|_{\dot{V}^s_{\infty,\infty,\theta}} = \sup_{N=1,2,\cdots}\left(\frac{1}{N}\sum_{|j|\le N} 2^{js\theta}\|\phi_j \star f\|_\infty^\theta\right)^{\frac{1}{\theta}} \ge \sup_{N=1,2,\cdots} C_1^{\frac{1}{\theta}} N^{\frac{1}{\theta}\left(\frac{\theta+2}{\theta+1}-1\right)} = \infty. \tag{2.4}$$

On the other hand, it follows that

$$\max_{|j|\le N} 2^{js}\|\phi_j \star g\|_\infty \le \max_{1\le k\le k_N+1}\ \max_{j=[k^{\theta+1}],[k^{\theta+1}]\pm1} 2^{js}\|\phi_j \star g\|_\infty$$
$$\le \max_{1\le k\le k_N+1}\ \max_{j=[k^{\theta+1}],[k^{\theta+1}]\pm1} 2^{js}2^{-s[k^{\theta+1}]}2^n k^{\frac{\theta+1}{\theta}}\|\hat{\phi}\|_1$$
$$\le C_2 \max_{1\le k\le k_N+1} k^{\frac{\theta+1}{\theta}} = C_2(k_N+1)^{\frac{\theta+1}{\theta}} \le C_2 k_N^{\frac{\theta+1}{\theta}} \le C_2 N^{\frac{1}{\theta}},$$

where C_2 is dependent only on n and s. Thus, we obtain

$$\|g\|_{\dot{U}^s_{\infty,1/\theta,\infty}} = \sup_{N=1,2,\cdots}\frac{\max_{|j|\le N} 2^{js}\|\phi_j \star f\|_\infty}{N^{\frac{1}{\theta}}} \le \sup_{N=1,2,\cdots}\frac{C_2^{\frac{1}{\theta}}N^{\frac{1}{\theta}}}{N^{\frac{1}{\theta}}} < \infty. \tag{2.5}$$

From (2.4) and (2.5), we get $g \in \dot{U}^s_{\infty,1/\theta,\infty} \setminus \dot{V}^s_{\infty,\infty,\theta}$.

2.2 Logarithmic interpolation inequalities and optimality

Theorem 2.5. (i) *Let $s_0, s_1, s_2 \in \mathbb{R}$ satisfy $s_1 < s_0 < s_2$, let $0 \le \beta < \infty$ and $1 \le p, \sigma \le \infty$. Then there exists a positive constant C depending only on s_0, s_1, s_2, but not on p, β, σ such that*

$$\|f\|_{\dot{B}^{s_0}_{p,\sigma}} \le C\left(1 + \|f\|_{\dot{U}^{s_0}_{p,\beta,\sigma}} \log^\beta(e + \|f\|_{\dot{B}^{s_1}_{p,\infty}\cap\dot{B}^{s_2}_{p,\infty}})\right) \tag{2.6}$$

for all $f \in \dot{B}^{s_1}_{p,\infty} \cap \dot{B}^{s_2}_{p,\infty}$.

(ii) *Let $s_0 \in \mathbb{R}$, $0 \le \beta < \infty$ and $1 \le p, \sigma \le \infty$, and let X be a normed space of distributions on $\mathcal{Z}$. Assume that X satisfies the following conditions:*

(C1) $X \hookrightarrow \mathcal{Z}'$;

(C2) *there exists a constant $K_1 > 0$ such that*

$$\|f(\cdot - y)\|_X \le K_1\|f\|_X \quad \text{for all } f \in X \text{ and all } y \in \mathbb{R}^n;$$

(C3) *there exists a constant $K_2 > 0$ such that*

$$\|\rho \star f\|_X \le K_2\|\rho\|_1\|f\|_X \quad \text{for all } \rho \in \mathcal{Z} \text{ and all } f \in X;$$

(C4) *there exist $s_1, s_2 \in \mathbb{R}$ satisfy $s_1 < s_0 < s_2$ and $K_3 > 0$ such that*

$$\|f\|_{\dot{B}^{s_0}_{p,\sigma}} \le K_3\left(1 + \|f\|_X \log^\beta\left(e + \|f\|_{\dot{B}^{s_1}_{p,\infty}\cap\dot{B}^{s_2}_{p,\infty}}\right)\right) \quad \text{for all } f \in X \cap \mathcal{Z}.$$

Then, $X \hookrightarrow \dot{U}^{s_0}_{p,\beta,\sigma}$ *holds.*

Remark 2.6. (1) In the first part of Theorem 2.5, the assumption $s_1 < s_0 < s_2$ is essential. If either of s_1 or s_2 tends to s_0, then the constant C appearing on the right hand side diverges to infinity.

(2) By Proposition 2.3 (ii), we observed that the following continuous embeddings hold for $s_1 < s_0 < s_2$ and $\beta \geq 0$:

$$\dot{B}^{s_1}_{p,\infty} \cap \dot{B}^{s_2}_{p,\infty} \subset \dot{B}^{s_0}_{p,\sigma} \subset \dot{U}^{s_0}_{p,\beta,\sigma}.$$

Thus, (2.6) may be regarded as an interpolation inequality.

(3) From Theorem 2.5 (i), we see that $\dot{U}^{s_0}_{p,q,\theta}$ satisfies conditions (C1)-(C4). Hence, Theorem 2.5 (ii) implies that $\dot{U}^{s_0}_{p,q,\theta}$ is the weakest normed space that satisfies (C1)-(C4).

