DE GRUYTER Adv. Nonlinear Anal. 2021; 10: 1061–1070

Research Article Open Access

Rahul Shukla and Andrzej Wiśnicki*

Iterative methods for monotone nonexpansive mappings in uniformly convex spaces

https://doi.org/10.1515/anona-2020-0170
Received August 17, 2020; accepted January 16, 2021.

Abstract: We show the nonlinear ergodic theorem for monotone 1-Lipschitz mappings in uniformly convex spaces: if C is a bounded closed convex subset of an ordered uniformly convex space $(X, \|\cdot\|, \preceq)$, $T : C \to C$ a monotone 1-Lipschitz mapping and $x \preceq T(x)$, then the sequence of averages $\frac{1}{n}\sum_{i=0}^{n-1} T^i(x)$ converges weakly to a fixed point of T. As a consequence, it is shown that the sequence of Picard's iteration $\{T^n(x)\}$ also converges weakly to a fixed point of T. The results are new even in a Hilbert space. The Krasnosel'skiĭ-Mann and the Halpern iteration schemes are studied as well.

Keywords: Monotone mapping; nonexpansive mapping; fixed point; ergodic theorem; Picard iteration; ordered Banach space

MSC: 47J26; 46B20; 47H07; 54F05

1 Introduction

Let C be a bounded closed convex subset of a Banach space X. A mapping $T : C \to X$ is said to be nonexpansive if for each pair of elements $x, y \in C$, we have

$$\left\|T(x) - T(y)\right\| \leq \|x - y\| .$$

A mappping T is accretive if for each $x, y \in C$ and $\lambda \geq 0$,

$$\|x - y\| \leq \left\|x - y + \lambda(T(x) - T(y))\right\| .$$

The study of fixed points of nonexpansive mappings and null points of accretive mappings extends the classical theory of successive approximations for strict contractions based on the Banach contraction principle. It is well-known that if $T : C \to C$ is a strict contraction, the Picard sequence $x_{n+1} = T(x_n)$ converges strongly to the unique fixed point of T. In the case of nonexpansive mappings, the Picard sequence need not converge nor need the fixed point be unique if it exists. However, if the space X is sufficiently smooth (for example, a Hilbert space), for any $\alpha \in (0, 1)$ and $x_1 \in C$ the iterative scheme defined by

$$x_{n+1} = (1 - \alpha)x_n + \alpha\, T(x_n)$$

converges weakly to a fixed point of a nonexpansive mapping $T : C \to C$. This, and a more general scheme, $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(x_n)$ is now called the Krasnosel'skiĭ–Mann (or Mann) formula for finding fixed points of nonexpansive mappings.

Rahul Shukla, Department of Mathematics & Applied Mathematics, University of Johannesburg, Kingsway Campus, Auckland Park 2006, South Africa, E-mail: rshukla@uj.ac.za; rshukla.vnit@gmail.com
***Corresponding Author: Andrzej Wiśnicki,** Department of Mathematics, Pedagogical University of Krakow, PL-30-084 Cracow, Poland, E-mail: andrzej.wisnicki@up.krakow.pl

Another iterative algorithm for nonexpansive mappings was proposed by B. Halpern [1]: select $u \in C$ and define

$$x_{n+1} = (1-\alpha_n)u + \alpha_n T(x_n).$$

It turns out that if the space X is sufficiently smooth and parameters $\{\alpha_n\}$ are appropriately chosen, the above sequence converges strongly to a fixed point of T which is the projection of the initial u to the set of fixed points of T.

The third kind of the approximation procedure that, in a sense, generalizes the Krasnosel'skiĭ-Mann method comes from the ergodic theory: the sequence of averages

$$x_n = \frac{1}{n}\sum_{i=0}^{n-1} T^i(x)$$

converges weakly to a fixed point of T. This is the nonlinear ergodic theorem, proved in a Hilbert space by J.B. Baillon [2].

A wide variety of well-known algorithms applied in practical and theoretical problems are based on the above iterative schemes and their perturbative modifications. These include the proximal point algorithm, the generalized Yosida approximation of the maximal monotone operator, the algorithms for solving split feasibility problems used frequently in signal processing and image reconstruction, and many others as described, for example, in [3–7].

The objective of this paper is to develop similar methods for mappings $T : C \to C$ that are monotone nonexpansive, that is,

$$\left\|T(x) - T(y)\right\| \le \|x-y\| \text{ and } T(x) \preceq T(y)$$

for $x, y \in C$ such that $x \preceq y$. Here $\preceq$ is a partial order on C that corresponds somehow to the norm of X. Very often, such ordering is given by a cone in X with certain properties. In this paper, following the recent works of M.A. Khamsi and his collaborators, we take a more general view and consider partial orders on X such that the order intervals are closed and, moreover, the partial order is compatible with the linear structure of X (for details, see Section 2). It turns out that many results in the theory of nonexpansive mappings are valid for monotone nonexpansive mappings too, despite the fact that those need not even be continuous. The interplay between the order and metrical structure of the space is very fruitful as observed, for example, by A. Ran and M. Reurings who applied a generalization of the Banach contraction principle to matrix equations (see [8]).

