Research article

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Analysis of a diffusive host-pathogen model with standard incidence and distinct dispersal rates

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Abstract: This paper concerns with detailed analysis of a reaction-diffusion host-pathogen model with space-dependent parameters in a bounded domain. By considering the fact the mobility of host individuals playing a crucial role in disease transmission, we formulate the model by a system of degenerate reaction-diffusion equations, where host individuals disperse at distinct rates and the mobility of pathogen is ignored in the environment. We first establish the well-posedness of the model, including the global existence of solution and the existence of the global compact attractor. The basic reproduction number is identified, and also characterized by some equivalent principal spectral conditions, which establishes the threshold dynamical result for pathogen extinction and persistence. When the positive steady state is confirmed, we investigate the asymptotic profiles of positive steady state as host individuals disperse at small and large rates. Our result suggests that small and large diffusion rate of hosts have a great impacts in formulating the spatial distribution of the pathogen.

Keywords: Host-pathogen model, Distinct dispersal rates, Global attractor, Basic reproduction number, Uniform persistence, Asymptotic profiles

MSC: 35K57, 92A15, 37N25, 35B40

1 Introduction

In recent years, the studies of some reaction-diffusion host-pathogen models have received much attentions, as the investigation of these systems allow us to get better understanding the interactions between host and pathogens and the mechanisms of the disease spread. Let $u_1(x,t)$, $u_2(x,t)$ and $u_3(x,t)$ be the densities of susceptible hosts, infected hosts and pathogen particles at the spatial location x and time t. Laplacian operator $d\Delta$ accounts for the host movement ($d > 0$ is the diffusion coefficient). In [6], under a one-dimensional and unbounded domain, the authors considered the following model,

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d\Delta u_1 + r\left(1 - \dfrac{u_1+u_2}{K}\right)u_1 - \beta u_1 u_3, & x \in \mathbb{R},\ t > 0, \\ \dfrac{\partial u_2}{\partial t} = d\Delta u_2 + \beta u_1 u_3 - \alpha u_2 - r\dfrac{u_1+u_2}{K}u_2, & x \in \mathbb{R},\ t > 0, \\ \dfrac{\partial u_3}{\partial t} = \lambda u_2 - \delta u_3, & x \in \mathbb{R},\ t > 0. \end{cases} \tag{1.1}$$

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where r and K represent respectively the reproductive rate and the carrying capacity of the susceptible and infected hosts; β and α are the transmission coefficient and disease-induced mortality rate; λ and δ are respectively the pathogens production rate from infected hosts and the pathogens' decay rate. All above mentioned coefficients in (1.1) are assumed to be positive constants. With the consideration of spatial spread of pathogens, the authors in [6] investigated the existence problem of traveling wave solution.

In reality, the habitats where hosts live should be a spatially bounded domain. Wang et al. [29] took the model (1.1) as a basis and extended it to a more general model in bounded spatial domain $\Omega \in \mathbb{R}^n$ with smooth boundary $\partial\Omega$,

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d\Delta u_1 + r\left(1 - \dfrac{u_1 + u_2}{K(x)}\right)u_1 - \beta(x)u_1u_3, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial u_2}{\partial t} = d\Delta u_2 + \beta(x)u_1u_3 - \alpha u_2 - r\dfrac{u_1 + u_2}{K(x)}u_2, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial u_3}{\partial t} = \lambda(x)u_2 - \delta u_3 - \beta(x)(u_1 + u_2)u_3, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \end{cases} \tag{1.2}$$

where $\frac{\partial u_1}{\partial n}$ stands for the differentiation along the unit outward normal n to $\partial\Omega$; $\beta(x)(u_1 + u_2)u_3$ represents the consumption of the pathogen due to the interaction with the hosts. Unlike in model (1.1), where parameters β, K, λ are constants, model (1.2) allows the space-dependent parameter functions, $\beta(\cdot)$, $K(\cdot)$ and $\lambda(\cdot)$, which are used to obey the spatial heterogeneity arising from the variance in environmental conditions (for example, temperature and humidity etc). Compared to model (1.1), the space-dependent functions $\beta(x)$, $K(x)$ and $\lambda(x)$ are assumed to be continuous and positive in Ω. Since the mobility of pathogen is ignored in the domain, the semiflow induced by solution lacks of compactness. The authors in [29] overcame this difficulty by verifying the k-contracting condition. The basic reproduction number (BRN) is proved to be a threshold index for the dynamics of (1.2), and defined by adopting the concept of next generation operator (NGO). Bifurcation analysis for steady state solution are also carried out when space-dependent parameters are used.

Note that the susceptible and infected hosts in (1.1) and (1.2) share the same diffusion rate. Wu and Zou [31] further generalized the model (1.2) with distinct diffusion rates d_1 and d_2,

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1\Delta u_1 + y(x) - \mu(x)u_1 - \beta(x)u_1u_3, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial u_2}{\partial t} = d_2\Delta u_2 + \beta(x)u_1u_3 - \upsilon(x)u_2, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial u_3}{\partial t} = \alpha(x)u_2 - \delta(x)u_3, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \end{cases} \tag{1.3}$$

Here, d_1 and d_2 are not necessarily equal. The simplest growth term for susceptible host, $y(x) - \mu(x)u_1$, is used. As pointed in [31] that distinct dispersal rates for hosts brings difficulty in proving the boundedness of the solution. The existence of global attractor needs appealing the general result in [8, Theorem 2.4.6] and Arezelà-Ascoli Theorem, so that the the asymptotic smoothness of semiflow is used instead of weak compactness by verifying the k-contraction condition. The authors studied the threshold dynamics of (1.3) and the effects of the spatial heterogeneity on disease dynamics. Wu and Zou in [31] also explored the asymptotic profiles of positive steady state for the case where $d_1 \to 0$ and $d_2 \to 0$.

Subsequently, inspired by the work [31], Shi et al. [23] revisited the model (1.2) by incorporating the horizontal disease transmission. The authors in [23] established the threshold dynamics, and performed the bifurcation analysis for steady state solution. Based on framework of [31], Wang and Wang [32] further extended the model (1.3) by adding the horizontal transmission term to (1.3). In [32], u_1, u_2 and u_3 are used to stand for respectively the density of susceptible, infected individuals and the concentration of vibrios in the environment. The horizontal transmission is termed as direct person-to-person transmission route in cholera dynamics. The complete analysis of (1.3) with horizontal transmission term demonstrated the threshold dynamics of (1.3) and the effect of the spatial heterogeneity on disease dynamics, and also revealed that ignoring horizontal transmission will underestimate the risk of disease transmission.

Our current work is also inspired by a series of works on diffusive models for disease dynamics in the spatial heterogeneous environment. These models are in the form of reaction-diffusion susceptible-infected-susceptible (SIS) equations in spatially bounded domain and the main concern are how the spatial heterogeneity and the diffusion affect the disease spread and control [1, 9, 10, 21, 31–33]. In an earlier article [1], the authors studied a SIS model with frequency-dependent interaction,

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_S \Delta S - \dfrac{\beta(x)SI}{S+I} + \gamma(x)I, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial I}{\partial t} = d_I \Delta I + \dfrac{\beta(x)SI}{S+I} - \gamma(x)I, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial S}{\partial n} = \dfrac{\partial I}{\partial n} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ \displaystyle\int_\Omega (S(x,0) + I(x,0))\mathrm{d}x \equiv N > 0, \end{cases} \tag{1.4}$$

where $S(x,t)$ and $I(x,t)$ represent respectively the density of susceptible and infected individuals. $\gamma(x)$ and $\beta(x)$ are respectively the space-dependent recovery rate and disease transmission rate. d_S and d_I represent respectively the diffusion rates of susceptible and infected individuals. The total number of human individuals remains constant N. The main concern in the aspect of biological implication is: limiting the flow of susceptible individuals ($d_S \to 0$) can eliminate the disease, provided that the disease is of low risk (i.e., $\beta(\cdot) < \gamma(\cdot)$ for $x \in \Omega$). This pioneering work start up the investigation that how the spatial heterogeneity and the diffusion affect the disease spread and control. Subsequently, the result that limiting the flow of the infected individuals can not eliminate the disease (see in [21]) revealed that d_S and d_I play different role in disease control. Motivated by meaningful and important aspect of spatial heterogeneity of environment and distinct dispersal rates, Wu and Zou [33] further modified the model (1.4) by replacing the frequency-dependent interaction with mass action mechanisms. They showed that an additional condition on the total population is needed for disease control if $d_S \to 0$. Additionally, disease can not be controlled when $d_S \to 0$ and the total population accounts large, and inversely, disease disappears in certain area when $d_I \to 0$. In contrast with [1, 21, 33], Li et al. [9, 10] analyzed an spatial SIS model with linear source and logistic source (which allows varying total population). With small and large diffusion rates, both of the works revealed that disease can not be controlled, which is not a good situation in disease control.

Based on the above mentioned works, we continue to explore how diffusion rates and the spatial heterogeneity affect the dynamics of (1.3) by incorporating the frequency-dependent interaction used in (1.4), and thus, can be considered as a continuation of the work [1, 9, 21, 31–33]. We shall explore the following system,

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \Delta u_1 + b(x) - \beta_1(x)\dfrac{u_1 u_2}{u_1+u_2} - \beta_2(x)u_1u_3 - \zeta(x)u_1, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial u_2}{\partial t} = d_2 \Delta u_2 + \beta_1(x)\dfrac{u_1 u_2}{u_1+u_2} + \beta_2(x)u_1u_3 - \gamma(x)u_2, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial u_3}{\partial t} = \varrho(x)u_2 - \delta(x)u_3, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \end{cases} \tag{1.5}$$

with the initial condition

$$u_i(x,0) = u_{i0}(x),\ x \in \Omega,\ i = 1,2,3, \tag{1.6}$$

where $\zeta(x)$ and $\gamma(x)$ represent the death rates of susceptible and infected host; $b(x)$ is the recruitment rate; $\beta_i(x)(i = 1,2)$ represent the disease transmission rate; $\delta(x)$ is the pathogens' decay rate; $\varrho(x)$ is the pathogens' production rate. By replacing the frequency-dependent interaction with mass action mechanisms (direct person-to-person transmission route), model (1.5) reduced to the model in [32]. Even though (1.5) bears a resemblance to that in [32], there is one major difference: frequency-dependent interaction $\frac{\beta_1(x)u_1u_2}{u_1+u_2}$ assumes bounded infection force, while unbounded infection mechanism for mass action term.

Note that when $d_1 = d_2 = 0$ and all parameters are space-independent, the reduced system of (1.5) is the same as the model [24] with standard incidence function for the direct disease transmission and bilinear incidence function for indirect disease transmission. It is mentioned in [24] that disease transmission within and without groups may be different. In this sense, it is natural to assume that the contact probability between a susceptible and an infected host is decreasing function of total host population (standard incidence function) and the contact probability between a susceptible host and pathogen is a constant (bilinear incidence function). Meanwhile, the model studied in [24] can also provide us a biological interpretation for our model (1.5) that: 1) In Zika virus transmission, u_1, u_2 can respectively stands for uninfected individuals, infected individuals, and u_3 stands for the infected mosquitoes in the local landscape; 2) In H1N1 and seasonal influenzas, u_1, u_2 represent respectively the uninfected and infected individuals, and u_3 represents the contaminated environment such as classrooms, or other public places; 3) In the transmission of avian influenza, u_1, u_2 can respectively stand for the uninfected migratory birds, infected migratory birds, and u_3 stands for infected domestic poultry. On the other hand, (1.5) can be used to describe the transmission of cholera in the sense that u_1, u_2 and u_3 stand for respectively the density of susceptible, infected individuals and the concentration of vibrios in the environment. However, it is well-known that those who recovered from cholera do lose immunity. A realistic excuse to justify the hypothesis that infected individuals do not lose immunity may be that if we care about this model during one outbreak, then those who recovered are very likely to remain immune throughout the outbreak.

We plan to proceed this paper as follows. Section 2 shall pay attention to the well-posedness of (1.5) with (1.6) such as, the global existence and uniqueness and ultimate boundedness of solution of (1.5), the asymptotic smoothness condition of semiflow, the existence of global compact attractor. In Section 3, we identify the basic reproduction number, $\Re_0$. We also establish that $\Re_0$ can be equivalently characterized as the principal spectral conditions. In section 4, with the $\Re_0$, detailed analysis are carried out on the threshold dynamics of (1.5), that is, $\Re_0$ predicts whether or not the disease persist. In a critical case that $\Re_0 = 1$, the global asymptotic stability of disease free steady state is also addressed. Section 5 is spent on the dynamics of (1.5) in homogeneous case. We addressed the existence of unique positive equilibrium and local stability. The global attractivity of positive equilibrium with additional condition is achieved by the technique of Lyapunov function. Section 6 is devoted to exploring the asymptotic profiles of the positive steady state for the case that $d_1 \to 0$, $d_1 \to \infty$, $d_2 \to 0$ and $d_2 \to \infty$. Finally, detailed conclusions are drawn and some discussion is presented.

2 Well-posedness of the problem

The main aim of this section is to confirm that (1.5) has a global compact attractor. Throughout of the paper,

- $\mathbb{X} := C(\bar{\Omega}, \mathbb{R}^3)$, $\mathbb{Y} := C(\bar{\Omega}, \mathbb{R}^2)$, $\mathbb{Z} := C(\bar{\Omega}, \mathbb{R})$ are respectively the Banach space with the supremum norm $\|\cdot\|$, and their positive cone are denoted by $\mathbb{X}^+ := C(\bar{\Omega}, \mathbb{R}^3_+)$, $\mathbb{Y}^+ := C(\bar{\Omega}, \mathbb{R}^2_+)$, $\mathbb{Z}^+ := C(\bar{\Omega}, \mathbb{R}_+)$, respectively.
- $$G^* := \max_{x\in\bar{\Omega}} G(\cdot), \ \ G_* = \min_{x\in\bar{\Omega}} G(\cdot),$$
  where $G(\cdot) = y(\cdot), \delta(\cdot), \varrho(\cdot), \zeta(\cdot), \beta_1(\cdot), \beta_2(\cdot)$, respectively.
- Γ is the Green function of $\frac{\partial u}{\partial t} = \Delta u$ in Ω with the Neumann boundary condition.
- $A_1(t),\ A_2(t) : C(\bar{\Omega}, \mathbb{R}) \to C(\bar{\Omega}, \mathbb{R})$ are respectively the C_0 semigroups of $d_1\Delta - \zeta(x)$ and $d_2\Delta - y(x)$ with the Neumann boundary condition, i.e., for any $\varphi \in C(\bar{\Omega}, \mathbb{R})$,
  $$(A_1(t)\varphi)(\cdot) = e^{-\zeta(\cdot)t} \int_\Omega \Gamma(d_1 t, \cdot, y)\varphi(y)\mathrm{d}y, \tag{2.1}$$
  and
  $$(A_2(t)\varphi)(\cdot) = e^{-y(\cdot)t} \int_\Omega \Gamma(d_2 t, \cdot, y)\varphi(y)\mathrm{d}y. \tag{2.2}$$
  We further, denote
  $$(A_3(t)\varphi)(x) = e^{-\delta(x)t}\varphi(x).$$

According to [22, Section 7.1], $A_1(t), A_2(t) : C(\bar{\Omega}, \mathbb{R}) \to C(\bar{\Omega}, \mathbb{R})$, $t > 0$, are strong positive and compact. Hence, for any $\varphi \in \mathbb{X}^+$,

$$(\mathcal{A}(t)\varphi)(\cdot) := ((A_1(t)\varphi)(\cdot), (A_2(t)\varphi)(\cdot), (A_3(t)\varphi)(\cdot))^T = \begin{pmatrix} e^{-\zeta(\cdot)t} \int_\Omega \Gamma(d_1 t, \cdot, y)\varphi_1(y)\mathrm{d}y \\ e^{-\gamma(\cdot)t} \int_\Omega \Gamma(d_2 t, \cdot, y)\varphi_2(y)\mathrm{d}y \\ e^{-\delta(\cdot)t}\varphi_3(\cdot) \end{pmatrix}, \ t \geq 0,$$

forms a C_0 semigroup on $\mathbb{X}$ preserving $\mathbb{X}^+$, namely, $\mathcal{A}(t)\mathbb{X}^+ \subset \mathbb{X}^+$, $t \geq 0$ (see, for example, [19]).

