Research article

Peter Bugiel, Stanisław Wędrychowicz, and Beata Rzepka*

Fixed point of some Markov operator of Frobenius-Perron type generated by a random family of point-transformations in $\mathbb{R}^d$

https://doi.org/10.1515/anona-2020-0163
Received July 26, 2020; accepted November 22, 2020.

Abstract: Existence of fixed point of a Frobenius-Perron type operator $P : L^1 \longrightarrow L^1$ generated by a family $\{\varphi_y\}_{y\in Y}$ of nonsingular Markov maps defined on a σ-finite measure space (I, Σ, m) is studied. Two fairly general conditions are established and it is proved that they imply for any $g \in G = \{f \in L^1 : f \geq 0, \text{ and } \|f\| = 1\}$, the convergence (in the norm of L^1) of the sequence $\{P^j g\}_{j=1}^{\infty}$ to a unique fixed point g_0. The general result is applied to a family of $C^{1+\alpha}$-smooth Markov maps in $\mathbb{R}^d$.

Keywords: Markov operator; Markov maps; fixed point; Radon-Nikodym derivative; Frobenius-Perron operator

MSC 2010: 37A30

1 Introduction

Let be given a randomly perturbed semi-dynamical system that evolves according to the rule:

$$x_j = \varphi_{\xi_j}(x_{j-1}) \quad \text{for} \quad j = 1, 2, \ldots,$$

where $\{\varphi_y\}_{y\in Y}$ is a family of nonsingular Markov maps defined on a subset $I \subseteq \mathbb{R}^d$ (bounded or not), $d \geq 1$, and $\{\xi_j\}_{j=1}^{\infty}$ is a sequence of identically distributed independent Y-valued random elements, where Y is a Polish metric space (i.e., a complete separable metric space).

Investigation of the asymptotic properties of such a semi-dynamical system leads to the study of the convergence of the sequence $\{P^j\}_{j=1}^{\infty}$ of iterates of some Frobenius-Perron type operator P which is Markov operator, i.e. $\|Pf\| = \|f\|$, and $Pf \geq 0$ if $f \geq 0$, acting in L^1 (Markov operator of F-P type, in short). More precisely, let $g \geq 0$, $\|g\| = 1$, if $\text{Prob}(x_0 \in B) \stackrel{\text{df}}{=} \int_B g\, dm$ (m denotes the Lebesgue measure on I), then $\text{Prob}(x_j \in B) = \int_B P^j g\, dm$, where P is the Markov operator of F-P type defined by (3.1) (see Proposition 3.1).

We establish two fairly general conditions: conditions (3.H1) and (3.H2), and prove that under those conditions the system in question evolves to a stationary distribution. That is, the sequence $\{P^j\}_{j=1}^{\infty}$ converges (in the norm of L^1) to a unique fixed point $g_0 \in G$ (Th. 3.3). The two conditions are probabilistic analogues of conditions (3.H1) and (3.H2) in [1], respectively. Actually, if

Peter Bugiel, Faculty of Mathematics and Computer Science, Jagiellonian University Cracow (Kraków), Poland, E-mail: uibugiel@cyf-kr.edu.pl
Stanisław Wędrychowicz, Department of Nonlinear Analysis, Ignacy Łukasiewicz Rzeszów University of Technology, al. Powstańców Warszawy 8, 35-959 Rzeszów, Poland, E-mail: swedrych@prz.edu.pl
***Corresponding Author: Beata Rzepka,** Department of Nonlinear Analysis, Ignacy Łukasiewicz Rzeszów University of Technology, al. Powstańców Warszawy 8, 35-959 Rzeszów, Poland, E-mail: brzepka@prz.edu.pl

$\varphi_y = \varphi$, $y \in Y$, that is if $x_j = \varphi_{\xi_j}(x_{j-1}) = \varphi(x_{j-1})$ is a deterministic semi-dynamical system (φ is a fixed Markov map), then we get two mentioned conditions given in [1].

As an application of this general result we show that the randomly perturbed semi-dynamical system generated by a family of $C^{1+\alpha}$-smooth Markov maps in $\mathbb{R}^d$ evolves to a unique stationary density (Th. 4.2).

Similar problems were considered by several authors: eg. [2–4], and the references therein.

2 Preliminaries

Let (I, Σ, m) be a σ-finite atomless (non-negative) measure space. Quite often the notions or relations occurring in this paper (in particular, the considered transformations) are defined or hold only up to the sets of m-measure zero. Henceforth we do not mention this explicitly.

The restriction of a mapping $\tau : X \to Y$ to a subset $A \subseteq X$ is denoted by $\tau_{|A}$ and the indicator function of a set A by $\mathbf{1}_A$.

Let $\tau : I \to I$ be a measurable transformation i.e., $\tau^{-1}(A) \in \Sigma$ for each $A \in \Sigma$. It is called nonsingular iff $m \circ \tau^{-1} \sim m$ i.e., for each $A \in \Sigma$, $m(\tau^{-1}(A)) = 0 \Leftrightarrow m(A) = 0$.

We give a few definitions. The following kind of transformations is considered in this paper:

Definition 2.1. A nonsingular transformation φ from I into itself is said to be a piecewise invertible iff

(2.M1) one can find a finite or countable partition $\pi = \{I_k : k \in K\}$ of I, which consists of measurable subsets (of I) such that $m(I_k) > 0$ for each $k \in K$, and $\sup\{m(I_k) : k \in K\} < \infty$, here and in what follows K is an arbitrary countable index set;

(2.M2) for each $I_k \in \pi$, the mapping $\varphi_k = \varphi_{|I_k}$ is one-to-one of I_k onto $J_k = \varphi_k(I_k)$ and its inverse φ_k^{-1} is measurable.