The existential part of the theory of monotone nonexpansive mappings was recently studied in [9] (see also [10–12]). It is shown there that if X is a topological space with a partial order for which order intervals are compact, then every directed subset of X has a supremum. It allows one to apply methods based on the Knaster–Tarski (Abian–Brown) theorem.

In this paper we concentrate on the iterative procedures for finding fixed points. One of the long-standing problems in the theory of nonexpansive iterations is whether the Krasnosel'skiĭ-Mann and the Halpern methods, as well as the nonlinear ergodic theorem, hold true in all uniformly convex Banach spaces. The usual requirements here are some kind of smoothness like the Fréchet or Gateaux differentiable norm, or possessing a weakly continuous duality map (see, eg., [13–17]).

We prove that in the case of monotone nonexpansive mappings, uniform convexity is sufficient to obtain weak convergence of the above algorithms to a fixed point of T. The paper is organized as follows. In Section 2 we show the demiclosedness principle for monotone nonexpansive mappings which is one of the main tools in our analysis. Sections 3 and 4 are devoted, respectively, to Krasnosel'skiĭ-Mann's and Halpern's iterations. Our main result–the nonlinear ergodic theorem–is proved in Section 5: if C is a bounded closed convex subset of an ordered uniformly convex space $(X, \|\cdot\|, \preceq)$, $T : C \to C$ a monotone nonexpansive mapping and $x \preceq T(x)$, then the sequence of averages $\frac{1}{n}\sum_{i=0}^{n-1} T^i(x)$ converges weakly to a fixed point of T. As a consequence, we show that the Picard iteration $\{T^n(x)\}$ also converges weakly to a fixed point of T. Our results, except the Krasnosel'skiĭ-Mann iteration (see [11, Theorem 3.3]), appear to be new even in a Hilbert space.

2 Demiclosedness principle in uniformly convex spaces

Let X be a Banach space with a partial order $\preceq$ compatible with the linear structure of X, that is,

$$\begin{aligned} x &\preceq y \text{ implies } x+z \preceq y+z, \\ x &\preceq y \text{ implies } \lambda x \preceq \lambda y \end{aligned}$$

for every $x, y, z \in X$ and $\lambda \geq 0$. It follows that all order intervals $[x, \rightarrow] = \{z \in X : x \preceq z\}$ and $[\leftarrow, y] = \{z \in X : z \preceq y\}$ are convex. Moreover, we will assume that each $[x, \rightarrow]$ and $[\leftarrow, y]$ is closed. We will say that $(X, \|\cdot\|, \preceq)$ is an ordered Banach space.

A sequence $\{x_n\}$ is said to be an approximate fixed point sequence (a.f.p.s. for short) for a mapping T if $\left\|T(x_n) - x_n\right\| \to 0$ as $n \to \infty$, see [18]. A sequence $\{x_n\}$ is monotone if $x_1 \preceq x_2 \preceq x_3 \preceq \cdots$.
The following lemma can be proved in much the same way as is the case of nonexpansive mappings (see, e.g., [19, Prop. 10.1]).

Lemma 1. *Suppose C is a bounded convex subset of an ordered uniformly convex Banach space $(X, \|\cdot\|, \preceq)$ and $T : C \to X$ is monotone nonexpansive. If $\{u_n\}$ and $\{v_n\}$ are approximate fixed point sequences such that $u_n \preceq v_n$ for each $n \in \mathbb{N}$, then $\{w_n\} = \{\frac{1}{2}(u_n + v_n)\}$ is an approximate fixed point sequence too.*

Proof. Suppose the assertion of the lemma is false. Then there exist sequences $\{u_n\}, \{v_n\}$ satisfying $\lim_{n\to\infty} \|u_n - T(u_n)\| = 0$, $\lim_{n\to\infty} \|v_n - T(v_n)\| = 0$, $u_n \preceq v_n$, such that $\|w_n - T(w_n)\| \geq \varepsilon$ for some $\varepsilon > 0$ and every $n \in \mathbb{N}$. Since $w_n = \frac{u_n+v_n}{2}$, by convexity of the order interval $[u_n, v_n]$, we have $u_n \preceq w_n \preceq v_n$ for each $n \in \mathbb{N}$. We can assume by passing to a subsequence that

$$\lim_{n\to\infty} \|u_n - v_n\| = 2r > 0.$$

It follows that

$$\lim_{n\to\infty} \|u_n - w_n\| = \lim_{n\to\infty} \|v_n - w_n\| = r.$$

Choose $s > 0$ such that $s < \frac{\varepsilon}{r}$. Hence, for sufficiently large n,

$$s < \frac{\varepsilon}{\|u_n - T(u_n)\| + \|u_n - w_n\|}$$

and

$$s < \frac{\varepsilon}{\|v_n - T(v_n)\| + \|v_n - w_n\|}.$$

By the triangle inequality and monotone nonexpansivity of T, we get

$$\begin{aligned} \|u_n - T(w_n)\| &\leq \|u_n - T(u_n)\| + \|T(u_n) - T(w_n)\| \\ &\leq \|u_n - T(u_n)\| + \|u_n - w_n\|. \end{aligned}$$