(4) By Proposition 2.3 (iii), we see that Theorem 2.5 covers the result given by the author [12]. Indeed, by setting $\beta = \frac{1}{\theta} - \frac{1}{q}$, $\sigma = \theta\,(1 \leq q \leq \infty,\ 1 \leq \theta \leq q)$ in (2.6), it holds that

$$\|f\|_{\dot{B}^{s_0}_{p,\theta}} \leq C\left(1 + \|f\|_{\dot{V}^{s_0}_{p,q,\theta}} \log^{\frac{1}{\theta}-\frac{1}{q}}\left(e + \|f\|_{\dot{B}^{s_1}_{p,\infty}\cap\dot{B}^{s_2}_{p,\infty}}\right)\right)$$

for all $f \in \dot{B}^{s_1}_{p,\infty} \cap \dot{B}^{s_2}_{p,\infty}$.

2.3 Serrin type regularity criteria for Navier-Stokes systems

Definition 2.7. *Let* $s > n/2 - 1$ *and let* $u_0 \in H^s_\sigma$. *A measurable function* u *on* $\mathbb{R}^n \times (0,T)$ *is called a strong solution to* (N-S) *in the class* $CL_s(0,T)$ *if*

(i) $u \in C([0,T); H^s_\sigma) \cap C^1((0,T); H^s_\sigma) \cap C((0,T); H^{s+2}_\sigma)$;

(ii) u *satisfies* (N-S) *with some distribution* π *such that* $\nabla\pi \in C((0,T); H^s)$.

Remark 2.8. For $s > n/2-1$, the existence of a strong solution to (N-S) in the class $CL_s(0,T)$ has been proven in Fujita-Kato [9], Kato [14] and Giga [10].

Our result on extension of strong solutions now reads as follows:

Theorem 2.9. (i) *Let* $0 < \alpha < 1,\ s > n/2 - \alpha$ *and let* $u_0 \in H^s_\sigma$. *Assume that* u *is a strong solution to* (N-S) *in the class* $CL_s(0,T)$. *If the solution* u *satisfies*

$$\int_0^T \|u(t)\|^{\theta}_{\dot{U}^{-\alpha}_{\infty,1/\theta,\infty}}\,\mathrm{d}t < \infty, \qquad \frac{2}{\theta} + \alpha = 1, \tag{2.7}$$

then u *can be extended to a strong solution to* (N-S) *in the class* $CL_s(0,T')$ *for some* $T' > T$.

(ii) *Let* $s > n/2$ *and let* $u_0 \in H^s_\sigma$. *Assume that* u *is a strong solution to* (N-S) *in the class* $CL_s(0,T)$. *If the solution* u *satisfies*

$$\int_0^T \|u(t)\|^2_{\dot{V}^0_{\infty,\infty,2}}\,\mathrm{d}t < \infty, \tag{2.8}$$

then u *can be extended to a strong solution to* (N-S) *in the class* $CL_s(0,T')$ *for some* $T' > T$.

Remark 2.10. (1) Let $0 < \alpha < 1$. As is mentioned Example 2.4, we have proper embeddings $\dot{B}^{-\alpha}_{\infty,\infty} \subset \dot{V}^{-\alpha}_{\infty,\infty,\theta} \subset \dot{U}^{-\alpha}_{\infty,1/\theta,\infty}$ and hence Thorem 2.9 (i) covers the extension criterion in $\dot{B}^{-\alpha}_{\infty,\infty}$ given by Kozono-Shimada [17] for $s > n/2 - \alpha$. Indeed, if the solution u satisfies either

$$\int_0^T \|u(\tau)\|^{\theta}_{\dot{B}^{-\alpha}_{\infty,\infty}}\,\mathrm{d}\tau < \infty, \qquad \frac{2}{\theta} + \alpha = 1,$$

or

$$\int_0^T \|u(\tau)\|^{\theta}_{\dot V^{-\alpha}_{\infty,\infty,\theta}}\,\mathrm{d}\tau < \infty, \qquad \frac{2}{\theta}+\alpha = 1,$$

then the estimate (2.8) is easily obtained, so that the solution can be extended beyond $t = T$.

(2) From Example 2.4, the proper embeddings $\dot B^0_{\infty,\infty} \subset \dot V^0_{\infty,\infty,2} \subset \dot U^0_{\infty,1/2,\infty}$ hold. Hence, Theorem 2.9 (ii) may be regarded as an extension of the $\dot B^0_{\infty,\infty}$-criterion given by Kozono-Ogawa-Taniuchi [16] for $s > n/2$. On the other hand, it seems to be difficult to obtain the same result as in Theorem 2.9 (ii) under the condition

$$\int_0^T \|u(\tau)\|^2_{\dot U^0_{\infty,1/2,\infty}}\,\mathrm{d}\tau < \infty.$$

This stems from inapplicability of Lemma 4.1 with $\alpha = 0$.

As an immediate consequence of the above Theorem 2.9, we have the following blow-up criteria of strong solutions:

Corollary 2.11. (i) *Let* $0 < \alpha < 1,\ s > n/2 - \alpha$ *and let* $u_0 \in H^s_\sigma$. *Assume that* u *is a strong solution to* (N-S) *in the class* $CL_s(0,T)$. *If* T *is maximal,* i.e., *u cannot be extended in the class* $CL_s(0,T')$ *for any* $T' > T$, *then it holds that*

$$\int_0^T \|u(t)\|^{\theta}_{\dot U^{-\alpha}_{\infty,1/\theta,\infty}}\,\mathrm{d}t = \infty, \qquad \frac{2}{\theta}+\alpha = 1.$$

In particular, we have $\limsup_{t\to T} \|u(t)\|_{\dot U^{-\alpha}_{\infty,1/\theta,\infty}} = \infty$.