Definition 2.2. A piecewise invertible transformation φ is said to be a Markov map iff its corresponding partition π satisfies the following two conditions:

(2.M3) π is a Markov partition i.e., for each $k \in K$,

$$\varphi(I_k) = \bigcup\{I_j : m(\varphi(I_k) \cap I_j) > 0\};$$

(2.M4) φ is indecomposable (irreducible) with respect to π i.e., for each $(j, k) \in K^2$ there exists an integer $n > 0$ such that $I_k \subseteq \varphi^n(I_j)$.

In what follows we denote by $\|\cdot\|$ the norm in $L^1 = L^1(I, \Sigma, m)$ and by $G = G(m)$ the set of all (probabilistic) densities i.e.,

$$G \stackrel{\text{df}}{=} \{g \in L^1 : g \geq 0, \text{ and } \|g\| = 1\}.$$

Let $\tau : I \to I$ be a nonsingular transformation. Then the formula

$$P_\tau f \stackrel{\text{df}}{=} \frac{d}{dm}(m_f \circ \tau^{-1}) \text{ for } f \in L^1, \tag{2.1}$$

where $dm_f = f\,dm$, and $\frac{d}{dm}$ denotes the Radon-Nikodym derivative, defines a linear operator from L^1 into itself. It is called the Frobenius-Perron operator (F-P operator, in short) associated with τ [5, 6].

Formula (2.1) is equivalent to the following one:

$$\int_A P_\tau f\,dm = m_f(\tau^{-1}(A)) = \int_{\tau^{-1}(A)} f\,dm.$$

From the definition of P_τ it follows that it is a Markov operator, i.e., P_τ is a linear operator and for any $f \in L^1(m)$ with $f \geq 0$, $P_\tau f \geq 0$ and $\|P_\tau f\| = \|f\|$.

The last equality follows immediately from the second formula equivalent to (2.1) if one puts $A = I$. Further, $P_\tau G \subseteq G$, and P_τ is a contraction, i.e. $\|P_\tau\| \leq 1$.

Let $\tau_1, \ldots, \tau_j$, $(j \geq 2)$, be some nonsingular transformations. Denote by $P_{(j,\ldots,1)}$, $P_{(j,\ldots,i+1)}$ and $P_{(i,\ldots,1)}$, $1 \leq i < j$, the F-P operators associated with the transformations $\tau_{(j,\ldots,1)} = \tau_j \circ \cdots \circ \tau_1$, $\tau_{(j,\ldots,i+1)} = \tau_j \circ \cdots \circ \tau_{i+1}$ and $\tau_{(i,\ldots,1)} = \tau_i \circ \cdots \circ \tau_1$, respectively. Then

$$P_{(j,\ldots,1)} = P_{(j,\ldots,i+1)} P_{(i,\ldots,1)}, \tag{2.2}$$

in particular $P_{(j,\ldots,1)} = P_j \ldots P_1$.

Definition 2.3. In what follows we consider a family $\{\varphi_y\}_{y\in Y}$ of Markov maps such that:

(2.M$_y$5) Y is a Polish metric space and the map $I \times Y \ni (x, y) \to \varphi_y(x) \in I$ is $\Sigma(I \times Y)/\Sigma(I)$-measurable;

(2.M$_y$6) there exists a partition π of I such that $\pi_y = \pi$ for each $y \in Y$, where π_y is a Markov partition associated with φ_y.

For $j \geq 1$, and $y_1, \ldots, y_j \in Y$, we denote $y(j) = (y_j, \ldots, y_1)$ and then we set

$$\varphi_{y(j)} \stackrel{\text{df}}{=} \varphi_{y_j} \circ \cdots \circ \varphi_{y_1}. \tag{2.3}$$

Clearly, $\varphi_{y(j)} : I \to I$ is a Markov map. Its Markov partition is given by

$$\pi_{y(j)} \stackrel{\text{df}}{=} \pi \vee \varphi_{y(1)}^{-1}(\pi) \vee \varphi_{y(2)}^{-1}(\pi) \vee \cdots \vee \varphi_{y(j-1)}^{-1}(\pi) \text{ provided } j \geq 2.$$

It consists of the sets of the form:

$$I_{k(j)}^{y(j-1)} \stackrel{\text{df}}{=} I_{k_0} \cap \varphi_{y(1)}^{-1}(I_{k_1}) \cap \varphi_{y(2)}^{-1}(I_{k_2}) \cap \cdots \cap \varphi_{y(j-1)}^{-1}(I_{k_{j-1}}), \tag{2.4}$$

where $k(j) = (k_0, k_1, \ldots, k_{j-1}) \in K^j$.

Let

$$\varphi_{y(j)k(j)} \stackrel{\text{df}}{=} (\varphi_{y(j)})_{|I_{k(j)}^{y(j-1)}},$$

then by condition (2.M2), $\varphi_{y(j)k(j)}$ is one-to-one mapping of $I_{k(j)}^{y(j-1)}$ onto

$$J_{k(j)}^{y(j)} \stackrel{\text{df}}{=} \varphi_{y(j)k(j)}(I_{k(j)}^{y(j-1)}) = \varphi_{y_j}(I_{k_{j-1}}).$$

It is nonsingular, and $\varphi_{y(j)k(j)}^{-1}$, the mapping inverse to $\varphi_{y(j)k(j)}$, is measurable.