Further, let

$$\mathcal{F}(\varphi) := (F_1(\varphi)(\cdot), F_2(\varphi)(\cdot), F_3(\varphi)(\cdot)) = \begin{pmatrix} b(\cdot) - \beta_1(\cdot)\frac{\varphi_1\varphi_2}{\varphi_1+\varphi_2} - \beta_2(\cdot)\varphi_1\varphi_3 \\ \beta_1(\cdot)\frac{\varphi_1\varphi_2}{\varphi_1+\varphi_2} + \beta_2(\cdot)\varphi_1\varphi_3 \\ \varrho(\cdot)\varphi_2 \end{pmatrix}.$$

By these settings, we can rewrite (1.5) as

$$u(t) = \mathcal{A}(t)\varphi + \int_0^t \mathcal{A}(t-s)\mathcal{F}(u(s))\mathrm{d}s. \tag{2.3}$$

We can easily check that

$$\lim_{h\to 0^+} \operatorname{dist}(\varphi + h\mathcal{F}(\varphi), \mathbb{X}^+) = 0, \ \forall\, \varphi \in \mathbb{X}^+.$$

In fact, it is easy to verify that for any $\varphi \in \mathbb{X}^+$ and small enough $h \geq 0$,

$$\begin{aligned} \varphi + h\mathcal{F}(\varphi) &= \begin{pmatrix} \varphi_1 + h\left(b(\cdot) - \beta_1(\cdot)\frac{\varphi_1\varphi_2}{\varphi_1+\varphi_2} - \beta_2(\cdot)\varphi_1\varphi_3\right) \\ \varphi_2 + h\left(\beta_1(\cdot)\frac{\varphi_1\varphi_2}{\varphi_1+\varphi_2} + \beta_2(\cdot)\varphi_1\varphi_3\right) \\ \varphi_3 + h\left(\varrho(\cdot)\varphi_2\right) \end{pmatrix} \\ &\geq \begin{pmatrix} \varphi_1\left[1 - h\left(\beta_1^\star\frac{\varphi_2}{\phi_1+\varphi_2} + \beta_2^\star\varphi_3\right)\right] \\ \varphi_2 \\ \varphi_3 \end{pmatrix}. \end{aligned}$$

As just a consequence of [22, Corollary 4], we have

Lemma 2.1. *For any $\phi \in \mathbb{X}^+$, (1.5) with (1.6) admits a unique nonnegative solution $u(\cdot, t) := (u_1(\cdot, t), u_2(\cdot, t), u_3(\cdot, t))$ on $\bar{\Omega} \times [0, t_{max})$ with $t_{max} \leq \infty$. Furthermore, $u(\cdot, t; \phi) \in \mathbb{X}^+$, $t \in [0, t_{max})$.*

In what follows, we shall confirm $t_{max} = \infty$ by verifying the boundedness of $u(\cdot, t)$ in $\Omega \times (0, t_{max})$.

2.1 Existence of the global solution

Theorem 2.1. *For any $\phi \in \mathbb{X}^+$, (1.5) with (1.6) admits a unique nonnegative solution defined on $\bar{\Omega} \times [0, \infty)$. Furthermore, the semiflow $Y(t) : \mathbb{X}^+ \to \mathbb{X}^+$ defined by*

$$Y(t)\phi = (u_1(\cdot, t; \phi), u_2(\cdot, t; \phi), u_3(\cdot, t; \phi)), \ \forall (x, t) \in \bar{\Omega} \times [0, \infty), \tag{2.4}$$

is ultimately bounded.

Proof. We first prove that the boundedness of u_1. From (1.5), $u_1(x, t)$ is governed by

$$\begin{cases} \dfrac{\partial \bar{u}_1}{\partial t} = d_1\Delta\bar{u}_1 + b(\cdot) - m(\cdot)\bar{u}_1, & (x, t) \in \Omega \times (0, \infty), \\ \dfrac{\partial \bar{u}_1}{\partial \nu} = 0, & (x, t) \in \partial\Omega \times (0, \infty). \end{cases} \tag{2.5}$$

By the comparison principle, we directly have

$$\limsup_{t\to\infty} u_1(\cdot,t) \le \lim_{t\to\infty} \bar{u}_1(\cdot,t) = \bar{u}_1^\star(\cdot), \text{ uniformly for } x \in \bar{\Omega}, \tag{2.6}$$

where $\bar{u}_1^\star(\cdot)$ is a unique global asymptotic stable steady state of (2.5) in $C(\bar{\Omega},\mathbb{R})$. It follows that

$$\| u_1(\cdot,t) \| \le M_0,$$

where $M_0 = \| \bar{u}_1^\star(\cdot) \|$, independent of initial conditions.

Next, we verify that the solution u_2 and u_3 of (1.5) are ultimately bounded. For this purpose, we first establish the uniform estimate in $L^1(\Omega)\times L^1(\Omega)$ and then pass it to the uniform estimate in $C(\Omega)\times C(\Omega)$.

- $u(\cdot,t)$ satisfies the L^1 bounded estimate, i.e.,

$$\limsup_{t\to\infty}(\|u_1(\cdot,t)\|_{L^1} + \|u_2(\cdot,t)\|_{L^1} + \|u_3(\cdot,t)\|_{L^1}) \le \mathbb{M}_1,$$

where $\mathbb{M}_1$ is a positive constant.
By (1.5), we have

$$\begin{aligned}\frac{d}{dt}\|u_1(\cdot,t)+u_2(\cdot,t))\|_{L^1} &= \int_\Omega b(\cdot)\mathrm{d}x - \int_\Omega \zeta(\cdot)u_1(\cdot,t)\mathrm{d}x - \int_\Omega y(\cdot)u_2(\cdot,t)\mathrm{d}x\\ &\le |\Omega|b^\star - h\|u_1(\cdot,t)+u_2(\cdot,t)\|_{L^1},\end{aligned}$$

where $h = \min_{x\in\bar{\Omega}}\{\zeta_\star, y_\star\}$ and $|\Omega|$ is the volume of Ω. Hence,

$$\limsup_{t\to\infty}(\|u_1(\cdot,t)\|_{L^1} + \|u_2(\cdot,t)\|_{L^1}) \le M_1, \text{ where } M_1 = |\Omega|b^\star/h.$$

Further, we have

$$\frac{d}{dt}\|u_3(\cdot,t)\|_{L^1} \le \varrho^\star M_1 - \delta_\star\|u_3(\cdot,t)\|_{L^1}.$$

It follows that

$$\limsup_{t\to\infty}\|u_3(\cdot,t)\|_{L^1} \le M_2, \text{ where } M_2 = \varrho^\star M_1/\delta_\star.$$

Hence, by taking $\mathbb{M}_1 = \max\{M_1, M_2\}$, the assertion directly follows.

- For $k \ge 0$, u_2 and u_3 satisfies the L^{2^k} bounded estimate, that is,

$$\limsup_{t\to\infty}\left(\|u_2(\cdot,t)\|_{2^k} + \|u_3(\cdot,t)\|_{2^k}\right) \le M_{2^k}, \tag{2.7}$$

where M_{2^k} is a positive constant.
We shall prove (2.7) by induction. Obviously, $k = 0$ holds. Suppose that (2.7) holds for $k-1$. Then for $M_{2^{k-1}} > 0$, we have

$$\limsup_{t\to\infty}\left(\|u_2(\cdot,t)\|_{2^{k-1}} + \|u_3(\cdot,t)\|_{2^{k-1}}\right) \le M_{2^{k-1}}. \tag{2.8}$$

We multiply u_2 by $u_2^{2^k-1}$, and then integrate it over Ω,

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega u_2^{2^k}\mathrm{d}x \le d_2\int_\Omega u_2^{2^k-1}\triangle u_2\mathrm{d}x + \int_\Omega \beta_2(\cdot)u_1u_3u_2^{2^k-1}\mathrm{d}x + \int_\Omega \beta_1(\cdot)u_2^{2^k}\mathrm{d}x - \int_\Omega y(\cdot)u_2^{2^k}\mathrm{d}x. \tag{2.9}$$

Recall that

$$d_2\int_\Omega u_2^{2^k-1}\triangle u_2\mathrm{d}x = -d_2\int_\Omega \nabla u_2\cdot\nabla u_2^{2^k-1}\mathrm{d}x = -(2^k-1)d_2\int_\Omega(\nabla u_2\cdot\nabla u_2)u_2^{2^k-2}\mathrm{d}x = -\frac{2^k-1}{2^{2k-2}}d_2\int_\Omega|\nabla u_2^{2^{k-1}}|^2\mathrm{d}x.$$

Hence, (2.9) becomes

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega u_2^{2^k}\mathrm{d}x \le -D_k\int_\Omega |\nabla u_2^{2^{k-1}}|^2\mathrm{d}x + \int_\Omega \beta_2(\cdot)u_1u_3u_2^{2^k-1}\mathrm{d}x + \int_\Omega \beta_1(\cdot)u_2^{2^k}\mathrm{d}x - \int_\Omega y(\cdot)u_2^{2^k}\mathrm{d}x. \tag{2.10}$$

where $D_k = \frac{2^k-1}{2^{2k-2}}d_2$.

By $\limsup_{t\to\infty}\|u_1(\cdot,t)\|_{L^1} \le M_0$, there is $t_0 > 0$ such that for $t \ge t_0$,

$$\int_\Omega \beta_1(\cdot)u_2^{2^k}\mathrm{d}x \le \beta_1^\star\int_\Omega u_2^{2^k}\mathrm{d}x \tag{2.11}$$

and

$$\int_\Omega \beta_2(\cdot)u_1u_2^{2^k-1}u_3\mathrm{d}x \le \beta_2^\star(M_0+1)\int_\Omega u_3u_2^{2^k-1}\mathrm{d}x. \tag{2.12}$$

Applying Young's inequality: $ab \le \epsilon a^p + \frac{1}{q}(\epsilon p)^{-\frac{q}{p}}b^q$, where $a, b, \epsilon > 0$, $1 < p, q < \infty$ and $\frac{1}{p}+\frac{1}{q} = 1$. One can estimate (2.12) by setting $\epsilon_1 = \frac{\delta_\star}{4\beta_2^\star(M_0+1)}$, $p = 2^k$ and $q = 2^k/(2^k-1)$ as follows,

$$\int_\Omega u_3u_2^{2^k-1}\mathrm{d}x \le \frac{\delta_\star}{4\beta_2^\star(M_0+1)}\int_\Omega u_3^{2^k}\mathrm{d}x + C_{\epsilon_1}\int_\Omega u_2^{2^k}\mathrm{d}x,\ \text{for } t \ge t_0, \text{where } C_{\epsilon_1} = \frac{2^k-1}{2^k}(2^k\epsilon_1)^{-\frac{1}{2^k-1}}. \tag{2.13}$$

Thus, (2.10) can be estimated by

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega u_2^{2^k}\mathrm{d}x \le -D_k\int_\Omega |\nabla u_2^{2^{k-1}}|^2\mathrm{d}x + \frac{\delta_\star}{4}\int_\Omega u_3^{2^k}\mathrm{d}x + C_k\int_\Omega u_2^{2^k}\mathrm{d}x, \tag{2.14}$$

where $C_k = \beta_1^\star + \beta_2^\star(M_0+1)C_{\epsilon_1}$.

We multiply u_3 by $u_3^{2^k-1}$, and then integrate it over Ω,

$$\begin{aligned}\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega u_3^{2^k}\mathrm{d}x &= \int_\Omega \varrho(x)u_3^{2^k-1}u_2\mathrm{d}x - \int_\Omega \delta(x)u_3^{2^k}\mathrm{d}x\\ &\le \varrho^\star\int_\Omega u_3^{2^k-1}u_2\mathrm{d}x - \delta_\star\int_\Omega u_3^{2^k}\mathrm{d}x.\end{aligned} \tag{2.15}$$

Again applying Young's inequality (by setting $\epsilon_2 = \frac{\delta_\star}{4\varrho^\star}$, $p = 2^k/(2^k-1)$ and $q = 2^k$), we have

$$\int_\Omega u_3^{2^k-1}u_2\mathrm{d}x \le \frac{\delta_\star}{4\varrho^\star}\int_\Omega u_3^{2^k}\mathrm{d}x + C_{\epsilon_2}\int_\Omega u_2^{2^k}\mathrm{d}x, \text{where } C_{\epsilon_2} = \epsilon_2^{1-2^k}\frac{(2^k-1)^{2^k-1}}{(2^k)^{2^k}}. \tag{2.16}$$

Hence (2.15) becomes

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega u_3^{2^k}\mathrm{d}x \le -\frac{3\delta_\star}{4}\int_\Omega u_3^{2^k}\mathrm{d}x + \varrho^\star C_{\epsilon_2}\int_\Omega u_2^{2^k}\mathrm{d}x. \tag{2.17}$$

Combined with (2.14) and (2.17), we obtain

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega (u_2^{2^k}+u_3^{2^k})\mathrm{d}x \le -D_k\int_\Omega |\nabla u_2^{2^{k-1}}|^2\mathrm{d}x + E_k\int_\Omega u_2^{2^k}\mathrm{d}x - \frac{\delta_\star}{2}\int_\Omega u_3^{2^k}\mathrm{d}x,\ \text{for } t \ge t_0, \tag{2.18}$$

where $E_k = C_k + \varrho^\star C_{\epsilon_2}$.

Applying interpolation inequality:

$$\|\xi\|_2^2 \le \epsilon\|\nabla\xi\|_2^2 + C_\epsilon\|\xi\|_1^2,\ \text{where } \epsilon > 0,\ C_\epsilon > 0,\ \xi \in W^{1,2}(\Omega).$$

Let $\epsilon_3 = D_k/(2E_k)$, $\xi = u_2^{2^{k-1}}$, then

$$-D_k \int_\Omega |\nabla u_2^{2^{k-1}}|^2 \mathrm{d}x \le -2E_k \int_\Omega u_2^{2^k} \mathrm{d}x + 2E_k C_{\epsilon_3} \left(\int_\Omega u_2^{2^{k-1}} \mathrm{d}x \right)^2 .$$

Thus, (2.18) becomes

$$\frac{1}{2^k} \frac{\partial}{\partial t} \int_\Omega (u_2^{2^k} + u_3^{2^k}) \mathrm{d}x \le -r_\star \int_\Omega (u_2^{2^k} + u_3^{2^k}) \mathrm{d}x + 2E_k C_{\epsilon_3} \left(\int_\Omega u_2^{2^{k-1}} \mathrm{d}x \right)^2 , \quad \text{for } t \ge t_0, \tag{2.19}$$

where $r_\star = \min\{E_k, \frac{\delta_\star}{2}\}$.

It then follows from (2.8) that $\limsup_{t\to\infty} \int_\Omega u_2^{2^{k-1}} \mathrm{d}x \le M_{2^{k-1}}^{2^{k-1}}$, which in turn implies that

$$\limsup_{t\to\infty} \left(\|u_2(\cdot,t)\|_{2^k} + \|u_3(\cdot,t)\|_{2^k} \right) \le M_{2^k}, \text{ with } M_{2^k} = \sqrt[2^k]{\frac{2E_k C_{\epsilon_3}}{r_\star}} M_{2^{k-1}}.$$

Thus, according to continuous embedding $L^q(\Omega) \subset L^p(\Omega)$, $q \ge p \ge 1$, we have

$$\limsup_{t\to\infty} (\|u_2(\cdot,t)\|_{L^p} + \|u_3(\cdot,t)\|_{L^p}) \le M_p, \tag{2.20}$$

where $M_p > 0$, independent of initial conditions. Denote by Y_a, $0 \le a \le 1$ the fractional power space. By [31, Lemma 2.4], we obtain that $Y_a \subset C(\bar\Omega)$ by selecting $p > n/2$ and $a \ge n/(2p)$. Hence $\limsup_{t\to\infty} \|u_2(\cdot,t)\| \le M_\infty$, where $M_\infty > 0$. Further, $\limsup_{t\to\infty} \|u_3(\cdot,t)\| \le \frac{\varrho^* M_\infty}{\delta_\star}$. This proves the ultimate boundedness of the solution of (1.5). The proof is cpmplete. □

2.2 Asymptotic smoothness of $Y(t)$

We refer the readers to consult [8, 36] for the definition of $\kappa(\cdot)$, the Kuratowski measure of noncompactness.

Lemma 2.2. *The semigroup $Y(t)$ is a κ-contraction on $\mathbb{X}^+$, that is, for any bounded set $\mathbb{B} \subseteq \mathbb{X}^+$,*

$$\kappa(Y(t)\mathbb{B}) \le e^{-\delta_\star t} \kappa(\mathbb{B}),$$

where $\delta_\star = \min_{x\in\bar\Omega} \delta(x)$.