Similarly,

$$\|v_n - T(w_n)\| \leq \|v_n - T(v_n)\| + \|v_n - w_n\|.$$

By the uniform convexity of X, we have

$$\begin{aligned} \left\|u_n - \frac{1}{2}(w_n + T(w_n))\right\| &\leq \left(1 - \delta\left(\frac{\varepsilon}{\|u_n - T(u_n)\| + \|u_n - w_n\|}\right)\right) \\ &\quad (\|u_n - T(u_n)\| + \|u_n - w_n\|) \\ &\leq (1 - \delta(s))(\|u_n - T(u_n)\| + \|u_n - w_n\|). \end{aligned} \tag{2.1}$$

Similarly,

$$\left\|v_n - \frac{1}{2}(w_n + T(w_n))\right\| \leq \left(1 - \delta\left(\frac{\varepsilon}{\|v_n - T(v_n)\| + \|v_n - w_n\|}\right)\right)$$

$$\begin{aligned} & (\|v_n - T(v_n)\| + \|v_n - w_n\|) \\ \leq\ & (1 - \delta(s))(\|v_n - T(v_n)\| + \|v_n - w_n\|). \end{aligned} \tag{2.2}$$

By the triangle inequality, (2.1) and (2.2), we obtain

$$\begin{aligned} \|u_n - v_n\| &\leq \left\| u_n - \frac{1}{2}(w_n + T(w_n)) \right\| + \left\| v_n - \frac{1}{2}(w_n + T(w_n)) \right\| \\ &\leq (1 - \delta(s))\{(\|u_n - T(u_n)\| + \|u_n - w_n\|) + (\|v_n - T(v_n)\| + \|v_n - w_n\|)\}. \end{aligned}$$

Letting $n \to \infty$, we get $2r \leq 2r(1 - \delta(s))$, a contradiction and this completes the proof. □

In what follows we will use the following observation (see [20, Lemma 3.1]). Suppose that $\{x_n\}$ is a monotone sequence that has a cluster point, i.e., there exists a subsequence $\{x_{n_j}\}$ that converges to g (with respect to the strong or weak topology). Since the order intervals are (weakly) closed, we have $g \in [x_n, \to)$ for each n, i.e., g is an upper bound for $\{x_n\}$. If g_1 is another upper bound for $\{x_n\}$, then $x_n \in (\leftarrow, g_1]$ for each n, and hence $g \preceq g_1$. It follows that $\{x_n\}$ converges to $g = \sup\{x_n\}$.

The following result is a basic tool in our consideration.

Theorem 1. *Suppose C is a weakly compact convex subset of an ordered uniformly convex Banach space $(X, \|\cdot\|, \preceq)$ and $\{x_n\}$ is a monotone a.f.p.s. for a monotone nonexpansive $T : C \to X$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Notice that on account of the above observation, $\{x_n\}$ converges weakly to a point x in C. Define

$$r = \inf \left\{ \liminf_{n\to\infty} \|u_n\| : \{u_n\} \text{ is a monotone a.f.p.s. for } T \text{ in } C \text{ that converges weakly to } x \right\}.$$

Since C is bounded, $r < \infty$. If $r = 0$, then $x = 0$ and $T(x) = x$. Thus, we suppose $r > 0$. From the definition of r, it is clear that there exists a monotone approximate fixed point sequence $\{v_n\}$ for T converging weakly to x such that $\|v_n\| \leq r + \frac{1}{n}$ for each n. If $\{v_n\}$ converges to x in norm, then $T(x) = x$. Therefore, we may assume that there exists $\varepsilon > 0$ and a subsequence $\{v_{n_j}\}$ of $\{v_n\}$ such that $\|v_{n_j} - v_{n_{j+1}}\| \geq \varepsilon$ for each $j = 1, 2, \ldots$. Let

$$w_{n_j} = \frac{1}{2}(v_{n_j} + v_{n_{j+1}}).$$

Let s be a real number such that $s < \frac{\varepsilon}{r}$, then $s < \frac{\varepsilon}{\|v_{n_j}\|}$ for sufficiently large j. By the definition of uniform convexity of X, for sufficiently large j, we have

$$\|w_{n_j}\| \leq \left(r + \frac{1}{n_j} \right)(1 - \delta(s)).$$

Then

$$\limsup_{n\to\infty} \|w_{n_j}\| \leq r(1 - \delta(s)) < r.$$

It is not difficult to see that $\{w_{n_j}\}$ is monotone and converges weakly to x. By Lemma 1, $\{w_{n_j}\}$ is an approximate fixed point sequence that contradicts the definition of r. □

3 Krasnosel'skiĭ-Mann iteration

Let C be a convex subset of a Banach space X and $T : C \to C$ a monotone nonexpansive mapping. The following iteration process is known as the Krasnosel'skiĭ-Mann iteration process (see [21, 22]):

$$\begin{cases} x_1 \in C \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T(x_n) \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ is a sequence in $[a, b]$ with $a, b \in (0, 1)$.