By Def. 2.3, $\pi_{y(1)} = \pi_{y_1} = \pi$ and therefore $I_{k(1)}^{y(0)} = I_{k_0}$; consequently $\varphi_{y(1)k(1)} = (\varphi_{y_1})_{|I_{k_0}} = \varphi_{y_1 k_0}$ and, according to (2.M2), φ_{yk} is one-to-one mapping of I_k onto $J_k^y = \varphi_{yk}(I_k)$.

We have to adjust the indecomposable condition (2.M4) to the new case when a single Markov map φ is replaced by a family $\{\varphi_y\}_{y\in Y}$ of Markov maps. We propose the following condition (see note at the end of Rem. 2.4):

(2.M$_y$4) for each $(j, k) \in K^2$ there exist an integer $s > 0$, and a subset $\widetilde{Y}^s \subseteq Y^s$ with $p^s(\widetilde{Y}^s) > 0$ such that $I_k \subseteq \varphi_{y(s)}(I_j)$ for all $y(s) \in \widetilde{Y}^s$. Here p is a probability measure on $\Sigma(Y)$, and $p^s = \underbrace{p \times \cdots \times p}_{s\text{-times}}$.

From the properties of $\varphi_{y(r)}$ it follows that the formula

$$m_{y(r)k(r)}(A) \stackrel{\text{df}}{=} m \circ \varphi_{y(r)k(r)}^{-1}(A) = m(\varphi_{y(r)k(r)}^{-1}(A)) \quad \text{for} \quad A \in \Sigma, \tag{2.5}$$

defines an absolutely continuous measure which is concentrated on $J_{k(r)}^{y(r)}$ (i.e., $m_{y(r)k(r)}(A) = m_{y(r)k(r)}(A \cap J_{k(r)}^{y(r)})$), and whose Radon-Nikodym derivative satisfies $\frac{d\, m_{y(r)k(r)}}{dm} > 0$ a.e. on $J_{k(r)}^{y(r)}$.

To see the latter property of the measure $m_{y(r)k(r)}$, note first that if $\frac{d\, m_{y(r)k(r)}}{dm} = 0$ on $A \subseteq J_{k(r)}^{y(r)}$, then $\varphi_{y(r)k(r)}^{-1}(A) \subseteq I \setminus I_{k(r)}^{y(r-1)}$ a.e., because

$m(\varphi_{y(r)k(r)}^{-1}(A) \cap I_{k(r)}^{y(r-1)}) = \int\limits_{A \cap J_{k(r)}^{y(r)}} \frac{d\, m_{y(r)k(r)}}{dm}\, dm = 0$. Therefore, $A = \emptyset$ a.e.

We put ($r = 1, 2, \ldots$)

$$\sigma_{y(r)k(r)} \stackrel{\text{df}}{=} \begin{cases} \frac{d\, m_{y(r)k(r)}}{dm}, & \text{on } J^{y(r)}_{k(r)}, \\ 0, & \text{on } I \setminus J^{y(r)}_{k(r)}, \end{cases} \tag{2.6}$$

next,

$$\overline{f}_{y(r)k(r)} \stackrel{\text{df}}{=} \begin{cases} f \circ \varphi^{-1}_{y(r)k(r)}, & \text{on } J^{y(r)}_{k(r)}, \\ 0, & \text{on } I \setminus J^{y(r)}_{k(r)}. \end{cases} \tag{2.7}$$

and finally

$$P_{y(r)} f \stackrel{\text{df}}{=} \frac{d}{dm}(m_f \circ \varphi^{-1}_{y(r)}), \quad \text{where} \quad dm_f = f dm.$$

Then the F-P operator $P_{y(r)}$ of the Markov map $\varphi_{y(r)k(r)}$ can be written in the following form

$$P_{y(r)} f = \sum_{k(r)} \overline{f}_{y(r)k(r)} \sigma_{y(r)k(r)}. \tag{2.8}$$

Indeed, from Def. 2.3, (2.3) and (2.5) it follows that for any $f \in L^1$, $f \geq 0$, the following equalities hold:

$$\int_A P_{y(r)} f \, dm = \int_A \frac{d}{dm}(m_f \circ \varphi^{-1}_{y(r)}) \, dm = m_f(\varphi^{-1}_{y(r)}(A)) = \sum_{k(r)} \int_{A_{y(r)k(r)}} f \, dm$$

$$= \sum_{k(r)} \int_A f \circ \varphi^{-1}_{y(r)k(r)} \, dm_{y(r)k(r)} = \int_A \left(\sum_{k(r)} \overline{f}_{y(r)k(r)} \sigma_{y(r)k(r)} \right) dm,$$

where $A_{y(r)k(r)} = \varphi^{-1}_{y(r)k(r)}(A)$, and $dm_{y(r)k(r)}$ is given by (2.5). Hence (2.8) follows.

Remark 2.4. The studies of this paper can be extended to the family $\{\varphi_y\}_{y \in Y}$ which satisfies condition (2.M$_y$5) of Def. 2.3 and, instead of (2.M$_y$6), the following less restrictive condition:

(2.$\widetilde{\text{M}}_y$6) there is a Markov map $\varphi_{\tilde{y}}$ such that for each $y \in Y$:
- (a) $\pi_y \prec \pi_{\tilde{y}}$, i.e., for each $V \in \pi_y$, there exists $U \in \pi_{\tilde{y}}$ which contains V, and
- (b) for each $V \in \pi_y$, $\varphi_y(V)$ is a union of a number of $U \in \pi_{\tilde{y}}$.