Proof. For $t \ge 0$, let $Y(t) = Y_1(t) + Y_2(t)$, where

$$Y_1(t)\phi = \left\{ u_1(\cdot,t;\phi), u_2(\cdot,t;\phi), \int_0^t e^{-\delta(\cdot)(t-s)} \varrho(\cdot) u_2(\cdot,s;\phi) \mathrm{d}s \right\},$$

and

$$Y_2(t)\phi = \left\{ 0, 0, e^{-\delta(\cdot)t} u_{30}(x) \right\}.$$

By [31, Lemma 2.5], $Y_1(t)\mathbb{B}$ is precompact. Thus, we have $\kappa(Y_1(t)\mathbb{B}) = 0$. We estimate $Y_2(t)$ as

$$\left\| Y_2(t) \right\| = \sup_{\psi\in\mathbb{X}} \frac{\left\| Y_2(t)\psi \right\|_{\mathbb{X}}}{\|\psi\|_{\mathbb{X}}} \le \mathrm{e}^{-\delta_\star t} \sup_{\psi\in\mathbb{X}} \frac{\|\psi\|_{\mathbb{X}}}{\|\psi\|_{\mathbb{X}}} = \mathrm{e}^{-\delta_\star t}.$$

Hence,

$$\kappa(Y(t)\mathbb{B}) \le \kappa(Y_1(t)\mathbb{B}) + \kappa(Y_2(t)\mathbb{B}) \le 0 + \left\| Y_2(t) \right\| \kappa(\mathbb{B}) \le \mathrm{e}^{-\delta_\star t} \kappa(\mathbb{B}),$$

that is, $Y(t)$ is κ-contraction on $\mathbb{X}^+$. This completes the proof. □

Theorem 2.2. *(1.5) possesses a global compact attractor in $\mathbb{X}^+$, denoted by $\mathcal{A}_0$.*

Proof. By Theorem 2.1, the solution of (1.5) exists globally and $Y(t)$ is point dissipative (see Lemma 2.2). Asymptotic smoothness condition of $Y(t)$ is implied by κ-contraction condition. Hence, by [8, Theorem 2.4.6], system (1.5) possesses a global compact attractor, which attracts every bounded set in $\mathbb{X}^+$. □

3 Basic reproduction number

Obviously, (1.5) has a disease-free steady state $E_0 = (u_1^0(x), 0, 0)$, where $u_1^0(x) = \frac{b(x)}{\zeta(x)}$ and satisfies $d\Delta u_1^0(x) + b(x) - m(x)u_1^0(x) = 0$, for $x \in \Omega$ and $\frac{\partial u_1^0(x)}{\partial \nu} = 0$, for $x \in \partial\Omega$. The linearization of (1.5) at E_0 leads to the following cooperative system,

$$\begin{cases} \begin{pmatrix} \frac{\partial u_2}{\partial t} \\ \frac{\partial u_3}{\partial t} \end{pmatrix} = \mathcal{B} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}, & (x, t) \in \Omega \times (0, \infty), \\ \dfrac{\partial u_2}{\partial n} = 0, & (x, t) \in \partial\Omega \times (0, \infty), \end{cases} \tag{3.1}$$

where

$$\mathcal{B} = \begin{pmatrix} d_2\Delta + \beta_1(\cdot) - \gamma(\cdot) & \beta_2(\cdot)u_1^0(\cdot) \\ \varrho(\cdot) & -\delta(\cdot) \end{pmatrix}. \tag{3.2}$$

Substituting $(u_2(\cdot, t), u_3(\cdot, t)) := e^{\lambda t}(\phi_2(\cdot), \phi_3(\cdot))$ into (3.1) gets

$$\begin{cases} \lambda \begin{pmatrix} \phi_2 \\ \phi_3 \end{pmatrix} = \mathcal{B} \begin{pmatrix} \phi_2 \\ \phi_3 \end{pmatrix}, & (x, t) \in \Omega \times (0, \infty), \\ \dfrac{\partial \phi_2}{\partial n} = 0, & (x, t) \in \partial\Omega \times (0, \infty). \end{cases} \tag{3.3}$$

We rewrite

$$\mathcal{B} := B + F = \begin{pmatrix} d_2\Delta + \beta_1(\cdot) - \gamma(\cdot) & 0 \\ \varrho(\cdot) & -\delta(\cdot) \end{pmatrix} + \begin{pmatrix} 0 & \beta_2(\cdot)u_1^0(\cdot) \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that both $\mathcal{B}$ and B are resolvent-positive operators [27]. Denote by $T(t)$(resp. $\tilde{T}(t)$) $: \mathbb{Y} \to \mathbb{Y}$ the positive semigroup generated by $\mathcal{B}$ (resp. B). Since both $\mathcal{B}$ and B are cooperative for any $x \in \Omega$, we get that $T(t)\mathbb{Y}_+ \subseteq \mathbb{Y}_+$ (resp. $\tilde{T}(t)\mathbb{Y}_+ \subseteq \mathbb{Y}_+$). Throughout of the paper, we denote by $s(Q) = \sup\{\mathrm{Re}\lambda, \lambda \in \sigma(\mathrm{Q})\}$ the spectral bound of Q and $r(Q) = \sup\{|\lambda|, \lambda \in \sigma(Q)\}$, the the spectral radius of Q.

Following the standard procedures in [27, 30], we define the NGO $\mathcal{L}$ as

$$\mathcal{L}[\phi](\cdot) = \int_0^\infty F(\cdot)\tilde{T}(t)\phi(\cdot)\mathrm{d}t = F(\cdot)\int_0^\infty \tilde{T}(t)\phi(\cdot)\mathrm{d}t, \quad \phi \in \mathbb{Y}, \quad x \in \bar{\Omega},$$

that is, within the infection period, $\mathcal{L}[\phi](\cdot)$ represents the total new infections distribution from initial distribution ϕ. Then, the BRN $\Re_0$ is defined as

$$\Re_0 := r(\mathcal{L}) \tag{3.4}$$

The following result is a consequence of [27, 30].

Lemma 3.1. *Let $\mathcal{B}$ and $\Re_0$ be defined in (3.4) and (3.2), respectively. $s(\mathcal{B})$ has the same sign as $\Re_0 - 1$.*

Proof. Due to the fact that B is resolvent-positive operator, we then have

$$(\lambda I - B)^{-1}\phi = \int_0^\infty e^{-\lambda t}\tilde{T}(t)\phi\mathrm{d}t, \forall\, \lambda > s(B), \phi \in \mathbb{X}^+. \tag{3.5}$$

Due to $s(B) < 0$, we can let $\lambda = 0$ in (3.5), leading to

$$-B^{-1}\phi = \int_0^\infty \tilde{T}(t)\phi\mathrm{d}t, \forall\, \lambda > s(B), \phi \in \mathbb{X}^+. \tag{3.6}$$

That is, $\mathcal{L} = -FB^{-1}$. Further, $\mathcal{B} = B + F$ can be viewed as the perturbation of B. According to [27, Theorem 3.5], $s(\mathcal{B})$ has the same sign as $r(-FB^{-1}) - 1 = \Re_0 - 1$. □

For the convenience of forthcoming discussions, we next claim that $\Re_0$ has relationship with other important indicators: $\tilde{\lambda}_0$ and η^0.

Lemma 3.2. *Let $\Re_0$ be defined by (3.4). Then we have*
(i) $\Re_0 = 1/\tilde{\lambda}_0$, *where $\tilde{\lambda}_0$ is the principal eigenvalue of*

$$\begin{cases} d_2\Delta\varphi - y(\cdot)\varphi + \tilde{\lambda}\left(\beta_1(\cdot) + \dfrac{\beta_2(\cdot)u_1^0(\cdot)\varrho(\cdot)}{\delta(\cdot)}\right)\varphi = 0, & x \in \Omega, \\ \dfrac{\partial\varphi}{\partial\nu} = 0, & x \in \partial\Omega; \end{cases} \tag{3.7}$$

(ii) $\Re_0 - 1$ *and* $s(\mathcal{B})$ *have the same sign as η^0, where η^0 is the principal eigenvalue of*

$$\begin{cases} d_2\Delta\phi + \left(\beta_1(\cdot) + \dfrac{\beta_2(\cdot)u_1^0(\cdot)\varrho(\cdot)}{\delta(\cdot)} - y(\cdot)\right)\phi = \eta\phi, & x \in \Omega, \\ \dfrac{\partial\phi}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.8}$$

Proof. We first prove (i). We rewrite

$$F = \begin{pmatrix} \beta_1(\cdot) & \beta_2(\cdot)u_1^0(\cdot) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}, \ B = \operatorname{diag}(d_2\Delta, 0) - V,$$

where

$$V = \begin{pmatrix} y(\cdot) & 0 \\ -\varrho(\cdot) & \delta(\cdot) \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}.$$

As $F_{21} = 0$, $F_{22} = 0$, by [30, Theorem 3.3], we know that $\Re_0 = r(-B^{-1}F) = r(-B_1^{-1}F_2)$, where

$$B_1 = d_2\Delta - V_{11} + V_{12}V_{22}^{-1}V_{21} = d_2\Delta - y(\cdot) \text{ and } F_2 = F_{11} - F_{12}V_{22}^{-1}V_{21} = \beta_1(\cdot) + \beta_2 u_1^0(\cdot)\varrho(\cdot)/\delta(\cdot).$$

Hence,

$$(-B_1^{-1}F_2)\varphi = -(d_2\Delta - y(\cdot))^{-1}\left(\beta_1(\cdot) + \frac{\beta_2(\cdot)u_1^0(\cdot)\varrho(\cdot)}{\delta(\cdot)}\right)\varphi.$$

Therefore, $\Re_0$ satisfies

$$-\left((d_2\Delta - y(\cdot))^{-1}\left(\beta_1(\cdot) + \frac{\beta_2(\cdot)u_1^0(\cdot)\varrho(\cdot)}{\delta(\cdot)}\right)\right)\varphi = \Re_0\varphi, \ \varphi \in C^2(\bar{\Omega}),$$

that is,

$$d_2\Delta\varphi - y(\cdot)\varphi + \left(\beta_1(\cdot) + \frac{\beta_2(\cdot)u_1^0(\cdot)\varrho(\cdot)}{\delta(\cdot)}\right)\frac{1}{\Re_0}\varphi = 0, \ \varphi \in C^2(\bar{\Omega}). \tag{3.9}$$

This proves (i).

We next prove (ii). In fact, eigenvalue problem (3.8) has a principle eigenvalue η^0, associated with a positive eigenfunction ϕ^* on Ω, i.e.,

$$\begin{cases} d_2\Delta\phi^* - y(\cdot)\phi^* + \left(\beta_1(\cdot) + \dfrac{\beta_2(\cdot)u_1^0(\cdot)\varrho(\cdot)}{\delta(\cdot)}\right)\phi^* = \eta^0\phi^*, \ x \in \Omega, \\ \dfrac{\partial\phi^*}{\partial\nu} = 0, \ x \in \partial\Omega. \end{cases} \tag{3.10}$$

We then multiply the first equation of (3.10) and (3.9) by φ and ϕ^*, respectively, then integrate and subtract the equation, obtaining

$$\left(1 - \frac{1}{\Re_0}\right)\int_\Omega \left(\beta_1(\cdot) + \frac{\beta_2(\cdot)u_1^0(\cdot)\varrho(\cdot)}{\delta(\cdot)}\right)\phi^*\varphi = \eta^0\int_\Omega \phi^*\varphi.$$

Since both $\int_\Omega \left(\beta_1(x) + \dfrac{\beta_2(x)u_1^0(x)\varrho(x)}{\delta(x)}\right)\phi^*\Phi$ and $\int_\Omega \phi^*\Phi$ are positive, we arrive at the conclusion that $\left(1 - \frac{1}{\Re_0}\right)$ and η^0 have the same sign, i.e., $\Re_0 > 1$ when $\eta^0 > 0$ and $\Re_0 < 1$ when $\eta^0 < 0$. This proves (ii). □

Due to the assertion in (i) of Lemma 3.2 and variational formula,

$$\Re_0 = \frac{1}{\tilde{\lambda}_0} = \sup_{\phi\in H^1(\Omega),\ \phi\neq 0} \left\{ \frac{\int\limits_\Omega \left(\beta_1(\cdot) + \frac{\beta_2(\cdot)u_1^0(\cdot)\varrho(\cdot)}{\delta(\cdot)}\right)\phi^2 \mathrm{d}x}{\int\limits_\Omega \left(d_2\,|\nabla\phi|^2 + y(\cdot)\phi^2\right)\mathrm{d}x} \right\}, \tag{3.11}$$

which indicate how $\Re_0$ depends on the diffusion coefficient d_2 (see also in [1, Theorem 2] and [31]).

Theorem 3.1. *Let $\Re_0$ be defined by (3.11). Then we have*

(i) Fix $d_1 > 0$, then $\Re_0$ is decreasing with respect to d_2, and satisfies

$$\lim_{d_2\to 0}\Re_0 = \max\left\{\frac{\left(\beta_1(\cdot) + \frac{\beta_2(\cdot)u_1^0(\cdot)\varrho(\cdot)}{\delta(\cdot)}\right)}{y(\cdot)} : x\in\bar{\Omega}\right\} \text{ and } \lim_{d_2\to\infty}\Re_0 = \frac{\int_\Omega \left(\beta_1(\cdot) + \frac{\beta_2(\cdot)u_1^0(\cdot)\varrho(\cdot)}{\delta(\cdot)}\right)\mathrm{d}x}{\int_\Omega y(\cdot)\mathrm{d}x}.$$

(ii) Fix $d_2 > 0$, then

$$\lim_{d_1\to 0}\Re_0 = \Re_{0,0} := \sup_{\phi\in H^1(\Omega),\ \phi\neq 0}\left\{\frac{\int_\Omega(\beta_1(\cdot) + \frac{\beta_2(\cdot)\varrho(\cdot)b(\cdot)}{\delta(\cdot)\zeta(\cdot)})\phi^2\mathrm{d}x}{\int_\Omega\left(d_2\,|\nabla\phi|^2 + y(\cdot)\phi^2\right)\mathrm{d}x}\right\}$$

and

$$\lim_{d_1\to\infty}\Re_0 = \Re_{0,\infty} := \sup_{\phi\in H^1(\Omega),\ \phi\neq 0}\left\{\frac{\int_\Omega(\beta_1(\cdot) + \frac{\beta_2(\cdot)\varrho(\cdot)\int_\Omega b(\cdot)\,\mathrm{d}x}{\delta(\cdot)\int_\Omega\zeta(\cdot)\,\mathrm{d}x})\phi^2\mathrm{d}x}{\int_\Omega\left(d_2\,|\nabla\phi|^2 + y(\cdot)\phi^2\right)\mathrm{d}x}\right\}.$$

(iii) If $\int\limits_\Omega \left(\beta_1(\cdot) + \frac{\beta_2(\cdot)u_1^0(\cdot)\varrho(\cdot)}{\delta(\cdot)}\right)\mathrm{d}x / \int\limits_\Omega y(\cdot)\mathrm{d}x > 1$, we have $\Re_0 > 1$ for all $d_1, d_2 > 0$.

(iv) If $\int\limits_\Omega \left(\beta_1(\cdot) + \frac{\beta_2(\cdot)u_1^0(\cdot)\varrho(\cdot)}{\delta(\cdot)}\right)\mathrm{d}x / \int\limits_\Omega y(\cdot)\mathrm{d}x < 1$ and $\left(\beta_1(\cdot) + \frac{\beta_2(\cdot)u_1^0(\cdot)\varrho(\cdot)}{\delta(\cdot)}\right)/y(\cdot) > 1$ for some $x\in\bar{\Omega}$, then there exists $d_2^\star\in(0,\infty)$ such that $\Re_0 > 1$ when $d_2 < d_2^\star$, and $\Re_0 < 1$ when $d_2 > d_2^\star$.

The following statements come from [13, Theorem 1.1].

Remark 1. *It is clear that the asymptotic behavior of the unique positive solution u_1^0 with d_1:*
- *$u_1^0(\cdot)\to b(\cdot)/\zeta(\cdot)$ in $C^1(\overline{\Omega})$ as $d_1\to 0$;*
- *$u_1^0(\cdot)\to \int_\Omega b(\cdot)\,\mathrm{d}x/\int_\Omega\zeta(\cdot)\,\mathrm{d}x$ in $C^1(\overline{\Omega})$ as $d_1\to\infty$.*

Remark 2. *Fix $d_2 > 0$, η^0 of the eigenvalue problem* (3.8) *satisfies*
- *$\eta^0\to\eta_0^0$ as $d_1\to 0$;*
- *$\eta^0\to\eta_\infty^0$ as $d_1\to\infty$,*

where η_0^0 and η_∞^0 are respectively the principal eigenvalue of (3.8) *with $u_1^0(\cdot) = b(\cdot)/\zeta(\cdot)$, and the principal eigenvalue of*

$$\begin{cases} d_2\Delta\phi + \left(\beta_1(\cdot) + \dfrac{\beta_2(\cdot)\varrho(\cdot)\int_\Omega b(x)\,\mathrm{d}x}{\delta(\cdot)\int_\Omega\zeta(\cdot)\,\mathrm{d}x} - y(\cdot)\right)\phi = \eta\phi, & x\in\Omega,\\ \dfrac{\partial\phi}{\partial\nu} = 0, & x\in\partial\Omega. \end{cases} \tag{3.12}$$

Remark 3. $\Re_{0,0} - 1$ *has the same sign as η_0^0, and* $\Re_{0,\infty} - 1$ *has the same sign as η_∞^0.*

Theorem 3.2. *If $\Re_0 \geq 1$ (or $s(\mathcal{B}) \geq 0$), $s(\mathcal{B})$ is the principal eigenvalue of (3.3).*

Proof. In fact, we see from (3.1) that

$$\begin{cases} u_2(\cdot,t,\phi) = A_2(t)\phi_2 + \int_0^t A_2(t-s)q(u_2(\cdot,s,\phi),u_3(\cdot,s,\phi))\mathrm{d}s, \\ u_3(\cdot,t,\phi) = A_3(t)\phi_3 + \int_0^t A_3(t)(t-s)(\varrho(\cdot)u_2(\cdot,s,\phi))\mathrm{d}s, \end{cases}$$

where $q(u_2,u_3) = \beta_1(\cdot)u_2 + \beta_2(\cdot)u_1^0(\cdot)u_3$, and A_1 and A_2 are respectively defined in (2.1) and (2.2). We decompose $T(t)$ as the sum of $T_2(t)$ and $T_3(t)$, $T(t) = T_2(t) + T_3(t)$, where

$$T_2(t)\phi = (0, A_3(t)\phi_3), \quad \phi = (\phi_2,\phi_3) \in \mathbb{Y}, \tag{3.13}$$

$$T_3(t)\phi = \left(u_2(\cdot,t,\phi), \int_0^t A_3(t)(t-s)[\varrho(\cdot)u_2(\cdot,s,\phi)]\mathrm{d}s\right), \quad \phi = (\phi_2,\phi_3) \in \mathbb{Y}.$$

By [31, Lemma 2.5], the compactness of $T_3(t)$ directly follows. Further,

$$\sup_{\phi\in C(\bar\Omega,\mathbb{R}^2),\|\phi\|\neq 0} \frac{\|T_2(t)\phi\|}{\|\phi\|} \le \sup_{\phi\in C(\bar\Omega,\mathbb{R}^2),\|\phi\|\neq 0} \frac{\|e^{-\delta(\cdot)t}\phi_3\|}{\|\phi\|} \le \sup_{\phi\in C(\bar\Omega,\mathbb{R}^2),\|\phi\|\neq 0} \frac{\|e^{-\delta_* t}\phi_3\|}{\|\phi\|} \le e^{-\delta_* t},$$

that is, $\|T_2(t)\| \le e^{-\delta_* t}$.

