Research Article

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Single peaked traveling wave solutions to a generalized μ-Novikov Equation

Abstract: In this paper, we study the existence of peaked traveling wave solution of the generalized μ-Novikov equation with nonlocal cubic and quadratic nonlinearities. The equation is a μ-version of a linear combination of the Novikov equation and Camassa-Hom equation. It is found that the equation admits single peaked traveling wave solutions.

Keywords: generalized μ-Novikov equation, Integrable system, peakons

MSC: 35Q35, 37K45

1 Introduction

We consider the following partial differential equation

$$m_t + k_1(u^2 m_x + 3uu_x m) + k_2(2mu_x + um_x) = 0, \tag{1.1}$$

where $u(t, x)$ is a function of time t and a single spatial variable x, and

$$m := \mu(u) - u_{xx}, \quad \mu(u) := \int_{\mathbb{S}} u(t, x)dx,$$

with $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ which denotes the unit circle on $\mathbb{R}^2$. Equation (1.1) can be reduced as μ-Novikov equation [39]

$$m_t + u^2 m_x + 3uu_x m = 0, \quad m = \mu(u) - u_{xx}, \tag{1.2}$$

for $k_1 = 1$ and $k_2 = 0$, and the μ-Camassa-Holm equation [28]

$$m_t + 2mu_x + um_x = 0, \quad m = \mu(u) - u_{xx}, \tag{1.3}$$

for $k_1 = 0$ and $k_2 = 1$, respectively.

It is known that the Camassa-Holm equation of the following form [2, 20]

$$m_t + 2mu_x + um_x = 0, \quad m = u - u_{xx}, \tag{1.4}$$

was proposed as a model for the unidirectional propagation of the shallow water waves over a flat bottom (see also [14, 25]), with $u(x, t)$ representing the height of the water's free surface in terms of non-dimensional variables. The Camassa-Holm equation (1.4) is completely integrable with a bi-Hamiltonian structure and an infinite number of conservation laws [2, 20], and can be solved by the inverse scattering method [5, 6, 30]. It is of interest to note that the Camassa-Holm equation (1.4) can also be derived by tri-Hamitonian duality

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from the Korteweg-de Vries equation (a number of additional examples of dual integrable systems derived applying the method of tri-Hamitonian duality can be found in [21, 42]). The Camassa-Holm equation (1.4) has two remarkable features: existence of peakon and multi-peakons [1–3] and breaking waves, i.e., the wave profile remains bounded while its slope becomes unbounded in finite time [7, 8, 10–12, 33]. Those peaked solitons were proved to be orbitally stable in the energy space [15, 16] and to be asymptotically stable under the Camassa-Holm flow [38] (see also [26, 27] for other equations). It is worth noting that solutions of this type are not mere abstractizations: the peakons replicate a feature that is characteristic for the waves of great height-waves of largest amplitude that are exact solutions of the governing equations for irrotational water waves [9, 13, 48]. Geometrically, the Camassa-Holm equation (1.4) describes the geodesic flows on the Bott-Virasoro group [37, 47] and on the diffeomorphism group of the unit circle under H^1 metric [29], respectively. The Camassa-Holm equation (1.4) also arises from a non-stretching invariant planar curve flow in the centro-equiaffine geometry [4, 41]. Well-posedness and wave breaking of the Camassa-Holm equation (1.4) were studied extensively, and many interesting results have been obtained, see [7, 10–12, 33], for example. The μ-Camassa-Holm equation (1.3) was originally proposed as the model for the evolution of rotators in liquid crystals with an external magnetic field and self interatction [28]. It is interesting to note that this equation is integrable in the sense that it admits the Lax-pair and bi-Hamiltonian structure, and also describes a geodesic flow on the diffeomorphism group of $\mathbb{S}$ with $H^\mu(\mathbb{S})$ metric (which is equivalent to $H^1(\mathbb{S})$ metric). Its integrability, well-posedness, blow-up and peakons were discussed in [19, 28].