In this situation each $\varphi_{y(j)} = \varphi_{y_j} \circ \varphi_{y_{j-1}} \circ \cdots \circ \varphi_1$ is defined on the interval of the form:

$$I^{y(j)}_{k(j)} \stackrel{\text{df}}{=} I^{y_1}_{k_0} \cap \varphi^{-1}_{y(1)}(I^{y_2}_{k_1}) \cap \varphi^{-1}_{y(2)}(I^{y_3}_{k_2}) \cap \cdots \cap \varphi^{-1}_{y(j-1)}(I^{y_j}_{k_{j-1}}) \in \pi_{y(j)}.$$

The following family can serve as a simple example: $\{\varphi_i\}_{i=1}^{\infty}$ where $\varphi_i = \varphi^i$, and φ is a Markov map. Note that conditions (2.M4) and (2.M$_y$4) are equivalent in this case, if $P(\{i\}) > 0$ for $i \geq 1$.

We close this section with the following criterion of the convergence in L^1 of the iterates P^n of Markov operator. It is used in the proof of Th. 3.3.

Theorem 2.5. *Let there exist $h \in L^1$, $h \geq 0$ with $\|h\| > 0$, and a dense subset $G_0 \subseteq G$ such that $\lim_{j\to\infty} \|(P^j g - h)^-\| = 0$, for $g \in G_0$, where $(P^j g - h)^- = \max\{0, -(P^j g - h)\}$. Then there exists exactly one P-fixed point $g_0 \in G$ such that*

$$\lim_{j\to\infty} P^j g = g_0, \quad \textit{for all } g \in G.$$

Proof. We refer to [8], Theorem 3. □

3 Convergence theorem

Let $\{\varphi_y\}_{y \in Y}$ be a family of Markov maps in the sense of Def. 2.3, $\Sigma(Y)$ – σ-algebra of all Borel-measurable subsets of Y (where Y is a Polish metric space), p a probability measure on $(Y, \Sigma(Y))$, and P_y the F-P operator of the Markov map φ_y.

We put

$$Pf \overset{\text{df}}{=} \int P_y f \, dp(y) \quad \text{for} \quad f \in L^1(m). \tag{3.1}$$

It follows from the definition, and the fact that the F-P operator $P_y : L^1(m) \to L^1(m)$ (given by (2.8)) is a Markov operator, that $P : L^1(m) \to L^1(m)$ is also a Markov operator.

Indeed, from (3.1) and Fubini's Theorem we have for $f \geq 0$:

$$\begin{aligned}\int Pf \, dm &= \int \left\{ \int P_y f \, dp(y) \right\} dm = \int \left\{ \int P_y f \, dm \right\} dp(y) \\ &= \int \left\{ \int f \, dm \right\} dp(y) = \int f \, dm.\end{aligned}$$

In general case we have $f = f^+ - f^-$, where $f^+ = \max\{0, f\}$, and $f^- = \max\{0, -f\}$, it concludes the proof.

Operator P is called, in this note, the Markov operator of F-P type.

Let $P^j = PP^{j-1} (j \geq 2)$, then from (3.1), (2.1) and (2.8) it follows that

$$P^j f = \int P_{y(j)} f \, dp^j(y_1, \ldots, y_j), \tag{3.2}$$

where $P_{y(j)}$ is the F-P operator corresponding to $\varphi_{y(j)}$ defined by (2.3), and $p^j = \underbrace{p \times \cdots \times p}_{j-times}$.

The Markov operator P of F-P type given by (3.1) has the following probabilistic interpretation:

Let $\xi_1, \xi_2, \ldots$ be a sequence of Y-valued random elements (indices) defined on a probability space $(\Omega, \Sigma(\Omega), p_1)$. For each $(x, \omega) \in I \times \Omega$ we put

$$x_j(x, \omega) \overset{\text{df}}{=} \begin{cases} x, & \text{for } j = 0, \\ \varphi_{\xi_j(\omega)}(x_{j-1}) = \varphi_{\xi(j)(\omega)}(x), & \text{for } j \geq 1, \end{cases} \tag{3.3}$$

where $\xi(j)(\omega) = (\xi_j(\omega), \ldots, \xi_1(\omega))$.

We assume that $\Omega = Y^\infty$ is a direct product space, $\Sigma(\Omega)$ – σ-algebra of all Borel measurable subsets of Y^∞, $p_1 = p^\infty = p \times p \times \ldots$ – direct product measure, and $\xi_j(\omega) = \omega_j$ for $\omega \in \Omega$ where ω_j is the j-th coordinate of $\omega = (\omega_1, \omega_2, \ldots) \in Y^\infty$. Thus $\{\xi_j\}_{j=1}^\infty$ is a sequence of identically p-distributed independent Y-valued random elements (indices).

Let $(\widetilde{\Omega}, \Sigma(\widetilde{\Omega}), \text{Prob})$ be a probability space with $\widetilde{\Omega} = I \times \Omega$ and $\text{Prob} = \widetilde{m} \times p_1$, where $\widetilde{m}$ is a probability measure on $\Sigma(I)$. Then the sequence $\{x_j\}_{j=0}^\infty$ defined by (3.3) is a sequence of random vectors over $(\widetilde{\Omega}, \Sigma(\widetilde{\Omega}), \text{Prob})$. Note that $\{ x_0 \in B \} = B \times \Omega$, hence $\text{Prob}(x_0 \in B) = \widetilde{m}(B)$ for any $B \in \Sigma(I)$.