Hence, for any bounded set $\mathbb{S}$ in $\mathbb{Y}$,

$$\kappa(T(t)\mathbb{S}) \le \|T_2(t)\|\kappa(\mathbb{S}) \le e^{-\delta_* t}\kappa(\mathbb{S}), \quad t > 0.$$

Thus, $T(t)$ is a κ-contraction on $\mathbb{Y}$, that is, the essential growth bound, $\omega_{ess}(\Pi(t)) \le -\delta$ and the essential spectral radius

$$r_e(\Pi(t)) \le e^{-\delta_* t} < 1, \ t > 0.$$

It is well-known that $\omega(T(t))$, the exponential growth bound (defined by $\omega(T(t)) := \lim_{t\to\infty} \frac{\ln\|T(t)\|}{t}$ such that $\|T(t)\| \le Me^{\omega(T(t))t}$, for some $M > 0$), satisfies

$$\omega(T(t)) = \max\{s(\mathcal{B}), \omega_{ess}(T(t))\}.$$

With the assumption that $s(\mathcal{B}) \ge 0$, the spectral radius of $T(t)$,

$$r(T(t)) = e^{s(\mathcal{B})t} \ge 1, \ t > 0.$$

Consequently, $r_e(T(t)) < r(T(t))$. With the help of the generalized Krein-Rutman Theorem (see, for example, [18]), we complete the proof. □

The following result indicates that for a special case, $s(\mathcal{B})$ is the principal eigenvalue of (3.3) without any limitations.

Lemma 3.3. *Suppose that $\delta(x) \equiv \delta$. $s(\mathcal{B})$ is the principal eigenvalue of (3.3).*

Proof. Let

$$L_\lambda\phi = d_2\Delta\phi + \left(\beta_1(\cdot) + \frac{\beta_2(\cdot)u_1^0(\cdot)\varrho(\cdot)}{\lambda+\delta}\right)\phi - y(\cdot)\phi, \ \lambda > -\delta,$$

Denote $C_1 := \min_{x\in\bar\Omega}\{\beta_1(\cdot)\}$ and $C_2 := \min_{x\in\bar\Omega}\{\varrho(\cdot)\beta_2(\cdot)u_1^0(\cdot)\}$. Note that the eigenvalue problem

$$\begin{cases} \eta\varphi(\cdot) = d_2\Delta\varphi(\cdot) - y(\cdot)\varphi(\cdot), & x \in \Omega \\ \dfrac{\partial\varphi(x)}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.14}$$

admits one principal eigenvalue, $\hat{\eta}$, (associated with a positive eigenvector $\varphi^\diamond \gg 0$). Denote by $\lambda^\star$ the larger root of

$$\lambda^2 + (\delta - C_1 - \hat{\eta})\lambda - (C_2 + \delta(C_1 + \hat{\eta})) = 0.$$

Then $\lambda^\star = \frac{1}{2}[(\hat{\eta} - \delta + C_1) + \sqrt{(\delta + C_1 + \hat{\eta})^2 + 4C_2}] > -\delta$. Hence,

$$L_{\lambda^\star}\varphi^\diamond = d_2\Delta\varphi^\diamond + \beta_1(\cdot)\varphi^\diamond + \frac{\varrho(\cdot)\beta_2(\cdot)u_1^0(\cdot)}{\lambda + \delta}\varphi^\diamond - y(\cdot)\varphi^\diamond \geq (\hat{\eta} + C_1 + \frac{C_2}{\lambda^\star + \delta})\varphi^\diamond = \lambda^\star\varphi^\diamond.$$

According to [30, Theorem 2.3], the assertion directly follows. We complete the proof. □

4 Threshold dynamics

We now study the threshold dynamics of model (1.5) as $\Re_0 \leq 1$ and $\Re_0 > 1$.

Theorem 4.1. *If $\Re_0 \leq 1$ (or $s(\mathcal{B}) \leq 0$), then E_0 is globally asymptotically stable.*

Proof. We divide the proof into two parts. One is $\Re_0 < 1$, the other is $\Re_0 = 1$. When $\Re_0 < 1$, we know from [30, Theorem 3.1] that E_0 is locally asymptotically stable. We are mow in a position to consider the global attractivity of E_0. Fix $\epsilon_0 > 0$. By (2.6), there is $t_1 > 0$ such that for all $(x, t) \in \bar{\Omega} \times [t_1, \infty)$,

$$0 \leq u_1(\cdot, t) \leq u_1^0(\cdot) + \epsilon_0.$$

Let $(\hat{u}_2(\cdot, t), \hat{u}_3(\cdot, t))$ is the solution of the following problem

$$\begin{cases} \begin{pmatrix} \frac{\partial \hat{u}_2}{\partial t} \\ \frac{\partial \hat{u}_3}{\partial t} \end{pmatrix} = \mathcal{B}_{\epsilon_0}\begin{pmatrix} \hat{u}_2 \\ \hat{u}_3 \end{pmatrix}, & (x, t) \in \bar{\Omega} \times [t_1, \infty), \\ \dfrac{\partial \hat{u}_2}{\partial n} = 0, & (x, t) \in \bar{\Omega} \times [t_1, \infty), \end{cases} \tag{4.1}$$

where

$$\mathcal{B}_{\epsilon_0} = \begin{pmatrix} d_2\Delta + \beta_1(\cdot) - y(\cdot) & \beta_2(\cdot)(u_1^0(\cdot) + \epsilon_0) \\ \varrho(\cdot) & -\delta(\cdot) \end{pmatrix}.$$

By the comparison principle (see, e.g. [16]), $(u_2(\cdot, t), u_3(\cdot, t)) \leq (\hat{u}_2(\cdot, t), \hat{u}_3(\cdot, t))$ on $\bar{\Omega} \times [t_1, \infty)$.

Let $T_{\epsilon_0}(t)$ be the positive semigroup induced by $\mathcal{B}_{\epsilon_0}$. We next aim to prove that the exponential growth bound of $T_{\epsilon_0}(t)$, ω_{ϵ_0} is negative. Let $\omega_{ess}(T_{\epsilon_0}(t)) := \lim_{t\to\infty} \frac{\alpha(T_{\epsilon_0}(t))}{t}$ be the essential growth bound of $T_{\epsilon_0}(t)$, where $\alpha(\cdot)$ is the measure of non-compactness. From Lemma 2.2, we have $\omega_{ess}(T_{\epsilon_0}) \leq -\delta_\star$. Recall that $\omega_{\epsilon_0} = \max\{s(\mathcal{B}_{\epsilon_0}), \omega_{ess}(T_{\epsilon_0}(t))\}$. This allow us to conclude that ω_{ϵ_0} has the same sign as $s(\mathcal{B}_{\epsilon_0})$. From Lemma 3.2, we directly obtain that $s(\mathcal{B}_{\epsilon_0})$ has the same sign to the principle eigenvalue η_{ϵ_0} of

$$\begin{cases} d_2\Delta\varphi - y(\cdot)\varphi + \left(\beta_1(\cdot) + \dfrac{\beta_2(\cdot)(u_1^0(\cdot) + \epsilon_0)\varrho(\cdot)}{\delta(\cdot)}\right)\varphi = \eta\varphi, & x \in \Omega, \\ \dfrac{\partial \varphi}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{4.2}$$

It follows from (ii) of Lemma 3.2, $\Re_0 < 1$ and the continuous dependence of $\eta_{\epsilon_0}^0$ on ϵ_0 that $\eta_{\epsilon_0}^0 < 0$ if ϵ_0 is small. Hence, $\omega_{\epsilon_0} < 0$. This together with $\|T_{\epsilon_0}(t)\| \leq Me^{\omega_{\epsilon_0}t}$, for some $M > 0$, imply that $(\hat{u}_2(\cdot, t), \hat{u}_3(\cdot, t)) \to (0, 0)$ as $t \to \infty$ uniformly for $x \in \bar{\Omega}$. Consequently, $(u_2(\cdot, t), u_3(\cdot, t)) \to (0, 0)$ as $t \to \infty$ uniformly for $x \in \bar{\Omega}$. Further, from (1.5), $u_1(\cdot, t) \to u_1^0(\cdot)$ as $t \to \infty$ uniformly for $x \in \bar{\Omega}$. This completes the first part.

When $\Re_0 = 1$, the global asymptotic stability result of E_0 needs to combine with the local stability and global attractivity of E_0 (see also in [3, 4, 31, 32]). We first confirm the local asymptotic stability of E_0. Let $\epsilon > 0$ be given. Choosing $\rho > 0$ such that for any $u_0 = (u_{10}, u_{20}, u_{30})$, we have $\| u_0 - E_0 \| \leq \rho$.

Let us introduce

$$\Sigma(x,t) := \frac{u_1(\cdot,t)}{u_1^0(\cdot)} - 1 \quad \text{and} \quad \tau(t) := \max_{x\in\bar{\Omega}}\{\Sigma(\cdot,t),0\}.$$

Hence, the first equation of (1.5) becomes

$$\frac{\partial\Sigma}{\partial t} - d_1\Delta\Sigma - 2d_1\frac{\nabla u_1^0(\cdot)\cdot\nabla\Sigma}{u_1^0(\cdot)} + \frac{b(\cdot)}{u_1^0(\cdot)}\Sigma = -\frac{\beta_1(\cdot)\frac{u_1u_2}{u_1+u_2} + \beta_2(\cdot)u_1u_3}{u_1^0(x)} \tag{4.3}$$

Solving (4.3) gives

$$\Sigma(\cdot,t) = \widehat{T}(t)(t)\Sigma_0 - \int_0^t \widehat{T}(t)(t-s)\frac{\beta_1(\cdot)\frac{u_1u_2}{u_1+u_2} + \beta_2(\cdot)u_1u_3}{u_1^0(\cdot)}\mathrm{d}s, \tag{4.4}$$

where $\Sigma_0 = \frac{u_{10}}{u_1^0} - 1$ and $\widehat{T}(t) \le \mathbb{M}e^{-rt}$ for some $r, \mathbb{M} > 0$, is the positive semigroup induced by $d_1\Delta + 2d_1\frac{\nabla u_1^0(\cdot)\cdot\nabla}{u_1^0(\cdot)} + \frac{b(\cdot)}{u_1^0(\cdot)}$. It follows that

$$\begin{aligned}\tau(t) &= \max_{x\in\bar{\Omega}}\left\{\widehat{T}(t)\Sigma_0 - \int_0^t \widehat{T}(t-s)\frac{\beta_1(\cdot)\frac{u_1u_2}{u_1+u_2} + \beta_2(\cdot)u_1u_3}{u_1^0(x)}\mathrm{d}s, 0\right\}\\ &\le \max_{x\in\bar{\Omega}}\left\{\widehat{T}(t)\Sigma_0, 0\right\} \le \|\widehat{T}(t)\Sigma_0\|\\ &\le \mathbb{M}e^{-rt}\|\frac{u_{10}}{u_1^0(\cdot)} - 1\| \le \rho\mathbb{M}e^{-rt}/\underline{u}_1^0,\end{aligned} \tag{4.5}$$

where $\underline{u}_1^0 = \min_{x\in\bar{\Omega}}\{u_1^0(\cdot)\}$.

By a zero trick and the definition of $T(t)$ (positive semigroup generated by $\mathcal{B}$), we have

$$\begin{aligned}\begin{pmatrix} u_2(\cdot,t)\\ u_3(\cdot,t)\end{pmatrix} &= T(t)\begin{pmatrix} u_{20}(\cdot)\\ u_{30}(\cdot)\end{pmatrix} + \int_0^t T(t-s)\begin{pmatrix} \beta_1(\cdot)\frac{u_1(\cdot,s)u_2(\cdot,s)}{u_1(\cdot,s)+u_2(\cdot,s)} - \beta_1(\cdot)u_2(\cdot,s)\\ +\beta_2(\cdot)(u_1(\cdot,s) - u_1^0)u_3(\cdot,s)\\ 0\end{pmatrix}\mathrm{d}s\\ &\le T(t)\begin{pmatrix} u_{20}(\cdot)\\ u_{30}(\cdot)\end{pmatrix} + \int_0^t T(t-s)\begin{pmatrix}\beta_2^\star(u_1(\cdot,s) - u_1^0(\cdot))u_3(\cdot,s)\\ 0\end{pmatrix}\mathrm{d}s.\end{aligned} \tag{4.6}$$

Under the condition that $\Re_0 = 1$ (or $s(\mathcal{B}) = 0$), we know that $\|T(t)\| \le M_\tau$ for $t \ge 0$ and $M_\tau > 0$. Hence, by (4.6), we have

$$\begin{aligned}\max\{\|u_2(\cdot,t)\|, \|u_3(\cdot,t)\|\} &\le M_\tau\rho + M_\tau\beta_2^\star\|u_1^0(\cdot)\|\int_0^t \tau(s)\|u_3(\cdot,s)\|\mathrm{d}s\\ &\le M_\tau\rho + M_\tau\beta_2^\star\|u_1^0(\cdot)\|\frac{\rho\mathbb{M}}{\underline{u}_1^0}\int_0^t e^{-rs}\|u_3(\cdot,s)\|\mathrm{d}s\\ &= M_\tau\rho + \overline{L}\rho\int_0^t e^{-rs}\|u_3(\cdot,s)\|\mathrm{d}s,\end{aligned} \tag{4.7}$$

where $\overline{L} = M_\tau\beta_2^\star\|u_1^0(\cdot)\|\mathbb{M}/\underline{u}_1^0$. With the help of the Gronwall's inequality, it is easy to get

$$\|u_3(x,t)\| \le M_\tau\rho e^{\overline{L}\rho/r}. \tag{4.8}$$

Further from (4.7) and (4.8), we know that $\|u_2(x,t)\| \le M_\tau\rho + \overline{L}\rho M_\tau\rho e^{\overline{L}\rho/r}/r$. This together with the first equation of system (1.5) and (4.8) imply that

$$\frac{\partial u_1}{\partial t} \ge d_1\Delta u_1 + b(\cdot) - \zeta(\cdot)u_1 - \beta_1^\star u_1 - \beta_2^\star M_\tau\rho e^{\overline{L}\rho/r}u_1,\ (x,t)\in\Omega\times(0,\infty).$$

Thus, we consider the system

$$\begin{cases} \dfrac{\partial \vartheta}{\partial t} = d_1 \Delta \vartheta + b(\cdot) - \zeta(\cdot)\vartheta - \beta_1^\star \vartheta - \beta_2^\star M_\tau \rho e^{\overline{L}\rho/r}\vartheta, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial \vartheta}{\partial n} = 0, & (x,t) \in \Omega \times (0,\infty), \\ \vartheta(\cdot, 0) = u_{10}(\cdot), & x \in \Omega. \end{cases} \tag{4.9}$$

Obviously, $u_1(\cdot,t) \geq \vartheta(\cdot,t)$, for all $(x,t) \in \Omega \times (0,\infty)$. Let $\widehat{\vartheta}(\cdot,t) = \vartheta(\cdot,t) - \vartheta^\rho(\cdot)$ and $\vartheta^\rho(\cdot)$ is the positive steady state of (4.9). Hence, $\widehat{\vartheta}(\cdot,t)$ satisfies

$$\begin{cases} \dfrac{\partial \widehat{\vartheta}}{\partial t} = d_1 \Delta \widehat{\vartheta} - (\zeta(\cdot) + \beta_1^\star + \beta_2^\star M_\tau \rho e^{\overline{L}\rho/r})\widehat{\vartheta}, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial \widehat{\vartheta}}{\partial n} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ \widehat{\vartheta}(\cdot, 0) = u_{10}(\cdot) - \vartheta^\rho(\cdot), & x \in \Omega. \end{cases} \tag{4.10}$$

Solving (4.10) yields

$$\widehat{\vartheta}(\cdot,t) = A_1(t)(u_{10}(\cdot) - \vartheta^\rho(\cdot)) - \int_0^t A_1(t-s)(\beta_1^\star + \beta_2^\star M_\tau \rho e^{\overline{L}\rho/r})\widehat{\vartheta}(\cdot,s)\mathrm{d}s.$$

where $A_1(t)$ is defined in (2.1) and satisfies $\|A_1(t)\| \leq K_0 e^{\zeta_\star t}$ for some constant K_0. Hence,

$$\|\widehat{\vartheta}(\cdot,t)\| \leq K_0 e^{\zeta_\star t}\|u_{10}(\cdot) - u_1^\rho(\cdot)\| + \int_0^t K_0 e^{\zeta_\star(t-s)}(\beta_1^\star + \beta_2^\star M_\tau \rho e^{\overline{L}\rho/r})\|\widehat{\vartheta}(\cdot,s)\|\mathrm{d}s.$$

Let $\overline{L}_1 = K_0(\beta_1^\star + \beta_2^\star M_\tau \rho e^{\overline{L}\rho/r})$. This together with the Gronwall's inequality imply that

$$\|\widehat{\vartheta}(\cdot,t)\| = \|\vartheta(\cdot,t) - \vartheta^\rho(\cdot)\| \leq K_0\|u_{10}(\cdot) - \vartheta^\rho(\cdot)\|e^{\overline{L}_1 t + \zeta_\star t}.$$

Choose $\rho > 0$ small enough that $\overline{L}_1 < -\zeta_\star/2$ and

$$\|\vartheta(\cdot,t) - \vartheta^\rho(\cdot)\| \leq K_0\|u_{10}(\cdot) - \vartheta^\rho(\cdot)\|e^{-\zeta_m t/2}.$$

By a zero trick,

$$\begin{aligned} u_1(\cdot,t) - u_1^0(\cdot) &\geq \vartheta(\cdot,t) - u_1^0(\cdot) = \widehat{\vartheta}(\cdot,t) - \vartheta^\rho(\cdot) + \vartheta^\rho(\cdot) - u_1^0(\cdot) \\ &\geq -K_0\|u_{10}(\cdot) - \vartheta^\rho(\cdot)\|e^{-\zeta_m t/2} + \vartheta^\rho(\cdot) - u_1^0(\cdot) \\ &\geq -K_0(\|u_{10}(\cdot) - u_1^0(\cdot)\| + \|u_1^0(\cdot) - \vartheta^\rho(\cdot)\|) - \|\vartheta^\rho(\cdot) - u_1^0(\cdot)\| \\ &\geq -K_0\rho - (K_0+1)\|\vartheta^\rho(\cdot) - u_1^0(\cdot)\|. \end{aligned} \tag{4.11}$$

From $\tau(t) \leq \rho \mathbb{M} e^{-rt}/\underline{u}_1^0$, we directly have

$$u_1(\cdot,t) - u_1^0(\cdot) = u_1^0(\cdot)\left(\frac{u_1(\cdot,t)}{u_1^0(\cdot)} - 1\right) \leq \rho\mathbb{M}\|u_1^0(\cdot)\|/\underline{u}_1^0. \tag{4.12}$$

Combined with (4.11) and (4.12), one can get

$$\|u_1(\cdot,t) - u_1^0(\cdot)\| \leq \max\{K_0\rho + (K_0+1)\|\vartheta^\rho(\cdot) - u_1^0(\cdot)\|,\ \rho\mathbb{M}\|u_1^0(\cdot)\|/\underline{u}_1^0\}. \tag{4.13}$$

By choosing ρ sufficiently small, $\lim_{\rho\to 0}\vartheta^\rho(\cdot) = u_1^0(\cdot)$, the inequalities (4.11) and (4.13) imply that

$$\|u_1(\cdot,t) - u_1^0(\cdot)\|, \|u_2(\cdot,t)\|, \|u_3(\cdot,t)\| \leq \varepsilon, \text{ for all } t > 0,$$

where ε can be chosen as a small number. This proves the local stability of E_0.