It is observed that all nonlinear terms in the Camassa-Holm equation (1.4) are quadratic. In contrast to the integrable modified Korteweg-de Vries equation with a cubic nonlinearity, it is of great interest to find integrable Camassa-Holm type equations with cubic or higher-order nonlinearity admitting peakon solitons. Recently, two integrable Camassa-Holm type equtions with cubic nonlinearities have been appeared in literature. One was introduced by Olver and Rosenau [42](called the modified Camassa-Holm equation, see also [18, 21]) by using the tri-Hamiltonian duality approach, which takes the form

$$m_t + [(u^2 - u_x^2)m]_x = 0, \quad m = u - u_{xx}. \tag{1.5}$$

It was shown that the modified Camassa-Holm equation is integrable with the Lax-pair and the bi-Hamiltonian structure. It has single and multi-peaked traveling waves with a different character than of the Camassa-Holm equation (1.4) [22], and it also has new features of blow-up criterion and wave breaking mechanism. The issue of the stability of peakons for the modified Camassa-Holm equation were investigated in [46]. Like μ-Camassa-Holm equation (1.3), μ-version of the modified Camassa-Holm equation

$$m_t + [(2u\mu(u) - u_x^2)m]_x = 0, \quad m = \mu(u) - u_{xx} \tag{1.6}$$

was introduced in [44]. Its integrability, wave breaking, existence of peaked traveling waves and their stability were discussed in [34, 44]. The second one is the Novikov equation

$$m_t + u^2 m_x + 3uu_x m = 0, \quad m = u - u_{xx}, \tag{1.7}$$

which is integrable with the Lax pair [40]. A matrix Lax pair reprsentation to the Novikov equation was founded in [23]. It is also noticed that the Novikov equation admits a bi-Hamiltonian structure [23]. Existence of peaked solitons and multi-peakons for Novikov equation were obtained in [24, 40]. Orbital stability of the peaked solitons to the Novikov equation were discussed in [35]. The μ-Novikov equation (1.2), regarded as a μ-version of the Novikov equation, was introduced first in [39]. The existence of its single peakons was established in [39].

More recently, the following generalized μ-Camassa-Holm equation

$$m_t + k_1((2\mu(u)u - u_x^2)m)_x + k_2(2mu_x + um_x) = 0, \quad m = \mu(u) - u_{xx} \tag{1.8}$$

was proposed in [45] as a μ-version of the generalized Camassa-Holm equation with quadradic and cubic nonlinearities

$$m_t + k_1((u^2 - u_x^2)m)_x + k_2(2mu_x + um_x) = 0, \quad m = u - u_{xx} \tag{1.9}$$

which was derived by Fokas [18] from the hydrodynamical wave, and can also obtained using the approach of tri-Hamiltonian duality [21, 42] to the bi-Hamiltonian Gardner equation

$$u_t + u_{xxx} + k_1 u^2 u_x + k_2 u u_x = 0. \tag{1.10}$$

Note that the Lax pair of equation (1.9) was obtained in [43]. It was shown in [45] that a scale limit of equation (1.8) yields the following integrable equation

$$v_{xt} - k_1 v_x^2 v_{xx} + k_2 \left(v v_{xx} + \frac{1}{2} v_x^2 \right) = 0, \tag{1.11}$$

which describes asymptotic dynamics of a short capillarty-gravity wave [17], where $v(t, x)$ denotes the fluid velocity on the surface. Notably, the generalized μ-Camassa-Holm equation (1.8) can be regarded as the integrable model that, in a sense, lies midway between equation (1.9) and its limiting version equation (1.11). It has been known that the generalized μ-Camassa-Holm equation (1.8) is formally integrable in the sense that it admits Lax formulation and bi-Hamiltonian form [45].

The existence of periodic peakons is of interest for nonlinear integrable equations because they are relatively new solitary waves (for most models the solitary waves are quite smooth). Applying the method of tri-Hamiltonian duality[21, 42] to the bi-Hamiltonian representation of the Korteweg-de Vries (KdV), modified Korteweg-de Vries (mKdV), and Gardner equation, the resulting dual systems, such as Camassa-Holm equation (1.4), the modified Camassa-Holm equation (1.5), and the generalized Camassa-Holm equation (1.9), exhibit nonlinear dispersion, and, in most cases, admit a remarkable variety of non-smooth soliton-like solutions, including peakons, compactons, tipons, rampons, mesaons, and so on [32]. It is known that Camassa-Holm equation (1.4), the modified Camassa-Holm equation (1.5), Novikov equation (1.7), and the generalized Camassa-Holm equation (1.9) [2, 22, 36, 40, 43] admit single peakons of the form