Lemma 2. [20]. *Let C be a bounded closed convex subset of an ordered uniformly convex space $(X, \|\cdot\|, \preceq)$ and $T : C \to C$ a monotone nonexpansive mapping. Suppose that $\{x_n\}$ is a sequence defined by* (3.1) *and $x_1 \preceq T(x_1)$. Then $\lim\limits_{n\to\infty} \|x_n - T(x_n)\| = 0$.*

Theorem 2. *Let C be a nonempty bounded closed convex subset of an ordered uniformly convex space $(X, \|\cdot\|, \preceq)$ and $T : C \to C$ a monotone nonexpansive mapping. Let $\{x_n\}$ be a sequence defined by* (3.1) *and $x_1 \preceq T(x_1)$. Then $\{x_n\}$ converges weakly to a fixed point of T. If, moreover, the range of C under T is contained in a compact subset of X, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. It is not difficult to see that the sequence $\{x_n\}$ is monotone. It follows from Lemma 2 that $\lim\limits_{n\to\infty} \|x_n - T(x_n)\| = 0$ and thus Theorem 1 yields that $\{x_n\}$ converges weakly to a fixed point of T. If the range of C under T is contained in a compact set then there exists a strongly convergent subsequence $\{T(x_{n_j})\}$ of $\{T(x_n)\}$ and it follows from the observation made before Theorem 1, $\{T(x_n)\}$ is strongly convergent. Thus $\{x_n\}$ is also strongly convergent. □

The above theorem should be compared with Theorem 2 in [23], which asserts the same conclusion in a uniformly convex space with a Fréchet differentiable norm without the assumption about monotonicity of the mapping T (it is required that $\sum\limits_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ there). In the case of monotone nonexpansive mappings we can drop the assumption about the Fréchet differentiability of the norm, thus uniform convexity is sufficient to obtain the weak convergence of Krasnosel'skiĭ-Mann iteration (3) in this case.

Theorem 2 complements [11, Theorem 3.3], where a similar result was proved in ordered Banach spaces with the Opial property (see also [24–26] for some generalizations).

4 Halpern iteration

Let C be a convex subset of a Banach space X and $T : C \to C$ a nonexpansive mapping. The following iteration process is known as the Halpern iteration process [1]:

$$\begin{cases} u, x_1 \in C \\ x_{n+1} = \alpha_n u + (1-\alpha_n)T(x_n) \end{cases} \tag{4.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. The first result extending Halpern's insights outside Hilbert space is Corollary 2 of the early paper by S. Reich [27]. A further extension was given by H.-K. Xu [7, Theorem 3.1], who proved that if C is a bounded closed convex subset of a uniformly smooth Banach space then the sequence defined by Halpern's iteration (4.1) converges strongly to a fixed point of T provided

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum\nolimits_{n=1}^{\infty} \alpha_n = \infty, \quad \lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n} = 0. \tag{4.2}$$

Using the tools of Section 2 we can prove a counterpart of Xu's result concerning weak convergence of Halpern's iterations for monotone nonexpansive mappings in ordered uniformly convex spaces.

Lemma 3. *Let C be a nonempty convex subset of an ordered uniformly convex space $(X, \|\cdot\|, \preceq)$ and $T : C \to C$ a monotone mapping. Let $\{x_n\}$ be a sequence defined by Halpern iteration* (4.1) *such that $x_1 \preceq u \preceq T(x_1)$. If the sequence $\{\alpha_n\}$ is decreasing, then*

$$u \preceq T(x_{n+1}) \ \textit{and}\ \ x_n \preceq x_{n+1} \text{ for all } n \in \mathbb{N}. \tag{4.3}$$

Proof. We shall use induction to prove our conclusion. By assumption, $x_1 \preceq u \preceq T(x_1)$. Since a partial order is compatible with the linear structure of X, we have

$$x_2 = \alpha_1 u + (1-\alpha_1)T(x_1) \succeq \alpha_1 x_1 + (1-\alpha_1)x_1 = x_1.$$

Since T is monotone, we have $u \preceq T(x_1) \preceq T(x_2)$. Thus (4.3) is true for $n = 1$. Now suppose it is true for any $k \in \mathbb{N}$, that is, $u \preceq T(x_{k+1})$ and $x_k \preceq x_{k+1}$. It follows that $T(x_k) \preceq T(x_{k+1})$. By assumption, $\{\alpha_n\}$ is decreasing and hence

$$\begin{aligned} x_{k+2} &= \alpha_{k+1}u + (1-\alpha_{k+1})T(x_{k+1}) \\ &= \alpha_{k+1}u + (\alpha_k - \alpha_{k+1})T(x_{k+1}) + (1-\alpha_k)T(x_{k+1}) \\ &\succeq \alpha_{k+1}u + (\alpha_k - \alpha_{k+1})u + (1-\alpha_k)T(x_k) = x_{k+1}. \end{aligned}$$

