

## Research article

Daniele Cassani\*, Luca Vilasi, and Youjun Wang

# Local versus nonlocal elliptic equations: short-long range field interactions

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**Abstract:** In this paper we study a class of one-parameter family of elliptic equations which combines local and nonlocal operators, namely the Laplacian and the fractional Laplacian. We analyze spectral properties, establish the validity of the maximum principle, prove existence, nonexistence, symmetry and regularity results for weak solutions. The asymptotic behavior of weak solutions as the coupling parameter vanishes (which turns the problem into a purely nonlocal one) or goes to infinity (reducing the problem to the classical semilinear Laplace equation) is also investigated.

**Keywords:** Nonlocal and local Laplacians, asymptotic behavior, variational methods.

**MSC:** 35A15

## 1 Introduction and main results

Consider the following class of mixed local–nonlocal elliptic equations

$$\begin{cases} (-\Delta)^s u - \epsilon \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1.1)$$

where  $\Omega$  is a bounded smooth domain of  $\mathbb{R}^N$ ,  $N > 2$ ,  $s \in (0, 1)$ ,  $\epsilon > 0$  is a real parameter and

$$(-\Delta)^s u(x) = c_{N,s} \text{PV} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy$$

is the fractional Laplacian operator. Here  $c_{N,s}$  is a positive normalizing constant and PV stands for the Cauchy Principal Value.

Integro-differential equations of the form (1.1) arise naturally in the study of stochastic processes with jumps. The generator of an  $N$ -dimensional Lévy process has the following general structure:

$$\mathcal{L}u = \sum_{i,j} a_{ij} \partial_{ij} u + \sum_j b_j \partial_j u + \int_{\mathbb{R}^N} (u(x+y) - u(x) - y \cdot \nabla u(x)) \chi_{B_1}(y) dv(y), \quad (1.2)$$

where  $\nu$  is the Lévy measure and satisfies  $\int_{\mathbb{R}^N} \min\{1, |y|^2\} dv(y) < +\infty$ , and  $\chi_{B_1}$  is the usual characteristic function of the unit ball  $B_1$  of  $\mathbb{R}^N$ . The first term of (1.2) on the right-hand side corresponds to the diffusion, the second one to the drift, and the third one to the jump part.

**\*Corresponding Author: Daniele Cassani**, Dip. di Scienza e Alta Tecnologia, Università degli Studi dell'Insubria, and RISM–Riemann International School of Mathematics Villa Toeplitz, Via G.B. Vico, 46 – 21100 Varese, E-mail: [daniele.cassani@uninsubria.it](mailto:daniele.cassani@uninsubria.it)

**Luca Vilasi**, Dip. di Scienza e Alta Tecnologia Università degli Studi dell'Insubria and RISM–Riemann International School of Mathematics Villa Toeplitz, Via G.B. Vico, 46 – 21100 Varese, E-mail: [luca.vilasi@uninsubria.it](mailto:luca.vilasi@uninsubria.it), [luca.vilasi@gmail.com](mailto:luca.vilasi@gmail.com)

**Youjun Wang**, Department of Mathematics South China University of Technology Guangzhou 510640, P. R. China and Dip. di Scienza e Alta Tecnologia Università degli Studi dell'Insubria via Valleggio 11, 22100 Como, Italy, E-mail: [scyjwang@scut.edu.cn](mailto:scyjwang@scut.edu.cn)

In the particular case when  $\nu = 0$ , namely when there are no jumps,  $\mathcal{L}$  turns into a classical and extensively studied second-order differential operator. On the other hand, the case when the process has no diffusion and no drift has attracted a lot of attention recently (we refer to the book [18] for the development of the existence theory for several nonlinear nonlocal problems; see also the survey [21] for related regularity properties of solutions). In particular, one of the most prominent operators belonging to this class is the fractional Laplacian  $(-\Delta)^s$ , which turns out to be the infinitesimal generator of isotropic  $s$ -stable Lévy processes and plays an important role in analysis and probability theory. The issue of the existence of solutions to fractional Laplacian problems, as well as the study of qualitative properties of such solutions, has been addressed by many authors and the literature devoted to the field grows up continuously (we mention, in particular, the following recent contributions [1, 4, 6, 7, 11, 17, 23, 25, 26] and the references therein). Finally, it is well-known, e.g. from [13, 19], that Markov processes with both diffusion and jump components are suitable to modeling many situations in finance and control theory, see also [8–10, 29].

For the above reasons, the study of the operator  $\mathcal{L}$  with diffusion, drift and jump components appears quite intriguing and, as far as we know, there are just a few contributions in this direction, mainly relying on a combination of probabilistic and analytic techniques. For instance in [12], under the assumption that the diffusion is uniformly elliptic and suitable conditions on  $\nu$ , the author established a Harnack-type inequality and a regularity result for functions that are nonnegative in  $\mathbb{R}^N$  and harmonic in some domain. In [14, 15] the author derived interior Schauder estimates for both classical and weak solutions of the equation  $\mathcal{L}u = f$ .

Here we focus on the case  $\mathcal{L} = \epsilon \Delta - (-\Delta)^s$ , obtained from (1.2) by setting  $a_{ij} = \epsilon \delta_{ij}$ ,  $\epsilon > 0$ ,  $b_j = 0$  and  $\nu$  given by the symmetric kernel  $|x - y|^{-N-2s}$ . We study this problem from a purely analytical point of view. Precisely, the paper is divided into two parts. Firstly, for fixed  $\epsilon > 0$  (without loss of generality, we may assume  $\epsilon = 1$ ), we provide some basic results related to (1.1), including spectral properties, maximum principles, existence, nonexistence, symmetry and regularity of weak solutions. With these preliminary tools, in the second part we will address further issues related to the asymptotic profile, as  $\epsilon \rightarrow 0$  as well as  $\epsilon \rightarrow +\infty$ , of ground states of the model problem

$$\begin{cases} (-\Delta)^s u - \epsilon \Delta u = |u|^{p-2}u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (1.3)$$

If  $\Omega$  is a bounded star-shaped domain, it was proved in [22] that (1.3) has no nontrivial bounded solutions if  $p \geq 2^* := \frac{2N}{N-2}$ . When  $\epsilon = 0$ , problem (1.3) possesses a positive solution for  $2 < p < 2_s^* := \frac{2N}{N-2s}$ ,  $N > 2s$ , and, if  $\Omega$  is a bounded,  $C^{1,1}$  and star-shaped domain, do not exist positive bounded solution for  $p = 2_s^*$  and there are no nontrivial bounded solution for  $p > 2_s^*$ , see [23, 24]. It is natural to investigate the behavior of the ground state  $u_\epsilon$  in (1.3), as  $\epsilon \rightarrow 0$  (corresponding to the fractional case), which may converge, blow up or vanish, depending on the range of  $p$ .

On the other hand, in the case  $\epsilon \rightarrow +\infty$ , setting  $v_\epsilon(x) = \epsilon^{-\frac{1}{p-2}} u_\epsilon(x)$ , the function  $v_\epsilon$  satisfies

$$\begin{cases} \epsilon^{-1}(-\Delta)^s v - \Delta v = |v|^{p-2}v & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1.4)$$

whose natural limit problem is

$$\begin{cases} -\Delta v = |v|^{p-2}v & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (1.5)$$

By Sobolev embedding theorems and regularity theory, classical solutions to (1.5) do exist for  $2 < p < 2^*$  with  $N > 2$ , and thus one may suspect in this case that  $u_\epsilon(x)$  blows up almost everywhere in  $\Omega$ , as  $\epsilon \rightarrow +\infty$ .

We mention that the operator  $\mathcal{L}$  can be given a local realization by means of a Caffarelli-Silvestre-type extension [6]. However, if one proceeds as in [5] by directly extending  $u \in H^s(\Omega)$  to the cylinder  $C_\Omega := \Omega \times (0, +\infty)$ , i.e. considering  $C_\Omega$  as the domain of the extension  $w$  of  $u$ , with  $w = 0$  on the lateral boundary  $\partial_L C_\Omega$  of  $C_\Omega$ , then one would end up with the fractional Laplacian operator defined by spectral decomposition, which on

bounded domains does not coincide with  $(-\Delta)^s$ . The idea is then to start from the half-space  $\mathbb{R}_+^{N+1}$  and to require the trace of the extension to vanish a.e. outside  $\Omega$ . This can be achieved as follows.

Define the space

$$Y^s := \left\{ w \in H_{\text{loc}}^1(\mathbb{R}_+^{N+1}) : \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w|^2 dx dy < +\infty \right\},$$

where  $\mathbb{R}_+^{N+1} := \mathbb{R}^N \times (0, +\infty)$ , equipped with the norm

$$\|w\|_{Y^s} = \left( \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w|^2 dx dy \right)^{1/2}.$$

It is known from [6] that, for all  $u \in H^s(\mathbb{R}^N)$  there exists a unique function  $E(u) \in Y^s$ , called the harmonic extension of  $u$  to  $\mathbb{R}_+^{N+1}$ , satisfying

$$\begin{cases} \operatorname{div}(y^{1-2s} \nabla E(u)(x, y)) = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ E(u)(x, 0) = u(x) & \text{in } \mathbb{R}^N, \end{cases} \quad (1.6)$$

and  $(-\Delta)^s u$  corresponds to the Dirichlet-to-Neumann map

$$(-\Delta)^s u(x) = k_s^{-1} \lim_{y \rightarrow 0} y^{1-2s} \frac{\partial E(u)}{\partial y}(x, y), \quad \forall x \in \mathbb{R}^N,$$

for a suitable normalizing constant  $k_s > 0$ . Next define the following subspace

$$Y_0^s := \left\{ w \in Y^s : w(x, 0) = w|_{\mathbb{R}^N \times \{0\}} \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega \text{ and } w(x, 0) \in H_0^1(\Omega) \right\}$$

whose elements have traces over  $\mathbb{R}^N$  which vanish outside  $\Omega$ . We say that  $w \in Y_0^s$  is a weak solution to

$$\begin{cases} \operatorname{div}(y^{1-2s} \nabla w(x, y)) = 0 & \text{in } \mathbb{R}_+^{N+1} \\ w(x, 0) = 0 & \text{for all } x \in \mathbb{R}^N \setminus \Omega, \\ k_s^{-1} \lim_{y \rightarrow 0} y^{1-2s} \frac{\partial w}{\partial y}(x, y) - \epsilon \Delta w(x, 0) = |w(x, 0)|^{p-2} w(x, 0) & \text{for all } x \in \Omega, \end{cases} \quad (1.7)$$

provided

$$\int_{\mathbb{R}_+^{N+1}} y^{1-2s} \nabla w \nabla \varphi dx dy + k_s \epsilon \int_{\Omega} \nabla w(x, 0) \nabla \varphi(x, 0) dx = k_s \int_{\Omega} |w(x, 0)|^{p-2} w(x, 0) \varphi(x, 0) dx \quad (1.8)$$

for all  $\varphi \in Y_0^s$ . It is clear that if  $w \in Y_0^s$  solves (1.7) in the weak sense, then its trace over  $\mathbb{R}^N$  weakly solves

$$\begin{cases} (-\Delta)^s u - \epsilon \Delta u = |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (1.9)$$

Before presenting our main results, let us introduce the functional framework. Set  $Q := (\mathbb{R}^N \times \mathbb{R}^N) \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ , with  $\mathcal{C}\Omega := \mathbb{R}^N \setminus \Omega$ . We denote by  $X^s(\Omega)$  the linear space of Lebesgue measurable functions  $u : \mathbb{R}^N \rightarrow \mathbb{R}$  such that the restriction of  $u$  to  $\Omega$  belongs to  $H^1(\Omega)$  and

$$[u]_{s,Q}^2 := \int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy < +\infty.$$

A norm in  $X^s(\Omega)$  is defined as follows

$$\|u\|_{X^s(\Omega)} := \left( \|u\|_{H^1(\Omega)}^2 + [u]_{s,Q}^2 \right)^{\frac{1}{2}}. \quad (1.10)$$

Let us also define the set

$$X_0^s := \{u \in X^s(\Omega) : u = 0 \text{ a.e. in } \mathcal{C}\Omega\}.$$

For  $u \in X_0^s$ , we can write the norm (1.10) as

$$\|u\|_{X^s(\Omega)} = \left( \|u\|_{H^1(\Omega)}^2 + [u]_{s, \mathbb{R}^N}^2 \right)^{\frac{1}{2}}, \quad (1.11)$$

and, by Poincaré's inequality, this norm in  $X_0^s$  is also equivalent to

$$\|u\|_s := \left( \int_{\Omega} |\nabla u|^2 dx + [u]_{s, \mathbb{R}^N}^2 \right)^{\frac{1}{2}}. \quad (1.12)$$

We claim that  $(X_0^s, \|\cdot\|_s)$  is an Hilbert space with scalar product

$$\langle u, v \rangle_s = \int_{\Omega} \nabla u \nabla v dx + \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy, \quad \forall u, v \in X_0^s.$$

For the proof, see Lemma 12.

Throughout this paper, we will use the symbol  $\|\cdot\|_p$ ,  $p \in [1, +\infty]$  to denote the usual  $L^p$ -norm in  $\mathbb{R}^N$ , while the letters  $C$ ,  $C_i$  will represent generic positive constants which may change from line to line.

## Preliminary results

Let us begin with some spectral properties of the operator  $\mathcal{L}$  which, it is well known, are crucial for the solvability of linear and resonance nonlinear problems.