(ii) *Let* $s > n/2$ *and let* $u_0 \in H^s_\sigma$. *Assume that u is a strong solution to* (N-S) *in the class* $CL_s(0,T)$. *If T is maximal, then it holds that*

$$\int_0^T \|u(t)\|^2_{\dot V^0_{\infty,\infty,2}}\,\mathrm{d}t = \infty.$$

In particular, $\limsup_{t\to T} \|u(t)\|_{\dot V^0_{\infty,\infty,2}} = \infty$.

3 Proof of Theorem 2.5

We first prove Theorem 2.5 (i). To this aim, we use arguments given in Kozono-Ogawa-Taniuchi [16], Nakao-Taniuchi [21] and Kanamaru [12].

Proof of Theorem 2.5 (i). We first consider the case $1 \le \sigma < \infty$. By the definition of the Besov space, we obtain

$$\begin{aligned}\|f\|_{\dot B^{s_0}_{p,\sigma}} &= \left(\sum_{j\in\mathbb{Z}} 2^{js_0\sigma}\|\phi_j * f\|^{\sigma}_q\right)^{\frac{1}{\sigma}} \\ &\le \sum_{j<-N} 2^{js_0}\|\phi_j * f\|_p + \sum_{j>N} 2^{js_0}\|\phi_j * f\|_p + \left(\sum_{|j|\le N} 2^{js_0\sigma}\|\phi_j * f\|^{\sigma}_p\right)^{\frac{1}{\sigma}} \qquad (3.1)\\ &=: S_1 + S_2 + S_3\end{aligned}$$

Concerning S_1, it holds that

$$\begin{aligned} S_1 &\le \sum_{j<-N} 2^{js_1} \|\phi_j \star f\|_p \, 2^{j(s_0-s_1)} \\ &\le \|f\|_{\dot B^{s_1}_{p,\infty}} \sum_{j<-N} 2^{j(s_0-s_1)} \\ &\le C_1 2^{-(s_0-s_1)N} \|f\|_{\dot B^{s_1}_{p,\infty}}, \end{aligned} \tag{3.2}$$

where C_1 is dependent only on s_0 and s_1. For S_2, in the same way as (3.2), we have

$$S_2 \le C_2 2^{-(s_2-s_0)N} \|f\|_{\dot B^{s_2}_{p,\infty}}, \tag{3.3}$$

where C_2 is dependent only on s_0 and s_2.

We finally estimate S_3. By Definition 2.2, it clearly follows that

$$S_3 \le N^\beta \|f\|_{\dot U^{s_0}_{p,\beta,\sigma}}. \tag{3.4}$$

Combining (3.2), (3.3) and (3.4) with (3.1), we obtain

$$\|f\|_{\dot B^{s_0}_{p,\sigma}} \le C \left(2^{-s_\star N} \|f\|_{\dot B^{s_1}_{p,\infty} \cap \dot B^{s_2}_{p,\infty}} + N^\beta \|f\|_{\dot U^{s_0}_{p,\beta,\sigma}} \right) \tag{3.5}$$

for $s_\star := \min(s_0 - s_1, s_2 - s_0)$ and $C = C(s_0, s_1, s_2)$. In the case $\|f\|_{\dot B^{s_1}_{p,\infty} \cap \dot B^{s_2}_{p,\infty}} \le 1$, we take $N = 1$ in (3.5). Then it holds that

$$\|f\|_{\dot B^{s_0}_{p,\sigma}} \le C \left(1 + \|f\|_{\dot U^{s_0}_{p,\beta,\sigma}}\right) \le C \left(1 + \|f\|_{\dot U^{s_0}_{p,\beta,\sigma}} \log^\beta (e + \|f\|_{\dot B^{s_1}_{p,\infty} \cap \dot B^{s_2}_{p,\infty}})\right);$$

this is the desired estimate (2.6). In the case $\|f\|_{\dot B^{s_1}_{p,\infty} \cap \dot B^{s_2}_{p,\infty}} > 1$, we take $N = 1 + \left[\log(e + \|f\|_{\dot B^{s_1}_{p,\infty} \cap \dot B^{s_2}_{p,\infty}})/(s_\star \log 2)\right]$ in (3.5), where $[\cdot]$ denotes the Gauß symbol. Then, we get (2.6) again.

In the case $\sigma = \infty$, we obtain, instead of (3.1),

$$\begin{aligned} \|f\|_{\dot B^{s_0}_{p,\infty}} &\le \sup_{j<-N} 2^{js_0} \|\phi_j \star f\|_p + \sup_{j>N} 2^{js_0} \|\phi_j \star f\|_p + \max_{|j|\le N} 2^{js_0} \|\phi_j \star f\|_p \\ &=: \tilde S_1 + \tilde S_2 + \tilde S_3 \end{aligned} \tag{3.6}$$

Therefore, using the same argument as in the previous case $1 \le \sigma < \infty$, we get (2.6). □

In order to prove the second part of Thorem 2.5, we use the following Lemma.