Secondly, we confirm the global stability of E_0. For any $u_0 \in \mathbb{X}_+$, we set

$$\partial\mathbb{X}_1 = \{(u_1, u_2, u_3) \in \mathbb{X}^+ : u_2 = u_3 = 0\},$$

and

$$\sigma(t; u_0) := \inf\{\tilde{c} \in \mathbb{R} : u_2(\cdot, t) \le \tilde{c}\phi_2 \text{ and } u_3(\cdot, t) \le \tilde{c}\phi_3\}.$$

where (ϕ_2, ϕ_3) is the positive eigenfunction of $s(\mathcal{B}) = 0$. Recall that (1.5) has a connected global compact attractor in $\mathbb{X}^+$, denoted by $\mathcal{A}_0$ (see Theorem 2.2).

We claim that for any $u_0 \in \mathcal{A}_0$, $\omega(u_0) \in \partial\mathbb{X}_1$. It is clear that for all $t > 0$, $\sigma(t; u_0) > 0$ is strictly decreasing. In fact, for $\tilde{t}_0 > 0$, let

$$\overline{u}_2(\cdot, t) = \sigma(\tilde{t}_0; u_0)\phi_2 \text{ and } \overline{u}_3(\cdot, t) = \sigma(\tilde{t}_0; u_0)\phi_3, \text{ for } t > \tilde{t}_0.$$

such that

$$(\overline{u}_2(\cdot, t), \overline{u}_3(\cdot, t)) \ge (u_2(\cdot, t), u_3(\cdot, t)), \text{ for } (x, t) \in \bar{\Omega} \times [\tilde{t}_0, \infty).$$

where $(\overline{u}_2(x, t), \overline{u}_3(x, t))$ satisfies

$$\begin{cases} \dfrac{\partial \overline{u}_2}{\partial t} = d_2 \Delta \overline{u}_2 + \beta_1(\cdot)\dfrac{u_1\overline{u}_2}{u_1 + \overline{u}_2} + \beta_2(\cdot)u_1\overline{u}_3 - \gamma(\cdot)\overline{u}_2, & (x, t) \in \Omega \times [\tilde{t}_0, \infty), \\ \dfrac{\partial \overline{u}_3}{\partial t} = \varrho(x)\overline{u}_2 - \delta(x)\overline{u}_3, & (x, t) \in \Omega \times [\tilde{t}_0, \infty), \\ \dfrac{\partial \overline{u}_2}{\partial n} = 0, & (x, t) \in \partial\Omega \times [\tilde{t}_0, \infty), \\ \overline{u}_2(\cdot, \tilde{t}_0) \ge u_2(\cdot, \tilde{t}_0),\ \overline{u}_3(\cdot, \tilde{t}_0) \ge u_3(\cdot, \tilde{t}_0), & x \in \Omega. \end{cases} \tag{4.14}$$

Furthermore, by (4.14) and comparison principle, one can see that

$$\sigma(t; u_0)\phi_2 = \overline{u}_2(\cdot, t) > u_2(\cdot, t) \text{ and } \sigma(t; u_0)\phi_3 = \overline{u}_3(\cdot, t) > u_3(\cdot, t), \text{ for } (x, t) \in \bar{\Omega} \times [\tilde{t}_0, \infty).$$

Since $\tilde{t}_0$ is arbitrary, $\sigma(t; u_0)$ is strictly decreasing.

Next, we pay attention to the the semiflow $Y(t)$, generated by (1.5) (see Theorem 2.1). Let $\{\tilde{t}_k\}$ be a sequence with $\tilde{t}_k \to \infty$, $\sigma_\star = \lim_{t\to\infty} \sigma(t; u_0)$ and $v = (v_1, v_2, v_3) \in \omega(u_0)$. We can conclude that

$$\lim_{\tilde{t}_k\to\infty} Y(t + \tilde{t}_k)u_0 = Y(t) \lim_{\tilde{t}_k\to\infty} Y(\tilde{t}_k)u_0 = Y(t)v.$$

This means that $\sigma(t; v) = \sigma_\star$. Repeating the previous arguments, $\sigma(t; v)$ is strictly decreasing when $v_2 \neq 0$ or $v_3 \neq 0$. This gives a contradiction. So, we have $v_2 = v_3 = 0$. When $t \to \infty$, we get $(u_2(\cdot, t), u_3(\cdot, t)) \to (0, 0)$. It follows that $u_1(\cdot, t) \to u_1^0(\cdot)$ eventually. This proves the second part. The proof is complete.

□

The following result indicates that (1.5) is uniformly persistent when $\Re_0 > 1$. Before going into details, we set

$$\begin{aligned} \mathbb{X}_0 &:= \left\{\phi(\cdot) = (\phi_1, \phi_2, \phi_3)(\cdot) \in \mathbb{X}^+ : \phi_2(\cdot) \not\equiv 0 \text{ and } \phi_3(\cdot) \not\equiv 0\right\}, \\ \partial\mathbb{X}_0 &:= \mathbb{X}^+ \setminus \mathbb{X}_0 = \left\{\phi(\cdot) = (\phi_1, \phi_2, \phi_3)(\cdot) \in \mathbb{X}^+ : \phi_2(\cdot) \equiv 0 \text{ or } \phi_3(\cdot) \equiv 0\right\}. \end{aligned}$$

and

$$M_\partial := \{\phi \in \partial\mathbb{X}_0 : Y(t)\phi \in \partial\mathbb{X}_0, \forall t \ge 0\},$$

where $Y(t)$ is the semiflow generated by (1.5).

Theorem 4.2. *If $\Re_0 > 1$, then (1.5) is uniformly persistent, i.e., for any $\phi \in \mathbb{X}^+$ with $\phi_2(\cdot) \not\equiv 0$, or $\phi_3(\cdot) \not\equiv 0$, there exists $\sigma > 0$,*

$$\liminf_{t\to\infty} u_i(\cdot, t) \ge \sigma_0, \quad \text{uniformly for } x \in \bar{\Omega}. \tag{4.15}$$

Moreover, (1.5) possesses at least one positive steady state.

Proof. Theorem 4.2 is achieved by the following claims. The first two claims are the direct consequences as those in [31, Theorem 3.10] with slight modifications. We present them here without proof.

Claim 1 $Y(t)\mathbb{X}_0 \subseteq \mathbb{X}_0$ for all $t \geq 0$.

Claim 2 $\omega(u_0) = \{E_0\}$, for any $u_0 \in M_\partial$, where $\omega(u_0)$ is the ω limit set of u_0.

Claim 3 $\limsup_{t\to\infty} \|Y(t)\phi - E_0\| \geq \sigma, \forall \phi \in \mathbb{X}_0$, where σ is a positive constant.

We prove Claim 3 by the way of contradiction. Assume to the contrary that for $\sigma > 0$, there exists $\bar{t}_1 > 0$ such that

$$u_1(\cdot, t, \phi) \geq u_1^0 - \sigma,\ u_2(\cdot, t; \phi) < \sigma,\ u_3(\cdot, t; \phi) < \sigma, \forall t \geq t_1.$$

Hence $(u_2(\cdot, t; \phi), u_3(\cdot, t; \phi))$ is the upper solution of

$$\begin{cases} \begin{pmatrix} \frac{\partial \tilde{u}_2}{\partial t} \\ \frac{\partial \tilde{u}_3}{\partial t} \end{pmatrix} = \mathcal{B}_\sigma \begin{pmatrix} \tilde{u}_2 \\ \tilde{u}_3 \end{pmatrix}, & (x, t) \in \Omega \times [\bar{t}_1, \infty), \\ \dfrac{\partial \tilde{u}_2}{\partial n} = 0, & (x, t) \in \partial\Omega \times [\bar{t}_1, \infty), \end{cases} \tag{4.16}$$

where

$$\mathcal{B}_\sigma = \begin{pmatrix} d_2\Delta + \beta_1(x) - y(x) & \beta_2(x)(u_1^0(x) - \sigma) \\ \varrho(x) & -\delta(x) \end{pmatrix}.$$

Combined with Lemma 3.2 and the continuous dependence of $s(\mathcal{B}) \geq 0$ on σ, choosing $\sigma > 0$ small enough that $s(\mathcal{B}_\sigma) > 0$, associated with positive eigenvalue function $(\phi_2^\sigma(\cdot), \phi_3^\sigma(\cdot))$. Choose $\xi > 0$ small enough that

$$(u_2(\cdot, \bar{t}_1; \phi), u_3(\cdot, \bar{t}_1; \phi) \geq \xi(\phi_2^\sigma(\cdot), \phi_3^\sigma(\cdot)).$$

It is easy to see that for $(\tilde{u}_2(\cdot, \bar{t}_1), \tilde{u}_3(\cdot, \bar{t}_1) = \xi(\phi_2^\sigma(\cdot), \phi_3^\sigma(\cdot))$, (4.16) has a solution

$$(\tilde{u}_2(\cdot, t; \phi), \tilde{u}_3(\cdot, t; \phi) = \xi e^{s(\mathcal{B}_\sigma)(t-\bar{t}_1)}(\phi_2^\sigma(\cdot), \phi_3^\sigma(\cdot)),\ t > \bar{t}_1.$$

Since $s(\mathcal{B}_\sigma) > 0$, $u_2(\cdot, t; \phi), u_3(\cdot, t; \phi) \to \infty$ as $t \to \infty$, a contradiction.

Similar to [25, Theorem 3] (see also in [31, Theorem 3.10]), let us define a distance function $\rho(\cdot) : \mathbb{X}^+ \to [0, \infty)$ by

$$\rho(\phi) = \min\{\min_{x\in\bar{\Omega}} \phi_2(\cdot), \min_{x\in\bar{\Omega}} \phi_3(\cdot)\}, \phi \in \mathbb{X}^+.$$

we conclude that

$$\liminf_{t\to\infty} u_2(\cdot, t, \phi) \geq \sigma_1 \text{ and } \liminf_{t\to\infty} u_3(\cdot, t, \phi) \geq \sigma_1,\ \forall \phi \in \mathbb{X}_0.$$

Recall from Theorem 2.1 that there exist $M_\infty > 0$ and $t_2 > 0$ such that $u_2(\cdot, t, \phi), u_3(\cdot, t, \phi) \leq M_\infty$, $(x, t) \in \Omega \times [t_2, \infty)$. It then follows that

$$\begin{cases} \dfrac{\partial u_1}{\partial t} \geq d_1 \triangle u_1 + b_\star - (\zeta^\star + \beta_1^\star M_\infty + \beta_2^\star M_\infty), & (x, t) \in \Omega \times [t_2, \infty) \\ \dfrac{\partial \tilde{u}_1}{\partial n} = 0, & (x, t) \in \partial\Omega \times [t_2, \infty), \end{cases}$$

Thus, there exists a constant σ_2, $\liminf_{t\to\infty} u_1(\cdot, t, \phi) \geq \sigma_2$. Let $\sigma_0 = \min\{\sigma_1, \sigma_2\}$. The uniform persistence is obtained. Further from [15, Theorem 4.7], (1.5) possesses at least one positive steady state. This completes the proof. □

5 Spatially homogeneous case

When

$$b(x) \equiv b,\ \beta_i(x) \equiv \beta_i,\ \zeta(x) \equiv \zeta,\ y(x) \equiv y,\ \delta(x) \equiv \delta, \varrho(x) \equiv \varrho,$$

system (1.5) becomes

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \Delta u_1 + b - \beta_1 \dfrac{u_1 u_2}{u_1 + u_2} - \beta_2 u_1 u_3 - \zeta u_1, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial u_2}{\partial t} = d_2 \Delta u_2 + \beta_1 \dfrac{u_1 u_2}{u_1 + u_2} + \beta_2 u_1 u_3 - y u_2, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial u_3}{\partial t} = \varrho u_2 - \delta u_3, & (x,t) \in \Omega \times (0,\infty), \end{cases} \tag{5.1}$$

subject to the same initial and boundary condition as in (1.5). Clearly, system (5.1) possesses a disease-free equilibrium $E_0 = (u_1^0, 0, 0)$ with $u_1^0 := b/\zeta$. From (3.4), we have the concrete formula of $\Re_0$,

$$[\Re_0] = \tfrac{\beta_1 \delta + \beta_2 \rho u_1^0}{y\delta}. \tag{5.2}$$

In this circumstance, Theorems 4.1 and 4.2 still hold for (5.1).

Theorem 5.1. *If $[\Re_0] < 1$, then E_0 is globally asymptotically stable; while if $[\Re_0] > 1$, then (5.1) is uniformly persistent.*

The positive equilibrium of (5.1) (whenever it exists) should satisfy

$$u_1^\star = \begin{cases} by\delta/(b\varrho\beta_2 + \beta_1 y\delta), & y = \zeta; \\ \dfrac{-(b\varrho\beta_2 + \beta_1 y\delta + y\delta\zeta - y^2\delta) + \sqrt{(b\varrho\beta_2 + \beta_1 y\delta + y\delta\zeta - y^2\delta)^2 + 4by\delta\varrho\beta_2(y-\zeta)}}{2\varrho\beta_2(y-\zeta)}, & y > \zeta, \end{cases}$$

$$u_2^\star = \frac{b - \zeta u_1^\star}{y} \text{ and } u_3^\star = \frac{(b - \zeta u_1^\star)\varrho}{y\delta}.$$

To make $u_2^\star > 0$ and $u_3^\star > 0$, we need $u_1^\star < b/\zeta$, which is equivalent with $[\Re_0] > 1$. The equivalence is obvious when $y = \zeta$. When $y > \zeta$, we rewrite $u_1^\star < b/\zeta$ as

$$\frac{-\mathbb{B} + \sqrt{\mathbb{B}^2 + 4\mathbb{A}\mathbb{C}}}{2\mathbb{A}} < \frac{b}{\zeta},$$

where $\mathbb{A} = \varrho\beta_2(y-\zeta) > 0$, $\mathbb{B} = b\varrho\beta_2 + \beta_1 y\delta + y\delta\zeta - y^2\delta$ and $\mathbb{C} = by\delta > 0$. Isolating the square root and squaring both sides gives $\mathbb{B} + 2\mathbb{A}b/\zeta > 0$ and

$$\mathbb{B}^2 + 4\mathbb{A}\mathbb{C} < \mathbb{B}^2 + \frac{4\mathbb{B}\mathbb{A}b}{\zeta} + \frac{4\mathbb{A}^2 b^2}{\zeta^2}.$$

Simplifying the last inequality yields $\mathbb{C}\zeta^2 < \mathbb{B}b\zeta + \mathbb{A}b^2$. We make use of the expressions of $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ and rewrite the inequality as

$$by\delta\zeta^2 < b\zeta(b\varrho\beta_2 + \beta_1 y\delta + y\delta\zeta - y^2\delta) + b^2\varrho\beta_2(y-\zeta).$$

By expanding and canceling, we obtain $y\delta\zeta < b\varrho\beta_2 + \beta_1\delta\zeta$, which is the same as $[\Re_0] > 1$. Furthermore, coupling $[\Re_0] > 1$ and $y > \zeta$ implies $\mathbb{B} + 2\mathbb{A}b/\zeta > 0$. Thus, if $[\Re_0] > 1$ and $y \geq \zeta$, (5.1) possesses a unique positive equilibrium.