**Theorem 1.** *Let  $s \in (0, 1)$ ,  $N > 2$ , and consider the following eigenvalue problem:*

$$\begin{cases} (-\Delta)^s u - \Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (1.13)$$

Then:

(1) *Problem (1.13) admits a positive eigenvalue  $\lambda_1$ , characterized by*

$$\lambda_1 = \min_{u \in X_0^s \setminus \{0\}} \frac{\|u\|_s^2}{\|u\|_2^2} \quad \text{or, equivalently,} \quad \lambda_1 = \min_{\substack{u \in X_0^s, \\ \|u\|_2 = 1}} \|u\|_s^2, \quad (1.14)$$

*and a nonnegative eigenfunction  $e_1 \in X_0^s$  corresponding to  $\lambda_1$ , satisfying  $\|e_1\|_2 = 1$  and  $\lambda_1 = \|e_1\|_s^2$ ;*

(2)  *$\lambda_1$  is simple;*

(3) *the set of the eigenvalues of (1.13) consists of a sequence  $\{\lambda_k\}$ ,  $k \in \mathbb{N}$ , with  $0 < \lambda_1 < \lambda_2 \leq \dots \leq \lambda_k \leq \dots$  and  $\lambda_k \rightarrow +\infty$  as  $k \rightarrow +\infty$ . The eigenvalue  $\lambda_{k+1}$  can be characterized by*

$$\lambda_{k+1} = \min_{u \in \mathcal{M}_{k+1} \setminus \{0\}} \frac{\|u\|_s^2}{\|u\|_2^2}, \quad \text{or, equivalently,} \quad \lambda_{k+1} = \min_{\substack{u \in \mathcal{M}_{k+1}, \\ \|u\|_2 = 1}} \|u\|_s^2, \quad k = 1, 2, \dots, \quad (1.15)$$

*where  $\mathcal{M}_{k+1} = \{u \in X_0^s : \langle u, e_j \rangle_s = 0, \forall j = 1, 2, \dots, k\}$ . Moreover, for any  $k \in \mathbb{N}$ , there exists  $e_{k+1} \in \mathcal{M}_{k+1}$  that is an eigenfunction of  $\mathcal{L}$  corresponding to  $\lambda_{k+1}$  and that verifies  $\|e_{k+1}\|_2 = 1$ ,  $\lambda_{k+1} = \|e_{k+1}\|_s^2$ ;*

(4) *the sequence  $\{e_k\}$  of eigenfunctions corresponding to  $\lambda_k$  is an orthonormal basis of  $L^2(\Omega)$  and an orthogonal basis of  $X_0^s$ ;*

(5) *each eigenvalue  $\lambda_k$  has finite multiplicity; more precisely, if  $\lambda_k$  is such that  $\lambda_{k-1} < \lambda_k = \dots = \lambda_{k+h} < \lambda_{k+h+1}$ , for  $k > 1$  and for some  $h \in \mathbb{N}$ , then the set of all the eigenfunctions corresponding to  $\lambda_k$  agrees with  $\text{span}\{e_k, \dots, e_{k+h}\}$ .*

Theorem 1 can be viewed as the generalization of similar eigenvalue problems for the Laplacian and the fractional Laplacian operators, see the monograph [18, Chapter 3].

Next, we focus on maximum principles, both for classical and for weak supersolutions of the problem  $\mathcal{L}u = 0$ ,  $u|_{\mathbb{R}^N \setminus \Omega} = 0$ . Define

$$L_s := \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ s.t. } \frac{u}{1 + |x|^{N+2s}} \in L^1(\mathbb{R}^N) \right\}.$$

**Theorem 2.** Assume that  $u \in L_s \cap C^2(\Omega)$  is lower semicontinuous on  $\bar{\Omega}$ . If

$$\begin{cases} (-\Delta)^s u - \Delta u \geq 0 & \text{in } \Omega, \\ u \geq 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1.16)$$

then,  $u(x) \geq 0$  in  $\mathbb{R}^N$ . Moreover, if  $u(x) = 0$  at some point in  $\Omega$ , then  $u(x) \equiv 0$  in  $\mathbb{R}^N$ .

**Theorem 3.** Assume that  $u \in H^1(\Omega)$ ,  $[u]_{s, \mathbb{R}^N} < +\infty$ , and  $u$  satisfies

$$\begin{cases} (-\Delta)^s u - \Delta u \geq 0 & \text{in } \Omega, \\ u \geq 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (1.17)$$

Then  $u(x) \geq 0$  a.e. in  $\Omega$ .

We have used the arguments of [7] to prove Theorem 2. Basically, if we have  $u(x_0) = \min_{\Omega} u(x) < 0$  for some  $x_0 \in \Omega$ , then  $(-\Delta)^s u(x_0) \geq (-\Delta)^s u(x_0) - \Delta u(x_0) \geq 0$  and we reduce the problem to the fractional setting. On the other hand, by Stampacchia's theorem the proof of Theorem 3 relies only on local arguments.

In a classical nonlinear setting, we derive the following existence result for problem (1.1), where we set for simplicity  $\epsilon = 1$ .

**Theorem 4.** Assume that  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  satisfies the following conditions:

- (f<sub>1</sub>)  $f(x, t) \in C^0(\bar{\Omega} \times \mathbb{R})$  and  $f(x, t) = 0$  for  $t \leq 0$ ;
- (f<sub>2</sub>) there exist constants  $C_1, C_2 > 0$  such that

$$|f(x, t)| \leq C_1 + C_2 |t|^q,$$

where  $1 < q < 2^* - 1$ ;

- (f<sub>3</sub>)  $f(x, t) = o(|t|)$  as  $t \rightarrow 0$ ;

- (f<sub>4</sub>) there exist  $\mu > 2$  and  $R \geq 0$  such that for  $t \geq R$ ,

$$0 < \mu F(x, t) \leq t f(x, t),$$

where  $F(x, t) := \int_0^t f(x, \xi) d\xi$ .

Then, the problem

$$\begin{cases} (-\Delta)^s u - \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \quad (1.18)$$

possesses a nonnegative solution  $u \in X_0^s$ .

**Theorem 5** ([22], Theorem 1.3). Assume that  $\Omega \subset \mathbb{R}^N$  is a bounded star-shaped domain,  $f \in C_{loc}^{0,1}(\bar{\Omega} \times \mathbb{R})$  and

$$\frac{N-2}{2} t f(x, t) \geq N F(x, t) + x \cdot F_x(x, t), \quad \text{for all } x \in \Omega \text{ and } t \in \mathbb{R}.$$

If  $u$  is a bounded solution of (1.18), then  $u \equiv 0$ .

A consequence of Theorems 4 and 5 we have

**Corollary 6.** Assume that  $\Omega \subset \mathbb{R}^N$  is a bounded domain,  $s \in (0, 1)$  and  $p \in (2, 2^*)$ . Then, the problem

$$\begin{cases} (-\Delta)^s u - \Delta u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \quad (1.19)$$

possesses a nonnegative bounded mountain pass solution  $u \in X_0^s$ . Moreover, if  $\Omega$  is a bounded star-shaped domain and  $p \geq 2^*$ , then (1.19) does not admit nontrivial bounded solutions.

The mountain pass theorem underlies the proof of Theorem 4 and we further show that this mountain pass solution of (1.19) has actually minimal energy via Nehari's method.

**Theorem 7.** The mountain pass solution  $u$  of Corollary 6 is a ground state, i.e.

$$\mathcal{F}(u) = \inf_{\mathcal{N}} \mathcal{F},$$

where

$$\mathcal{F}(u) := \frac{1}{2} \|u\|_s^2 - \frac{1}{p} \int_{\Omega} |u|^p dx, \quad u \in X_0^s,$$

$$\mathcal{N} := \{u \in X_0^s \setminus \{0\} : \langle \mathcal{F}'(u), u \rangle = 0\}.$$

The remaining results are related to symmetry properties and regularity of weak (classical) solutions.

**Theorem 8.** Assume that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz continuous and  $u \in L_s \cap C^2(B(0, 1))$  is a positive solution of

$$\begin{cases} (-\Delta)^s u - \Delta u = f(u) & \text{in } B(0, 1), \\ u = 0 & \text{in } \mathbb{R}^N \setminus B(0, 1). \end{cases} \quad (1.20)$$

Then  $u$  is radially symmetric, i.e.  $u(x) = v(|x|)$  for some decreasing function  $v : [0, 1] \rightarrow [0, +\infty)$ .

**Theorem 9.** Assume that  $u \in X_0^s$  is a weak solution of the problem

$$\begin{cases} (-\Delta)^s u - \Delta u = f(x) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (1.21)$$

Then the following facts hold:

- (1) if  $f \in L^q(\Omega)$  with  $q > \frac{N}{2}$ , then there exists a constant  $C$ , depending only on  $N$ ,  $\|u\|_s$  and  $\|f\|_q$ , such that the positive weak solution of (1.21) satisfies

$$\|u\|_{\infty} \leq C;$$

- (2) if  $f \in L^q(\Omega)$  with  $q = \frac{N}{2}$ , then there exists a constant  $\alpha > 0$  such that the positive weak solution of (1.21) satisfies

$$\int_{\Omega} e^{\alpha u} dx < +\infty.$$

In particular,  $u \in L^q(\Omega)$  for every  $q < +\infty$ ;

- (3) if  $f$  is a positive function and  $f \in L^q(\Omega)$  with  $\frac{2N}{N+2} \leq q < \frac{N}{2}$ , then there exists a constant  $C > 0$  such that the positive weak solution of (1.21) satisfies

$$\|u\|_{q^*} \leq C \|f\|_q, \quad q^* = \frac{qN}{N-2q}.$$

Note that the results in Theorem 9 are consistent with the ones known for the Laplace operator. As we will see in the proof, this is due to the choice of appropriate truncated functions which downplay the effects of the nonlocal term.

In the second part of this paper we aim at investigating the asymptotic behavior of ground states (weak solutions) of problem (1.3) when  $2 < p < 2^*$ ,  $N > 2$ . For this purpose set

$$S_\epsilon := \inf\{\|u\|_\epsilon^2 : \|u\|_p = 1\}.$$

Note that  $S_\epsilon$  can be also written in the form

$$S_\epsilon = \inf_{u \in X_0^s \setminus \{0\}} P_\epsilon(u),$$

where  $P_\epsilon(u) = \frac{\|u\|_\epsilon^2}{\|u\|_p^2}$ . Hereafter let us denote

$$\|u\|_\epsilon := \left( \epsilon \|\nabla u\|_2^2 + [u]_{s, \mathbb{R}^N}^2 \right)^{\frac{1}{2}}.$$

The compactness of Sobolev embeddings yields the existence of non-negative minimizers  $w_\epsilon$  of  $S_\epsilon$ . Then, by Lagrange's multiplier rule, there exists some  $\lambda_\epsilon > 0$  such that

$$(-\Delta)^s w_\epsilon - \epsilon \Delta w_\epsilon = \lambda_\epsilon |w_\epsilon|^{p-2} w_\epsilon \quad \text{in } \Omega. \quad (1.22)$$

By multiplying both sides of (1.22) by  $w_\epsilon$  and then integrating, we get  $\lambda_\epsilon = S_\epsilon$ . Moreover, by maximum principles,  $w_\epsilon > 0$ .

The energy associated to (1.22) is

$$J(u) = \frac{1}{2} \|u\|_\epsilon^2 - \frac{1}{p} S_\epsilon \|u\|_p^p, \quad u \in X_0^s. \quad (1.23)$$

Since  $w_\epsilon$  is a solution of (1.22), we get  $\|w_\epsilon\|_\epsilon^2 = S_\epsilon \|w_\epsilon\|_p^p$  and  $\|w_\epsilon\|_p = 1$ , and thus

$$J(w_\epsilon) = \frac{1}{2} \|w_\epsilon\|_\epsilon^2 - \frac{1}{p} S_\epsilon \|w_\epsilon\|_p^p = \frac{p-2}{2p} S_\epsilon. \quad (1.24)$$

On the other hand, any nontrivial solution  $v$  of (1.22) necessarily satisfies

$$J(v) = \frac{1}{2} \|v\|_\epsilon^2 - \frac{1}{p} S_\epsilon \|v\|_p^p = \frac{p-2}{2p} S_\epsilon \|v\|_p^p. \quad (1.25)$$

We have also

$$S_\epsilon \leq \frac{\|v\|_\epsilon^2}{\|v\|_p^2} = \frac{S_\epsilon \|v\|_p^p}{\|v\|_p^2} = S_\epsilon \|v\|_p^{p-2}, \quad (1.26)$$

which yields  $\|v\|_p \geq 1$ . This means that, by (1.25),  $v$  should satisfy

$$J(v) \geq \frac{p-2}{2p} S_\epsilon.$$

This fact, together with (1.24), implies that  $w_\epsilon$  is a ground state of equation (1.22) and so  $u_\epsilon = S_\epsilon^{-\frac{1}{2-p}} w_\epsilon$  is a ground state of (1.3). Note that  $u_\epsilon$  also satisfies  $P_\epsilon(u_\epsilon) = S_\epsilon$ .

Let us stress the fact that we do not know whether the mountain-pass solution obtained from Theorem 4 and the minimal solution  $u_\epsilon$  obtained above agree. Next we will focus on the minimal solution  $u_\epsilon$ .

Before stating our main results, we introduce the following best constants

$$S := \inf_{u \in H_0^s \setminus \{0\}} \frac{[u]_s^2}{\|u\|_p^2} \quad \text{for } 2 < p < 2_s^*, \quad (1.27)$$

and

$$\tilde{S} := \inf_{u \in D^{s,2}(\mathbb{R}^N) \setminus \{0\}} \frac{[u]_s^2}{\|u\|_{2_s^*}^2} \quad \text{for } p = 2_s^*, \quad (1.28)$$

where  $H_0^s$  is the linear space of Lebesgue measurable functions  $u : \mathbb{R}^N \rightarrow \mathbb{R}$  such that  $u(x) = 0$  a.e. in  $\mathcal{C}\Omega$  and  $[u]_s := [u]_{s, \mathbb{R}^N} < +\infty$ , while  $D^{s,2}(\mathbb{R}^N)$  is the closure of  $C_0^\infty(\mathbb{R}^N)$  with respect to the semi-norm  $[\cdot]_{s, \mathbb{R}^N}$ .