Lemma 3.1. *Let $\rho \in \mathcal{Z}$ and Let X be a normed space. Assume that X satisfies conditions* (C1) *and* (C2) *given in Theorem* 2.5 (ii). *Then, it holds that*

$$\rho \star g \in L^\infty \quad \textit{for all } g \in X. \tag{3.7}$$

Proof. By (C1), we get that for all $\phi \in \mathcal{Z}$, there exists a constant $C = C(\phi) > 0$ such that

$$|g(\phi)| \le C \|g\|_X \quad \text{for all } g \in X. \tag{3.8}$$

Assume that (3.8) does not hold. Then, there is $\phi_0 \in \mathcal{Z}$ with the following property: for each positive integer N, there is a $g_N \in X$ such that

$$|g_N(\phi_0)| > N \|g_N\|_X. \tag{3.9}$$

Letting $h_N := \frac{g_N}{N^{\frac12} \|g_N\|_X} (\in X)$, we obtain $\|h_N\|_X = N^{-\frac12} \to 0$ as $N \to \infty$, which implies $h_N \to 0$ in X. By (C1), this convergence holds in $\mathcal{Z}'$. On the other hand, by (3.9),

$$|h_N(\phi_0)| = \frac{|g_N(\phi_0)|}{N^{\frac12} \|g_N\|_X} > N^{\frac12} \to \infty \quad \text{as } N \to \infty,$$

which contradicts $h_N \to 0$ in $\mathcal{Z}'$. Thus we get (3.8).

We finally prove (3.7). Note that

$$\rho \star g(x) = g(\tau_x \tilde{\rho}) = \tau_{-x} g(\tilde{\rho}),$$

where $\tau_x f(y) = f(y-x)$ and $\tilde{f}(y) = f(-y)$. Hence, from (3.8) and (C2), we obtain

$$|\rho \star g(x)| \le C(\rho)\|\tau_{-x} g\|_X \le C'(\rho, K_1)\|g\|_X \quad \text{for all } x \in \mathbb{R}^n,$$

which means (3.7). □

We are now in position to prove the second part of Theorem 2.5 and follow arguments given by Nakao-Taniuchi [21] and the author [12].

Proof of Theorem 2.5 (ii). Subsituting $f = \frac{h}{\varepsilon\|h\|_{\dot{B}^{s_1}_{p,\infty}\cap \dot{B}^{s_2}_{p,\infty}}}$ into the inequality given in (C4), we obtain

$$\|h\|_{\dot{B}^{s_0}_{p,\sigma}} \le K_3\left(\varepsilon\|h\|_{\dot{B}^{s_1}_{p,\infty}\cap \dot{B}^{s_2}_{p,\infty}} + \|h\|_X \log^\beta\left(e + \frac{1}{\varepsilon}\right)\right) \tag{3.10}$$

for all $h \in X \cap \mathcal{Z}$ and all $\varepsilon > 0$. Let $g \in X$ and $\Phi_N := \sum_{|j|\le N} \phi_j \,(\in \mathcal{Z})$ for $N = 1, 2, \cdots$. By Lemma 3.1, $\Phi_N \star g \in L^\infty$. Hence, since $\Phi_N \star g = \Phi_{N+1} \star \Phi_N \star g$, we have $\Phi_N \star g \in \mathcal{Z}$. On the other hand, it holds from (C3) that

$$\|\Phi_N \star g\|_X \le K_2\|\Phi_N\|_1\|g\|_X \le K_2(\|\psi_N\|_1 + \|\psi_{-N-1}\|_1)\|g\|_X \le 2K_2\|\psi\|_1\|g\|_X, \tag{3.11}$$

where $\psi_j(x) := 2^{jn}\psi(2^j x)$. Thus, we also get $\Phi_N \star g \in X$. Substituting $h = \Phi_N \star g \,(\in X \cap \mathcal{Z})$ into (3.10), we obtain

$$\|\Phi_N \star g\|_{\dot{B}^{s_0}_{p,\sigma}} \le K_3\varepsilon\|\Phi_N \star g\|_{\dot{B}^{s_1}_{p,\infty}\cap \dot{B}^{s_2}_{p,\infty}} + K_3\|\Phi_N \star g\|_X \log^\beta\left(e + \frac{1}{\varepsilon}\right). \tag{3.12}$$

We first consider the case $1 \le \sigma < \infty$.

The left-hand side of (3.12) can be estimated from below as follows. Noting that $\operatorname{supp}\hat{\Phi}_N \subset \{2^{-N-1} \le |\xi| \le 2^{N+1}\}$, we get

$$\begin{aligned}\|\Phi_N \star g\|^\sigma_{\dot{B}^{s_0}_{p,\sigma}} &= \sum_{|j|\le N+1} 2^{js_0\sigma}\|\phi_j \star \Phi_N \star g\|_p^\sigma \\ &= \left(\sum_{|j|\le N-1} + \sum_{j=N,N+1} + \sum_{j=-N,-N-1}\right) 2^{js_0\sigma}\|\phi_j \star \Phi_N \star g\|_p^\sigma.\end{aligned} \tag{3.13}$$

Concerning the second term on the right-hand side of (3.13), we obtain

$$\begin{aligned}\sum_{j=N,N+1} 2^{js_0\sigma}\|\phi_j \star \Phi_N \star g\|_p^\sigma &\ge 2^{-|s_0|\sigma}2^{Ns_0\sigma}\sum_{j=N,N+1}\|\phi_j \star \Phi_N \star g\|_p^\sigma \\ &\ge 2^{-|s_0|\sigma}2^{Ns_0\sigma}2^{-\sigma}\left(\sum_{j=N,N+1}\|\phi_j \star \Phi_N \star g\|_p\right)^\sigma \\ &\ge 2^{-(|s_0|+1)\sigma}2^{Ns_0\sigma}\left\|\sum_{j=N,N+1}\phi_j \star \Phi_N \star g\right\|_p^\sigma \\ &= 2^{-(|s_0|+1)\sigma}2^{Ns_0\sigma}\|\phi_N \star g\|_p^\sigma.\end{aligned} \tag{3.14}$$