Remark 4. *Note that y and ζ are the death rates of infected and susceptible host, respectively. Due to the disease burden, it is realistic to assume that $y \geq \zeta$. Combined with the analysis above, (5.1) possesses a unique positive equilibrium.*

Theorem 5.2. *Suppose that $y \geq \zeta$ hold. If $[\Re_0] > 1$, then unique positive equilibrium $E^\star_{(5.1)} = (u_1^\star, u_2^\star, u_3^\star)$ is locally asymptotically stable.*

Proof. Linearizing system (5.1) at $E^\star_{(5.1)}$ gives

$$\frac{\partial \chi}{\partial t} = D\Delta\chi(\cdot, t) + Q\chi(\cdot, t),$$

where $\chi = (u_1, u_2, u_3)$, $D = \mathrm{diag}(d_1, d_2, 0)$ and

$$Q = \begin{pmatrix} -\zeta - \beta_1 \frac{u_2^{\star 2}}{(u_1^\star + u_2^\star)^2} - \beta_2 u_3^\star & -\beta_1 \frac{u_1^{\star 2}}{(u_1^\star + u_2^\star)^2} & -\beta_2 u_1^\star \\ \beta_1 \frac{u_2^{\star 2}}{(u_1^\star + u_2^\star)^2} + \beta_2 u_3^\star & \beta_1 \frac{u_1^{\star 2}}{(u_1^\star + u_2^\star)^2} - y & \beta_2 u_1^\star \\ 0 & \varrho & -\delta \end{pmatrix}.$$

Hence, the characteristic equation of $E^\star_{(5.1)}$ is

$$\lambda^3 + a_1(l^2)\lambda^2 + a_2(l^2)\lambda + a_3(l^2) = 0, \tag{5.3}$$

where l is the wavenumber,

$$a_1(l^2) := l^2 d_1 + \zeta + \beta_2 u_3^\star + \beta_1 \frac{u_2^{\star 2}}{(u_1^\star + u_2^\star)^2} + l^2 d_2 + \frac{\beta_1 u_1^\star u_2^{\star 2} + \beta_2 u_1^\star u_3^\star (u_1^\star + u_2^\star)^2}{u_2^\star (u_1^\star + u_2^\star)^2} + \delta,$$

$$\begin{aligned} a_2(l^2) :=& \left(l^2 d_2 + \beta_1 \frac{u_1^\star u_2^{\star 2}}{(u_1^\star + u_2^\star)^2}\right)\delta + \left(\beta_2 u_3^\star + \beta_1 \frac{u_2^{\star 2}}{(u_1^\star + u_2^\star)^2}\right)\beta_1 \frac{u_1^{\star 2}}{(u_1^\star + u_2^\star)^2} \\ &+ \left(l^2 d_1 + \zeta + \beta_2 u_3^\star + \beta_1 \frac{u_2^{\star 2}}{(u_1^\star + u_2^\star)^2}\right)\left(l^2 d_2 + \frac{\beta_1 u_1^\star u_2^{\star 2} + \beta_2 u_1^\star u_3^\star (u_1^\star + u_2^\star)^2}{u_2^\star (u_1^\star + u_2^\star)^2} + \delta\right), \end{aligned}$$

and

$$\begin{aligned} a_3(l^2) :=& (l^2 d_1 + \zeta)\left(l^2 d_2 + \beta_1 \frac{u_1^\star u_2^{\star 2}}{(u_1^\star + u_2^\star)^2}\right)\delta + \beta_1 \frac{u_1^{\star 2}}{(u_1^\star + u_2^\star)^2}\left(\beta_1 \frac{u_2^{\star 2}}{(u_1^\star + u_2^\star)^2} + \beta_2 u_3^\star\right)\delta \\ &+ \left(l^2 d_2 + \frac{\beta_1 u_1^\star u_2^{\star 2} + \beta_2 u_1^\star u_3^\star (u_1^\star + u_2^\star)^2}{u_2^\star (u_1^\star + u_2^\star)^2}\right)\left(\beta_1 \frac{u_2^{\star 2}}{(u_1^\star + u_2^\star)^2} + \beta_2 u_3^\star\right)\delta. \end{aligned}$$

It then follows that $a_1(l^2) > 0$, $a_2(l^2) > 0$ and $a_3(l^2) > 0$. Further, after elementary calculations, $a_1(l^2)a_2(l^2) - a_3(l^2) > 0$. Hence, the local stability of $E^\star_{(5.1)}$ directly follows from the Routh-Hurwitz criterion. □

Theorem 5.3. *Assume that* $(\boldsymbol{H}1):\ \beta_1(u_2^\star - u_1^\star)^2 \leq 4\zeta u_1^\star(u_1^\star + u_2^\star)$. *If* $[\Re_0] > 1$, *then* $E^\star_{(5.1)}$ *is globally asymptotically stable.*

Proof. Define

$$\mathbf{V}(t) := \int_\Omega \left(u_1 - u_1^\star \ln u_1 + u_2 - u_2^\star \ln u_2 + \frac{\beta_2 u_1^\star u_3^\star}{\rho u_2^\star}(u_3 - u_3^\star \ln u_3)\right)\mathrm{d}x,$$

Since

$$\beta_1 \frac{u_1^\star u_2^\star}{u_1^\star + u_2^\star} + \beta_2 u_1^\star u_3^\star + \zeta u_1^\star = b,\ \beta_1 \frac{u_1^\star u_2^\star}{u_1^\star + u_2^\star} + \beta_2 u_1^\star u_3^\star = y u_2^\star \text{ and } \rho u_2^\star = \delta u_3^\star,$$

after elementary calculations, we can obtain

$$\begin{aligned} \frac{d\mathbf{V}(t)}{dt} =& d_1 \int_\Omega \left(1 - \frac{u_1^\star}{u_1}\right)\Delta u_1 dx + d_2 \int_\Omega \left(1 - \frac{u_2^\star}{u_2}\right)\Delta u_2 \mathrm{d}x \\ &+ \int_\Omega \left(-\frac{\zeta(u_1 - u_1^\star)^2}{u_1} - \frac{\beta_1 u_2^\star (u_1 - u_1^\star)^2}{u_1(u_1^\star + u_2^\star)} + \frac{\beta_1(u_1 - u_1^\star)[u_2^\star(u_1 - u_1^\star) - u_1^\star(u_2 - u_2^\star)]}{(u_1 + u_2)(u_1^\star + u_2^\star)}\right)\mathrm{d}x \\ &+ \int_\Omega \left(\frac{\beta_1(u_2 - u_2^\star)[u_2 r^\star(u_1 - u_1^\star) - u_1^\star(u_2 - u_2^\star)]}{(u_1 + u_2)(u_1^\star + u_2^\star)}\right)\mathrm{d}x \\ &+ \beta_2 u_1^\star u_3^\star \int_\Omega \left(3 - \frac{u_1^\star}{u_1} - \frac{u_2^\star u_1 u_3}{u_2 u_1^\star u_3^\star} - \frac{u_2 u_3^\star}{u_2^\star u_3}\right)\mathrm{d}x. \end{aligned}$$

By using $u_1 \le u_1 + u_2$, we have

$$\frac{d\mathbf{V}(t)}{dt} \le \int_{\Omega} \left(-\frac{\beta_1}{(u_1+u_2)(u_1^\star+u_2^\star)}\right)\left(\frac{\zeta(u_1^\star+u_2^\star)}{\beta_1}(u_1-u_1^\star)^2-(u_2^\star-u_1^\star)(u_1-u_1^\star)(u_2-u_2^\star)+u_1^\star(u_2-u_2^\star)^2\right)dx$$
$$+\beta_2 u_1^\star u_3^\star \int_{\Omega}\left(3-\frac{u_1^\star}{u_1}-\frac{u_2^\star u_1 u_3}{u_2 u_1^\star u_3^\star}-\frac{u_2 u_3^\star}{u_2^\star u_3}\right)\mathrm{d}x.$$

By assumption (**H**1), we have

$$\left(\sqrt{\frac{\zeta(u_1^\star+u_2^\star)}{\beta_1}}(u_1-u_1^\star)-\sqrt{u_1^\star}(u_2-u_2^\star)\right)^2 \le \frac{\zeta(u_1^\star+u_2^\star)}{\beta_1}(u_1-u_1^\star)^2-(u_2^\star-u_1^\star)(u_1-u_1^\star)(u_2-u_2^\star)+u_1^\star(u_2-u_2^\star)^2.$$

Hence,

$$\frac{d\mathbf{V}(t)}{dt} \le \int_{\Omega}\left(-\frac{\beta_1}{(u_1+u_2)(u_1^\star+u_2^\star)}\right)\left(\sqrt{\frac{\zeta(u_1^\star+u_2^\star)}{\beta_1}}(u_1-u_1^\star)-\sqrt{u_1^\star}(u_2-u_2^\star)\right)^2 \mathrm{d}x \le 0.$$

Hence, the largest invariant subset $\{(u_1,u_2,u_3)\in \mathbb{X}_+ : \frac{d\mathbf{V}(t)}{dt}=0\}$ consists just one singleton $\{E^\star_{(5.1)}\}$. According to the LaSalle's invariance principle, the global stability of $E^\star_{(5.1)}$ is confirmed. This completes the proof. □

Remark 5. *In Theorem 5.3, (**H**1) is a technical condition such that the derivation of Lyapunov function is less than zero. It is significant to establish the global asymptotic stability of endemic equilibrium by constructing a suitable Lyapunov function in epidemiology. Our model (1.5) includes two types of infection: frequency dependent and mass action, which leads to challenging issue in proving global asymptotic stability of endemic equilibrium by Lyapunov function. Similar arguments can be found in (C1)-(C3) in the proof of Theorem 2.3 of [24].*

6 Asymptotic profiles of the positive steady state

Theorem 4.2 indicates that when $\Re_0 > 1$, (6.1) admits at least one positive steady state $E^\star$, which is the positive solution of

$$\begin{cases} d_1\Delta u_1 + b(x) - \beta_1(x)\dfrac{u_1u_2}{u_1+u_2} - \beta_2(x)u_1u_3 - \zeta(x)u_1 = 0, & x\in\Omega,\\ d_2\Delta u_2 + \beta_1(x)\dfrac{u_1u_2}{u_1+u_2} + \beta_2(x)u_1u_3 - y(x)u_2 = 0, & x\in\Omega,\\ \varrho(x)u_2 - \delta(x)u_3 = 0, & x\in\Omega,\\ \dfrac{\partial u_1}{\partial\nu} = \dfrac{\partial u_2}{\partial\nu} = 0, & x\in\partial\Omega. \end{cases} \tag{6.1}$$

This section is devoted to the investigation of the asymptotic profiles of $E^\star$ for the cases that $d_1 \to 0$, $d_1 \to \infty$, $d_2 \to 0$ and $d_2 \to \infty$. As exploration of such problem can achieve better understanding the spatial distribution of disease.

We begin with the preliminary estimates of solutions of (6.1).

Lemma 6.1. *Let (u_1,u_2,u_3) be any positive solution of (6.1). We then directly have*
(i) *For any $d_1, d_2 > 0$, we have the upper bound of u_1:*

$$u_1(x) \le \frac{b^\star}{\zeta_\star}, \quad \forall x\in\overline{\Omega}; \tag{6.2}$$

(ii) *Fix $d_2 > 0$, for any $d_1 > 0$, we have the upper bound of u_2 and lower bound of u_1:*

$$u_2(x) \le C_1, \quad u_1(x) \ge C_2, \quad \forall x\in\overline{\Omega}, \tag{6.3}$$

where the positive constants C_i $(i = 1, 2)$ do not depend on $d_1 > 0$.

Proof. (i) Set $u_1(x_0) = \max_{x\in\overline{\Omega}} u_1(x)$. By applying the maximum principle (see, e.g. [14, Proposition 2.2]) to u_1-equation of (6.1), we get

$$b(x_0) - \beta_1(x_0)\frac{u_1(x_0)u_2(x_0)}{u_1(x_0)+u_2(x_0)} - \frac{\beta_2(x_0)\varrho(x_0)}{\delta(x_0)}u_1(x_0)u_2(x_0) - \zeta(x_0)u_1(x_0) \geq 0.$$

Thus, we know that

$$\max_{x\in\overline{\Omega}} u_1(\cdot) = u_1(x_0) \leq \frac{b^*}{\zeta_\star},$$

this gives the upper bound of u_1.

(ii) Making the sum of u_1, u_2 of (6.1) and integrating over Ω lead to

$$y_\star \int_\Omega u_2\,\mathrm{d}x \leq \int_\Omega \zeta u_1\,\mathrm{d}x + \int_\Omega y u_2\,\mathrm{d}x = \int_\Omega b\,\mathrm{d}x \leq b^*|\Omega|. \tag{6.4}$$

Notice that u_2 solves

$$\begin{cases} \Delta u_2 + \left[\dfrac{\beta_1(\cdot)u_1}{d_2(u_1+u_2)} + \dfrac{\beta_2(\cdot)\varrho(\cdot)}{d_2\delta(\cdot)}u_1 - \dfrac{y(\cdot)}{d_2}\right]u_2 = 0, & x \in \Omega, \\ \dfrac{\partial u_2}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{6.5}$$

Fix $d_2 > 0$, it follows from statement (i) that

$$\left|\frac{\beta_1(\cdot)u_1}{(u_1+u_2)d_2} + \frac{\beta_2(\cdot)\varrho(\cdot)}{d_2\delta(\cdot)}u_1 - \frac{y(\cdot)}{d_2}\right| \leq \frac{\beta_1^*}{d_2} + \frac{\beta_2^*\varrho^*b^*}{d_2\delta_\star\zeta_\star} + \frac{y^*}{d_2}.$$

By using [20, Lemma 2.2] (see also [12]), we get a Harnack-type inequality of u_2 as follows

$$\max_{\overline{\Omega}} u_2 \leq C\min_{\overline{\Omega}} u_2. \tag{6.6}$$

where $C > 0$ does not depend on d_1. In what follows, we permit it changing from place to place.

Based on (6.4) and (6.6), we have

$$u_2(\cdot) \leq C\min_{\overline{\Omega}} u_2 \leq \frac{C}{|\,\Omega\,|}\int_\Omega u_2\,\mathrm{d}x \leq C, \text{ for } \forall x \in \overline{\Omega}. \tag{6.7}$$

Set $u_1(x_1) = \min_{x\in\overline{\Omega}} u_1(x)$. By applying the maximum principle, we have

$$b(x_1) - \beta_1(x_1)\frac{u_1(x_1)u_2(x_1)}{u_1(x_1)+u_2(x_1)} - \frac{\beta_2(x_1)\varrho(x_1)}{\delta(x_1)}u_1(x_1)u_2(x_1) - \zeta(x_1)u_1(x_1) \leq 0.$$

From (6.7), we have

$$b_\star \leq \left(\beta_1^* + \frac{\beta_2^*\varrho^*C}{\delta_\star} + \zeta^*\right)u_1(x_1).$$

Hence, we get the lower bound of u_1. □

Remark 6. *Note that the statement (ii) hold for any $d_1 > 0$ and $d_2 \geq 1$. Actually, we can get* (6.6) *for any $d_2 \geq 1$.*

6.1 Profile as $d_1 \to 0$

In view of Theorems 3.1 and 4.2, (6.1) possesses at least a positive solution for sufficiently small d_1 when $\Re_{0,0} > 1$. This subsection is spent on exploring the asymptotic profile of the positive solution of (6.1) with $d_1 \to 0$.

Theorem 6.1. *Fix $d_2 > 0$, and let $d_1 \to 0$. If $\Re_{0,0} > 1$, then up to a subsequence of d_1, every positive solution $(u_1(\cdot; d_1), u_2(\cdot; d_1), u_3(\cdot; d_1))$ of (6.1) satisfies*

$$(u_1(\cdot; d_1), u_2(\cdot; d_1), u_3(\cdot; d_1)) \to \left(\Phi_{u_1}, \Phi_{u_2}, \frac{\rho}{\delta}\Phi_{u_2}\right) \text{ uniformly for } x \in \overline{\Omega},$$

where

$$\begin{aligned}\Phi_{u_1}(\cdot) =& G(x\cdot, \Phi_{u_2}(\cdot))\\ :=& \frac{\left[b\delta - (\beta_1\delta + \beta_2\varrho\Phi_{u_2} + \zeta\delta)\,\Phi_{u_2} + \sqrt{[b\delta - (\beta_1\delta + \beta_2\varrho\Phi_{u_2} + \zeta\delta)\,\Phi_{u_2}]^2 + 4b\delta\Phi_{u_2}\,(\beta_2\varrho\Phi_{u_2} + \zeta\delta)}\right]}{2\,(\beta_2\varrho\Phi_{u_2} + \zeta\delta)},\end{aligned}$$

and Φ_{u_2} is a positive solution to

$$\begin{cases} d_2\Delta\Phi_{u_2} + \beta_1(x)\dfrac{G(x,\Phi_{u_2})\Phi_{u_2}}{G(x,\Phi_{u_2}) + \Phi_{u_2}} + \dfrac{\beta_2(x)\varrho(x)}{\delta(x)}G(x,\Phi_{u_2})\Phi_{u_2} - y(x)\Phi_{u_2} = 0, & x \in \Omega,\\ \dfrac{\partial\Phi_{u_2}}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{6.8}$$

Proof. We deal with the proof by two steps.