## Main results

**Theorem 10.** *The following facts hold:*

- (i) if  $2 < p < 2_s^*$ , then  $S_\epsilon \rightarrow S$  as  $\epsilon \rightarrow 0$ . Moreover, there exists a minimizer  $u_0$  of  $S$  and  $u_\epsilon \rightarrow u_0$  in  $H_0^s$  as  $\epsilon \rightarrow 0$ ;
- (ii) if  $p = 2_s^*$ , then  $S_\epsilon \rightarrow \tilde{S}$  as  $\epsilon \rightarrow 0$ ;
- (iii) if  $2_s^* < p < 2^*$ , then  $S_\epsilon \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Moreover, there exist a minimizer  $u_\epsilon$  such that  $\|u_\epsilon\|_\epsilon \rightarrow 0$  as  $\epsilon \rightarrow 0$ .

**Theorem 11.** *If  $2 < p < 2^*$ , then  $S_\epsilon \rightarrow 0$  and  $\|u_\epsilon\|_\epsilon \rightarrow +\infty$  as  $\epsilon \rightarrow +\infty$ .*

Finally let us make a few comments on Theorems 10 and 11. We expect the minimizers  $u_\epsilon$  obtained in (i) to be uniformly bounded with respect to  $\epsilon$  and thus that the convergence should hold in the classical sense. In this context, the existence or nonexistence of positive solutions of the problem

$$(-\Delta)^s u - \Delta u = |u|^{p-2} u \quad \text{in } \mathbb{R}^N \text{ or } \mathbb{R}_+^N \quad (1.29)$$

with  $2 < p < 2_s^*$  plays a very important role. However, the moving plane methods can not be applied directly here because one of the most important tools - the Kelvin transform - is not available. On the other hand, the classical minimizers  $u_\epsilon$  obtained in (ii) blow up as  $\epsilon \rightarrow 0$ , up to a subsequence if necessary. Indeed, if not one has  $u_\epsilon \rightarrow u_0$  in classical sense for  $\text{supp}(u_0) \subset \Omega$ , and then  $u_0$  is a minimizer of  $\tilde{S}$ , thus a contradiction. Obviously, for the case (iii), the minimizers  $u_\epsilon$  vanishes a.e. in  $\Omega$ . We stress that the constant

$$\bar{S} := \inf_{u \in D^{s,2}(\mathbb{R}^N) \cap D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\|\nabla u\|_2^2 + [u]_s^2}{\|u\|_p^2} \quad \text{for } 2_s^* < p < 2^*$$

plays a crucial role to get (iii), while to prove Theorem 11, we need the constant

$$\hat{S} := \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_p^2} \quad \text{for } 2 < p < 2^*.$$

The rest of this paper is organized as follows. We prove Theorems 1–9 in Section 2 whence Section 3 is devoted to prove Theorems 10 and 11.

## 2 Proofs of theorems 1–4 and 7–9

Let us prove first the following preliminary result

**Lemma 12.**  $(X_0^s, \|\cdot\|_s)$  is a Hilbert space with scalar product

$$\langle u, v \rangle_s = \int_{\Omega} \nabla u \nabla v dx + \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy, \quad \forall u, v \in X_0^s.$$

*Proof.* It is straightforward to see that  $\langle \cdot, \cdot \rangle_s$  is a scalar product on  $X_0^s$  which induces the norm  $\|\cdot\|_s$ . So, it suffices to prove that  $(X_0^s, \|\cdot\|_s)$  is complete. Let  $\{u_k\}$  be a Cauchy sequence in  $X_0^s$ , that is, for any  $\varepsilon > 0$ , there exists  $k_0 > 0$  such that for  $k, j \geq k_0$ ,

$$\|u_j - u_k\|_s < \varepsilon. \quad (2.1)$$

This implies that  $\|u_j - u_k\|_{H_0^1(\Omega)} < \varepsilon$ . Since  $H_0^1(\Omega)$  is complete, there exists  $u \in H_0^1(\Omega)$  such that  $u_k \rightarrow u$  in  $H_0^1(\Omega)$  as  $k \rightarrow +\infty$ . Moreover, since  $u_k = 0$  a.e. in  $\mathcal{C}\Omega$ , we may define  $u = 0$  a.e. in  $\mathcal{C}\Omega$  and then  $u_k \rightarrow u$  in  $H^1(\mathbb{R}^N)$  as  $k \rightarrow +\infty$ . Thus, up to a subsequence, still denoted by  $\{u_k\}$ ,  $u_k \rightarrow u$  a.e. in  $\mathbb{R}^N$ . Fatou's lemma and (2.1) yield

$$[u]_{s, \mathbb{R}^N}^2 \leq \lim_{k \rightarrow +\infty} [u_k]_{s, \mathbb{R}^N}^2 \leq 2 \lim_{k \rightarrow +\infty} (\|u_k - u_{k_0}\|_s^2 + \|u_{k_0}\|_s^2) < +\infty, \quad (2.2)$$



and hence  $u \in X_0^s$ . Finally, by using again (2.1), we get

$$\|u_j - u\|_s \leq \liminf_{k \rightarrow +\infty} \|u_j - u_k\|_s \leq \varepsilon, \quad (2.3)$$

and  $u_j \rightarrow u$  in  $X_0^s$  as  $j \rightarrow +\infty$ , as desired.  $\square$

*Proof of Theorem 1.* The proof is modeled on the one of the nonlocal operator  $-\mathcal{L}_K$  studied in [18]. However, for reader's convenience, we give the detailed proof.

For simplicity set  $I(u) := \|u\|_s^2$  and  $\mathcal{M} := \{u \in X_0^s : \|u\|_2 = 1\}$ .

(1) Let  $\{u_j\}$  be a minimizing sequence for  $I$ , that is,

$$\lim_{j \rightarrow +\infty} I(u_j) = \inf_{u \in \mathcal{M}} I(u) \geq 0.$$

Then  $\{I(u_j)\}$  is bounded, i.e.  $\{u_j\}$  is bounded in  $X_0^s$ . Up to a subsequence, still denoted by  $\{u_j\}$ ,  $u_j \rightharpoonup u^*$  in  $X_0^s$  for some  $u^* \in X_0^s$  and  $u_j \rightarrow u^*$  in  $L^2(\Omega)$ . Since  $\|u_j\|_2 = 1$ , we get  $\|u^*\|_2 = 1$ , that is,  $u^* \in \mathcal{M}$ . On the other hand,

$$\lim_{j \rightarrow +\infty} I(u_j) \geq I(u^*) \geq \inf_{u \in \mathcal{M}} I(u). \quad (2.4)$$

and this implies that  $I(u^*) = \inf_{u \in \mathcal{M}} I(u)$ .

Let  $t \in (-1, 1)$ ,  $v \in X_0^s$ ,  $C_t = \|u^* + tv\|_2$  and  $u_t = (u^* + tv)/C_t$ . Then  $u_t \in \mathcal{M}$  and one has

$$\begin{aligned} I(u_t) &= \frac{\|u^*\|_s^2 + 2t\langle u^*, v \rangle_s + o(t)}{1 + 2t \int_{\mathbb{R}^N} u^* v dx + o(t)} \\ &= (\|u^*\|_s^2 + 2t\langle u^*, v \rangle_s + o(t)) \left( 1 - 2t \int_{\mathbb{R}^N} u^* v dx + o(t) \right) \\ &= I(u^*) + 2t \left( \langle u^*, v \rangle_s - I(u^*) \int_{\mathbb{R}^N} u^* v dx \right) + o(t). \end{aligned} \quad (2.5)$$

By the very definition of  $u^*$  we get

$$\langle u^*, v \rangle_s = I(u^*) \int_{\mathbb{R}^N} u^* v dx. \quad (2.6)$$

Now, assume that  $\lambda_1$  is attained at some  $e_1 \in X_0^s$  with  $\|e_1\|_2 = 1$ . By (2.6), we get  $I(e_1) = \lambda_1$ . To prove that  $e_1 \geq 0$ , we first show that if  $e$  is an eigenfunction related to  $\lambda_1$  with  $\|e\|_2 = 1$ , then both  $e$  and  $|e|$  realize the minimum in (1.14). In fact, if  $x_1 \in \{x \in \mathbb{R}^N : e(x) > 0\}$  and  $x_2 \in \{x \in \mathbb{R}^N : e(x) < 0\}$ , then

$$\|e(x_1) - |e(x_2)|\| = \|e(x_1) + e(x_2)\| < e(x_1) - e(x_2) = |e(x_1) - e(x_2)|.$$

Thus, if both sets  $\{x \in \mathbb{R}^N : e(x) > 0\}$  and  $\{x \in \mathbb{R}^N : e(x) < 0\}$  have positive measure, we have  $I(|e|) < I(e)$ . Note that  $|e| \in X_0^s$  and  $\||e|\|_2 = \|e\|_2 = 1$ . Thus, since  $e_1$  is a minimizer, we get  $I(|e|) = I(e) = I(e_1) = \lambda_1$ . This also implies that  $e \geq 0$  or  $e \leq 0$  a.e. in  $\Omega$ . By replacing  $e_1$  by  $|e_1|$ , we may conclude that  $e_1 \geq 0$ .

(2) Suppose by contradiction that there exists another eigenfunction  $\tilde{e}_1 \in X_0^s$ ,  $\tilde{e}_1 \neq e_1$ , corresponding to  $\lambda_1$ . Then, by (1) we may assume that  $\tilde{e}_1 \geq 0$ . Setting  $u_1 := \frac{\tilde{e}_1}{\|\tilde{e}_1\|_2}$  and  $v_1 := e_1 - u_1$ , we claim that  $v_1 = 0$ . In fact, direct calculations show that  $v_1$  is also an eigenfunction related to  $\lambda_1$ , and so from (1) we deduce  $v_1 \geq 0$  or  $v_1 \leq 0$  a.e. in  $\Omega$ , that is,  $e_1^2 \geq u_1^2$  or  $e_1^2 \leq u_1^2$  a.e. in  $\Omega$ . However, since  $\|e_1\|_2^2 - \|u_1\|_2^2 = 0$ , we get  $e_1^2 = u_1^2$  and hence  $v_1 = 0$  a.e. in  $\Omega$ . This also implies that  $v_1 = 0$  a.e. in  $\mathbb{R}^N$  and therefore, if  $w \in X_0^s$  is an eigenfunction related to  $\lambda_1$ , then  $w = \pm \|w\|_2 e_1$  a.e. in  $\mathbb{R}^N$ .

(3) Since  $\mathcal{M}_k$  is a weakly closed subspace of  $X_0^s$ , arguing as in step (1), we deduce that  $\lambda_{k+1}$  is attained at some  $e_{k+1} \in X_0^s$ . On the other hand, being  $\mathcal{M}_{k+1} \subseteq \mathcal{M}_k$ , we get  $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots$ . Now, we claim that

$\lambda_1 \neq \lambda_2$ . If not,  $e_2 \in \mathcal{M}_2$  would be also an eigenfunction related to  $\lambda_1$ . By (2),  $e_2 = \pm \|e_2\|_2 e_1$  a.e. in  $\mathbb{R}^N$ . Thus, we would get  $0 = \langle e_1, e_2 \rangle_s = \pm \|e_2\|_2 \|e_1\|_s^2$ , which yields  $e_1 \equiv 0$  a.e. in  $\mathbb{R}^N$ , a contradiction.

Analogously to the proof of (2.6), we have

$$\langle e_{k+1}, w \rangle_s = \lambda_{k+1} \int_{\mathbb{R}^N} e_{k+1} w dx, \quad \forall w \in \mathcal{M}_{k+1}. \quad (2.7)$$

We claim that equality (2.7) holds for any  $w \in X_0^s$ , i.e., that  $e_{k+1}$  is an eigenfunction related to  $\lambda_{k+1}$ . Let us consider the decomposition  $X_0^s = \text{span}\{e_1, \dots, e_k\} \oplus (\text{span}\{e_1, \dots, e_k\})^\perp = \text{span}\{e_1, \dots, e_k\} \oplus \mathcal{M}_{k+1}$ , and write any given  $v \in X_0^s$  as  $v = v_1 + v_2$ , with  $v_1 = \sum_{j=1}^k c_j e_j$ ,  $c_j \in \mathbb{R}$ , and  $v_2 \in \mathcal{M}_{k+1}$ . Testing (2.7) with  $w = v_2$  we then obtain

$$\begin{aligned} \langle e_{k+1}, v \rangle_s - \lambda_{k+1} \int_{\Omega} e_{k+1} v dx &= \langle e_{k+1}, v_1 \rangle_s - \lambda_{k+1} \int_{\Omega} e_{k+1} v_1 dx \\ &= \sum_{j=1}^k c_j \left[ \langle e_{k+1}, e_j \rangle_s - \lambda_{k+1} \int_{\Omega} e_{k+1} e_j dx \right] \\ &= 0, \end{aligned}$$

which proves the claim.