As in (3.14), similar estimates hold when replacing N and $N+1$ by $-N$ and $-N-1$, respectively. Summarizing (3.13), (3.14) we obtain that

$$\|\Phi_N \star g\|_{\dot{B}^{s_0}_{p,\sigma}} \ge 2^{-(|s_0|+1)}\left(\sum_{|j|\le N} 2^{js_0\sigma}\|\phi_j \star g\|_p^\sigma\right)^{\frac{1}{\sigma}}. \tag{3.15}$$

Next, we estimate the first term on the right-hand side of (3.12). From Young's inequality and Hölder's inequality, it holds that

$$\begin{aligned}\|\Phi_N \star g\|_{\dot B^{s_1}_{p,\infty}} &= \sup_{|j|\le N+1} 2^{js_1}\|\phi_j \star \Phi_N \star g\|_p \\ &\le \sup_{|j|\le N+1} 2^{js_1}\|\phi_j\|_1\|\Phi_N \star g\|_p \\ &\le C_1 2^{|s_1|N}\sum_{|j|\le N} 2^{-js_0}2^{js_0}\|\phi_j \star g\|_p \\ &\le C_1 2^{(|s_0|+|s_1|)N}\Big(\sum_{|j|\le N} 1\Big)^{\frac{1}{\sigma'}}\Big(\sum_{|j|\le N} 2^{js_0\sigma}\|\phi_j \star g\|_p^\sigma\Big)^{\frac{1}{\sigma}} \\ &\le C_1 2^{(|s_0|+|s_1|+1)N}\Big(\sum_{|j|\le N} 2^{js_0\sigma}\|\phi_j \star g\|_p^\sigma\Big)^{\frac{1}{\sigma}},\end{aligned} \tag{3.16}$$

where C_1 depends only on n and s_1. In the same way as (3.16), we have

$$\|\Phi_N \star g\|_{\dot B^{s_2}_{p,\infty}} \le C_2 2^{(|s_0|+|s_2|+1)N}\Big(\sum_{|j|\le N} 2^{js_0\sigma}\|\phi_j \star g\|_p^\sigma\Big)^{\frac{1}{\sigma}}, \tag{3.17}$$

where C_2 depends only on n and s_2. In the end, from (3.16) and (3.17), we get that

$$\|\Phi_N \star g\|_{\dot B^{s_1}_{p,\infty}\cap \dot B^{s_2}_{p,\infty}} \le C_3 2^{s^\star N}\Big(\sum_{|j|\le N} 2^{js_0\sigma}\|\phi_j \star g\|_p^\sigma\Big)^{\frac{1}{\sigma}} \tag{3.18}$$

for $s^\star := |s_0| + \max(|s_1|, |s_2|) + 1$ and $C_3 = C_3(n, s_1, s_2)$.

Thus, combining (3.11), (3.15) and (3.18) with (3.12), we obtain

$$\Big(\sum_{|j|\le N} 2^{js_0\sigma}\|\phi_j \star g\|_p^\sigma\Big)^{\frac{1}{\sigma}} \le C\varepsilon 2^{s^\star N}\Big(\sum_{|j|\le N} 2^{js_0\sigma}\|\phi_j \star g\|_p^\sigma\Big)^{\frac{1}{\sigma}} + C\|g\|_X \log^\beta\Big(e + \frac{1}{\varepsilon}\Big)$$

for all $N = 1, 2, \cdots$, all $\varepsilon > 0$ and $C = C(n, s_0, s_1, s_2, K_2, K_3)$. Taking $\varepsilon = \frac{1}{2C2^{s^\star N}}$, from the above inequality, we get

$$\Big(\sum_{|j|\le N} 2^{js_0\sigma}\|\phi_j \star g\|_p^\sigma\Big)^{\frac{1}{\sigma}} \le CN^\beta\|g\|_X \text{ for all } N = 1, 2, \cdots.$$

This implies

$$\|g\|_{\dot U^{s_0}_{p,\beta,\sigma}} \le C\|g\|_X \ \text{ for all } g \in X,$$

i.e., the embedding $X \hookrightarrow \dot U^{s_0}_{p,\beta,\sigma}$.

In the case $\sigma = \infty$, we obtain, instead of (3.13),

$$\|\Phi_N \star g\|_{\dot B^{s_0}_{p,\infty}} = \max\Big(\max_{|j|\le N-1} 2^{js_0}\|\phi_j \star g\|_p, \max_{j=N,N+1} 2^{js_0}\|\phi_j \star \Phi_N \star g\|_p, \max_{j=-N,-N-1} 2^{js_0}\|\phi_j \star \Phi_N \star g\|_p\Big).$$

Therefore, by using the same argument as in the case $1 \le \sigma < \infty$, we get

$$\|g\|_{\dot U^{s_0}_{p,\beta,\infty}} \le C\|g\|_X \text{ for all } g \in X.$$

This proves Theorem 2.5 (ii). □

4 Proof of Theorem 2.9

In order to prove Theorem 2.9, we need bilinear estimates which are related to Leibniz' rule. Therefore, we first recall the following two lemmata.