Since T is monotone, $u \preceq T(x_{k+1}) \preceq T(x_{k+2})$. Therefore, (4.3) is true for all $n \in \mathbb{N}$. □

Theorem 3. *Let C be a bounded closed convex subset of an ordered uniformly convex space $(X, \|\cdot\|, \preceq)$ and $T : C \to C$ a monotone nonexpansive mapping. Let $\{x_n\}$ be a sequence defined by Halpern iteration* (4.1) *satisfying conditions* (4.2) *such that $x_1 \preceq u \preceq T(x_1)$. If the sequence $\{\alpha_n\}$ is decreasing then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. We show that $\|T(x_n) - x_n\| \to 0$ as $n \to \infty$. From (4.1) and (4.2), we get

$$\lim_{n\to\infty} \|x_{n+1} - T(x_n)\| = \lim_{n\to\infty} \alpha_n \|u - T(x_n)\| = 0. \tag{4.4}$$

It follows from Lemma 3 that the sequence $\{x_n\}$ is monotone and hence

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(\alpha_n - \alpha_{n-1})(u - T(x_{n-1})) + (1-\alpha_n)(T(x_n) - T(x_{n-1}))\| \\ &\le (1-\alpha_n)\|x_n - x_{n-1}\| + \alpha_n \frac{\alpha_{n-1} - \alpha_n}{\alpha_n} \operatorname{diam} C. \end{aligned}$$

By assumption, $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n} = 0$, and it follows from [7, Lemma 2.5] that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Thus

$$\|T(x_n) - x_n\| \le \|T(x_n) - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty.$$

Theorem 1 now shows that $\{x_n\}$ converges weakly to a fixed point of T. □

Notice that our result covers a natural iteration scheme

$$\begin{cases} x_1 \in C, \quad x_1 \preceq T(x_1), \\ x_{n+1} = \frac{1}{n}x_1 + (1-\frac{1}{n})T(x_n). \end{cases}$$

5 Nonlinear ergodic theory

A standard way to regularize a nonconvergent sequence $\{y_n\}$ is to consider its averages $x_n = \frac{1}{n}\sum_{i=1}^{n} y_i$. Baillon showed in [2] that if T is a nonexpansive mapping acting on a nonempty bounded closed convex subset of a Hilbert space then the sequence of averages

$$x_n = \frac{1}{n}\sum_{i=0}^{n-1} T^i(x)$$

converges weakly to a fixed point of T, see also [28–30]. This result, known as the nonlinear ergodic theorem, was generalized to L^p spaces by Baillon [31] and to uniformly convex spaces with a Fréchet differentiable norm by R.E. Bruck [32] and Reich [23, 33]. A long-standing problem is to drop the assumption about a Fréchet differentiability of the norm. In this section we are able to do it in the case of monotone nonexpansive mappings.

We will need the following variant of Lemma 1 (see also [34, Lemma 3.17]).

Lemma 4. *Suppose C is a bounded convex subset of an ordered uniformly convex space $(X, \|\cdot\|, \preceq)$ and $T : C \to X$ is monotone nonexpansive. If $\{u_n\}, \{v_n\}$ are sequences such that* $\lim_{n\to\infty} \sup_{m\ge1} \|u_{n+m} - T^m(u_n)\| = 0$,

$\lim_{n\to\infty} \sup_{m\geq 1} \|v_{n+m} - T^m(v_n)\| = 0$ *and* $u_n \preceq v_n$ *for each* $n \in \mathbb{N}$, *then* $\{w_n\} = \{\alpha u_n + \beta v_n\}$, *where* $\alpha, \beta \in (0,1)$, $\alpha + \beta = 1$, *satisfies the condition* $\lim_{n\to\infty} \sup_{m\geq 1} \|w_{n+m} - T^m(w_n)\| = 0$, *too.*

Proof. Since $u_n \preceq v_n$ for each $n \in \mathbb{N}$ and by convexity of the order interval $[u_n, v_n]$, we have $u_n \preceq \alpha u_n + \beta v_n \preceq v_n$. For a fixed $m \in \mathbb{N}$ and for each $n \in \mathbb{N}$, take

$$x_n = \frac{T^m(v_n) - T^m(\alpha u_n + \beta v_n)}{\alpha\|u_n - v_n\|}$$

and

$$y_n = \frac{T^m(\alpha u_n + \beta v_n) - T^m(u_n)}{\beta\|u_n - v_n\|}.$$

Since T is a monotone nonexpansive mapping, $\|x_n\|, \ \|y_n\| \leq 1$ for all $n \in \mathbb{N}$. By the definition of the modulus of convexity, we have

$$2\alpha\beta\delta(\|x-y\|) \leq 2\min\{\alpha,\beta\}\delta(\|x-y\|) \leq 1 - \|\alpha x + \beta y\|$$

for every $x, y \in C$ with $\|x\|, \ \|y\| \leq 1$. Therefore,

$$2\alpha\beta\delta\left(\frac{\|\alpha T^m(u_n) + \beta T^m(v_n) - T^m(\alpha u_n + \beta v_n)\|}{\alpha\beta\|u_n - v_n\|}\right) \leq 1 - \frac{\|T^m(u_n) - T^m(v_n)\|}{\|u_n - v_n\|}.$$