Next we show that  $\lambda_k \rightarrow +\infty$  as  $k \rightarrow +\infty$ . Suppose, by contradiction, that  $\lambda_k \rightarrow C \in \mathbb{R}^+$  as  $k \rightarrow +\infty$ . Since  $\|e_k\|_s^2 = \lambda_k$ , up a subsequence,  $e_{k_j} \rightarrow e$  in  $L^2(\Omega)$  as  $k_j \rightarrow +\infty$ . On the other hand, being  $\langle e_{k_i}, e_{k_j} \rangle_s = 0$  for  $k_i \neq k_j$ , we get  $\int_{\Omega} e_{k_i} e_{k_j} dx = 0$  for  $k_i \neq k_j$  and thus

$$\|e_{k_i} - e_{k_j}\|_2^2 = \|e_{k_i}\|_2^2 + \|e_{k_j}\|_2^2 = 1,$$

which is a contradiction, as

$$\|e_{k_i} - e_{k_j}\|_2^2 \leq \|e_{k_i} - e\|_2^2 + \|e_{k_j} - e\|_2^2 \rightarrow 0 \quad \text{as } k_i, k_j \rightarrow +\infty.$$

Finally, let us show that any eigenvalue of (1.13) can be written in the form (1.15). Arguing again by contradiction, assume that there exists an eigenvalue  $\lambda \notin \{\lambda_k\}$  and let  $e$  be an eigenfunction corresponding to  $\lambda$  with  $\|e\|_2 = 1$ . Then  $\lambda = \|e\|_s^2 \geq \|e_1\|_s^2 = \lambda_1$  and this implies the existence of  $k \in \mathbb{N}$  such that  $\lambda_k < \lambda < \lambda_{k+1}$ . Thus,  $e \notin \mathcal{M}_{k+1}$  and so there exists  $j \in \{1, 2, \dots, k\}$  such that  $\langle e, e_j \rangle_s \neq 0$ , a contradiction.

(4) Let us prove that  $\{e_k\}$  is a basis of  $X_0^s$  by a standard Fourier analysis technique. Given  $u \in X_0^s$ , we define

$$u_j = \sum_{i=1}^j \langle u, \tilde{e}_i \rangle_s \tilde{e}_i,$$

where  $\tilde{e}_i = \frac{e_i}{\|e_i\|_s}$ . Letting  $v_j = u - u_j$ , we get

$$0 \leq \|v_j\|_s^2 = \|u\|_s^2 - \sum_{i=1}^j \langle u, \tilde{e}_i \rangle_s^2,$$

that is,

$$\rho_j := \sum_{i=1}^j \langle u, \tilde{e}_i \rangle_s^2 \leq \|u\|_s^2.$$

Now, for  $l > j$ , we get

$$\|v_l - v_j\|_s^2 = \sum_{i=j+1}^l \langle u, \tilde{e}_i \rangle_s^2 = \rho_l - \rho_j,$$

which implies that  $\{v_j\}$  is a Cauchy sequence in  $\mathbb{R}$ , and thus there exists  $v^* \in X_0^s$  such that  $v_j \rightarrow v^*$  in  $X_0^s$  as  $j \rightarrow +\infty$ . For  $j \geq i$  we also have

$$\langle v_j, \tilde{e}_i \rangle_s = \langle u, \tilde{e}_i \rangle_s - \langle v, \tilde{e}_i \rangle_s = \langle u, \tilde{e}_i \rangle_s - \langle v, \tilde{e} \rangle_s = 0.$$

Passing to the limit, we get  $\langle v^*, \tilde{e}_i \rangle_s = 0$  for any  $i \in \mathbb{N}$ . This means that  $v^* = 0$  and therefore  $u_j = u - v_j \rightarrow u$  in  $X_0^s$  as  $j \rightarrow +\infty$ . So,  $u = \sum_{i=1}^{\infty} \langle u, \tilde{e}_i \rangle_s \tilde{e}_i$ , as we wanted.

In order to show that  $\{e_i\}$  is a basis for  $L^2(\Omega)$ , too, chosen  $v \in L^2(\Omega)$ , we let  $v_j \in C_0^2(\Omega)$  such that  $\|v - v_j\|_{L^2(\Omega)} < \frac{1}{j}$ . Since  $\{e_i\}$  is a basis of  $X_0^s$  and  $X_0^s \subset L^2(\Omega)$ , there exist  $k_j \in \mathbb{N}$  and a function  $w_j \in \text{span}\{e_1, \dots, e_{k_j}\}$  such that  $\|v_j - w_j\|_s \leq \frac{1}{j}$ . Thus, by Sobolev embeddings,

$$\|v - w_j\|_2 \leq \|v - v_j\|_2 + \|v_j - w_j\|_2 \leq \|v - v_j\|_2 + C\|v_j - w_j\|_s \leq \frac{1+C}{j},$$

which yields the claim.

(5)  $X_0^s = \text{span}\{e_k, \dots, e_{k+h}\} \oplus (\text{span}\{e_k, \dots, e_{k+h}\})^\perp$ . So, any eigenfunction  $u \in X_0^s \setminus \{0\}$  corresponding to  $\lambda_k$  admits the representation  $u = u_1 + u_2$ , with  $u_1 = \sum_{i=k}^{k+h} c_i e_i \in \text{span}\{e_k, \dots, e_{k+h}\}$  for some  $c_i \in \mathbb{R}$ , and  $u_2 \in (\text{span}\{e_k, \dots, e_{k+h}\})^\perp$ . Obviously,  $\langle u_1, u_2 \rangle_s = 0$ , which yields

$$\lambda_k \|u\|_2^2 = \|u\|_s^2 = \|u_1\|_s^2 + \|u_2\|_s^2. \quad (2.8)$$

By definition,  $u_1$  is also an eigenfunction corresponding to  $\lambda_k$ . Thus,

$$\lambda_k \int_{\Omega} u_1 u_2 dx = \langle u_1, u_2 \rangle_s = 0,$$

and

$$\|u\|_2^2 = \|u_1\|_2^2 + \|u_2\|_2^2.$$

On the other hand, we have

$$\|u_1\|_s^2 = \sum_{i=k}^{k+h} c_i^2 \|e_i\|_s^2 = \lambda_k \sum_{i=k}^{k+h} c_i^2 = \lambda_k \|u_1\|_2^2. \quad (2.9)$$

Recalling that  $u$  and  $u_1$  are eigenfunctions corresponding to  $\lambda_k$ , we deduce that  $u_2$  enjoys the same property and thus,  $u_2 \in (\text{span}\{e_1, \dots, e_{k+h}\})^\perp = \mathcal{M}_{k+h+1}$ .

We claim that  $u_2 \equiv 0$ . If not, we would have

$$\lambda_k < \lambda_{k+h+1} = \min_{u \in \mathcal{M}_{k+h+1} \setminus \{0\}} \frac{\|u\|_s^2}{\|u\|_2^2} \leq \frac{\|u_2\|_s^2}{\|u_2\|_2^2}. \quad (2.10)$$

Collecting (2.8), (2.9) and (2.10), we would obtain

$$\lambda_k \|u\|_2^2 > \lambda_k \|u_1\|_2^2 + \lambda_k \|u_2\|_2^2 = \lambda_k \|u\|_2^2,$$

clearly a contradiction. Therefore,  $u = u_1 \in \text{span}\{e_k, \dots, e_{k+h}\}$  and this completes the proof.  $\square$

*Proof of Theorem 2.* Inspired by [7], assume that  $u(x) < 0$  in  $\Omega$ . Owing to the lower semi-continuity of  $u$  on  $\bar{\Omega}$ , there exists some  $x_0 \in \bar{\Omega}$  such that

$$u(x_0) = \min_{\bar{\Omega}} u(x) < 0.$$

Moreover, since  $u \geq 0$  in  $\mathbb{R}^N \setminus \Omega$ , we deduce that  $x_0$  is an interior point of  $\Omega$ . Taking into account the fact that  $\Delta u(x_0) \geq 0$ , we get

$$\begin{aligned} (-\Delta)^s u(x_0) - \Delta u(x_0) &\leq c_{N,s} \text{PV} \int_{\mathbb{R}^N} \frac{u(x_0) - u(y)}{|x_0 - y|^{N+2s}} dy \\ &\leq c_{N,s} \text{PV} \int_{\mathbb{R}^N \setminus \Omega} \frac{u(x_0) - u(y)}{|x_0 - y|^{N+2s}} dy \\ &< 0, \end{aligned}$$

which contradicts the assumptions. Moreover, if  $u(x_0) = 0$  at some point  $x_0 \in \Omega$ , then

$$0 \leq (-\Delta)^s u(x_0) - \Delta u(x_0) \leq c_{N,s} \text{PV} \int_{\mathbb{R}^N} \frac{-u(y)}{|x_0 - y|^{N+2s}} dy, \quad (2.11)$$

which forces  $u(x) \equiv 0$  in  $\mathbb{R}^N$ .  $\square$

As a consequence of the previous result one has

**Corollary 13.** Assume that  $u \in L_s \cap C^2(\Omega)$  is upper semi-continuous on  $\bar{\Omega}$ . If

$$\begin{cases} (-\Delta)^s u - \Delta u \leq 0 & \text{in } \Omega, \\ u \leq 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (2.12)$$

Then,  $u(x) \leq 0$  in  $\Omega$ . Moreover, if  $u(x) = 0$  at some point in  $\Omega$ , then  $u(x) \equiv 0$  in  $\mathbb{R}^N$ .

*Proof of Theorem 3.* Let  $G : \mathbb{R} \rightarrow \mathbb{R}$  be the truncation defined by  $G(t) = 0$  for  $t \leq 0$  and  $G(t) = t$  for  $t > 0$ . Note that  $G$  is clearly a Lipschitz function and so, if  $v \in H^1(\Omega)$ , we get  $G(v) \in H^1(\Omega)$  thanks to Stampacchia's theorem. Moreover,  $[G(v)]_s < +\infty$ .

Choosing  $v = -u$ , then  $v \in H^1(\Omega)$ ,  $[v]_{s,\mathbb{R}^N} < +\infty$ , and

$$\begin{cases} (-\Delta)^s v - \Delta v \leq 0 & \text{in } \Omega, \\ v \leq 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Using  $G(v)$  as a test function, we get

$$\int_{\Omega} \nabla v \nabla G(v) dx + \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(v(x) - v(y))(G(v(x)) - G(v(y)))}{|x - y|^{N+2s}} dx dy \leq 0, \quad (2.13)$$

and since  $(v(x) - v(y))(G(v(x)) - G(v(y))) \geq 0$ , we get

$$\int_{\Omega} \nabla v \nabla G(v) dx \leq 0. \quad (2.14)$$

Now, by Stampacchia's theorem (cf.[3]), the proof is standard. In fact, by (2.14) we get  $G'(v)|\nabla v|^2 = 0$  a.e. in  $\Omega$  and hence  $\nabla G(u) = 0$  a.e. in  $\Omega$ . Since  $G(v) \in H_0^1(\Omega)$ , we deduce  $G(v) = 0$  a.e. in  $\Omega$  and, as a consequence,  $v \leq 0$  and  $u \geq 0$  a.e. in  $\Omega$ .  $\square$

Similarly, one has

**Corollary 14.** Assume that  $u \in H^1(\Omega)$ ,  $[u]_{s,\mathbb{R}^N} < +\infty$ , and  $u$  satisfies

$$\begin{cases} (-\Delta)^s u - \Delta u \leq 0 & \text{in } \Omega, \\ u \leq 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then,  $u(x) \leq 0$  a.e. in  $\Omega$ .

The proof of the existence result established in Theorem 4 requires some preliminary steps. Since we focus on nonnegative solutions, we consider the truncated energy functional associated with problem (1.18), namely

$$\mathcal{F}(u) := \frac{1}{2} \|u\|_s^2 - \int_{\Omega} F(x, u^+) dx, \quad u \in X_0^s, \quad (2.15)$$

where  $u^+ := \max\{u, 0\}$ . Thanks to  $(f_2)$ ,  $(f_3)$  and Sobolev inequalities,  $\mathcal{F}$  is well-defined, Fréchet differentiable and

$$\langle \mathcal{F}'(u), v \rangle = \langle u, v \rangle_s - \int_{\Omega} f(x, u^+) v dx, \quad \forall u, v \in X_0^s. \quad (2.16)$$

In order to exploit the mountain pass theorem, let us first prove that  $\mathcal{F}$  exhibits the suitable geometry.

**Lemma 15.** *There exist  $\rho, \delta > 0$  such that, for any  $u \in X_0^s$  satisfying  $\|u\|_s = \rho$ , one has  $\mathcal{F}(u) \geq \delta$ .*

*Proof.* For any  $u \in X_0^s$ , by  $(f_2)$ ,  $(f_3)$  and Sobolev embeddings, we get

$$\mathcal{F}(u) \geq \frac{1}{4} \|u\|_s^2 - C \int_{\Omega} |u|^{q+1} dx \geq \left( \frac{1}{4} - C \|u\|_s^{q-1} \right) \|u\|_s^2.$$

By the definition of  $q$ , we can choose  $\rho > 0$  sufficiently small such that  $\frac{1}{4} - C \|u\|_s^{q-1} > 0$  and the result follows by setting  $\delta = \frac{1}{4} \rho^2 - C \rho^{q+1}$ .  $\square$

**Lemma 16.** *There exists  $e \in X_0^s$  satisfying  $\|e\|_s > \rho$  and  $\mathcal{F}(e) < 0$ .*

*Proof.* Fix  $v \in X_0^s$ , with  $\|v\|_s = 1$  and  $v > 0$  a.e. in  $\mathbb{R}^N$ , and let  $u = tv$  for  $t > 0$ . Then, by  $(f_4)$  we obtain

$$\mathcal{F}(u) \leq \frac{1}{2} t^2 - \frac{1}{\mu} t^\mu \int_{\Omega} |v|^{\mu} dx - C \rightarrow -\infty, \quad \text{as } t \rightarrow +\infty.$$

The assertion follows by taking  $e = tv$ , with  $t$  sufficiently large.  $\square$

A key-ingredient is the Palais-Smale condition, (PS) for short.