$$u(t, x) = \varphi_c(x - ct) = a e^{-|x - ct|}, \tag{1.12}$$

where the amplitude a is given by $c, \sqrt{3c/2}, \sqrt{c}$, and

$$\frac{-3k_2 \pm 3\sqrt{k_2^2 + \frac{8}{3} c k_1}}{4k_1} \quad \text{with} \quad k_2^2 + \frac{8}{3} c k_1 \geq 0 (k_1 \neq 0),$$

for the Camassa-Holm equation, the modified Camassa-Holm equation, Novikov equation, and the generalized Camassa-Holm equation, respectively. Their corresponding periodic peakons take the form

$$u(t, x) = \varphi_c(x - ct) = a \frac{\cosh(x - ct - [x - ct] - \frac{1}{2})}{\cosh(\frac{1}{2})}, \tag{1.13}$$

where the amplitude a is also given by $c, \sqrt{3c} \cosh(\frac{1}{2}) / \sqrt{(1 + 2 \cosh^2(\frac{1}{2}))}, \sqrt{c}$, and

$$\frac{-3k_2 \cosh(\frac{1}{2}) \pm 3\sqrt{k_2^2 \cosh^2(\frac{1}{2}) + \frac{4}{3} k_1 c (1 + 2 \cosh^2(\frac{1}{2}))}}{2k_1 (1 + 2 \cosh^2(\frac{1}{2}))}$$

with

$$k_2^2 \cosh^2(\frac{1}{2}) + \frac{4}{3} k_1 c (1 + 2 \cosh^2(\frac{1}{2})) \geq 0,$$

for the Camassa-Holm equation, the modified Camassa-Holm equation, Novikov equation, and the generalized Camassa-Holm equation, respectively.

It is worth noting that the periodic peakons of the μ-integrable equation are of a manifestly different character. For example, in [28, 31, 39, 44], the authors showed that the μ-Camassa-Holm equatioin (1.3), μ-Novikov equation (1.2), modified μ-Camassa-Holm equation (1.6), and generalized μ-Camassa Holm equation (1.8) admit periodic peakons of the following form

$$u(t, x) = \varphi_c(x - ct) = a \varphi(x - ct), \tag{1.14}$$

where

$$\varphi(x) = \frac{1}{2}\left(x^2 + \frac{23}{24}\right), \quad x \in \left[-\frac{1}{2}, \frac{1}{2}\right], \tag{1.15}$$

and φ is extended periodically to the real line, the constant a takes value $\frac{12c}{13}, \frac{12\sqrt{c}}{13}, \frac{2\sqrt{3c}}{5}$ and

$$\frac{-13k_2 \pm \sqrt{169k_2^2 + 1200ck_1}}{50k_1} \quad \text{with} \quad 169k_x^2 + 1200ck_1 \geq 0,$$

respectively, for the μ-Camassa-Holm equation, μ-Novikov equation, modified μ-Camassa-Holm equation, and generalized μ-Camassa-Holm equation.

Motivated by the recent work [28, 44, 45], the aim of this paper is to investigate the existence of periodic peaked solution of the generalized μ-Novikov equation (1.1). Indeed, in Section 2, we give a short review on the notion of a strong and weak solution of the generalized μ-Novikov equation (1.1) and then show that equation (1.1) admits the periodic peakon, which is given by (1.15) with a replaced by

$$\frac{-6k_2 \pm 6\sqrt{k_2^2 + 4ck_1}}{13k_1} \tag{1.16}$$

where the wave speed c satisfies $k_2^2 + 4ck_1 \geq 0$.

2 Peaked Traveling Waves

We first introduce the initial value problem of Equation (1.1) on the unit circle $\mathbb{S}$, that is

$$\begin{cases} m_t + k_1(u^2 m_x + 3uu_x m) + k_2(2mu_x + um_x) = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), \quad m := \mu(u) - u_{xx}, \quad x \in \mathbb{R}, \\ u(t, x + 1) = u(t, x), \quad t \geq 0, \quad x \in \mathbb{R}. \end{cases} \tag{2.1}$$

We then formalize the notion of a strong (or classical) and weak solutions of the Equation (1.1) used throughout this paper.