It turns out that if the initial probability distribution is absolutely continuous, then the probability distribution of each random vector x_j, defined by (3.3), is also absolutely continuous:

Proposition 3.1. *If* $\text{Prob}(x_0 \in B) \overset{\text{df}}{=} \int_B g \, dm$ *for all* $B \in \Sigma(I)$, *where* $g \in L^1(m)$ *and* $g \geq 0$, $\|g\| = 1$, *then*

$$\text{Prob}(x_j \in B) = \int_B P^j g \, dm \quad (j = 1, 2, \ldots)$$

for all $B \in \Sigma(I)$ *where* P^j *is the* j*-th iterate of the Markov operator* P *of* F-P *type defined by* (3.1).

Proof. We refer to [7], Prop. 3.1. □

The convergence of the sequence $\{P^j\}$ of the iterates of the Markov operator P of F-P type associated with $\{x_j\}$ is established under two general conditions. We are going now to formulate the first of them.

Let $\varphi_{y(j)}$ be a Markov map given by (2.3) and let $P_{y(j)}$ be its F-P operator given by (2.8). We put

$$A_{y(j)k}(g) \overset{\text{df}}{=} \text{ess sup}\{P_{y(j)} g(x) : x \in I_k \cap \text{spt}(P_{y(j)} g)\},$$

$$a_{y(j)k}(g) \stackrel{\text{df}}{=} \text{ess inf}\{P_{y(j)}g(x) : x \in I_k \cap \text{spt}(P_{y(j)}g)\},$$

where $\text{spt}(g) \stackrel{\text{df}}{=} \{x : g(x) > 0\}$.

Now we give the following:

Definition 3.2. A density $g \in G$, belongs to $G(C^*)$, $0 < C^* < \infty$, iff there exist constants $C_{y(j)}(g) \geq 1$, $y(j) \in Y^j$, $j \geq j_1(g)$, such that the following two conditions are satisfied:

(a) $A_{y(j)k}(g) \leq C_{y(j)}(g)a_{y(j)k}(g)$ a.e. $[p^j]$, $j \geq j_1(g)$, and

(b) $\limsup_{j\to\infty} \int \ln C_{y(j)}(g)\, dp^j < C^*$.

Having defined the set $G(C^*)$ we are in a position to formulate the first condition:

(3.H1) (Distortion Inequality for the family $P_{y(j)}$, $y(j) \in Y^j$) There exists a constant $0 < C^* < \infty$ such that the set $G(C^*)$ defined by Def. 3.2 contains a subset dense in G.

To formulate the second condition we define first the following auxiliary function:

$$u_{w(2r)}(x) \stackrel{\text{df}}{=} \inf\{g_{w(2r)k(r)}(x) : k(r) \in K^r, \text{ and } I^{w(r-1)}_{k(r)} \neq \emptyset\}, \tag{3.4}$$

where

$$g_{w(2r)k(r)} \stackrel{\text{df}}{=} \sum_{\tilde{k}(r)} \tilde{\sigma}_{\tilde{w}(r)\tilde{k}(r)} \int_{I^{\tilde{w}(r-1)}_{\tilde{k}(r)}} \tilde{\sigma}_{w(r)k(r)}\, dm, \tag{3.5}$$

$w(2r) = (\tilde{w}(r), w(r)) \in Y^r \times Y^r$, (in (3.5) we put $\tilde{w}(r) = (\tilde{w}_r, \ldots, \tilde{w}_1)$, and $\tilde{k}(r) = (\tilde{k}_0, \ldots, \tilde{k}_{r-1})$); further

$$\tilde{\sigma}_{w(r)k(r)} \stackrel{\text{df}}{=} \frac{\sigma_{w(r)k(r)}}{m(I^{w(r-1)}_{k(r)})}, \tag{3.6}$$

where $I^{w(r-1)}_{k(r)}$, and $\sigma_{w(r)k(r)}$ are defined, respectively by (2.4) and (2.6).

The second condition reads as follows:

(3.H2) There exists $\tilde{r} \geq 1$ such that $0 < \| \int u_{w(2\tilde{r})}\, dp^{2\tilde{r}} \| < \infty$.

The theorem below states that the semi-dynamical system given by (3.3) evolves to a stationary distribution under the above two conditions.

Theorem 3.3. (Convergence Theorem) *Assume that a family* $\{\varphi_y\}_{y\in Y}$ *of Markov maps satisfies* (3.H1) *and* (3.H2). *Then there exists exactly one P-fixed point* $g_0 \in G$, *that is* $Pg_0 = g_0$, *such that*

$$\lim_{j\to\infty} P^j g = g_0, \quad \textit{for all } g \in G.$$

Proof. The point is to show that for each $r \geq 1$, the function

$$u_{2r} \stackrel{\text{df}}{=} \widehat{C} \exp\left(\int \ln u_{w(2r)}\, dp^{2r}\right), \quad \text{where } \widehat{C} = \exp(-2C^*), \tag{3.7}$$

is a function for P which, under condition (3.H1), plays the role of h from Theorem 2.5. That is, it satisfies the relation

$$\lim_{j\to\infty} \|(P^{j+2r}g - u_{2r})^-\| = 0 \quad \text{for all } g \in G. \tag{3.8}$$