**Lemma 17.** *The functional  $\mathcal{F}$  satisfies (PS).*

*Proof.* Let  $c \in \mathbb{R}$ , and let  $\{u_k\}$  be a sequence in  $X_0^s$  such that

$$\lim_{k \rightarrow +\infty} \mathcal{F}(u_k) = c \quad (2.17)$$

and

$$\lim_{k \rightarrow +\infty} \langle \mathcal{F}'(u_k), v \rangle = 0, \quad \forall v \in X_0^s. \quad (2.18)$$

By  $(f_4)$ , we have

$$\mu \mathcal{F}(u_k) - \langle \mathcal{F}'(u_k), u_k \rangle \geq \frac{\mu - 2}{2} \|u_k\|_s^2 - C(R), \quad (2.19)$$

where  $C(R) > 0$ , and hence  $\{u_k\}$  is bounded in  $X_0^s$ . Then, up to a subsequence still denoted by  $\{u_k\}$ , there exists  $u \in X_0^s$  such that  $u_k \rightharpoonup u$  in  $X_0^s$ ,  $u_k \rightarrow u$  in  $L^m(\Omega)$ ,  $1 < m < 2^*$ , and  $u_k \rightarrow u$  a.e. in  $\Omega$  as  $k \rightarrow +\infty$  (note that, since  $u_k = 0$  a.e. in  $\mathcal{C}\Omega$ , we may define  $u = 0$  a.e. in  $\mathcal{C}\Omega$ ). By (2.18) and the boundedness of the sequence  $\{u_k - u\}$ , we get

$$\lim_{k \rightarrow +\infty} \langle \mathcal{F}'(u_k), (u_k - u) \rangle = \lim_{k \rightarrow +\infty} \left( \langle u_k, u_k - u \rangle_s - \int_{\Omega} f(x, u_k^+)(u_k - u) dx \right) = 0. \quad (2.20)$$

On the other hand, by  $(f_2)$  and the compactness of the embeddings  $X_0^s \hookrightarrow L^1(\mathbb{R}^N)$ ,  $X_0^s \hookrightarrow L^{q+1}(\mathbb{R}^N)$ , we deduce

$$\lim_{k \rightarrow +\infty} \int_{\Omega} f(x, u_k^+)(u_k - u) dx = 0. \quad (2.21)$$

A joint use of (2.20), (2.21) and the fact that  $u_k \rightharpoonup u$  in  $X_0^s$ , lead to  $\|u_k\|_s \rightarrow \|u\|_s$  as  $k \rightarrow +\infty$  and, by classical arguments, to  $\|u_k - u\|_s \rightarrow 0$  as  $k \rightarrow +\infty$ , as desired.  $\square$

*Proof of Theorem 4.* It is straightforward by the mountain pass theorem and joining Lemmas 15, 16 and 17. By Theorem 3, the solution is nonnegative.  $\square$

*Proof of Theorem 7.* For any  $u \in X_0^s$ , define the fibering map  $\phi_u : [0, +\infty) \rightarrow \mathbb{R}$  by  $\phi_u(t) := \mathcal{F}(tu)$ . Of course,  $u \in \mathcal{N}$  if and only if

$$\|u\|_s^2 - \int_{\Omega} |u|^p dx = 0, \quad (2.22)$$

and  $tu \in \mathcal{N}$  iff  $\phi'_u(t) = 0$ . By direct computations,

$$\begin{aligned}\phi'_u(t) &= t\|u\|_s^2 - t^{p-1} \int_{\Omega} |u|^p dx, \\ \phi''_u(t) &= \|u\|_s^2 - (p-1)t^{p-2} \int_{\Omega} |u|^p dx,\end{aligned}$$

and so  $\phi_u$  has a unique critical point (a local maximum), namely

$$t = t(u) = \left( \frac{\|u\|_s^2}{\int_{\Omega} |u|^p dx} \right)^{\frac{1}{p-2}}. \quad (2.23)$$

Moreover  $\phi_u$  is increasing in  $(0, t(u))$ , decreasing in  $(t(u), +\infty)$  and  $t(u)u \in \mathcal{N}^-$ .

Let us split  $\mathcal{N}$  as follows

$$\mathcal{N}^+ := \{u \in \mathcal{N} : \phi''_u(t) > 0\}, \quad \mathcal{N}^- := \{u \in \mathcal{N} : \phi''_u(t) < 0\}, \quad \mathcal{N}^0 := \{u \in \mathcal{N} : \phi''_u(t) = 0\}$$

to obtain  $\mathcal{N} = \mathcal{N}^-$ .  $\mathcal{F}$  is of course bounded below on  $\mathcal{N}^-$ , as

$$\mathcal{F}(u) = \left( \frac{1}{2} - \frac{1}{p} \right) \|u\|_p^p \geq 0, \quad \forall u \in \mathcal{N}^-.$$

Setting

$$\bar{c} := \inf_{u \in \mathcal{N}^-} \mathcal{F}(u),$$

we easily derive that  $\bar{c} > 0$ . Indeed, if  $v = u/\|u\|_s$  and  $t(v)$  is as in (2.23), we get

$$\begin{aligned}\mathcal{F}(u) &= \mathcal{F}(t(v)v) = \left( \frac{1}{2} - \frac{1}{p} \right) t^2(v) \|v\|_s^2 \\ &\geq \left( \frac{1}{2} - \frac{1}{p} \right) \frac{\|v\|_s^{\frac{2p}{p-2}}}{\|v\|_p^{\frac{2p}{p-2}}} \\ &\geq \left( \frac{1}{2} - \frac{1}{p} \right) c_p^{-\frac{2p}{p-2}} \\ &> 0,\end{aligned}$$

for any  $u \in \mathcal{N}^-$ , being  $c_p > 0$  the best constant of the embedding  $X_0^s \hookrightarrow L^p(\mathbb{R}^n)$ .

The next step is to show that  $\bar{c}$  is attained at some function within  $\mathcal{N}^-$ . Let  $\{u_k\} \subset \mathcal{N}^-$  be a minimizing sequence, i.e. such that  $\mathcal{F}(u_k) \rightarrow \bar{c}$  as  $k \rightarrow +\infty$ . Clearly  $\{u_k\}$  is bounded, so, up to a subsequence,  $u_k \rightharpoonup u_0 \in X_0^s$  and  $u_k \rightarrow u_0 \in L^p(\mathbb{R}^N)$ . So,

$$0 < \bar{c} = \lim_{k \rightarrow +\infty} \mathcal{F}(u_k) = \lim_{k \rightarrow +\infty} \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\Omega} |u_k|^p dx = \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\Omega} |u_0|^p dx,$$

which yields  $u_0 \neq 0$ . If  $u_k \not\rightarrow u_0$  in  $X_0^s$ , we would obtain

$$\phi'_{u_0}(1) = \|u_0\|_s^2 - \int_{\Omega} |u_0|^p dx < \liminf_{k \rightarrow +\infty} \left( \|u_k\|_s^2 - \int_{\Omega} |u_k|^p dx \right) = 0.$$

We know that there exists a unique  $t_0 \in (0, 1)$  such that  $t_0 u_0 \in \mathcal{N}^-$  and that  $t \mapsto \phi_{u_k}(t)$  admits its maximum at  $t = 1$ , so

$$\mathcal{F}(t_0 u_0) < \liminf_{k \rightarrow +\infty} \mathcal{F}(t_0 u_k) \leq \lim_{k \rightarrow +\infty} \mathcal{F}(u_k) = \bar{c},$$

a contradiction. So  $u_k \rightarrow u_0$  and  $\phi'_{u_0}(1) = 0$ , namely  $u_0 \in \mathcal{N}^-$ . We finally get

$$\mathcal{F}(u_0) = \lim_{k \rightarrow +\infty} \mathcal{F}(u_k) = \bar{c}$$

and  $u_0$  is the minimizer sought for.

To conclude the proof, we need to give a minimax characterization of  $\bar{c}$ . If  $\psi_1 : X_0^s \setminus \{0\} \rightarrow (0, +\infty)$ ,  $\psi_2 : \{u \in X_0^s : \|u\|_\epsilon = 1\} \rightarrow \mathbb{N}$  are the maps defined by  $\psi_1(u) := t(u)$ ,  $\psi_2(u) := t(u)u$ , it is easy to show that  $\psi_1$  is continuous, while  $\psi_2$  is a homeomorphism.

Then, arguing as in [28, Theorem 4.1], define

$$c^* := \inf_{u \in X_0^s \setminus \{0\}} \max_{t \geq 0} \phi_u(t),$$

$$c := \inf_{y \in \Gamma} \max_{t \in [0,1]} \mathcal{F}(y(t)),$$

where

$$\Gamma := \{y \in C^0([0, 1], X_0^s) : y(0) = 0, \mathcal{F}(y(1)) < 0\}.$$

The above considerations yield  $\bar{c} = c^*$ . Moreover, if  $u \in X_0^s \setminus \{0\}$ ,

$$\mathcal{F}(tu) = \frac{t^2}{2} \|u\|_s^2 - \frac{t^p}{p} \int_{\Omega} |u|^p dx \rightarrow -\infty$$

as  $t \rightarrow +\infty$  and hence  $c \leq c^*$ . If  $X_1$  and  $X_2$  are the two components of  $X_0^s$  induced by  $\mathcal{N}$ , with  $0 \in X_2$ , there will exist  $r > 0$  such that  $B(0, r) \subset X_2$ . Since  $\phi'_u(t) = \langle \mathcal{F}'(tu), u \rangle \geq 0$  for all  $t \in [0, t(u)]$ , one has  $\mathcal{F}(u) \geq 0$  for all  $u \in X_2$ . As a result, any  $y \in \Gamma$  will have a non-empty intersection with  $\mathcal{N}$  and then  $\bar{c} \leq c$ .

This implies that  $\bar{c} = c^* = c$  and the proof is complete.  $\square$

**Remark 18.** It is worth pointing out that each ground state solution  $u_0$  of (1.19) does not change sign. Indeed, denoting as usual  $u_0^+ = \max\{u_0, 0\}$  and  $u_0^- = \min\{u_0, 0\}$  the positive and the negative part of  $u_0$ , respectively, multiplying (1.19) by  $u_0^+$  and integrating, we get  $u_0^+ \in \mathcal{N}$ . As a result

$$\begin{aligned} \bar{c} &= \mathcal{F}(u_0) = \left(\frac{1}{2} - \frac{1}{p}\right) \|u_0\|_s^2 = \left(\frac{1}{2} - \frac{1}{p}\right) \langle u_0^+ - u_0^-, u_0^+ - u_0^- \rangle_s \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) (\|u_0^+\|_s^2 + \|u_0^-\|_s^2) \\ &= \mathcal{F}(u_0^+) + \mathcal{F}(u_0^-) \\ &\geq 2\bar{c}, \end{aligned}$$

a contradiction.

Before proving Theorem 8, it is necessary to introduce some notations and some preliminary results, namely an antisymmetric maximum principle holding for “narrow” regions. This is needed to apply the moving plane method directly to problem (1.20) (cf. [7]). For  $\lambda \in \mathbb{R}$ , denote by

$$T_\lambda := \{x \in \mathbb{R}^n : x_1 = \lambda\}$$

the moving planes, by

$$\Sigma_\lambda := \{x \in \mathbb{R}^n : x_1 < \lambda\}$$

the region to the left of the plane and by

$$x^\lambda := (2\lambda - x_1, x_2, \dots, x_n)$$

the reflection of  $x \in \mathbb{R}^N$  with respect to  $T_\lambda$ , and set

$$w_\lambda(x) := u(x^\lambda) - u(x).$$

**Lemma 19.** Let  $V$  be a bounded subset of  $\Sigma_\lambda$  contained in  $\{x \in \mathbb{R}^n : \lambda - l < x_1 < \lambda\}$ ,  $l > 0$  sufficiently small. Suppose that  $u \in L_s \cap C^2(V)$  is lower semicontinuous on  $\bar{V}$ . If  $c : \mathbb{R}^n \rightarrow \mathbb{R}$  is bounded from below in  $V$  and

$$\begin{cases} (-\Delta)^s u(x) - \Delta u(x) + c(x)u(x) \geq 0 & \text{in } V, \\ u(x) \geq 0 & \text{in } \Sigma_\lambda \setminus V, \\ u(x^\lambda) = -u(x) & \text{in } \Sigma_\lambda, \end{cases} \quad (2.24)$$

then  $u(x) \geq 0$  in  $V$  for  $l$  sufficiently small. Moreover, if  $u(x) = 0$  for some  $x \in V$ , then  $u(x) \equiv 0$  for a.e.  $x \in \mathbb{R}^n$ .