Definition 2.1. *If* $u \in C([0, T), H^s(\mathbb{S})) \cap C^1([0, T), H^{s-1}(\mathbb{S}))$ *with* $s > \frac{5}{2}$ *and some* $T > 0$ *satisfies (2.1), then* u *is called a strong solution on* $[0, T)$. *If* u *is a strong solution on* $[0, T)$ *for every* $T > 0$, *then it is called a global strong solution.*

Note that the inverse operator $(\mu - \partial_x^2)^{-1}$ can be obtained by convolution with the corresponding Green's function, so that

$$u = (\mu - \partial_x^2)^{-1}m = g \star m, \tag{2.2}$$

where g is given by [28]

$$g(x) := \frac{1}{2}\left(x - [x] - \frac{1}{2}\right)^2 + \frac{23}{24}. \tag{2.3}$$

Here $[x]$ denote the greatest integer for $x \in [-\frac{1}{2}, \frac{1}{2}]$. Its derivative at $x = 0$ can be assigned to zero, so one has [31]

$$g_x(x) := \begin{cases} 0, & x = 0 \\ x - \frac{1}{2}, & 0 < x < 1. \end{cases} \tag{2.4}$$

Plugging the formula for $m := \mu(u) - u_{xx}$ in terms of u into Equation (1.1) results in the following fully nonlinear partial differential equation:

$$u_t + k_1 \left[u^2 u_x + \frac{3}{2}(\mu - \partial_x^2)^{-1} \partial_x \left(\mu(u)u^2 + uu_x^2 \right) + \frac{1}{2}(\mu - \partial_x^2)^{-1} \left(u_x^3 \right) \right] \tag{2.5}$$

$$+k_2 \left[uu_x + (\mu - \partial_x^2)^{-1} \partial_x (2u\mu(u) + \frac{1}{2}u_x^2) \right] = 0.$$

The formulation (2.5) allows us to define the notion of a weak solutions as follows.

Definition 2.2. *Given the initial data $u_0 \in W^{1,3}(\mathbb{S})$, the function $u \in L^\infty([0, T); W^{1,3}(\mathbb{S}))$ is said to be a weak solution to (2.1) if it satisfies the following identity:*

$$\int_0^T \int_{\mathbb{S}} \left[u\psi_t + k_1 \left(\frac{1}{3}u^3 \psi_x - \frac{3}{2} g_x \star \left(\mu(u)u^2 + uu_x^2 \right) \psi - \frac{1}{2}(g \star u_x^3)\psi \right) \right.$$

$$\left. +k_2 \left(\frac{1}{2}u^2 \psi_x - g_x \star (2u\mu(u) + \frac{1}{2}u_x^2)\psi \right) \right] dxdt + \int_{\mathbb{S}} u_0(x)\psi(0, x)dx = 0,$$

for any smooth test function $\psi(t, x) \in C_c^\infty([0, T) \times \mathbb{S})$. If u is a weak solution on $[0, T)$ for every $T > 0$, then it is called a global weak solution.

Our main theorem is in the following.

Theorem 2.1. *For any $c \geq -\frac{k_2^2}{4k_1}$, Equation (1.1) admits the peaked periodic-one traveling wave solution $u_c = \phi_c(\xi)$, $\xi = x - ct$, where $\phi_c(\xi)$ is given by*

$$\phi_c(\xi) = a \left[\frac{1}{2} \left(\xi - \frac{1}{2} \right)^2 + \frac{23}{24} \right], \quad \xi \in \left[-\frac{1}{2}, \frac{1}{2} \right], \tag{2.6}$$

where the amplitude

$$a = \begin{cases} \dfrac{-6k_2 \pm 6\sqrt{k_2^2 + 4ck_1}}{13k_1}, & k_1 \neq 0 \\ \\ \dfrac{12c}{13k_2}, & k_1 = 0, k_2 \neq 0 \end{cases} \tag{2.7}$$

and $\phi_c(\xi)$ is extended periodically to the real line with period one.