This implies

$$2\alpha\beta\|u_n - v_n\|\delta\left(\frac{\|\alpha T^m(u_n) + \beta T^m(v_n) - T^m(\alpha u_n + \beta v_n)\|}{\alpha\beta\|u_n - v_n\|}\right) \leq \|u_n - v_n\| - \|T^m(u_n) - T^m(v_n)\|.$$

Let $\Theta = \operatorname{diam} C$, noting that $\alpha\beta\|u_n - v_n\| \leq \frac{\Theta}{4}$ and $\frac{\delta(s)}{s}$ is nondecreasing [34, Proposition A.4.], from the above inequality, we get

$$\begin{aligned}\frac{\Theta}{2}\delta\left(\frac{4}{\Theta}\|\alpha T^m(u_n) + \beta T^m(v_n) - T^m(\alpha u_n + \beta v_n)\|\right) &\leq \|u_n - v_n\| \\ &\quad - \|T^m(u_n) - T^m(v_n)\|. \qquad (5.1)\end{aligned}$$

Choose a sequence of nonnegative numbers $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0$ and $\|u_{n+m} - T^m(u_n)\|, \ \|v_{n+m} - T^m(v_n)\| \leq \varepsilon_n$ for every $n, m \in \mathbb{N}$. Thus, for each n and m, we have

$$\begin{aligned}\|u_{n+m} - v_{n+m}\| &\leq \|u_{n+m} - T^m(u_n)\| + \|v_{n+m} - T^m(v_n)\| + \|T^m(v_n) - T^m(u_n)\| \\ &\leq 2\varepsilon_n + \|u_n - v_n\|.\end{aligned}$$

Thus, $\lim_{n\to\infty} \|u_n - v_n\|$ exists. Now, using (5.1), we obtain

$$\begin{aligned}\|w_{n+m} - T^m(w_n)\| &\leq \alpha\|u_{n+m} - T^m(u_n)\| + \beta\|v_{n+m} - T^m(v_n)\| + \|\alpha T^m(u_n) + \beta T^m(v_n) - T^m(w_n)\| \\ &\leq \varepsilon_n + \frac{\Theta}{4}\delta^{-1}\left(\frac{2}{\Theta}(\|u_n - v_n\| - \|T^m(u_n) - T^m(v_n)\|)\right) \\ &\leq \varepsilon_n + \frac{\Theta}{4}\delta^{-1}\left(\frac{2}{\Theta}(\|u_n - v_n\| - \|u_{n+m} - v_{n+m}\| + 2\varepsilon_n)\right).\end{aligned}$$

Therefore, $\|w_{n+m} - T^m(w_n)\| \to 0$ as $n \to \infty$ because $\lim_{n\to\infty} \|u_n - v_n\|$ exists. □

We are now in a position to prove the nonlinear ergodic theorem for monotone nonexpansive mappings in any ordered uniformly convex Banach space.

Theorem 4. *Let C be a bounded closed convex subset of an ordered uniformly convex space $(X, \|\cdot\|, \preceq)$ and $T : C \to C$ a monotone nonexpansive mapping. If $x \preceq T(x)$ then the sequence $\left\{\frac{1}{n}\sum_{i=0}^{n-1} T^i(x)\right\}$ converges weakly to a fixed point of T.*

Proof. For any integer $p \geq 1$, let

$$w_n^{(p)} = \frac{1}{p}\sum_{i=0}^{p-1} T^{i+n}(x).$$

Since a partial order is compatible with the linear structure of X, it is not difficult to see that

$$w_1^{(p)} \preceq w_2^{(p)} \preceq w_3^{(p)} \preceq \ldots. \tag{5.2}$$

for each $p \geq 1$. Moreover,

$$\begin{aligned} w_n^{(p)} &= \frac{1}{p}\sum_{i=0}^{p-1} T^{i+n}(x) = \frac{1}{p+1}\sum_{i=0}^{p-1} T^{i+n}(x) + \left(\frac{1}{p} - \frac{1}{p+1}\right)\sum_{i=0}^{p-1} T^{i+n}(x) \\ &\preceq \frac{1}{p+1}\sum_{i=0}^{p-1} T^{i+n}(x) + \frac{1}{p(p+1)}\sum_{i=0}^{p-1} T^{p+n}(x) = \frac{1}{p+1}\sum_{i=0}^{p} T^{i+n}(x) = w_n^{(p+1)} \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} w_n^{(p+1)} &= \left(\frac{1}{p} - \frac{1}{p+1}\right) pT^n(x) + \frac{1}{p+1}\sum_{i=1}^{p} T^{i+n}(x) \\ &\preceq \left(\frac{1}{p} - \frac{1}{p+1}\right)(T^{1+n}(x) + \ldots + T^{p+n}(x)) + \frac{1}{p+1}\sum_{i=1}^{p} T^{i+n}(x) \\ &= \frac{1}{p}\sum_{i=0}^{p-1} T^{i+1+n}(x) = w_{n+1}^{(p)} \end{aligned} \tag{5.4}$$