*Proof.* Arguing by contradiction, assume that  $u$  solves (2.24) and  $u < 0$  somewhere in  $V$ . Since  $u$  is lower semicontinuous, there exists  $x_0 \in \text{int}(V)$  such that

$$u(x_0) = \min_{u \in \bar{V}} u(x) < 0.$$

Since  $u$  is anti-symmetric in  $\Sigma_\lambda$ , following [7] one has

$$\begin{aligned} (-\Delta)^s u(x_0) &= c_{N,s} \text{PV} \int_{\mathbb{R}^n} \frac{u(x_0) - u(y)}{|x_0 - y|^{N+2s}} dy \\ &= c_{N,s} \text{PV} \left( \int_{\Sigma_\lambda} \frac{u(x_0) - u(y)}{|x_0 - y|^{N+2s}} dy + \int_{\{x \in \mathbb{R}^N : x^\lambda \in \Sigma_\lambda\}} \frac{u(x_0) - u(y)}{|x_0 - y|^{N+2s}} dy \right) \\ &= c_{N,s} \text{PV} \left( \int_{\Sigma_\lambda} \frac{u(x_0) - u(y)}{|x_0 - y|^{N+2s}} dy + \int_{\Sigma_\lambda} \frac{u(x_0) + u(y)}{|x_0 - y^\lambda|^{N+2s}} dy \right) \\ &\leq c_{N,s} \int_{\Sigma_\lambda} \frac{2u(x_0)}{|x_0 - y^\lambda|^{N+2s}} dy, \end{aligned}$$

and, after a change of variables, it is not difficult to see that the last term diverges to  $-\infty$  as  $l \rightarrow 0^+$ . Being  $\Delta u(x_0) \geq 0$  and  $c$  bounded below in  $V$ , we get

$$(-\Delta)^s u(x_0) - \epsilon \Delta u(x_0) + c(x_0)u(x_0) < 0 \quad \text{as } l \rightarrow 0^+,$$

which contradicts (2.24).  $\square$

*Proof of Theorem 8.* Since  $u$  is a positive solution of (1.20), we get

$$(-\Delta)^s w_\lambda(x) - \Delta w_\lambda(x) + c_\lambda(x)w_\lambda(x) = 0, \quad (2.25)$$

for every  $x \in \Sigma_\lambda$ , where

$$c_\lambda(x) := \frac{f(u^\lambda(x)) - f(u(x))}{u^\lambda(x) - u(x)},$$

and  $|c_\lambda(x)| \leq L$ , for some  $L > 0$ . Applying Theorem 2.24, for  $\lambda \rightarrow -1^+$  one has  $w_\lambda(x) \geq 0$  in  $\Sigma_\lambda$ . Now, setting

$$\lambda_0 := \sup\{\lambda \leq 0 : w_\mu(x) \geq 0, x \in \Sigma_\mu, \mu \leq \lambda\},$$

we claim that  $\lambda_0 = 0$ . Assuming the contrary, then it should be

$$w_{\lambda_0}(x) > 0 \quad \forall x \in \Sigma_{\lambda_0},$$

otherwise there would exist  $x_0 \in \Sigma_{\lambda_0}$ :

$$w_{\lambda_0}(x_0) = \min_{u \in \Sigma_{\lambda_0}} w_{\lambda_0}(x) = 0.$$

Let us estimate

$$(-\Delta)^s w_{\lambda_0}(x_0) = -c_{N,s} \text{PV} \int_{\mathbb{R}^N} \frac{w_{\lambda_0}(y)}{|x_0 - y|^{N+2s}} dy$$



$$\begin{aligned}
&= -c_{N,s} \operatorname{PV} \left( \int_{\Sigma_{\lambda_0}} \frac{w_{\lambda_0}(y)}{|x_0 - y|^{N+2s}} dy + \int_{\mathbb{R}^N \setminus \Sigma_{\lambda_0}} \frac{w_{\lambda_0}(y)}{|x_0 - y|^{N+2s}} dy \right) \\
&= c_{N,s} \operatorname{PV} \int_{\Sigma_{\lambda_0}} \left( \frac{1}{|x_0 - y^{\lambda_0}|^{N+2s}} - \frac{1}{|x_0 - y|^{N+2s}} \right) w_{\lambda_0}(y) dy \\
&< 0
\end{aligned}$$

which together with the fact that  $\Delta w_{\lambda_0}(x_0) \geq 0$  obviously contradicts (2.25). So, there exists  $\delta > 0$  such that, for any  $\lambda \in (\lambda_0, \lambda_0 + \delta)$  we have  $w_\lambda(x) \geq 0$ ,  $x \in \Sigma_\lambda$ , which contradicts the definition of  $\lambda_0$ .

Therefore

$$w_0(x) \geq 0, \quad x \in \Sigma_0,$$

or equivalently

$$u(-x_1, x_2, \dots, x_n) \leq u(x_1, x_2, \dots, x_n), \quad 0 < x_1 < 1.$$

Since the  $x_1$ -direction is arbitrary, the previous relation implies that  $u$  is radially symmetric with respect to the origin. The monotonicity follows from the fact that  $w_\lambda(x) \geq 0$  for all  $\lambda \in (-1, 0]$ .  $\square$

Finally, we deal with the regularity of solutions to the problem

$$\begin{cases} (-\Delta)^s u - \Delta u = f(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (2.26)$$

where  $\Omega \subset \mathbb{R}^N$  is a bounded domain,  $N > 2$ . To be precise, we will study the  $L^p$  regularity, including the boundedness of solution via Stampacchia's method, exponential summability and a Calderón-Zygmund type result (for the purely nonlocal case we refer to [16]).

Let us recall following lemma by Stampacchia

**Lemma 20.** Let  $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a nonincreasing function such that

$$\psi(h) \leq \frac{M\psi(k)^\delta}{(h-k)^y}, \quad \forall h > k > 0,$$

where  $M > 0$ ,  $\delta > 1$  and  $y > 0$ . Then  $\psi(d) = 0$ , where  $d^y = M\psi(0)^{\delta-1} 2^{\frac{\delta y}{\delta-1}}$ .

*Proof of Theorem 9 (1)* We follow Stampacchia's method. For  $k \geq 0$ , we define the function

$$G_k(\sigma) = \begin{cases} \sigma - k, & \text{if } \sigma \geq k, \\ 0, & \text{if } |\sigma| < k, \\ \sigma + k, & \text{if } \sigma \leq -k. \end{cases}$$

Clearly,  $G_k(0) = 0$  and  $G_k(\sigma)$  is Lipschitz continuous. For  $u \in X_0^s$ , we get  $G_k(u) \in X_0^s$ . Using  $G_k(u)$  as a test function, we get

$$\int_{\Omega} \nabla u \nabla G_k(u) dx + \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(G_k(u(x)) - G_k(u(y)))}{|x - y|^{N+2s}} dx dy = \int_{\Omega} f(x) G_k(u) dx. \quad (2.27)$$

Since  $(u(x) - u(y))(G_k(u(x)) - G_k(u(y))) \geq (G_k(u(x)) - G_k(u(y)))^2$ , from (2.27) we get

$$\int_{\Omega} |\nabla G_k(u)|^2 dx \leq \int_{A_k} |f(x) G_k(u(x))| dx,$$

where  $A_k := \{x \in \Omega : |u(x)| \geq k\}$ . By applying Sobolev and Hardy inequalities, we have

$$s^{-2} \|G_k(u)\|_{2^*}^2 \leq \|f\|_q \|G_k(u)\|_{2^*} |A_k|^{1-\frac{1}{2^*}-\frac{1}{q}},$$

where  $\mathcal{S}$  is the best constant of the embedding  $X_0^s \hookrightarrow L^{2^*}(\mathbb{R}^N)$ .

For any  $h > k$ , we have  $A_h \subset A_k$  and  $G_k(\sigma)\chi_{A_h} \geq h - k$ , thus

$$\mathcal{S}^{-2}(h-k)|A_h|^{\frac{1}{2^*}} \leq \|f\|_q |A_k|^{1-\frac{1}{2^*}-\frac{1}{q}},$$

which implies

$$|A_h| \leq \frac{\mathcal{S}^{2 \cdot 2^*} \|f\|_q^{2^*} |A_k|^{2^* \left(1-\frac{1}{2^*}-\frac{1}{q}\right)}}{(h-k)^{2^*}}.$$

The assumption  $q > \frac{N}{2}$  implies  $2^* \left(1 - \frac{1}{2^*} - \frac{1}{q}\right) > 1$  and so, by Lemma 20, there exists  $k_0 > 0$  such that  $\psi(k_0) = |A_{k_0}| = 0$ . Consequently, we get  $|u(x)| \leq k_0$  a.e. in  $\Omega$ .

(2) For any  $T > 0$ , we consider the function

$$\Phi(\sigma) = \begin{cases} e^{a\sigma} - 1 & \text{if } 0 \leq \sigma < T, \\ ae^{aT}(\sigma - T) + e^{aT} - 1 & \text{if } \sigma \geq T, \end{cases}$$

where  $a > 0$  will be fixed later. Note that  $\Phi(0) = 0$ ,  $\Phi \in C^1$ , is convex and its first derivative is Lipschitz continuous. Thus, if  $u \in X_0^s$ , then  $\Phi(u) \in X_0^s$  as well.

From

$$\Phi(u(x)) - \Phi(u(y)) \leq \Phi'(u(x))[u(x) - u(y)],$$

one has

$$\begin{aligned} \int_{\Omega} [-\Delta \Phi(u) + (-\Delta)^s \Phi(u)] \Phi(u) dx &\leq \int_{\Omega} [-\Delta u + (-\Delta)^s u] \Phi'(u) \Phi(u) dx \\ &= \int_{\Omega} f(x) \Phi'(u) \Phi(u) dx, \end{aligned}$$

which implies

$$\begin{aligned} \mathcal{S}^{-2} \|\Phi(u)\|_{2^*}^2 &\leq \alpha \int_{\{x \in \Omega: u(x) < T\}} [\Phi^2(u)f + \Phi(u)f] dx + ae^{aT} \int_{\{x \in \Omega: u(x) \geq T\}} \Phi(u)f dx \\ &\leq \alpha \|\Phi(u)\|_{2^*}^2 \|f\|_{\frac{N}{2}} \left(1 + |\Omega|^{\frac{N-2}{2N}}\right) + C_f \\ &\quad + ae^{aT} \Phi^{-1}(T) \int_{\{x \in \Omega: u(x) \geq T\}} \Phi^2(u)f dx \\ &\leq \alpha \|\Phi(u)\|_{2^*}^2 \|f\|_{\frac{N}{2}} \left(3 + |\Omega|^{\frac{N-2}{2N}}\right) + C_f, \end{aligned}$$

provided that  $aT > 1$ . So, fixing  $\alpha$  small enough and taking  $T$  large, we obtain

$$\|\Phi(u)\|_{2^*}^2 \leq C,$$

and the conclusion follows.

(3) For any  $T > 0$ , consider the function

$$\Phi(\sigma) = \begin{cases} \sigma^\beta & \text{if } 0 \leq \sigma < T, \\ \beta T^{\beta-1}(\sigma - T) + T^\beta & \text{if } \sigma \geq T, \end{cases}$$

where  $\beta = \frac{m^*}{2^*} > 1$ . It is straightforward to observe that  $\Phi$  enjoys the same properties as in item (2). Then, following the same steps of item (2), we have

$$\int_{\Omega} [-\Delta \Phi(u) + (-\Delta)^s \Phi(u)] \Phi(u) dx \leq \int_{\Omega} f(x) \Phi'(u) \Phi(u) dx,$$

and so

$$\begin{aligned} S^{-2} \|\Phi(u)\|_{2^*}^2 &\leq \|f\|_q \|\Phi'(u)\Phi(u)\|_{q'} \\ &\leq C \|f\|_q \left( \int_{\Omega} |\Phi(u)|^{2^*} dx \right)^{\frac{1}{q'}}, \end{aligned}$$

where  $1/q + 1/q' = 1$  and  $C > 0$  is independent of  $T$ . Thus,

$$S^{-2} \left( \int_{\Omega} |\Phi(u)|^{2^*} dx \right)^{\frac{2}{2^*} - \frac{1}{q'}} \leq C \|f\|_q. \quad (2.28)$$

and letting  $T \rightarrow +\infty$  and recalling that  $2^* \beta = q'(2\beta - 1)$  we arrive at

$$\|u\|_{m^*} \leq C \|f\|_q,$$

as desired.  $\square$

Finally, we prove the boundedness of the solution  $u$  obtained in Corollary 6.

**Lemma 21.** *The solution  $u$  of (1.19) stated in Corollary 6 satisfies  $\|u\|_{\infty} < C$  for some constant  $C > 0$ .*

*Proof.* Let  $\beta, L > 0$  and let  $Y : \mathbb{R}_0^+ \rightarrow \mathbb{R}$  be the function defined by

$$Y(t) = \begin{cases} t^{2\beta+1} & \text{if } 0 \leq t < L^{\frac{1}{\beta}}, \\ (2\beta+1)L^2 t - 2\beta L^{2+\frac{1}{\beta}} & \text{if } t \geq L^{\frac{1}{\beta}}. \end{cases} \quad (2.29)$$

Note that  $Y$  is convex, differentiable,  $Y(u) \in X_0^S$  and

$$(-\Delta)^S Y(u) \leq Y'(u)(-\Delta)^S u.$$

Besides, we get

$$Y'Y = \begin{cases} (2\beta+1)u^{4\beta+1} & \text{if } 0 \leq t < L^{\frac{1}{\beta}}, \\ (2\beta+1)^2 u L^4 - 2\beta(2\beta+1)L^{4+\frac{1}{\beta}} & \text{if } t \geq L^{\frac{1}{\beta}}. \end{cases} \quad (2.30)$$

Clearly,  $Y'(u)Y(u) \in H_0^1(\Omega)$ . Multiplying  $Y'(u)Y(u)$  both sides of (1.19) and integrating, we get

$$\begin{aligned} 0 &\leq \int_{\Omega} Y(u)(-\Delta)^S Y(u) dx \\ &\leq \int_{\Omega} Y'(u)Y(u)(-\Delta)^S u dx \\ &= \int_{\Omega} Y'(u)Y(u)\Delta u dx + \int_{\Omega} Y'(u)Y(u)|u|^{p-2} u dx, \end{aligned} \quad (2.31)$$

which yields

$$\int_{\Omega} \nabla(Y'(u)Y(u)) \nabla u dx \leq \int_{\Omega} Y'(u)Y(u)|u|^{p-2} u dx. \quad (2.32)$$

Direction calculations show that

$$\int_{\Omega} |\nabla(u \min\{u^{2\beta}, L^2\})|^2 dx \leq \int_{\Omega} Y'(u)Y(u)|u|^{p-2} u dx. \quad (2.33)$$

Then, arguing as in [27, Lemma B.3], we get  $u \in L^q(\Omega)$  for any  $q < +\infty$  and, in particular, for  $q > \frac{N}{2}$ . The conclusion then is a consequence of Theorem 9–(1).  $\square$

### 3 Proof of Theorems 10 and 11

Here we study the asymptotic behavior of ground states of problem (1.3). Let  $u_\epsilon$  be a minimizer of  $S_\epsilon$ , as well as a ground state of (1.3), as shown in Section 1.

It is immediate to estimate the  $\epsilon$ -norm and the  $L^p$ -norm of such minimizers.