Proof. Inspired by the forms of periodic peakons for the μ-CH equation [28](See also [44, 45]), we assume that the peaked periodic traveling wave of Equation (1.1) is given by

$$u_c(t, x) = a \left[\frac{1}{2} \left(\xi - [\xi] - \frac{1}{2} \right)^2 + \frac{23}{24} \right].$$

According to Definition 2.2 it is found that $u_c(t, x)$ satisfies the following equation

$$\sum_{j=1}^6 I_j := \int_0^T \int_{\mathbb{S}} u_{c,t}\psi dxdt + k_1 \int_0^T \int_{\mathbb{S}} u_c^2 u_{c,x}\psi dxdt + \frac{3}{2}k_1 \int_0^T \int_{\mathbb{S}} g_x \star (\mu(u_c)u_c^2 + u_c u_{c,x}^2)\psi dxdt$$

$$+ \frac{1}{2}k_1 \int_0^T \int_{\mathbb{S}} g \star (u_{c,x}^3)\psi dxdt + k_2 \int_0^T \int_{\mathbb{S}} u_c u_{c,x}\psi dxdt$$

$$+ k_2 \int_0^T \int_{\mathbb{S}} g_x \star (2\mu(u_c)u_c + \frac{1}{2}u_{c,x}^2)\psi dxdt = 0, \tag{2.8}$$

for some $T > 0$ and every test function $\psi(t, x) \in C_c^\infty([0, T) \times \mathbb{S})$. For any $x \in \mathbb{S}$, one finds that

$$\mu(u_c) = a \int_0^{ct} \left[\frac{1}{2}\left(x - ct + \frac{1}{2}\right)^2 + \frac{23}{24} \right] dx + a \int_{ct}^1 \left[\frac{1}{2}\left(x - ct - \frac{1}{2}\right)^2 + \frac{23}{24} \right] dx = a.$$

To evaluate $I_j, j = 1, \cdots, 6$, we need to consider two cases: (i) $x > ct$, and (ii) $x \le ct$.
For $x > ct$, we have

$$\mu(u_c)u_c^2 + u_c u_{c,x}^2 = a^3 \left(\frac{3}{4}\left(\xi - \frac{1}{2}\right)^4 + \frac{23}{12}\left(\xi - \frac{1}{2}\right)^2 + \frac{529}{576} \right),$$

$$u_c^2 u_{c,x} = a^3 \left(\frac{1}{4}\left(\xi - \frac{1}{2}\right)^5 + \frac{23}{24}\left(\xi - \frac{1}{2}\right)^3 + \frac{529}{576}\left(\xi - \frac{1}{2}\right) \right),$$

$$2\mu(u_c)u_c + \frac{1}{2}u_{c,x}^2 = a^2 \left(\frac{3}{2}\left(\xi - \frac{1}{2}\right)^2 + \frac{23}{12} \right) \quad \text{and} \quad u_c u_{c,x} = a^2 \left(\frac{1}{2}\left(\xi - \frac{1}{2}\right)^3 + \frac{23}{24}\left(\xi - \frac{1}{2}\right) \right).$$

On the other hand,

$$\frac{3}{2}k_1 g_x \star \left(\mu(u_c)u_c^2 + u_c u_{c,x}^2 \right)$$

$$= \frac{3}{2}k_1 a^3 \int_{\mathbb{S}} \left(x - y - [x - y] - \frac{1}{2} \right) \left(\frac{3}{4}\left(y - ct - [y - ct] - \frac{1}{2}\right)^4 + \frac{23}{12}\left(y - ct - [y - ct] - \frac{1}{2}\right)^2 + \frac{529}{576} \right) dy$$

$$= \frac{3}{2}k_1 a^3 \int_0^{ct} \left(x - y - \frac{1}{2} \right) \left(\frac{3}{4}\left(y - ct + \frac{1}{2}\right)^4 + \frac{23}{12}\left(y - ct + \frac{1}{2}\right)^2 + \frac{529}{576} \right) dy$$

$$+ \frac{3}{2}k_1 a^3 \int_{ct}^x \left(x - y - \frac{1}{2} \right) \left(\frac{3}{4}\left(y - ct - \frac{1}{2}\right)^4 + \frac{23}{12}\left(y - ct - \frac{1}{2}\right)^2 + \frac{529}{576} \right) dy$$

$$+ \frac{3}{2}k_1 a^3 \int_x^1 \left(x - y + \frac{1}{2} \right) \left(\frac{3}{4}\left(y - ct - \frac{1}{2}\right)^4 + \frac{23}{12}\left(y - ct - \frac{1}{2}\right)^2 + \frac{529}{576} \right) dy$$

$$= k_1 a^3 \left(-\frac{9}{40}\left(\xi - \frac{1}{2}\right)^5 - \frac{23}{24}\left(\xi - \frac{1}{2}\right)^3 + \frac{487}{1920}\left(\xi - \frac{1}{2}\right) \right),$$