Lemma 4.1 ([13], Proposition 2.2). *Let $1 \le p, q \le \infty$, $s_0 > 0$, $\alpha > 0$ and $\beta > 0$. Moreover, assume that $1 \le p_1, p_2, p_3, p_4 \le \infty$ satisfy $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$. Then, there exists a constant $C(n, s_0, \alpha, \beta) > 0$ such that*

$$\|f \cdot g\|_{B^{s_0}_{p,q}} \le C\left(\|f\|_{B^{s_0+\alpha}_{p_1,q}}\|g\|_{B^{-\alpha}_{p_2,\infty}} + \|f\|_{B^{-\beta}_{p_3,\infty}}\|g\|_{B^{s_0+\beta}_{p_4,q}}\right) \tag{4.1}$$

for all $f \in B^{s_0+\alpha}_{p_1,q} \cap B^{-\beta}_{p_3,\infty}$ and $g \in B^{-\alpha}_{p_2,\infty} \cap B^{s_0+\beta}_{p_4,q}$.

Lemma 4.2 ([18], Lemma 1). *Let $1 < p < \infty$ and Let $\alpha, \beta \in \mathbb{N}^n$. Then, there exists a constant $C(n, p, \alpha, \beta) > 0$ such that*

$$\|\partial^\alpha f \cdot \partial^\beta g\|_p \le C\left(\|f\|_{BMO}\|(-\Delta)^{\frac{|\alpha|+|\beta|}{2}} g\|_p + \|(-\Delta)^{\frac{|\alpha|+|\beta|}{2}} f\|_p\|g\|_{BMO}\right) \tag{4.2}$$

for all $f, g \in BMO \cap W^{|\alpha|+|\beta|,p}$.

We are now in a position to prove Theorem 2.9 and follow arguments given by Kozono-Ogawa-Taniuchi [16], Kozono-Shimada [17], Kozono-Taniuchi [18] and the author [12].

Proof of Theorem 2.9. (i) It is well-known that the local existence time $T_\star$ of the strong solution to (N-S) can be estimated from below as

$$T_\star \ge \frac{C(n, s)}{\|u_0\|_{H^s}^{\frac{2}{s-(n/2-1)}}},$$

see *e.g.* [10] and [14]. Hence by the standard argument of continuation of local solutions, it suffices to establish the following *a priori* estimate:

$$\sup_{\varepsilon_0 \le t < T}\|u(t)\|_{H^{[s]+1}} \le C\left(n, s, \alpha, T, \|u(\varepsilon_0)\|_{H^{[s]+1}}, \int_{\varepsilon_0}^{T}\|u(\tau)\|^\theta_{\dot{U}^{-\alpha}_{\infty,\infty,\theta}}\,\mathrm{d}\tau\right) \tag{4.3}$$

for some $\varepsilon_0 \in (0, T)$, where $[\cdot]$ denotes the Gauß symbol.

Applying ∂^k with $|k| = 0, 1, \cdots, [s] + 1$ to (N-S), we have

$$\partial_t v_k - \Delta v_k + \nabla q_k = F_k, \tag{4.4}$$

where $v_k := \partial^k u$, $q_k := \partial^k \pi$ and $F_k := -\partial^k(u \cdot \nabla u) = -\partial^k \nabla \cdot u \otimes u$. Taking the inner product in L^2 between (4.4) and $2v_k$, and then integrating the resulting identity on the time interval (ε_0, t), we obtain

$$\|v_k(t)\|_2^2 + 2\int_{\varepsilon_0}^{t}\|\nabla v_k\|_2^2\,\mathrm{d}\tau \le \|v_k(\varepsilon_0)\|_2^2 + 2\int_{\varepsilon_0}^{t}|(F_k, v_k)|\,\mathrm{d}\tau, \quad \varepsilon_0 \le t < T, \tag{4.5}$$

where

$$|(F_k, v_k)| = |((-\Delta)^{-\frac{\alpha}{2}}\partial^k \nabla \cdot u \otimes u, (-\Delta)^{\frac{\alpha}{2}} v_k)| \le C\|u \otimes u\|_{\dot{B}^{1+|k|-\alpha}_{2,2}}\|v_k\|_{\dot{H}^\alpha}.$$

By the bilinear estimate Lemma 4.1 (4.1) with $p = q = 2$, $p_1 = p_4 = 2$, $p_2 = p_3 = \infty$, $s_0 = 1 + |k| - \alpha$, $\beta = \alpha$, it follows that

$$\|u \otimes u\|_{\dot{B}^{1+|k|-\alpha}_{2,2}} \le C\|u\|_{\dot{B}^{-\alpha}_{\infty,\infty}}\|u\|_{\dot{B}^{1+|k|}_{2,2}}.$$