for every integer $n \geq 0, p \geq 1$. Hence the sequences $\{w_n^{(p)}\}_n$ converge weakly to some $w \in C$ if $n \to \infty$ for any integer $p \geq 1$. Fix a functional $f \in X^*$ and notice that $f(T^n(x) - w) \to 0$ if $n \to \infty$ since $\{T^n(x)\} = \{w_n^{(1)}\}$ converges weakly to w. Hence

$$f\left(\frac{1}{p}\sum_{i=0}^{p-1} T^i(x) - w\right) = \frac{1}{p}\sum_{i=0}^{p-1} f(T^i(x) - w) \to 0$$

if $p \to \infty$ for every $f \in X^*$, that is, $\{w_0^{(p)}\} = \left\{\frac{1}{p}\sum_{i=0}^{p-1} T^i(x)\right\}$ converges weakly to w.

We show that w is a fixed point of T. Since $\lim_{n\to\infty} \sup_{m\geq 1} \|T^{i+n+m}(x) - T^m(T^{i+n}(x))\| = 0$ for each i, it follows by induction from Lemma 4 that $\lim_{n\to\infty} \sup_{m\geq 1} \|w_{n+m}^{(p)} - T^m(w_n^{(p)})\| = 0$ for every p. Therefore, we can choose a sequence $M_1 < M_2 < \ldots$ such that $\|w_{n+1}^{(p)} - T(w_n^{(p)})\| < \frac{1}{p}$ for every $n \geq M_p$. Furthermore $\|w_{n+1}^{(p)} - w_n^{(p)}\| \leq \frac{2}{p}$ for every n, p. Hence

$$\|T(w_n^{(p)}) - w_n^{(p)}\| < \tfrac{3}{p}$$

for every p and $n \geq M_p$. In particular, $\{w_{M_p}^{(p)}\}_p$ is an approximate fixed point sequence for T. It follows from (5.2) and (5.4) that

$$w_0^{(p)} \preceq w_{M_p}^{(p)} \preceq w_{M_p+p}^{(1)}.$$

Since both $\{w_0^{(p)}\}_p$ and $\{w_{M_p+p}^{(1)}\}_p$ converge weakly to w, $\{w_{M_p}^{(p)}\}_p$ converges weakly to w, too. Moreover, $\{w_{M_p}^{(p)}\}_p$ is monotone and thus Theorem 1 yields $T(w) = w$, which proves the theorem. □

Notice that the monotonicity of the mapping T forces the Picard iteration to converge weakly to a point. Therefore, quite surprisingly, the proof above gives more, namely the Picard sequence $\{T^n(x)\}$ also converges weakly to a fixed point of T.

Theorem 5. *Let C be a bounded closed convex subset of an ordered uniformly convex space $(X, \|\cdot\|, \preceq)$ and $T : C \to C$ a monotone nonexpansive mapping. If $x \preceq T(x)$ then the sequence of Picard iteration $\{T^n(x)\}$ converges weakly to a fixed point of T.*

Proof. In the notation of Theorem 4, the sequences $\{w_n^{(p)}\}_n$, $p \geq 1$, and $\{w_n^{(p)}\}_p$, $n \geq 0$, converge weakly to w that is a fixed point of T. In particular, $\{w_n^{(1)}\} = \{T^n(x)\}$. □

Open Problem

Finally, we raise the following question: is there a direct proof of Theorem 5?

Acknowledgments: We are very much thankful to the reviewer for a very thorough reading and many helpful corrections and suggestions, which have been incorporated into this version of the paper.

References

[1] B. Halpern, Fixed points of nonexpanding maps, Bull. Amer. Math. Soc., **73** (1967), 957–961.

[2] J.B. Baillon, Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert, C. R. Acad. Sci. Paris Sér. A-B, **280** (1975), no. 22, Aii, A1511–A1514.

[3] J. Borwein, S. Reich and I. Shafrir, Krasnosel'skiĭ iterations in normed spaces. Canad. Math. Bull. **35** (1992), no. 1, 21–28.

[4] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, Inverse Problems, **20** (2004), no. 1, 103–120.

[5] G. Lopez, V. Martin, and H. K. Xu, Perturbation techniques for nonexpansive mappings with applications, Nonlinear Anal. Real World Appl., **10** (2009), no. 4, 2369–2383.