$$\frac{1}{2}k_1 g \star (u_{c,x}^3) = \frac{1}{2}a^3 \int_{\mathbb{S}} \left(\frac{1}{2}\left(x - y - [x - y] - \frac{1}{2}\right)^2 + \frac{23}{24} \right) \left(\left(y - ct - [y - ct] - \frac{1}{2}\right)^3 \right) dy$$

$$= \frac{1}{2}k_1 a^3 \int_0^{ct} \left(\frac{1}{2}\left(x - y - \frac{1}{2}\right)^2 + \frac{23}{24} \right) \left(\left(y - ct + \frac{1}{2}\right)^3 \right) dy$$

$$+ \frac{1}{2}k_1 a^3 \int_{ct}^x \left(\frac{1}{2}\left(x - y - \frac{1}{2}\right)^2 + \frac{23}{24} \right) \left(\left(y - ct - \frac{1}{2}\right)^3 \right) dy$$

$$+ \frac{1}{2}k_1 a^3 \int_x^1 \left(\frac{1}{2}\left(x - y + \frac{1}{2}\right)^2 + \frac{23}{24} \right) \left(\left(y - ct - \frac{1}{2}\right)^3 \right) dy$$

$$= k_1 a^3 \left(-\frac{1}{40}\left(\xi - \frac{1}{2}\right)^5 + \frac{1}{640}\left(\xi - \frac{1}{2}\right) \right),$$

and

$$k_2 g_x \star \left(2\mu(u_c)u_c + \frac{1}{2}u_{c,x}^2 \right)$$

$$= k_2 a^2 \int_{\mathbb{S}} \left(x - y - [x - y] - \frac{1}{2} \right) \left(\frac{3}{2} \left(y - ct - [y - ct] - \frac{1}{2} \right)^2 + \frac{23}{12} \right) dy$$

$$= k_2 a^2 \int_0^{ct} \left(x - y - \frac{1}{2} \right) \left(\frac{3}{2} \left(y - ct + \frac{1}{2} \right)^2 + \frac{23}{12} \right) dy$$

$$+ k_2 a^2 \int_{ct}^x \left(x - y - \frac{1}{2} \right) \left(\frac{3}{2} \left(y - ct - \frac{1}{2} \right)^2 + \frac{23}{12} \right) dy$$

$$+ k_2 a^2 \int_x^1 \left(x - y + \frac{1}{2} \right) \left(\frac{3}{2} \left(y - ct - \frac{1}{2} \right)^2 + \frac{23}{12} \right) dy$$

$$= k_2 a^2 \left(-\frac{1}{2} \left(\xi - \frac{1}{2} \right)^3 + \frac{1}{8} \left(\xi - \frac{1}{2} \right) \right).$$

It follows that

$$I_1 = \int_0^T \int_{\mathbb{S}} u_{c,t} \psi dx dt = -ca \int_0^T \int_{\mathbb{S}} \left(\xi - \frac{1}{2} \right) \psi(x,t) dx dt,$$

$$I_2 = k_1 a^3 \int_0^T \int_{\mathbb{S}} \left(\frac{1}{4} \left(\xi - \frac{1}{2} \right)^5 + \frac{23}{24} \left(\xi - \frac{1}{2} \right)^3 + \frac{529}{576} \left(\xi - \frac{1}{2} \right) \right) \psi(x,t) dx dt,$$

$$I_3 = k_1 a^3 \int_0^T \int_{\mathbb{S}} \left(-\frac{9}{40} \left(\xi - \frac{1}{2} \right)^5 - \frac{23}{24} \left(\xi - \frac{1}{2} \right)^3 + \frac{487}{1920} \left(\xi - \frac{1}{2} \right) \right) \psi(x,t) dx dt,$$

$$I_4 = k_1 a^3 \int_0^T \int_{\mathbb{S}} \left(-\frac{1}{40} \left(\xi - \frac{1}{2} \right)^5 + \frac{1}{640} \left(\xi - \frac{1}{2} \right) \right) \psi(x,t) dx dt,$$

$$I_5 = k_2 a^2 \int_0^T \int_{\mathbb{S}} \left(\frac{1}{2} \left(\xi - \frac{1}{2} \right)^3 + \frac{23}{24} \left(\xi - \frac{1}{2} \right) \right) \psi(x,t) dx dt,$$

$$I_6 = k_2 a^2 \int_0^T \int_{\mathbb{S}} \left(-\frac{1}{2} \left(\xi - \frac{1}{2} \right)^3 + \frac{1}{8} \left(\xi - \frac{1}{2} \right) \right) \psi(x,t) dx dt.$$