**Lemma 22.**  $\|u_\epsilon\|_\epsilon = S_\epsilon^{\frac{p}{2(p-2)}}$  and  $\|u_\epsilon\|_p = S_\epsilon^{\frac{1}{p-2}}$ .

*Proof.* By direct calculations, we get

$$\|u_\epsilon\|_\epsilon = S_\epsilon^{-\frac{1}{2-p}} \|w_\epsilon\|_\epsilon = S_\epsilon^{-\frac{1}{2-p} + \frac{1}{2}} = S_\epsilon^{\frac{p}{2(p-2)}}$$

and

$$\|u_\epsilon\|_p = S_\epsilon^{-\frac{1}{2-p}} \|w_\epsilon\|_p = S_\epsilon^{\frac{1}{p-2}}.$$

□

Next, we prove that the function  $\{S_\epsilon\}$  is bounded. To be more precise, we show that  $\{S_\epsilon\}$  converges to different constants according to the range of  $p$ . We consider three different cases.

#### 3.1 The case $2 < p < 2_s^*$

Let

$$S = \inf_{u \in H_0^s \setminus \{0\}} \frac{[u]_s^2}{\|u\|_p^2} = \inf\{[u]_s^2 : \|u\|_p = 1\}.$$

By Sobolev embedding theorems, for  $2 < p < 2_s^*$  there exists a non-negative minimizer  $w_0$  of  $S$ . Analogously to the proof carried out in Section 1, the function  $u_0 = S^{-\frac{1}{2-p}} w_0$  is a non-negative ground state of

$$\begin{cases} (-\Delta)^s u = |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (3.1)$$

We point out that (3.1) also admits a mountain pass type solution, see e.g. [24].

In the same flavor of Lemma 22, we also have

**Lemma 23.**  $[u_0]_s = S^{\frac{p}{2(p-2)}}$ ,  $\|u_0\|_p = S^{\frac{1}{p-2}}$ .

The asymptotic behavior of  $S_\epsilon$  as  $\epsilon \rightarrow 0$  is established in the next

**Lemma 24.**  $\lim_{\epsilon \rightarrow 0} S_\epsilon = S$ .

*Proof.* Since  $X_0^s \subset H_0^s$ , we get

$$S \leq \inf_{u \in X_0^s \setminus \{0\}} \frac{[u]_s^2}{\|u\|_p^2} \leq S_\epsilon,$$

and thus

$$S \leq \liminf_{\epsilon \rightarrow 0} S_\epsilon.$$

Assume  $w_0$  is a minimizer of  $S$ . By density, there exists a sequence  $\{w_k\} \subseteq C_0^\infty(\mathbb{R}^N)$  such that  $w_k \rightarrow w_0$  in  $H_0^s(\mathbb{R}^N)$  and  $L^p(\mathbb{R}^N)$ . Thus, we have

$$S_\epsilon \leq \frac{\|w_k\|_\epsilon^2}{\|w_k\|_p^2} \quad (3.2)$$

and

$$\limsup_{\epsilon \rightarrow 0} S_\epsilon \leq \lim_{k \rightarrow \infty} \limsup_{\epsilon \rightarrow 0} \frac{\|w_k\|_\epsilon^2}{\|w_k\|_p^2} = \lim_{k \rightarrow \infty} \frac{[w_k]_s^2}{\|w_k\|_p^2} = S. \quad (3.3)$$

This completes the proof.  $\square$

From Lemmas 22, 23 and 24, we get

$$\|u_\epsilon\|_\epsilon \rightarrow [u_0]_s, \quad \|u_\epsilon\|_p \rightarrow \|u_0\|_p, \quad \text{as } \epsilon \rightarrow 0.$$

Note that since  $[u_\epsilon]_s$  is bounded, up to a subsequence we may assume that  $u_\epsilon \rightharpoonup \tilde{u}$  in  $H_0^s(\Omega)$  and thus  $u_\epsilon \rightarrow \tilde{u}$  in  $L^p(\Omega)$ . On the other hand, from  $\|u_\epsilon\|_p \rightarrow \|u_0\|_p$ , we obtain  $u_\epsilon \rightarrow u_0$  in  $L^p(\Omega)$ . This gives  $\tilde{u} = u_0$  a.e. in  $\mathbb{R}^N$ . Therefore, we obtain

**Lemma 25.**  $\lim_{\epsilon \rightarrow 0} \epsilon \|\nabla u_\epsilon\|_2 = 0$ ,  $\lim_{\epsilon \rightarrow 0} [u_\epsilon]_s = [u_0]_s$ ,  $\lim_{\epsilon \rightarrow 0} \|u_\epsilon\|_p = \|u_0\|_p$ .

### 3.2 The case $2_s^* \leq p < 2^*$

For a fixed  $\epsilon > 0$ , let us consider the problem

$$\begin{cases} (-\Delta)^s u - \Delta u = |u|^{p-2}u & \text{in } \Omega_\epsilon, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega_\epsilon, \end{cases} \quad (3.4)$$

where  $\Omega_\epsilon := \epsilon^{\frac{1}{2s-2}} \Omega$  and  $2_s^* \leq p < 2^*$ . Note that  $\Omega_\epsilon \rightarrow \mathbb{R}^N$  as  $\epsilon \rightarrow 0$ .

Define

$$\tilde{S}_\epsilon = \inf\{\|u\|_s^2 : u \in X_0^s(\Omega_\epsilon), \|u\|_{L^p(\Omega_\epsilon)} = 1\} = \inf_{u \in X_0^s(\Omega_\epsilon) \setminus \{0\}} \frac{\|u\|_s^2}{\|u\|_p^2}.$$

Let  $\tilde{w}_\epsilon$  be a minimizer of  $\tilde{S}_\epsilon$  satisfying  $\|\tilde{w}_\epsilon\|_s^2 = \tilde{S}_\epsilon$  and  $\|\tilde{w}_\epsilon\|_{L^p(\Omega_\epsilon)} = 1$ . Then,  $\tilde{u}_\epsilon = \tilde{S}_\epsilon^{-\frac{1}{2-p}} \tilde{w}_\epsilon$  is a ground state of (3.4). Moreover, one has

$$S_\epsilon = \epsilon^{\frac{p(2s-N)+2N}{2p(s-1)}} \tilde{S}_\epsilon. \quad (3.5)$$

Now let us introduce the scaled function  $v_\epsilon(x) = \epsilon^{\frac{s}{(s-1)(p-2)}} \tilde{u}_\epsilon(\epsilon^{\frac{1}{2s-2}} x)$ ,  $x \in \Omega$ . We show that it is a solution of problem (1.3). In fact, one has

$$\begin{aligned} \int_{\Omega} |v_\epsilon|^p dx &= \epsilon^{\frac{sp}{(s-1)(p-2)}} \int_{\Omega} |\tilde{u}_\epsilon(\epsilon^{\frac{1}{2s-2}} x)|^p dx \\ &= \epsilon^{\frac{sp}{(s-1)(p-2)} - \frac{N}{2s-2}} \int_{\Omega_\epsilon} |\tilde{u}_\epsilon(y)|^p dy \\ &= \epsilon^{\frac{2sp-N(p-2)}{2(s-1)(p-2)}} \int_{\Omega_\epsilon} |\tilde{u}_\epsilon(y)|^p dy \\ &= \begin{cases} \tilde{S}_\epsilon^{-\frac{p}{2-p}} & \text{if } p = 2_s^*, \\ \epsilon^{\frac{2N+p(2s-N)}{2(s-1)(p-2)}} \tilde{S}_\epsilon^{-\frac{p}{2-p}} & \text{if } p > 2_s^*, \end{cases} \\ &= S_\epsilon^{-\frac{p}{2-p}} \quad \text{for } p \geq 2_s^* \end{aligned} \quad (3.6)$$

and, clearly,

$$\begin{aligned}
 \|v_\epsilon\|_\epsilon^2 &= \epsilon^{\frac{2s}{(s-1)(p-2)} + \frac{2s-N}{2s-2}} \left[ \int_{\Omega_\epsilon} |\nabla \tilde{u}_\epsilon|^2 dx + [\tilde{u}_\epsilon]_s^2 \right] \\
 &= \epsilon^{\frac{2N+p(2s-N)}{2(s-1)(p-2)}} \|\tilde{u}_\epsilon\|_s^2 \\
 &= \begin{cases} \tilde{S}_\epsilon^{-\frac{p}{2-p}} & \text{if } p = 2_s^*, \\ \epsilon^{\frac{2N+p(2s-N)}{2(s-1)(p-2)}} \tilde{S}_\epsilon^{-\frac{p}{2-p}} & \text{if } p > 2_s^*, \end{cases} \\
 &= S_\epsilon^{-\frac{p}{2-p}} \quad \text{for } p \geq 2_s^*.
 \end{aligned} \tag{3.7}$$

By (3.5), (3.6) and (3.7), we have

$$P_\epsilon(v_\epsilon) = S_\epsilon \quad \text{for } p \geq 2_s^*, \tag{3.8}$$

and thus  $v_\epsilon$  is a minimizer of  $S_\epsilon$  and also a ground state of (1.3), with  $J(v_\epsilon) = \frac{p-2}{2p} S_\epsilon^{\frac{p}{p-2}}$ .

### 3.2.1 The case $p = 2_s^*$

**Lemma 26.** For  $p = 2_s^*$ ,  $\tilde{S}_\epsilon \rightarrow \tilde{S}$ , where  $\tilde{S}$  is the Sobolev constant

$$\tilde{S} := \inf_{u \in D^{s,2}(\mathbb{R}^N) \setminus \{0\}} \frac{[u]_{s,\mathbb{R}^N}^2}{\|u\|_{L^{2_s^*}(\mathbb{R}^N)}^2},$$

or equivalently

$$\tilde{S} := \inf \left\{ [u]_{s,\mathbb{R}^N}^2 : \|u\|_{L^{2_s^*}(\mathbb{R}^N)} = 1 \right\}.$$

*Proof.* It is well-known that  $\tilde{S}$  is achieved by the function

$$U(x) = C_{N,s}(1 + |x|^2)^{\frac{2s-N}{2}},$$

where  $C_{N,s} = 2^{\frac{N-2s}{2}} [\Gamma((N+2s)/2)\Gamma((N-2s)/2)^{-1}]^{\frac{N-2s}{4s}}$ . Direct calculations show that  $\|U\|_{2_s^*} = 1$ . Define

$$U_\epsilon(x) = \epsilon^{\frac{2s-N}{8(s-1)}} U(\epsilon^{\frac{1}{4-4s}} x), \quad \phi_\epsilon(x) = \phi(\epsilon^{\frac{1}{2-2s}} x),$$

where  $\phi \in C_0^\infty(\mathbb{R}^N)$  satisfies  $\text{supp}(\phi) \subset \Omega$ ,  $\phi = 1$  in  $\frac{1}{2}\Omega \Subset \Omega$  with  $\text{diam}(\frac{1}{2}\Omega) = \frac{1}{2}\text{diam}(\Omega)$ ,  $0 \leq \phi \leq 1$ ,  $|\nabla \phi| \leq \frac{C}{\text{diam}(\Omega)}$ . Setting  $v_\epsilon(x) = U_\epsilon(x)\phi_\epsilon(x)$  and  $\hat{v}_\epsilon(x) = \frac{v_\epsilon(x)}{\|v_\epsilon\|_{2_s^*}}$  for all  $x \in \Omega_\epsilon$ , then  $\|\hat{v}_\epsilon\|_{2_s^*} = 1$  and, by definition,

$$\tilde{S}_\epsilon \leq \|\hat{v}_\epsilon\|_{2_s^*}^2.$$

Now, we deduce that

$$\begin{aligned}
 \lim_{\epsilon \rightarrow 0} \int_{\Omega_\epsilon} v_\epsilon^{2_s^*} dx &= \lim_{\epsilon \rightarrow 0} \epsilon^{\frac{N}{4(1-s)}} \int_{\Omega_\epsilon} U^{2_s^*}(\epsilon^{\frac{1}{4-4s}} x) \phi^{2_s^*}(\epsilon^{\frac{1}{2-2s}} x) dx \\
 &= \lim_{\epsilon \rightarrow 0} \int_{\epsilon^{\frac{1}{4s-4}} \Omega} U^{2_s^*}(x) \phi^{2_s^*}(\epsilon^{\frac{1}{4-4s}} x) dx \\
 &= \lim_{\epsilon \rightarrow 0} \int_{\frac{1}{2}\epsilon^{\frac{1}{4s-4}} \Omega} U^{2_s^*}(x) dx + \lim_{\epsilon \rightarrow 0} \int_{\epsilon^{\frac{1}{4s-4}} \Omega \setminus \frac{1}{2}\epsilon^{\frac{1}{4s-4}} \Omega} U^{2_s^*}(x) \phi^{2_s^*}(\epsilon^{\frac{1}{4-4s}} x) dx \\
 &= \lim_{\epsilon \rightarrow 0} \int_{\frac{1}{2}\epsilon^{\frac{1}{4s-4}} \Omega} U^{2_s^*}(x) dx + C_{N,s}^{2_s^*} \lim_{\epsilon \rightarrow 0} \epsilon^{\frac{N}{4-4s}} \int_{\Omega \setminus \frac{1}{2}\Omega} (\epsilon^{\frac{1}{2-2s}} + |x|^2)^{-N} \phi^{2_s^*}(x) dx \\
 &= \int_{\mathbb{R}^N} U^{2_s^*}(x) dx = 1.
 \end{aligned} \tag{3.9}$$

Next, one has

$$\begin{aligned} \int_{\Omega_\epsilon} |\nabla \widehat{v}_\epsilon|^2 dx &= \frac{1}{\|v_\epsilon\|_{2_s^*}^2} \int_{\Omega_\epsilon} |\nabla v_\epsilon|^2 dx \\ &\leq \frac{2}{\|v_\epsilon\|_{2_s^*}^2} \left[ \int_{\Omega_\epsilon} |\nabla U_\epsilon|^2 \phi_\epsilon^2(x) dx + \int_{\Omega_\epsilon} |\nabla \phi_\epsilon|^2 U_\epsilon^2(x) dx \right]. \end{aligned} \quad (3.10)$$

By direct calculations, we also have

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{\Omega_\epsilon} |\nabla U_\epsilon|^2 \phi_\epsilon^2(x) dx &= \lim_{\epsilon \rightarrow 0} \epsilon^{\frac{1}{2}} \int_{\epsilon^{\frac{1}{4s-4}} \Omega} |\nabla U|^2 \phi^2(\epsilon^{\frac{1}{4s-4}} x) dx \\ &\leq \lim_{\epsilon \rightarrow 0} \epsilon^{\frac{1}{2}} \int_{\mathbb{R}^N} |\nabla U|^2 dx = 0, \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{\Omega_\epsilon} |\nabla \phi_\epsilon|^2 U_\epsilon^2(x) dx &= \lim_{\epsilon \rightarrow 0} \epsilon^{\frac{4-2s-N}{4(1-s)}} \int_{\Omega} |\nabla \phi|^2 U^2(\epsilon^{\frac{1}{4s-4}} x) dx \\ &= C_{N,s}^2 \lim_{\epsilon \rightarrow 0} \epsilon^{\frac{4-6s+N}{4(1-s)}} \int_{\Omega} |\nabla \phi|^2 (\epsilon^{\frac{1}{2-2s}} + |x|^2)^{2s-N} dx \\ &= 0, \end{aligned} \quad (3.12)$$

where we used the facts that  $\|\nabla U\|_2 < \infty$  and  $N > 2s$ .