To this end note that by condition (3.H1) there exists a subset $\widetilde{G} \subseteq G(C^*)$ dense in G. Thus for any $g \in \widetilde{G}$ there exists, by Def. 3.2 (a), $j_1 = j_1(g)$ such that the following inequalities hold:

$$C^{-1}_{y(j)}(g) \leq P_{y(j)}g(y)/P_{y(j)}g(x) \leq C_{y(j)}(g) \tag{3.9}$$

for each $j \geq j_1$, all $y(j) \in Y^j$, any $I_k \in \pi$ and $m \times m$ a.e. $(x,\ y) \in I_k \times I_k$.
These inequalities imply the following estimate:

$$C^{-1}_{y(j)}(g)F_{rw(r)}(P_{y(j)}g) \leq P_{w(r)}P_{y(j)}g \leq C_{y(j)}(g)F_{rw(r)}(P_{y(j)}g) \tag{3.10}$$

for every $r \geq 1$, $j \geq j_1$ and all $w(r) \in Y^r$, $y(j) \in Y^j$; where $C_{y(j)}(g)$ are constants involved in Def. 3.2, and $F_{rw(r)}$ is defined by the following formula:

$$F_{rw(r)}(g) \stackrel{\text{df}}{=} \sum_{k(r)} \tilde{\sigma}_{w(r)k(r)} \int\limits_{I^{w(r-1)}_{k(r)}} g\, dm. \tag{3.11}$$

In the last formula $\tilde{\sigma}_{w(r)k(r)}$ and $I^{w(r-1)}_{k(r)}$ are defined by (3.6) and (2.4), respectively.

To see it note that from (3.9) we obtain

$$\begin{aligned} C^{-1}_{y(j)}(g)\overline{(P_{y(j)}g)}_{w(r)k(r)}(x)\sigma_{w(r)k(r)}(x) \leq & \overline{(P_{y(j)}g)}_{w(r)k(r)}(y)\sigma_{w(r)k(r)}(x) \\ \leq & C_{y(j)}(g)\overline{(P_{y(j)}g)}_{w(r)k(r)}(x)\sigma_{w(r)k(r)}(x), \end{aligned}$$

for each $J^{w(r)}_{k(r)} = \varphi_{w(r)k(r)}(I^{w(r-1)}_{k(r)})$, all $x, y \in J^{w(r)}_{k(r)}$, and $j \geq j_1(g)$; where
$\overline{(P_{y(j)}g)}_{w(r)k(r)}(x) \stackrel{\text{df}}{=} (P_{y(j)}g) \circ \varphi^{-1}_{w(r)k(r)}(x)$, or 0, according as $x \in J^{w(r)}_{k(r)}$, or $x \in I \setminus J^{w(r)}_{k(r)}$.

Integrating the above inequalities with respect to x on $J^{w(r)}_{k(r)}$ and multiplying by $\tilde{\sigma}_{w(r)k(r)}(y)$, then summing the resulting inequalities with respect to all $k(r)$ and finally using equality (2.8) one gets the desired double inequality (3.10).

Let $w(2r) = (\tilde{w}(r),\ w(r)) \in Y^r \times Y^r$, then iterating the first of the double inequality (3.10), by using the equalities

$$P_{w(2r)} = P_{\tilde{w}(r)}P_{w(r)},$$

$$\sum_{k(r)} \|\mathbf{1}_{I^{w(r-1)}_{k(r)}} P_{y(j)}g\| = 1, \tag{3.12}$$

and the formula (3.11), one gets for every $r \geq 1$, $j \geq j_1(g)$, and all $\tilde{w}(r)$, $w(r) \in Y^r$, and $y(j) \in Y^j$:

$$\begin{aligned} P_{w(2r)}P_{y(j)}g \geq & C^{-1}_{z(r+j)}(g)C^{-1}_{y(j)}(g)F_{r\tilde{w}(r)}(F_{rw(r)}P_{y(j)}g) \\ \geq & C^{-1}_{z(r+j)}(g)C^{-1}_{y(j)}(g)u_{w(2r)}, \end{aligned} \tag{3.13}$$

where $z(r+j) = (w(r), y(j))$ and $u_{w(2r)}$ is defined by formulas (3.4), and (3.5).

Integrating the above inequalities with respect to $w(2r) = (\tilde{w}(r), w(r))$, and $y(j)$, using Jensen's inequality and condition (b) of Def. 3.2, and applying formulas (2.2) together with (3.2), give:

$$P^{j+2r}g \geq u_{2r},$$

where u_{2r} is defined by (3.7).

The last inequality implies that (3.8) holds. This is so because $\widetilde{G} \subseteq G$ is dense, and P is a contraction.