Plugging above expressions into (2.8), we deduce that for any $\psi(t,x) \in C_c^\infty([0,T] \times \mathbb{S})$

$$\sum_{j=1}^6 I_j = \int_0^T \int_{\mathbb{S}} a \left(\xi - \frac{1}{2} \right) \left(\frac{169}{144} k_1 a^2 + \frac{13}{12} k_2 a - c \right) \psi(t,x) dx dt.$$

A similar computation yields for $x \le ct$ that

$$\mu(u_c)u_c^2 + u_c u_{c,x}^2 = a^3 \left(\frac{3}{4} \left(\xi + \frac{1}{2} \right)^4 + \frac{23}{12} \left(\xi + \frac{1}{2} \right)^2 + \frac{529}{576} \right),$$

$$u_c^2 u_{c,x} = a^3 \left(\frac{1}{4} \left(\xi + \frac{1}{2} \right)^5 + \frac{23}{24} \left(\xi + \frac{1}{2} \right)^3 + \frac{529}{576} \left(\xi + \frac{1}{2} \right) \right),$$

$$2\mu(u_c)u_c + \frac{1}{2}u_{c,x}^2 = a^2 \left(\frac{3}{2} \left(\xi + \frac{1}{2} \right)^2 + \frac{23}{12} \right), \, u_c u_{c,x} = a^2 \left(\frac{1}{2} \left(\xi + \frac{1}{2} \right)^3 + \frac{23}{24} \left(\xi + \frac{1}{2} \right) \right),$$

and

$$\frac{3}{2}k_1 g_x \star \left(\mu(u_c)u_c^2 + u_c u_{c,x}^2\right) = k_1 a^3 \left(-\frac{9}{40}\left(\xi + \frac{1}{2}\right)^5 - \frac{23}{24}\left(\xi + \frac{1}{2}\right)^3 + \frac{487}{1920}\left(\xi + \frac{1}{2}\right)\right),$$

$$\frac{1}{2}k_1 g \star (u_{c,x}^3) = k_1 a^3 \left(-\frac{1}{40}\left(\xi + \frac{1}{2}\right)^5 + \frac{1}{640}\left(\xi + \frac{1}{2}\right)\right),$$

$$k_2 g_x \star \left(2\mu(u_c)u_c + \frac{1}{2}u_{c,x}^2\right) = k_2 a^2 \left(-\frac{1}{2}\left(\xi + \frac{1}{2}\right)^3 + \frac{1}{8}\left(\xi + \frac{1}{2}\right)\right).$$

This allows us to evaluate

$$\sum_{j=1}^{4} I_j = \int_0^T \int_{\mathbb{S}} \left(\xi + \frac{1}{2}\right)\left(-ac + \frac{169}{144}k_1 a^3\right)\psi(t,x)dxdt,$$

$$I_5 = k_2 a^2 \int_0^T \int_{\mathbb{S}} \left(\frac{1}{2}\left(\xi + \frac{1}{2}\right)^3 + \frac{23}{24}\left(\xi + \frac{1}{2}\right)\right)\psi(x,t)dxdt,$$

$$I_6 = k_2 a^2 \int_0^T \int_{\mathbb{S}} \left(-\frac{1}{2}\left(\xi + \frac{1}{2}\right)^3 + \frac{1}{8}\left(\xi + \frac{1}{2}\right)\right)\psi(x,t)dxdt.$$

Hence we arrive at

$$\sum_{j=1}^{6} I_j = \int_0^T \int_{\mathbb{S}} a\left(\xi + \frac{1}{2}\right)\left(\frac{169}{144}k_1 a^2 + \frac{13}{12}k_2 a - c\right)\psi(t,x)dxdt.$$

Since $\psi(t,x)$ is an arbitrary, both cases imply that the parameter a fulfills the equation

$$\frac{169}{144}k_1 a^2 + \frac{13}{12}k_2 a - c = 0.$$

Clearly, its solutions are given by which gives (2.7). Thus the theorem is proved. □

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