By combining (3.9), (3.13), (3.11) and (3.12), we deduce

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega_\epsilon} |\nabla \widehat{v}_\epsilon|^2 dx = 0. \quad (3.13)$$

Finally, let us estimate the term  $[v_\epsilon]_s^2$ . To this end, we rewrite it as follows

$$[v_\epsilon]_s^2 = I_\epsilon^1 + I_\epsilon^2 + I_\epsilon^3, \quad (3.14)$$

where

$$\begin{aligned} I_\epsilon^1 &:= \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|U_\epsilon(x) - U_\epsilon(y)|^2}{|x - y|^{N+2s}} \phi_\epsilon^2(x) dy dx, \\ I_\epsilon^2 &:= \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\phi_\epsilon(x) - \phi_\epsilon(y)|^2}{|x - y|^{N+2s}} U_\epsilon^2(y) dy dx, \\ I_\epsilon^3 &:= \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(\phi_\epsilon(x) - U_\epsilon(y))(\phi_\epsilon(x) - \phi_\epsilon(y))U_\epsilon(y)\phi_\epsilon(x)}{|x - y|^{N+2s}} dy dx. \end{aligned}$$

Then, as in the proof of [2, Theorem A.1], we get

$$\lim_{\epsilon \rightarrow 0} I_\epsilon^1 = \widetilde{S}, \quad \lim_{\epsilon \rightarrow 0} I_\epsilon^2 = \lim_{\epsilon \rightarrow 0} I_\epsilon^3 = 0. \quad (3.15)$$

Thus, from (3.9), (3.14) and (3.15), we obtain

$$\lim_{\epsilon \rightarrow 0} [\widehat{v}_\epsilon]_s^2 = \lim_{\epsilon \rightarrow 0} [v_\epsilon]_s^2 = \widetilde{S} \quad (3.16)$$

and therefore

$$\limsup_{\epsilon \rightarrow 0} \widetilde{S}_\epsilon \leq \lim_{\epsilon \rightarrow 0} [\widehat{v}_\epsilon]_s^2 = \widetilde{S}. \quad (3.17)$$

We have

$$\tilde{S} \leq \liminf_{\epsilon \rightarrow 0} \tilde{S}_\epsilon. \quad (3.18)$$

By definition of  $\tilde{S}_\epsilon$ , for any  $\delta > 0$ , there exists  $w_{\epsilon,\delta} \in X_0^S(\Omega_\epsilon)$  with  $\|w_{\epsilon,\delta}\|_{L^p(\Omega_\epsilon)} = 1$  such that

$$\|w_{\epsilon,\delta}\|_S^2 < \tilde{S}_\epsilon + \delta. \quad (3.19)$$

Let  $\eta$  be the standard mollifier function,  $\eta_\sigma(x) = \sigma^{-N} \eta(\frac{x}{\sigma})$ . Define  $w_{\epsilon,\delta}^\sigma = w_{\epsilon,\delta} \star \eta_\sigma$  and  $v_{\epsilon,\sigma}^\sigma = \frac{w_{\epsilon,\delta}^\sigma}{\|w_{\epsilon,\delta}^\sigma\|_{L^{2_s^*}(\mathbb{R}^N)}}$ . Then  $v_{\epsilon,\sigma}^\sigma \rightarrow w_{\epsilon,\delta}$  in  $D^{s,2}(\mathbb{R}^N)$  as  $\sigma \rightarrow 0$ . Thus, we have

$$\tilde{S} \leq [v_{\epsilon,\sigma}^\sigma]_{S,\mathbb{R}^N}^2 \rightarrow [w_{\epsilon,\delta}]_{S,\mathbb{R}^N}^2, \quad \text{as } \sigma \rightarrow 0. \quad (3.20)$$

This gives

$$\tilde{S} < \tilde{S}_\epsilon + \delta$$

and the arbitrariness of  $\delta$  yields (3.18). This ends the proof.  $\square$

Combining (3.6) and (3.7), one has

**Lemma 27.**  $\lim_{\epsilon \rightarrow 0} \|v_\epsilon\|_\epsilon^2 = \lim_{\epsilon \rightarrow 0} \|v_\epsilon\|_{2_s^*}^2 = \tilde{S}^{-\frac{2_s^*}{2-2_s^*}}.$

### 3.2.2 The case $2_s^* < p < 2^*$

Now, we consider the minimization problem

$$\bar{S} = \inf \left\{ I(u) : u \in D^{s,2}(\mathbb{R}^N) \cap D^{1,2}(\mathbb{R}^N) : \|u\|_p = 1 \right\} \quad (3.21)$$

with  $2_s^* < p < 2^*$ , where

$$I(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy. \quad (3.22)$$

**Lemma 28.**  $\bar{S}$  is attained.

*Proof.* Let  $u_n \in D^{s,2}(\mathbb{R}^N) \cap D^{1,2}(\mathbb{R}^N)$  be a minimizing sequence with  $\|u_n\|_p = 1$  such that

$$\lim_{n \rightarrow \infty} I(u_n) = \bar{S}.$$

Thus,  $\{u_n\}$  is bounded in  $D^{s,2}(\mathbb{R}^N) \cap D^{1,2}(\mathbb{R}^N)$  and there exists  $u \in D^{s,2}(\mathbb{R}^N) \cap D^{1,2}(\mathbb{R}^N)$  such that  $u_n \rightarrow u$  in  $D^{s,2}(\mathbb{R}^N) \cap D^{1,2}(\mathbb{R}^N)$  and  $L^p(\mathbb{R}^N)$  with  $2_s^* < p < 2^*$ .

By means of symmetric rearrangement techniques, we may assume that  $u_n$  is radially symmetric (cf. [20]). Thus, we get  $u_n \rightarrow u$  in  $L^p(\mathbb{R}^N)$ , which yields that  $\|u\|_p = 1$ .  $\square$

It is easy to check that  $u$  is radially symmetric. Moreover, applying the Lagrange multiplier rule,  $u$  satisfies

$$(-\Delta)^s u - \Delta u = \mu |u|^{p-2} u \quad \text{in } \mathbb{R}^N \quad (3.23)$$

for some  $\mu > 0$ .

**Lemma 29.** For  $2_s^* < p < 2^*$ , then  $\lim_{\epsilon \rightarrow 0} \tilde{S}_\epsilon = \bar{S}$ .

*Proof.* Let  $w$  be a minimizer of  $\bar{S}$  and set  $w_\epsilon(x) = w(x)\phi_\epsilon(x)$ ,  $\widehat{w}_\epsilon(x) = \frac{w_\epsilon}{\|w_\epsilon\|_p}$  for all  $x \in \Omega_\epsilon$ . Then  $\|\widehat{w}_\epsilon\|_p = 1$  and by definition,

$$\tilde{S}_\epsilon \leq \|\widehat{w}_\epsilon\|_S^2.$$



Now,

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega_\epsilon} |w_\epsilon|^p dx = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} |w\phi_\epsilon|^p dx - \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N \setminus \Omega_\epsilon} |w\phi_\epsilon|^p dx = \|w\|_p^p = 1. \quad (3.24)$$

and

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{\Omega_\epsilon} |\nabla w_\epsilon|^2 dx &= \lim_{\epsilon \rightarrow 0} \left[ \int_{\Omega_\epsilon} |\nabla w|^2 \phi_\epsilon^2 dx + 2 \int_{\Omega_\epsilon} \nabla \phi_\epsilon \nabla w dx + \int_{\Omega_\epsilon} |\nabla \phi_\epsilon|^2 w^2 dx \right] \\ &= \int_{\mathbb{R}^N} |\nabla w|^2 dx. \end{aligned} \quad (3.25)$$

Similarly to the proof of (3.14), (3.15) and (3.16), we deduce that

$$\lim_{\epsilon \rightarrow 0} [w_\epsilon]_s^2 = [w]_s^2. \quad (3.26)$$

Thus, we have

$$\limsup_{\epsilon \rightarrow 0} \tilde{S}_\epsilon \leq \limsup_{\epsilon \rightarrow 0} \|\widehat{w}_\epsilon\|_2^2 = \lim_{\epsilon \rightarrow 0} \frac{\|\nabla w_\epsilon\|_2^2 + [w_\epsilon]_{s, \mathbb{R}^N}^2}{\|w_\epsilon\|_p^2} = \bar{S}, \quad (3.27)$$

while the inequality  $\bar{S} \leq \liminf_{\epsilon \rightarrow 0} \tilde{S}_\epsilon$  can be obtained by the same arguments as Lemma 26.  $\square$

From (3.5), (3.6) and (3.7), one has

**Lemma 30.** For  $2_s^* < p < 2^*$ ,  $\lim_{\epsilon \rightarrow 0} S_\epsilon = \lim_{\epsilon \rightarrow 0} \|v_\epsilon\|_\epsilon = \lim_{\epsilon \rightarrow 0} \|v_\epsilon\|_p = 0$ .

### 3.3 The case $\epsilon \rightarrow +\infty$

Setting  $v_\epsilon(x) = \epsilon^{-\frac{1}{p-2}} u_\epsilon(x)$ , then  $v_\epsilon$  satisfies

$$\begin{cases} \epsilon^{-1}(-\Delta)^s v - \Delta v = |v|^{p-2}v & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

and is a minimizer of

$$S_\epsilon := \inf\{Q_\epsilon(v) : \|v\|_p = 1\},$$

or, equivalently, of

$$S_\epsilon = \inf_{v \in X_0^s \setminus \{0\}} \frac{Q_\epsilon(v)}{\|v\|_p^2},$$

where

$$Q_\epsilon(v) = \|\nabla v\|_2^2 + \epsilon^{-1}[v]_s^2.$$

Then,  $S_\epsilon = \epsilon^{-1}S_\epsilon$ . We also define

$$\widehat{S} = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_p^2} \quad \text{for } 2 < p < 2^*. \quad (3.28)$$

It is well known that  $\widehat{S}$  is achieved by a positive classical minimizer.

**Lemma 31.**  $\lim_{\epsilon \rightarrow +\infty} S_\epsilon = \widehat{S}$ .

*Proof.* It closely follows the one of Lemma 24. Since  $X_0^s \subset H_0^1$ , we get

$$\widehat{S} \leq \inf_{v \in X_0^s \setminus \{0\}} \frac{\|\nabla v\|_2^2}{\|v\|_p^2} \leq S_\epsilon,$$

which yields that

$$\widehat{S} \leq \liminf_{\epsilon \rightarrow +\infty} S_\epsilon.$$

Assume that  $w_0$  is a minimizer of  $\widehat{S}$ . By density, there exists a sequence  $\{w_n\} \subseteq C_0^\infty(\Omega)$  such that  $w_n \rightarrow w_0$  in  $H_0^1(\Omega)$  and  $L^p(\Omega)$ . Thus, we have

$$S_\epsilon \leq \frac{\|\nabla w_n\|_2^2 + \epsilon^{-1}[w_n]_s^2}{\|w_n\|_p^2}. \quad (3.29)$$

Then,

$$\limsup_{\epsilon \rightarrow +\infty} S_\epsilon \leq \lim_{n \rightarrow \infty} \limsup_{\epsilon \rightarrow +\infty} \frac{\|\nabla w_n\|_2^2 + \epsilon^{-1}[w_n]_s^2}{\|w_n\|_p^2} = \lim_{n \rightarrow \infty} \frac{\|\nabla w_n\|_2^2}{\|w_n\|_p^2} = \widehat{S}, \quad (3.30)$$

and this completes the proof.  $\square$

**Lemma 32.**  $\lim_{\epsilon \rightarrow +\infty} S_\epsilon = 0$  and  $\lim_{\epsilon \rightarrow +\infty} \|u_\epsilon\|_\epsilon = +\infty$ .

*Proof.* This is a direct consequence of Lemma 31, as

$$\|u_\epsilon\|_\epsilon^2 = \epsilon \|\nabla u_\epsilon\|_2^2 + [u_\epsilon]_s^2 = \epsilon^{\frac{p}{p-2}} Q_\epsilon(v_\epsilon) = \epsilon^{\frac{p}{p-2}} S_\epsilon. \quad \square$$

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