Thus we have proved that for each $r \geq 1$, u_{2r} indeed plays the role of h from Theorem 2.5 for P; possibly the trivial one, if $\| \int u_{w(2r)}\, dp^{2r} \| = 0$. To exclude the trivial possibility we have to assume the existence of a nontrivial function u_{2r} for P, for some $r \geq 1$, that is condition (3.H2). Then by Theorem 2.5 we have $\lim\limits_{j\to\infty} P^j g = g_0$, for all $g \in G$. From this and the inequality

$$\|g_0 - Pg_0\| \leq \|g_0 - P^{j+1}g\| + \|P^j g - g_0\| \quad \text{for all} \quad g \in G,$$

it follows that $Pg_0 = g_0$, i.e. the density g_0 is P-invariant. This finishes the proof of the theorem. □

4 An application to a family $\{\varphi_y\}_{y\in Y}$ of $C^{1+\alpha}$, $0 < \alpha \le 1$ in $\mathbb{R}^d$

We use the following notation: $\mathbb{R}^d$ – d-dimensional Euclidean space ($d \ge 1$); $|\cdot|$ – the Euclidean norm; I – a domain in $\mathbb{R}^d$, i.e. an open, connected subset of $\mathbb{R}^d$; Σ – σ-algebra of all Borel-measurable subsets of I; m – the Lebesgue measure on $\mathbb{R}^d$; $\operatorname{diam}(A)$ – the diameter of the set A.

A $C^{1+\alpha}$-smooth Markov map φ, $0 < \alpha \le 1$, means a Markov map in the sense of Def. 2.2 and such that: the partition π of φ consists of domains, and the restriction φ_k, of φ to any $I_k \in \pi$, is a $C^{1+\alpha}$-diffeomorphism.

In this section we consider a family $\{\varphi_y\}_{y\in Y}$ of $C^{1+\alpha}$-smooth Markov maps which satisfy the following $C^{1+\alpha}$-variant of the so-called Reńyi's Condition (see e.g. [9] or [10]):

($4.\mathrm{H}_y4$) Let $\{\varphi_y\}_{y\in Y}$ be a family of $C^{1+\alpha}$-smooth Markov maps. There exist constants $C_{10,y(r)} > 0$, $y(r) \in Y^r$, such that for $k(r) \in K^r$, $r = 1, 2, \ldots$, and all $I_k \in \pi$ one has:

(a) $$|\sigma_{y(r)k(r)}(x) - \sigma_{y(r)k(r)}(y)| \le C_{10,y(r)}\sigma_{y(r)k(r)}(y)|x-y|^\alpha$$
for all $x, y \in J^{y(r)}_{k(r)} \cap I_k$, where $\sigma_{y(r)k(r)}$ is defined by (2.6), and $J^{y(r)}_{k(r)} = \varphi_{y(r)k(r)}(I^{y(r-1)}_{k(r)})$.

Furthermore, the constants $C_{10,y(r)} > 0$ satisfy the following condition:

(b) $$\limsup_{j\to\infty} \int C_{10,y(j)}\, dp^j < \infty.$$

Let $\{\varphi_y\}_{y\in Y}$ be a given family of $C^{1+\alpha}$-smooth Markov maps, and let $\{\pi_{y(r)} : y(r) \in Y^r, r = 1, 2, \ldots\}$ be a family of partitions whose elements are defined by (2.4). We assume that this family has the following generating property:

($4.\mathrm{H}_y7$) (Generating Condition on $\{\pi_{y(r)} : y(r) \in Y^r, r = 1, 2, \ldots\}$)

$$\lim_{j\to\infty} \int \left\{ \sup_{k(j+1)} \operatorname{diam}(I^{y(j)}_{k(j+1)})^\alpha \right\} dp^j = 0.$$

We are going now to examine the convergence of $\{P^j g\}$ under conditions (3.H2) ($4.\mathrm{H}_y4[a, b]$), and ($4.\mathrm{H}_y7$). We show that condition ($4.\mathrm{H}_y4[a, b]$) together with condition ($4.\mathrm{H}_y7$) implies condition (3.H1). Then under (3.H2) one gets the thesis of Th. 3.3. It turns out that one can take as a dense subset occurring in condition (3.H1) the following:

Definition 4.1. We denote by G_α, $0 < \alpha \le 1$, the class of all densities $g \in G$ satisfying the following three conditions:

(a) $\operatorname{spt}(g) \overset{\mathrm{df}}{=} \{x \in I : g(x) > 0\}$ is a sum of a number of $I_k \in \pi$;

(b) for each $I_k \in \pi$, $g_{|I_k} \in C^{0+\alpha}(I_k)$, and

$$|g(x) - g(y)| \le C(g)\, g(y)|x-y|^\alpha \text{ for all } x, y \in \operatorname{spt}(g) \cap I_k;$$

where $C(g)$ is a constant depending on g.

The following theorem is a consequence of Th. 3.3:

Theorem 4.2. *Let a family $\{\varphi_y\}_{y\in Y}$ of $C^{1+\alpha}$-smooth Markov maps satisfy conditions* ($4.\mathrm{H}_y4[a, b]$), ($4.\mathrm{H}_y7$), *and* (3.H2). *Then there exists exactly one P-fixed point $g_0 \in G$ such that*

$$\lim_{j\to\infty} P^j g = g_0, \text{ for all } g \in G.$$

Proof. We show that

$$G_\alpha \subseteq G(C^*) \quad \text{for an arbitrary} \quad C^* > \limsup_{j\to\infty} \int \ln C_{y(j)}(g)\, dp^j, \tag{4.1}$$

here in (4.1), for $g \in G_\alpha$, we define

$$C_{y(j)}(g) \stackrel{\text{df}}{=} \left\{1 + C(g) \sup_{k(j+1)\in K^{j+1}} \operatorname{diam}(I^{y(j)}_{k(j+1)})^\alpha\right\}\left\{1 + \widetilde{C}^\alpha_0 C_{10,y(j)}\right\}, \tag{4.2}$$

where $\widetilde{C}^\alpha_0 = \sup\{(\operatorname{diam}(I_k))^\alpha : k \in K\}$.