Step 1. Convergence of u_2. Observe that u_2 satisfies

$$\begin{cases} -d_2\Delta u_2 + y(\cdot)u_2 = \beta_1(\cdot)\dfrac{u_1u_2}{u_1 + u_2} + \dfrac{\beta_2(\cdot)\varrho(x)}{\delta(\cdot)}u_1u_2, & x \in \Omega,\\ \dfrac{\partial u_2}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{6.9}$$

From the estimates in Lemmma 6.1, we know that $\forall\, p > 1$,

$$\left\|\beta_1(x)\frac{u_1u_2}{u_1 + u_2} + \frac{\beta_2(x)\varrho(x)}{\delta(x)}u_1u_2\right\|_{L^p(\Omega)} \le C.$$

Combined with the standard L^p-estimate for elliptic equations and Sobolev embedding theorem (see, e.g. [7]),

$$\| u_2 \|_{C^{1+\alpha}(\overline{\Omega})} \le C, \text{ for some } 0 < \alpha < 1.$$

From the third equation of (6.1), it is clear that

$$\| u_3 \|_{C^{1+\alpha}(\overline{\Omega})} \le C, \text{ for some } 0 < \alpha < 1.$$

Denoted by d_{1n} the subsequence of $d_1 \to 0$, satisfying $d_{1n} \to 0$ as $n \to \infty$. With this subsequence, $(u_{1n}, u_{2n}, u_{3n}) := (u_1(x; d_{1n}), u_2(x; d_{1n}), u_3(x; d_{1n}))$ of (6.1) with $d_1 = d_{1n}$ satisfies

$$(u_{2n}, u_{3n}) \to \left(\Phi_{u_2}, \frac{\rho}{\delta}\Phi_{u_2}\right) \text{ uniformly for } x \in \overline{\Omega} \text{ and } n \to \infty, \tag{6.10}$$

where $\Phi_{u_2} \in C^1(\overline{\Omega})$ and $\Phi_{u_2} \ge 0$. Taking into account of Harnack-type inequality of u_2 (6.6), it is clear that

$$\text{either } \Phi_{u_2} \equiv 0 \text{ on } \overline{\Omega} \text{ or } \Phi_{u_2} > 0 \text{ on } \overline{\Omega}. \tag{6.11}$$

We next prove that $\Phi_{u_2} > 0$ on $\overline{\Omega}$. Assume to the contrary that $\Phi_{u_2} \equiv 0$ on $\overline{\Omega}$, i.e., as $n \to \infty$,

$$u_{2n} \to 0 \text{ uniformly on } \overline{\Omega}. \tag{6.12}$$

Choose $0 < \epsilon < b_\star/\beta_1^\star$ small enough that

$$0 \le u_{2n}(\cdot) \le \epsilon \text{ on } \overline{\Omega}, \text{ for all large } n.$$

Hence, u_{1n} satisfies

$$-d_{1n}\Delta u_{1n} \le b(\cdot) - \zeta(\cdot)u_{1n}, \; x \in \Omega,$$

and

$$-d_{1n}\Delta u_{1n} \geq b(\cdot) - \beta_1^\star \epsilon - \frac{\beta^\star \rho^\star}{\delta_\star}\epsilon u_{1n} - \zeta(\cdot)u_{1n},\ x \in \Omega,$$

respectively, with $\frac{\partial u_{1n}}{\partial \nu} = 0,\ x \in \partial\Omega$. Fix sufficiently large n, denoted by U_n and V_n respectively the unique positive solution of the following two auxiliary systems:

$$-d_{1n}\Delta U = b(\cdot) - \zeta(\cdot)U,\ x \in \Omega;\ \frac{\partial U}{\partial \nu} = 0,\ x \in \partial\Omega, \tag{6.13}$$

and

$$-d_{1n}\Delta V = b(\cdot) - \beta_1^\star \epsilon - \frac{\beta^\star \rho^\star}{\delta_\star}\epsilon V - \zeta(\cdot)V,\ x \in \Omega;\ \frac{\partial V}{\partial \nu} = 0,\ x \in \partial\Omega. \tag{6.14}$$

It follows from the super-subsolution argument that

$$V_n \leq u_{1n} \leq U_n \text{ on } \overline{\Omega}. \tag{6.15}$$

By applying the singular perturbation theory technique (see, e.g. [5, Lemma 2.4] or [13, Lemma 2.1]), we know that as $n \to \infty$,

$$U_n(x) \to \frac{b(\cdot)}{\zeta(\cdot)} \text{ and } V_n(\cdot) \to \frac{b(\cdot) - \beta_1^\star \epsilon}{\frac{\beta^\star \rho^\star}{\delta_\star}\epsilon + \zeta(\cdot)} \text{ uniformly on } \overline{\Omega}.$$

Hence,

$$\frac{b(\cdot) - \beta_1^\star \epsilon}{\frac{\beta^\star \rho^\star}{\delta_\star}\epsilon + \zeta(\cdot)} \leq \liminf_{n\to\infty} u_{1n}(\cdot) \leq \limsup_{n\to\infty} u_{1n}(\cdot) \leq \frac{b(\cdot)}{\zeta(\cdot)} \text{ on } \overline{\Omega}. \tag{6.16}$$

Since ϵ is arbitrary, (6.16) further implies that

$$u_{1n}(x) \to \frac{b(\cdot)}{\zeta(\cdot)} \text{ uniformly on } \overline{\Omega}, \text{ as } n \to \infty. \tag{6.17}$$

We now pay attention to u_2 equation of (6.1) with u_{2n}:

$$-d_2\Delta u_{2n} = \beta_1(\cdot)\frac{u_{1n}u_{2n}}{u_{1n}+u_{2n}} + \frac{\beta_2(\cdot)\varrho(\cdot)}{\delta(\cdot)}u_{1n}u_{2n} - y(\cdot)u_{2n},\ x \in \Omega;\quad \frac{\partial u_{2n}}{\partial \nu} = 0,\ x \in \partial\Omega. \tag{6.18}$$

For all $n \geq 1$, define $\widetilde{u}_{2n} := \frac{u_{2n}}{\|u_{2n}\|_{L^\infty(\Omega)}}$ that $\|\widetilde{u}_{2n}\|_{L^\infty(\Omega)} = 1$. In this setting, $\widetilde{u}_{2n}$ satisfies

$$-d_2\Delta \widetilde{u}_{2n} = \left[\frac{\beta_1(\cdot)u_{1n}}{u_{1n}+u_{2n}} + \frac{\beta_2(\cdot)\varrho(\cdot)}{\delta(\cdot)}u_{1n} - y(\cdot)\right]\widetilde{u}_{2n},\ x \in \Omega;\quad \frac{\partial \widetilde{u}_{2n}}{\partial \nu} = 0,\ x \in \partial\Omega. \tag{6.19}$$

As above, with the aid of standard compactness argument, as $n \to \infty$,

$$\widetilde{u}_{2n} \to \widetilde{u}_2 \text{ in } C^1(\overline{\Omega}),$$

where $\widetilde{u}_2 \geq 0$ belongs to $C^1(\overline{\Omega})$, and satisfies $\|\widetilde{u}_2\|_{L^\infty(\Omega)} = 1$. From (6.12) and (6.17), by sending $n \to \infty$ in (6.19), it is clear that $\widetilde{u}_2$ satisfies

$$-d_2\Delta \widetilde{u}_2 = \left[\beta_1(\cdot) - y(\cdot) + \frac{\beta_2(\cdot)\varrho(\cdot)b(\cdot)}{\delta(\cdot)\zeta(\cdot)}\right]\widetilde{u}_2,\ x \in \Omega;\quad \frac{\partial \widetilde{u}_2}{\partial \nu} = 0,\ x \in \partial\Omega.$$

From [20, Lemma 2.2] (see also [12]), it follows that the Harnack-type inequality of $\widetilde{u}_2$ remains true and then $\widetilde{u}_2 > 0$ on $\overline{\Omega}$. Therefore, 0 is the principal eigenvalue of (3.8) with $u_1^0(x) = b(x)/\zeta(x)$. This contradicts with our assumption that $\Re_{0,0} > 1$. Thus, $\Phi_{u_2} > 0$ on $\overline{\Omega}$, which means that

$$u_{2n} \to \Phi_{u_2} > 0 \text{ uniformly for } x \in \overline{\Omega}. \tag{6.20}$$

Step 2. Convergence of u_1. From (6.1), we know that u_{1n} solves:

$$\begin{cases} -d_{1n}\Delta u_{1n} = b(\cdot) - \beta_1(\cdot)\dfrac{u_{1n}u_{2n}}{u_{1n}+u_{2n}} - \dfrac{\beta_2(\cdot)\rho(\cdot)}{\delta(\cdot)}u_{1n}u_{2n} - \zeta(\cdot)u_{1n}, & x \in \Omega, \\ \dfrac{\partial u_{1n}}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

By (6.20), choose $\epsilon > 0$ small enough that $0 < \Phi_{u_2} - \epsilon \le u_{2n} \le \Phi_{u_2} + \epsilon$ on $\overline{\Omega}$, for all large n. Then,

$$\begin{aligned} b(\cdot) - \beta_1(\cdot)\frac{u_{1n}u_{2n}}{u_{1n}+u_{2n}} - \frac{\beta_2(\cdot)\rho(\cdot)}{\delta(\cdot)}u_{1n}u_{2n} - \zeta(\cdot)u_{1n} &\le b(\cdot) - \beta_1(\cdot)\frac{u_{1n}(\Phi_{u_2}-\epsilon)}{u_{1n}+(\Phi_{u_2}-\epsilon)} - \frac{\beta_2(\cdot)\varrho(\cdot)}{\delta(\cdot)}u_{1n}(\Phi_{u_2}-\epsilon) - \zeta(\cdot)u_{1n} \\ &= \frac{(h_+^\epsilon(x,\Phi_{u_2}) - u_{1n})(u_{1n} - h_-^\epsilon(x,\Phi_{u_2}))}{u_{1n}+(\Phi_{u_2}-\epsilon)}, \end{aligned}$$

where

$$h_\pm^\epsilon(x,\Phi_{u_2}) = \left\{ \frac{b(\cdot) - (\beta_1(\cdot) + \frac{\beta_2(\cdot)\varrho(\cdot)}{\delta(\cdot)}(u_2^\star - \epsilon) + \zeta(\cdot))(u_2^\star - \epsilon) \pm \mathbb{D}_1}{2\left[\frac{\beta_2(\cdot)\varrho(\cdot)}{\delta(\cdot)}(u_2^\star-\epsilon) + \zeta(\cdot)\right]} \right\}$$

and $\mathbb{D}_1 = \sqrt{[b(\cdot) - (\beta_1(\cdot) + \frac{\beta_2(\cdot)\varrho(\cdot)}{\delta(\cdot)}(u_2^\star - \epsilon) + \zeta(\cdot))(u_2^\star-\epsilon)]^2 + 4b(u_2^\star-\epsilon)(\frac{\beta_2(\cdot)\varrho(\cdot)}{\delta(\cdot)}(u_2^\star-\epsilon)+\zeta(\cdot))}$. Obviously,

$$h_+^\epsilon(x,\Phi_{u_2}) > 0 \text{ and } h_-^\epsilon(x,\Phi_{u_2}) < 0 \text{ on } \overline{\Omega}.$$

Consider the following problem

$$\begin{cases} -d_n\Delta z = \dfrac{(h_+^\epsilon(x,\Phi_{u_2}) - z)(z - h_-^\epsilon(x,\Phi_{u_2}))}{(z + (\Phi_{u_2}-\epsilon))}, & x \in \Omega, \\ \dfrac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{6.21}$$

Then, for sufficiently large $C > 0$, (u_{1n}, C) is a pair of sub-supersolution of (6.21) on $\overline{\Omega}$. Hence, (6.21) admits at least one positive solution, z_n, and z_n satisfies $u_{1n} \le z_n \le C$ on $\overline{\Omega}$. Further by the singular perturbation theory technique,

$$z_n \to h_+^\epsilon(x,\Phi_{u_2}(x)) \text{ uniformly on } \overline{\Omega}, \text{ as } n \to \infty.$$

Hence,

$$\limsup_{n\to\infty} u_{1n}(x) \le h_+^\epsilon(x,\Phi_{u_2}(x)) \text{ uniformly on } \overline{\Omega}. \tag{6.22}$$

On the other hand,

$$\begin{aligned} b(\cdot) - \beta_1(\cdot)\frac{u_{1n}u_{2n}}{u_{1n}+u_{2n}} - \frac{\beta_2(\cdot)\varrho(\cdot)}{\delta(\cdot)}u_{1n}u_{2n} - \zeta(\cdot)u_{1n} &\ge b(\cdot) - \beta_1(\cdot)\frac{u_{1n}(\Phi_{u_2}+\epsilon)}{u_{1n}+(\Phi_{u_2}+\epsilon)} - \frac{\beta_2(\cdot)\varrho(\cdot)}{\delta(\cdot)}u_{1n}(\Phi_{u_2}+\epsilon) - \zeta(\cdot)u_{1n} \\ &= \frac{(h_\epsilon^+(x,\Phi_{u_2}) - u_{1n})(u_{1n} - h_\epsilon^-(x,\Phi_{u_2}))}{u_{1n}+(\Phi_{u_2}+\epsilon)}, \end{aligned}$$

with

$$h_\epsilon^\pm(x,\Phi_{u_2}) = \left\{ \frac{b(\cdot) - (\beta_1(\cdot) + \frac{\beta_2(\cdot)\varrho(\cdot)}{\delta(\cdot)}(\Phi_{u_2} + \epsilon) + \zeta(\cdot))(\Phi_{u_2} + \epsilon) \pm \mathbb{D}_2}{2\left[\frac{\beta_2(\cdot)\varrho(\cdot)}{\delta}(\Phi_{u_2}+\epsilon) + \zeta(\cdot)\right]} \right\},$$

where $\mathbb{D}_2 = \sqrt{[b(\cdot) - (\beta_1(\cdot) + \frac{\beta_2(\cdot)\varrho(\cdot)}{\delta(\cdot)}(\Phi_{u_2} + \epsilon) + \zeta(\cdot))(\Phi_{u_2}+\epsilon)]^2 + 4b(\Phi_{u_2}+\epsilon)(\frac{\beta_2(\cdot)\varrho(\cdot)}{\delta(\cdot)}(\Phi_{u_2}+\epsilon)+\zeta(\cdot))}$. By a similar argument as before, we further have

$$\liminf_{n\to\infty} u_{1n}(x) \ge h_\epsilon^+(x,\Phi_{u_2}(x)) \text{ uniformly on } \overline{\Omega}. \tag{6.23}$$

Since

$$\lim_{\epsilon\to 0} h_+^\epsilon(x,\Phi_{u_2}) = G(x,\Phi_{u_2}(x)) = \lim_{\epsilon\to 0} h_\epsilon^+(x,\Phi_{u_2}),$$

then as $n \to \infty$,

$$u_{1n}(x) \to G(x,\Phi_{u_2}(x)) \text{ uniformly on } \overline{\Omega}.$$

As a result, by (6.18), Φ_{u_2} satisfies (6.9). This proves Theorem 6.1. □

6.2 Profile as $d_2 \to 0$

This subsection is spent on exploring the asymptotic profile of the positive solution of (6.1) with $d_2 \to 0$. In this case, because of mathematical difficulty, we only consider the limiting profile of positive solutions in 1-D space. Without loss of generality, we take $\Omega = (0, 1)$.

Theorem 6.2. *Assume that* $\{x \in [0,1] : \delta(\cdot)\beta_1(\cdot) + \beta_2(\cdot)\varrho(\cdot)u_1^0(\cdot) > \delta(\cdot)y(\cdot)\}$ *is non-empty. Fix* $d_1 > 0$, *and let* $d_2 \to 0$, *then every positive solution* $(u_1(\cdot; d_2), u_2(\cdot; d_2), u_3(\cdot; d_2))$ *of (6.1) satisfies*

$$(u_1(\cdot; d_2), u_2(\cdot; d_2), u_3(\cdot; d_2)) \to \left(\Psi_{u_1}, \Psi_{u_2}, \frac{\rho}{\delta}\Psi_{u_2}\right) \text{ uniformly on } [0,1],$$

where

$$\Psi_{u_2}(x) := \left(\frac{(\beta_1(\cdot)\delta(\cdot) + \beta_2(\cdot)\rho(\cdot)\Psi_{u_1}(\cdot) - y(\cdot)\delta(\cdot))\Psi_{u_1}(\cdot)}{y(\cdot)\delta(\cdot) - \beta_2(\cdot)\rho(\cdot)\Psi_{u_1}(\cdot)}\right)_+, \tag{6.24}$$

and Ψ_{u_1} *solves*

$$\begin{cases} d_1\Psi''_{u_1} + b(\cdot) - \beta_1(\cdot)\dfrac{\Psi_{u_1}\Psi_{u_2}}{\Psi_{u_1} + \Psi_{u_2}} - \dfrac{\beta_2(\cdot)\rho(\cdot)}{\delta(\cdot)}\Psi_{u_1}\Psi_{u_2} - \zeta(\cdot)\Psi_{u_1} = 0, & x \in (0,1), \\ \Psi'_{u_1}(0) = \Psi'_{u_1}(1) = 0. \end{cases}$$

Proof. From the estimats (6.2) and (6.4), together with the elliptic L^1-estimate theory (see, e.g. [2] or [12, Lemma 2.2]),

$$\|u_1\|_{W^{1,q}((0,1))} \le C, \ \forall q > 0,$$

Choose p large enough, combined with the Sobolev embedding theorem, that

$$\| u_1 \|_{C^\alpha([0,1])} \le C \text{ for some } 0 < \alpha < 1.$$

Denoted by d_{2k} the subsequence of $d_2 \to 0$, satisfying $d_{2k} \to 0$ as $k \to \infty$. With this subsequence, $(u_{1k}, u_{2k}, u_{3k}) := (u_1(\cdot; d_{2k}), u_2(\cdot; d_{2k}), u_3(\cdot; d_{2k}))$ of (6.1) with $d_2 = d_{2k}$ satisfies $u_{1k} \to \Psi_{u_1}$ in $C([0,1])$ as $k \to \infty$.