[6] E. Masad and S. Reich, A note on the multiple-set split convex feasibility problem in Hilbert space, J. Nonlinear Convex Anal. **8** (2007), no. 3, 367–371.

[7] H.K. Xu, Iterative algorithms for nonlinear operators, J. London Math. Soc. (2), **66** (2002), no. 1, 240–256.

[8] A.C.M. Ran and M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc., **132** (2004), no. 5, 1435–1443.

[9] R. Espínola and A. Wiśnicki, The Knaster-Tarski theorem versus monotone nonexpansive mappings, Bull. Pol. Acad. Sci. Math., **66** (2018), no. 1, 1–7.

[10] M.R. Alfuraidan and M.A. Khamsi, A fixed point theorem for monotone asymptotically nonexpansive mappings, Proc. Amer. Math. Soc., **146** (2018), no. 6, 2451–2456.

[11] B.A. Bin Dehaish and M.A. Khamsi, Browder and Göhde fixed point theorem for monotone nonexpansive mappings, Fixed Point Theory Appl., **2016**, Paper No. 20, 9 pp.

[12] S. Reich and A.J. Zaslavski, Generic well-posedness of the fixed point problem for monotone nonexpansive mappings, Mathematics almost everywhere, 169–179, World Sci. Publ., Hackensack, NJ, 2018.

[13] C. Chidume, *Geometric properties of Banach spaces and nonlinear iterations*, volume 1965 of Lecture Notes in Mathematics, Springer-Verlag London, Ltd., London, 2009, xviii+326 pp. ISBN: 978-1-84882-189-7.

[14] S. Reich, Some problems in nonlinear functional analysis, The Altgeld Book 6 (1975), University of Illinois Functional Analysis Seminar, pp. xii.1–xii.18.

[15] S. Reich, Some fixed point problems, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **57** (1974), no. 3-4, 194–198 (1975).

[16] H.K. Xu, Viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl., **298** (2004), no. 1, 279–291.

[17] H.K. Xu, Strong convergence of an iterative method for nonexpansive and accretive operators, J. Math. Anal. Appl., **314** (2006), no. 2, 631–643.

[18] E. Matoušková and S. Reich, Reflexivity and approximate fixed points, Studia Math. **159** (2003), no. 3, 403–415.

[19] K. Goebel and W.A. Kirk. *Topics in metric fixed point theory*, volume 28 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990, viii+244 pp. ISBN: 0-521-38289-0.

[20] B.A. Bin Dehaish and M.A. Khamsi, Mann iteration process for monotone nonexpansive mappings, Fixed Point Theory Appl., **2015**, 2015:177, 7 pp.

[21] M.A. Krasnosel'skiĭ, Two remarks on the method of successive approximations, Uspehi Mat. Nauk (N.S.), **10** (1955), 123–127.

[22] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc., **4** (1953), 506–510.

[23] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, J. Math. Anal. Appl., **67** (1979), 274–276.

[24] R. Shukla, R. Pant, Z. Kadelburg, and H.K. Nashine, Existence and convergence results for monotone nonexpansive type mappings in partially ordered hyperbolic metric spaces, Bull. Iranian Math. Soc., **43** (2017), no. 7, 2547–2565.

[25] R. Shukla, R. Pant and P. Kumam, On the α-nonexpansive mapping in partially ordered hyperbolic metric spaces, J. Math. Anal. **8** (2017), no. 1, 1–15.

[26] Y. Song, K. Promluang, P. Kumam, and Y. Je Cho, Some convergence theorems of the Mann iteration for monotone α-nonexpansive mappings, Appl. Math. Comput., **287/288** (2016), 74–82.

[27] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, J. Math. Anal. Appl. **75** (1980), no. 1, 287–292.

[28] K. Goebel and S. Reich, *Uniform convexity, hyperbolic geometry, and nonexpansive mappings,* Monographs and Textbooks in Pure and Applied Mathematics, 83. Marcel Dekker, Inc., New York, 1984, ix+170 pp. ISBN: 0-8247-7223-7.
[29] S. Reich, Almost convergence and nonlinear ergodic theorems. J. Approx. Theory **24** (1978), no. 4, 269–272.
[30] R.E. Bruck and S. Reich, Accretive operators, Banach limits, and dual ergodic theorems, Bull. Acad. Polon. Sci. Sér. Sci. Math. **29** (1981), no. 11-12, 585–589 (1982).
[31] J.B. Baillon, Comportement asymptotique des itérés de contractions nonlinéaires dans les espaces L^p, C. R. Acad. Sci. Paris Sér. A-B, **286** (1978), no. 3, A157–A159.
[32] R.E. Bruck, A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces, Israel J. Math., **32** (1979), no. 2-3, 107–116.
[33] S. Reich, Nonlinear ergodic theory in Banach spaces, Argonne National Lab., IL (USA), (1979), 69–79.
[34] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis*, Vol. 1, volume 48 of American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, RI, 2000, xii+488 pp. ISBN: 0-8218-0835-4.