Note that by conditions (4.H$_y$4[b]) and (4.H$_y$7) we have

$$\limsup_{j\to\infty} \int \ln C_{y(j)(g)}\, dp^j \le \widetilde{C}^\alpha_0 \limsup_{j\to\infty} \int C_{10,y(j)}\, dp^j < \infty, \tag{4.3}$$

that is condition (b) of Def. 3.2 holds.

It remains to show the first condition of that definition holds. Let $g \in G_\alpha$, then for any $y(j) \in Y^j$, $k(j) \in K^j$, $j = 1, 2, \ldots$, and for any $x, z \in I_r$ the following inequality holds:

$$g \circ \varphi^{-1}_{y(j)k(j)}(x)/g \circ \varphi^{-1}_{y(j)k(j)}(z) \le 1 + C(g) \sup_{k(j+1)} \operatorname{diam}(I^{y(j)}_{k(j+1)})^\alpha.$$

Next, by condition (4.H$_y$4[a]), we have the following inequality (for any $y(j) \in Y^j$, $k(j) \in K^j$, $j = 1, 2, \ldots$, and for any $x, z \in I_r$):

$$\sigma_{y(j)k(j)}(x)/\sigma_{y(j)k(j)}(z) \le 1 + \widetilde{C}^\alpha_0 C_{10,y(j)},$$

where $\widetilde{C}^\alpha_0 = \sup\{(\operatorname{diam}(I_k))^\alpha : k \in K\}$.

Therefore for any $y(j) \in Y^j$, $j = 1, 2, \ldots$, $I_k \in \pi$, and for any $x, z \in I_k$ we have

$$P_{y(j)}g(x) \le C_{y(j)}(g) P_{y(j)}g(z),$$

where $C_{y(j)}$ are defined by (4.2). Hence condition (a) of Def. 3.2 holds for $g \in G_\alpha$ too.

The last inequality, and relations (4.2), (4.3) show that (4.1) holds. This implies condition (3.H1) because G_α is dense in G.

□

Remark 4.3. (Final Remark) We present two cases of particular nature of the system (3.3) (for more details see [7], Examples (5.1), and (5.3)).

Example 4.4. The first particular case is the following $x_j(x, \omega) = \varphi^{\xi_j(\omega)}(x_{j-1})$, for $j = 1, 2, \ldots$. The stochastic perturbation of the system arises from not knowing the precise number of iterations. That kind of stochastic perturbation has no influence on the statistical behaviour of the deterministic system $x_j = \varphi^j(x_{j-1})$, for $j = 1, 2, \ldots$.

Example 4.5. The second case of stochastic perturbation is the following $x_j(x, \omega) = \zeta_j(\omega)\varphi(x_{j-1})$, for $j = 1, 2, \ldots$. In that case stochastic perturbation appears in a multiplicative way (it is the so called parametric noise). Such a perturbation changes essentially the statistical behaviour of the system. It illustrates the example: Let $\varphi_y(x) = y\tan(x)$, $y \in Y = \{b, 1\}$; and $p_1(\zeta_j = b) = 1 - a$, and $p_1(\zeta_j = 1) = a$, for $j = 1, 2, \ldots$, where $b > 1$, and $0 < a < 1$.

Here $\varphi_1(x) = \tan(x)$ is (a Markov map) without any invariant density [11]. However, the considered random system has P-invariant density, but its deterministic counterpart, i.e. when $Y = \{1\}$ with $p_1(\zeta_j = 1) = 1$, not.

Acknowledgement: The authors thank the referees for their valuable remarks and comments on this paper.

References

[1] P. Bugiel, Distortion inequality for the Frobenius-Perron operator and some its consequences in ergodic theory of Markov maps in $\mathbb{R}^d$, *Annal. Polonici Math.* **LXVIII.2** (1998), 125–157.

[2] T. Bogenschütz and Z.S. Kowalski, A condition for mixing of skew products, *Aequationes Math.* **59** (2000), 222–234.
[3] A. Lasota and M.C. Mackey, Stochastic perturbation of dynamical systems: The weak convergence of measures, *J. Math. Anal. Appl.* **138** (1989), 232–248.
[4] S. Wędrychowicz and A. Wiśnicki, On some results on the stability of Markov operators, *Studia Math.* **241(1)** (2018), 41–55.
[5] O. Rechard, Invariant measures for many-one transformations, *Duke Math. J.* **23** (1956), 477–488.
[6] S.M. Ulam, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Appl. Math. no. 8, Interscience, New York, 1960.
[7] P. Bugiel, Ergodic properties of a randomly perturbed family of piecewise C^2-diffeomorphisms in $\mathbb{R}^d$, *Math. Z.* **224** (1997), 289–311.
[8] A. Lasota and J. Yorke, Exact dynamical systems and the Frobenius-Perron operator, *Trans. Amer. Math. Soc.* **273** (1982), 375–384.
[9] R. Mañé, *Ergodic Theory and Differentiable Dynamics*, Ergebnisse der Mathematic und ihrer Grenzgebiete, Springer, Berlin and New York, 1987.
[10] A. Rényi, Representation of real numbers and their ergodic properties, *Acta Math. Akad. Sc. Hungar* **8** (1957), 477–493.
[11] F. Schweiger, $\tan x$ is ergodic, *Proc. Amer. Math. Soc.* **71** (1978), 54–56.