We note that u_{2k} satisfies

$$\begin{cases} d_{2k}u''_{2k} + \left[\beta_1(\cdot)\dfrac{u_{1k}}{u_{1k} + u_{2k}} + \dfrac{\beta_2(\cdot)\varrho(\cdot)}{\delta(\cdot)}u_{1k} - y(\cdot)\right]u_{2k} = 0, & x \in (0,1), \\ u'_{2k}(0) = u'_{2k}(1) = 0. \end{cases} \tag{6.25}$$

By a similar super-subsolution argument (see Theorem 6.1), we get

$$u_{2k} \to \Psi_{u_2} \text{ uniformly on } [0,1] \text{ as } k \to \infty,$$

where Ψ_{u_2} is given by (6.24). Moreover, from the expression of Ψ_{u_2}, it is clear that Ψ_{u_1} solves (6.2). The proof is complete. □

6.3 Profile as $d_1 \to \infty$

In view of Theorems 3.1 and 4.2, (6.1) possesses at least a positive solution for sufficiently small d_1 when $\Re_{0,\infty} > 1$. This subsection is spent on exploring the asymptotic profile of the positive solution of (6.1) with $d_1 \to \infty$.

Theorem 6.3. *Assume that* $\Re_{0,\infty} > 1$. *Fix* $d_2 > 0$, *and let* $d_1 \to \infty$, *then every positive solution* $(u_1(\cdot; d_1), u_2(\cdot; d_1), u_3(\cdot; d_1))$ *of (6.1), up to a subsequence of* d_1, *satisfies*

$$(u_1(\cdot; d_1), u_2(\cdot; d_1), u_3(\cdot; d_1)) \to \left(u_1^\infty, u_2^\infty, \frac{\rho}{\delta}u_2^\infty\right) \text{ uniformly on } \overline{\Omega},$$

where (u_1^∞, u_2^∞) *solves*

$$\begin{cases} \int\limits_\Omega \left[b(\cdot) - \zeta(\cdot)u_1^\infty - y(\cdot)u_2^\infty\right] \mathrm{d}x = 0, & x \in \Omega, \\ -d_2 \Delta u_2^\infty = \beta_1(\cdot)\dfrac{u_1^\infty u_2^\infty}{u_1^\infty + u_2^\infty} + \dfrac{\beta_2(\cdot)\varrho(\cdot)}{\delta(\cdot)} u_1^\infty u_2^\infty - y(\cdot)u_2^\infty, & x \in \Omega, \\ \dfrac{\partial u_2^\infty}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{6.26}$$

and u_1^∞ *is a positive constant and* $u_2^\infty > 0$ *on* $\overline{\Omega}$.

Proof. Rewriting u_1 equation of (6.1) as

$$\begin{cases} -\Delta u_1 = \dfrac{1}{d_1}\left[b(\cdot) - \dfrac{\beta_1(\cdot)u_1u_2}{u_1 + u_2} - \dfrac{\beta_2(\cdot)\varrho(\cdot)}{\delta(\cdot)}u_1u_2 - \zeta(\cdot)u_1\right], & x \in \Omega, \\ \dfrac{\partial u_1}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{6.27}$$

From the estimates of u_1 and u_2 in Lemma 6.1, we know that

$$\left\| \frac{1}{d_1}\left[b(\cdot) - \frac{\beta_1(\cdot)u_1u_2}{u_1 + u_2} - \frac{\beta_2(\cdot)\varrho(\cdot)}{\delta(\cdot)}u_1u_2 - \zeta(\cdot)u_1\right] \right\|_{L^p(\Omega)} \le C, \quad \forall p \ge 1,$$

where $C > 0$ does not dependent on $d_1 \ge 1$. Combined with the standard L^p-estimate for elliptic equations and Sobolev embedding theorem,

$$\| u_1 \|_{C^{1+\alpha}(\overline{\Omega})} \le C \text{ for some } 0 < \alpha < 1.$$

Denoted by d_{1i} the subsequence of $d_1 \to \infty$, satisfying $d_{1i} \to 0$ as $i \to \infty$. With this subsequence, $(u_{1i}, u_{2i}, u_{3i}) := (u_1(\cdot; d_{1i}), u_2(\cdot; d_{1i}), u_3(\cdot; d_{1i}))$ of (6.1) with $d_1 = d_{1i}$ satisfies $u_{1i} \to u_1^\infty$ in $C^1(\overline{\Omega})$ as $i \to \infty$, where $u_1^\infty > 0$ on $\overline{\Omega}$ due to the estimate of u_1. Moreover, $u_{1\infty}$ solves $-\Delta u_1^\infty = 0$, $x \in \Omega$ with homogeneous Neumann boundary condition implies that $u_1^\infty > 0$ is a positive constant.

Similar arguments as before, as $i \to \infty$,

$$(u_{2i}, u_{3i}) \to \left(u_2^\infty, \frac{\rho}{\delta}u_2^\infty\right) \text{ in } C^1(\overline{\Omega}),$$

where $u_2^\infty \in C^1(\overline{\Omega})$ and $u_2^\infty \ge 0$ on $\overline{\Omega}$. Since the Harnack-type inequality of u_2, then either $u_2^\infty > 0$ on $\overline{\Omega}$ or $u_2^\infty \equiv 0$ on $\overline{\Omega}$. Similar to the arguments in Section 6.1, we suppose that $u_2^\infty \equiv 0$ on $\overline{\Omega}$. Define $\tilde{u}_{2i} := u_{2i}/\|u_{2i}\|_{L^\infty(\Omega)}$ that $\|\tilde{u}_{2i}\|_{L^\infty(\Omega)} = 1$ for all $i \ge 1$, and $\tilde{u}_{2i}$ satisfies

$$\begin{cases} d_2\Delta\tilde{u}_{2i} + \left[\beta_1(\cdot)\dfrac{u_{1i}}{u_{1i} + u_{2i}} + \dfrac{\beta_2(\cdot)\varrho(\cdot)}{\delta(\cdot)}u_{1i} - y(\cdot)\right]\tilde{u}_{2i} = 0, & x \in \Omega, \\ \dfrac{\partial \tilde{u}_{2i}}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

As a result, $\tilde{u}_{2i} \to \hat{u}_2^\infty$ in $C^1(\overline{\Omega})$ as $i \to \infty$, where $\hat{u}_2^\infty \ge 0$ on $\overline{\Omega}$, $\|\hat{u}_2^\infty\|_{L^\infty(\Omega)} = 1$. As $u_{2i} \to 0$ as $i \to \infty$, it follows from a super-subsolution argument that $u_{1i} \to \int_\Omega b(x)\,\mathrm{d}x / \int_\Omega \zeta(x)\,\mathrm{d}x$. Thus, $\hat{u}_2^\infty$ solves

$$\begin{cases} d_2\Delta\hat{u}_2^\infty + \left[\beta_1(\cdot) + \dfrac{\beta_2(\cdot)\varrho(\cdot)\int_\Omega b(\cdot)\,\mathrm{d}x}{\delta(\cdot)\int_\Omega \zeta(\cdot)\,\mathrm{d}x} - y(\cdot)\right]\hat{u}_2^\infty = 0, & x \in \Omega, \\ \dfrac{\partial \hat{u}_2^\infty}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

It can be seen that the Harnack-type inequality of $\hat{u}_2^\infty$ also remains true and then $\hat{u}_2^\infty > 0$ on $\overline{\Omega}$. Consequently, 0 is the principal eigenvalue of (3.12), a contradiction with $\Re_{0,\infty} > 1$. Hence, $u_2^\infty > 0$ satisfies (6.26). This proves Theorem 6.3. □

6.4 Profile as $d_2 \to \infty$

This subsection is spent on exploring the asymptotic profile of the positive solution of (6.1) with $d_2 \to \infty$.

Theorem 6.4. *Assume that* $\int_\Omega \left(\beta_1(\cdot) + \frac{\beta_2(\cdot)u_1^0(\cdot)\varrho(\cdot)}{\delta(\cdot)}\right) dx / \int_\Omega y(\cdot) dx > 1$. *Fix* $d_1 > 0$, *and let* $d_2 \to \infty$, *then every positive solution* $(u_1(\cdot; d_2), u_2(\cdot; d_2), u_3(\cdot; d_2))$ *of (6.1), up to a subsequence of* d_2, *satisfies*

$$(u_1(\cdot; d_2), u_2(\cdot; d_2), u_3(\cdot; d_2)) \to \left(u_{1\infty}, u_{2\infty}, \frac{\rho}{\delta} u_{2\infty}\right) \text{ uniformly on } \overline{\Omega},$$

where $(u_{1\infty}, u_{2\infty})$ *solves*

$$\begin{cases} -d_1 \Delta u_{1\infty} = b(\cdot) - \beta_1(\cdot)\dfrac{u_{1\infty}u_{2\infty}}{u_{1\infty} + u_{2\infty}} - \dfrac{\beta_2(\cdot)\varrho(\cdot)}{\delta(\cdot)} u_{1\infty}u_{2\infty} - \zeta(\cdot)u_{1\infty}, & x \in \Omega, \\ \displaystyle\int_\Omega \left[b(\cdot) - \zeta(\cdot)u_{1\infty} - y(\cdot)u_{2\infty}\right] dx = 0, & x \in \Omega, \\ \dfrac{\partial u_{1\infty}}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{6.28}$$

and $u_{2\infty}$ *is a positive constant and* $u_{1\infty} > 0$ *on* $\overline{\Omega}$.

Proof. From the discussion in Lemma 6.1 and Remark 6, it is clear that the estimats (6.6) and (6.7) hold for all large d_2, and $C > 0$ does not depend on d_1 and $d_2 \geq 1$.

Denoted by d_{2j} the subsequence of $d_2 \to \infty$, satisfying $d_{2j} \to 0$ as $j \to \infty$. With this subsequence, combined with the standard L^p-estimate for elliptic equations and Sobolev embedding theorem, $(u_{1j}, u_{2j}, u_{3j}) := (u_1(\cdot; d_{2j}), u_2(\cdot; d_{2j}), u_3(\cdot; d_{2j}))$ of (6.1) with $d_2 = d_{2j}$ satisfies

$$(u_{1j}, u_{2j}, u_{3j}) \to \left(u_{1\infty}, u_{2\infty}, \frac{\rho(\cdot)}{\delta(\cdot)} u_{2\infty}\right) \text{ in } C^1(\overline{\Omega}), \text{ as } j \to \infty,$$

where $u_{1\infty} > 0$ on $\overline{\Omega}$ and $u_{2\infty}$ is a nonnegative constant.

We next show that $u_{2\infty}$ is a positive constant. By a similar argument as before, suppose that $u_{2\infty} \equiv 0$. For all $j \geq 1$, define $\tilde{u}_{2j} := u_{2j}/\|u_{2j}\|_{L^\infty(\Omega)}$ that $\|\tilde{u}_{2j}\|_{L^\infty(\Omega)} = 1$, and $\tilde{u}_{2j}$ satisfies

$$\begin{cases} d_{2j}\Delta \tilde{u}_{2j} + \left[\beta_1(\cdot)\dfrac{u_{1j}}{u_{1j} + u_{2j}} + \dfrac{\beta_2(\cdot)\varrho(\cdot)}{\delta(\cdot)} u_{1j} - y(\cdot)\right] \tilde{u}_{2j} = 0, & x \in \Omega, \\ \dfrac{\partial \tilde{u}_{2j}}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{6.29}$$

Thus, if necessary, by passing to a further subsequence of d_{2j}, one can get

$$\tilde{u}_{2j} \to 1 \quad \text{in} \quad C^1(\overline{\Omega}), \quad \text{as} \quad j \to \infty. \tag{6.30}$$

On the other hand, since $u_{2j} \to 0$ as $j \to \infty$, it follows from a super-subsolution argument that $u_{1j} \to u_1^0$ as $j \to \infty$. Then, integrating (6.29) and sending $j \to \infty$, we have

$$\int_\Omega \left[\beta_1(\cdot) + \frac{\beta_2(\cdot)\varrho(\cdot)u_1^0(\cdot)}{\delta(\cdot)} - y(\cdot)\right] dx = 0,$$

a contradiction. Therefore, $u_{2\infty}$ is a positive constant, and $(u_{1\infty}, u_{2\infty})$ satisfies (6.28). The proof is complete. □

7 Conclusion and Discussion

In current work, we explore the global dynamics and asymptotic profiles of a host-pathogen model in a spatially bounded domain. We formulate the model by a reaction-diffusion system with space dependent parameters, where host individuals disperse at distinct rates and the mobility of pathogen is ignored. Complete analysis of the system allow us to investigate how large or small flows of hosts affect the spatial spread of disease, and what is the role of spatial heterogeneity on disease transmission.

We firstly establish that the existence of global solution, which is achieved by extending the local solution to a global one (see Lemma 2.1 and Theorem 2.1). To cope with the non-compactness of solution semiflow $Y(t)$, we utilize the Kuratowski measure of noncompactness to verify the asymptotic smoothness condition. Hence, by [8, Theorem 2.4.6], (1.5) possesses a global compact attractor in $\mathbb{X}^+$, denoted by $\mathcal{A}_0$ (see Theorem 2.2).

The BRN, $\Re_0$, is identified as the spectral radius of NGO, and also characterized by some equivalent principal spectral conditions, which establishes the threshold dynamical result for pathogen extinction and persistence (see Theorem 4.1 and 4.2). Specifically, we demonstrate that how $\Re_0$ depends on the diffusion coefficient d_1 for $d_1 \to 0$ and $d_1 \to \infty$ (see Theorem 3.1). We have also confirmed the global stability of E_0 in a critical case that $\Re_0 = 1$. It should be pointed here that the method used in Theorem 4.1 can also be applied in Avian influenza dynamical model [28] and Ebola transmission model [35], where the dynamic behaviors for the case that $\Re_0 = 1$ are still open. We left it as future investigation. In a homogeneous case and additional condition, we explore the global attractivity of PSS (positive equilibrium) by the technique of Lyapunov function.

When $\Re_0 > 1$, (1.5) possesses at lease one positive steady state. To achieve better understanding the effects of the host's movements on the spatial distribution of pathogen, we explore the asymptotic profiles of positive steady state for the cases that $d_1 \to 0$, $d_1 \to \infty$, $d_2 \to 0$ and $d_2 \to \infty$. When $d_1 \to 0$, our result (Theorem 6.1) demonstrate that host individuals distribute on $\overline{\Omega}$ in a non-homogeneous way. Under the assumption that 1-D space, $\Omega = (0, 1)$, and the condition that $\left\{x \in [0, 1] : \delta(\cdot)\beta_1(\cdot) + \beta_2(\cdot)\varrho(\cdot)u_1^0(\cdot) > \delta(\cdot)y(\cdot)\right\}$ is non-empty (of which locations can be termed as the favorite or not favorite sites for pathogens), we see from Theorem 6.2 that the infected hosts will vanish in some place and distribute in the remaining place (due to 6.24) and the susceptible hosts stay inhomogeneously on the whole habitat. When $d_1 \to \infty$ ($d_2 \to \infty$), susceptible (resp. infected) hosts distribute eventually over Ω, and infective (resp. susceptible) hosts distribute on Ω in a non-homogeneous way (see Theorems 6.3 and Theorem 6.4). The condition in Theorem 6.4 that $\int_\Omega [\beta_1(\cdot) + \beta_2(\cdot)\varrho(\cdot)u_1^0(\cdot)/\delta(\cdot)]dx > \int_\Omega y(\cdot)dx$ is usually termed as the favorable domain for pathogens and also ensure the existence of the positive solution in Theorem 6.4). In summary, our result suggests that slow or fast movement of host individuals have a great impacts on the spatial distribution of the pathogens, which may help to design strategies for disease control and prevention.

On the other hand, our results on asymptotic profiles are established when positive steady state exists. With the same arguments as those in [31, 32], $u_1^0(x) \to b(x)/\zeta(x)$ in $C^1(\overline{\Omega})$ as $d_1 \to 0$ and η^0 defined in 3.8 ensures that $\eta^0 < 0$. It follows that $\Re_0 < 1$ if $d_1 < \tilde{d}_1$, for some $\tilde{d}_1 > 0$. This together with Theorem 4.1 indicate that pathogens still can be eliminated with $d_1 \to 0$. Compared to the results in [31, 32], our results enrich the dynamical results on the asymptotic profiles. In fact, our results also reveal that disease can not be eliminated by fast movement of host individuals. It also remains an challenging problem to revisit Theorem 6.2 as $d_2 \to 0$ without spatial dimension limitations. In a homogeneous case, it also remains an interesting problem to perform the bifurcation analysis on steady state solutions with specific bifurcation parameter.

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