Together with an interpolation inequality applied to $\|v_k\|_{\dot{H}^\alpha}$ we conclude from Young's inequality that

$$\begin{aligned}
|(F_k, v_k)| &\le C\|u\|_{\dot{B}^{-\alpha}_{\infty,\infty}}\|u\|_{\dot{H}^{1+|k|}}\|v_k\|^\alpha_{\dot{H}^1}\|v_k\|_2^{1-\alpha} \\
&\le C\|u\|_{\dot{B}^{-\alpha}_{\infty,\infty}}\|\nabla v_k\|_2^{1+\alpha}\|v_k\|_2^{1-\alpha} \\
&\le C\|u\|^\theta_{\dot{B}^{-\alpha}_{\infty,\infty}}\|v_k\|_2^2 + \frac{1+\alpha}{2}\|\nabla v_k\|_2^2,
\end{aligned} \tag{4.6}$$

where $\theta = \frac{2}{1-\alpha}$, C depends on n, s, α. Inserting (4.6) to the right-hand side of (4.5), summing for $|k| = 0, 1, \cdots, [s]+1$, and absorbing the terms $\|\nabla v_k\|_2^2$ from the right-hand side by the left-hand side, we obtain that

$$\|u(t)\|_{H^{[s]+1}}^2 \le \|u(\varepsilon_0)\|_{H^{[s]+1}}^2 + C\int_{\varepsilon_0}^{t} \|u(\tau)\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\theta} \|u(\tau)\|_{H^{[s]+1}}^2 \, d\tau,$$

for all $\varepsilon_0 \le t < T$. By using Gronwall's inequality, we get

$$\|u(t)\|_{H^{[s]+1}} \le \|u(\varepsilon_0)\|_{H^{[s]+1}} \exp\left(C\int_{\varepsilon_0}^{t} \|u(\tau)\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\theta} \, d\tau \right). \tag{4.7}$$

Now, applying the logarithmic interpolation inequality (2.6) with $s_0 = -\alpha$, $s_1 = -n/2\ (\le -1)$, $s_2 = s - n/2\ (> -\alpha)$, $\beta = 1/\theta$, $p = \sigma = \infty$ to $f = u(\tau)$, it follows that

$$\|u(\tau)\|_{\dot{B}^{-\alpha}_{\infty,\infty}} \le C\left(1 + \|u(\tau)\|_{\dot{U}^{-\alpha}_{\infty,1/\theta,\infty}} \log^{\frac{1}{\theta}}\left(e + \|u(\tau)\|_{\dot{B}^{-n/2}_{\infty,\infty}\cap\dot{B}^{s-n/2}_{\infty,\infty}}\right)\right). \tag{4.8}$$

By the embeddings $\dot{B}^0_{2,\infty} \subset \dot{B}^{-n/2}_{\infty,\infty}$, $\dot{B}^s_{2,\infty} \subset \dot{B}^{s-n/2}_{\infty,\infty}$ and $H^s \subset B^s_{2,\infty} = L^2 \cap \dot{B}^s_{2,\infty} \subset \dot{B}^0_{2,\infty} \cap \dot{B}^s_{2,\infty}$, we have

$$\|u(\tau)\|_{\dot{B}^{-n/2}_{\infty,\infty}\cap\dot{B}^{s-n/2}_{\infty,\infty}} \le C\|u(\tau)\|_{\dot{B}^0_{2,\infty}\cap\dot{B}^s_{2,\infty}} \le C\|u(\tau)\|_{B^s_{2,\infty}} \le C\|u(\tau)\|_{H^s}. \tag{4.9}$$

Hence, by (4.7), (4.8) and (4.9), it holds that

$$\|u(t)\|_{H^{[s]+1}} \le \|u(\varepsilon_0)\|_{H^{[s]+1}} \exp\left(C\int_{\varepsilon_0}^{t} \left(1 + \|u(\tau)\|_{\dot{U}^{-\alpha}_{\infty,1/\theta,\infty}}^{\theta} \log(e + \|u(\tau)\|_{H^{[s]+1}})\right) d\tau \right),$$

where $C = C(n, s, \alpha)$. Therefore, with $g(t) \equiv \log(e + \|u(t)\|_{H^{[s]+1}})$, we obtain

$$g(t) \le g(\varepsilon_0) + C\int_{\varepsilon_0}^{t} \left(1 + \|u(\tau)\|_{\dot{U}^{-\alpha}_{\infty,1/\theta,\infty}}^{\theta} g(\tau)\right) d\tau.$$

Then Gronwall's inequality implies that

$$g(t) \le g(\varepsilon_0) \exp\left(C\int_{\varepsilon_0}^{t} \left(1 + \|u(\tau)\|_{\dot{U}^{-\alpha}_{\infty,1/\theta,\infty}}^{\theta}\right) d\tau \right)$$

for all $\varepsilon_0 \le t < T$. Thus, we get the estimate (4.3) in the form

$$\sup_{\varepsilon_0 \le t < T} \|u(t)\|_{H^{[s]+1}} \le \left(e + \|u(\varepsilon_0)\|_{H^{[s]+1}}\right)^{\exp\left(CT + C\int_{\varepsilon_0}^{T} \|u(\tau)\|_{\dot{U}^{-\alpha}_{\infty,1/\theta,\infty}}^{\theta} d\tau\right)}.$$

(ii) By the same argument as in the above proof, it suffices to establish the following *a priori* estimate:

$$\sup_{\varepsilon_0 \le t < T} \|u(t)\|_{H^{[s]+1}} \le C\left(n, s, T, \|u(\varepsilon_0)\|_{H^{[s]+1}}, \int_{\varepsilon_0}^{T} \|u(\tau)\|_{\dot{V}^0_{\infty,\infty,2}}^2 \, d\tau\right) \tag{4.10}$$

for some $\varepsilon_0 \in (0, T)$.

Applying ∂^k with $|k| = 0, 1, \cdots, [s]+1$ to (N-S), we have

$$\partial_t v_k - \Delta v_k + u \cdot \nabla v_k + \nabla q_k = G_k, \tag{4.11}$$

where $v_k := \partial^k u$, $q_k := \partial^k \pi$ and $G_k := -\sum_{l \le k, |l| \le |k|-1} \binom{k}{l} \partial^{k-l} u \cdot \nabla(\partial^l u)$. Testing (4.11) with v_k and integrating the resulting identity on the time interval (ε_0, t), we obtain

$$\|v_k(t)\|_2^2 + 2\int_{\varepsilon_0}^{t} \|\nabla v_k\|_2^2 \, d\tau \le \|v_k(\varepsilon_0)\|_2^2 + 2\int_{\varepsilon_0}^{t} |(G_k, v_k)| \, d\tau, \quad \varepsilon_0 \le t < T. \tag{4.12}$$

Now the bilinear estimate (4.2) with $p = 2,\ |\alpha| = |k| - |l|,\ |\beta| = |l| + 1$, implies that

$$\|G_k\|_2 \le C\|u\|_{BMO}\|(-\Delta)^{\frac{|k|+1}{2}}u\|_2. \tag{4.13}$$

From (4.13) and Young's inequality we conclude that

$$\begin{aligned}|(G_k, v_k)| \le \|G_k\|_2\|v_k\|_2 &\le C\|u\|_{BMO}\|(-\Delta)^{\frac{|k|+1}{2}}u\|_2\|v_k\|_2\\ &\le C\|u\|_{BMO}^2\|v_k\|_2^2 + \frac{1}{2}\|\nabla v_k\|_2^2,\end{aligned} \tag{4.14}$$

with $C = C(n, s)$. Inserting (4.14) to the right-hand side of (4.12) and summing for $|k| = 0, 1, \cdots, [s] + 1$, we obtain that

$$\|u(t)\|_{H^{[s]+1}}^2 \le \|u(\varepsilon_0)\|_{H^{[s]+1}}^2 + C\int_{\varepsilon_0}^{t}\|u(\tau)\|_{BMO}^2\|u(\tau)\|_{H^{[s]+1}}^2\,\mathrm{d}\tau,$$

for all $\varepsilon_0 \le t < T$. By using Gronwall's inequality and then the continuous embedding $\dot{B}^0_{\infty,2} \subset BMO$, we get

$$\begin{aligned}\|u(t)\|_{H^{[s]+1}} &\le \|u(\varepsilon_0)\|_{H^{[s]+1}}\exp\left(C\int_{\varepsilon_0}^{t}\|u(\tau)\|_{BMO}^2\,\mathrm{d}\tau\right)\\ &\le \|u(\varepsilon_0)\|_{H^{[s]+1}}\exp\left(C\int_{\varepsilon_0}^{t}\|u(\tau)\|_{\dot{B}^0_{\infty,2}}^2\,\mathrm{d}\tau\right)\end{aligned} \tag{4.15}$$

Now, by applying the logarithmic interpolation inequality (2.6) with $s_1 = -n/2 < s_0 = 0 < s_2 = s - n/2$, $\beta = 1/2,\ p = \infty$ and $\sigma = 2$ to $f = u(\tau)$, it follows that

$$\|u(\tau)\|_{\dot{B}^0_{\infty,2}} \le C\left(1 + \|u(\tau)\|_{\dot{V}^0_{\infty,\infty,2}}\log^{\frac{1}{2}}\left(e + \|u(\tau)\|_{\dot{B}^{-n/2}_{\infty,\infty}\cap\dot{B}^{s-n/2}_{\infty,\infty}}\right)\right). \tag{4.16}$$

Here, we note that $\dot{U}^0_{\infty,1/2,2} = \dot{V}^0_{\infty,\infty,2}$ holds due to Proposition 2.3 (iii). Hence, combining (4.15), (4.16) and (4.9), it holds that

$$\|u(t)\|_{H^{[s]+1}} \le \|u(\varepsilon_0)\|_{H^{[s]+1}}\exp\left(C\int_{\varepsilon_0}^{t}\left(1 + \|u(\tau)\|_{\dot{V}^0_{\infty,\infty,2}}^2\log(e + \|u(\tau)\|_{H^{[s]+1}})\right)\mathrm{d}\tau\right),$$

where $C = C(n, s)$. Therefore, letting $g(t) \equiv \log(e + \|u(t)\|_{H^{[s]+1}})$, we obtain

$$g(t) \le g(\varepsilon_0) + C\int_{\varepsilon_0}^{t}\left(1 + \|u(\tau)\|_{\dot{V}^0_{\infty,\infty,2}}^2 g(\tau)\right)\mathrm{d}\tau,$$

which by Gronwall's inequality implies that

$$g(t) \le g(\varepsilon_0)\exp\left(C\int_{\varepsilon_0}^{t}(1 + \|u(\tau)\|_{\dot{V}^0_{\infty,\infty,2}}^2)\,\mathrm{d}\tau\right)$$

for all $\varepsilon_0 \le t < T$. Thus, we get the estimate

$$\sup_{\varepsilon_0\le t<T}\|u(t)\|_{H^{[s]+1}} \le \left(e + \|u(\varepsilon_0)\|_{H^{[s]+1}}\right)^{\exp\left(CT + C\int_{\varepsilon_0}^{T}\|u(\tau)\|_{\dot{V}^0_{\infty,\infty,2}}^2\,\mathrm{d}\tau\right)},$$

which is the desired estimate (4.10). □

Acknowledgement. We acknowledge support by the German Research Foundation and the Open Access Publishing Fund of Technical University of Darmstadt.

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